55
Geodesy

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55.1 Introduction

This chapter covers the basic mathematical and physical aspects of modeling the size and shape of the earth and its gravity field. Terrestrial and space geodetic measurement techniques are reviewed. Extra attention is paid to the relatively new technique of satellite surveying using the Global Positioning System (GPS). GPS surveying has not only revolutionized the art of navigation, but also brought about an efficient positioning technique for a variety of users, engineers not the least. It is safe to say that any geometry-based data collecting scheme profits in some sense from the full constellation of 24 GPS satellites. Except for the obvious applications in geodesy, surveying, and photogrammetry, the use of GPS is applied in civil engineering areas such as transportation (truck and emergency vehicle monitoring, intelligent vehicle and highway systems, etc.) and structures (monitoring of deformation of such structures as water dams). Even in areas such as forestry and agriculture (crop yield management), GPS provides the geometric backbone to the (geographic) information systems.

Modern geodetic measurement techniques, using signals from satellites orbiting the earth, necessitate a new look at the science of geodesy. Classical measurement techniques divided the theoretical problem of mapping small or large parts of the earth into a horizontal issue and a vertical issue. Three-dimensional measurement techniques "solve" the geodetic problem at once. However, careful interpretation of these three-dimensional results is still warranted, probably even more so than before. This chapter will center around this three-dimensional approach. Less attention has been devoted to classical issues such as the computation of a geodesic on an ellipsoid of revolution. Although this issue still has some importance, the reader is referred to the textbooks listed at the end of this chapter.

More than in classical texts, three-dimensional polar (spherical) coordinate representations are used, because the fundamental issues pertaining to various geodetic models are easier to illustrate by spherical coordinates than by ellipsoidal coordinates. Moreover, the increased influence of the satellite techniques in everyday surveying revives the use of three-dimensional polar coordinate representations, because the three-dimensional location of a point is equally accurately represented by Cartesian, spherical, or ellipsoidal coordinates.

Throughout this chapter all coordinate frames are treated as right-handed orthogonal trihedrals. Because this also applies to curvilinear coordinates, the well-known geographic coordinates of latitude and longitude are presented in the following order:

1. Longitude (positive in east direction), λ
2. Latitude, ψ or ϕ
3. Height, h

In short, $\{\lambda, \psi, h\}$ or $\{\lambda, \phi, h\}$. Local Cartesian and curvilinear coordinates are treated in a similar fashion.

55.2 Coordinate Representations

For a detailed discussion on coordinate frames and transformations, the reader is referred to Chapter 53.

Two-Dimensional

In surveying and mapping, two-dimensional frames are widely used. The different representations are all dependent, because only two numbers suffice to define the location of a point in 2-space. Cartesian frames consist of two often perpendicular reference axes, denoted as x and y, or e (easting) and n (northing). Points in two-dimensional frames are equally well represented by polar coordinates r (distance from an origin) and α (polar angle, counted positive counterclockwise from a reference axis).

We have

$$\begin{aligned} x &= r\cos\alpha \\ y &= r\sin\alpha \end{aligned} \tag{55.1}$$

The polar coordinates $\{r, \alpha\}$ are expressed in terms of the Cartesian counterparts by

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \alpha &= \arctan(y/x) \end{aligned} \tag{55.2}$$

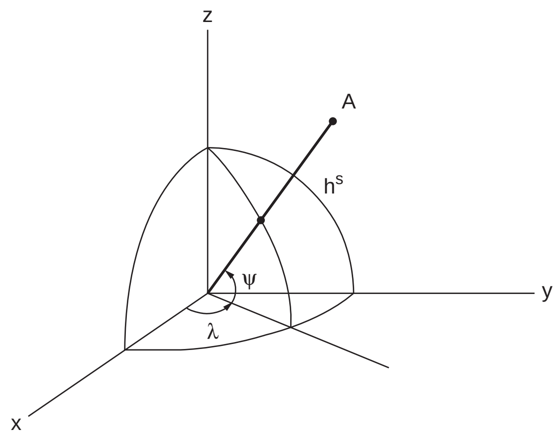

FIGURE 55.1 (Geographic) spherical coordinates: longitude λ, latitude ψ, and height *h*.

Three-Dimensional

Three-Dimensional Cartesian Coordinates

There are various ways to represent points in a three-dimensional space. One of the most well known is the representation by the so-called Cartesian coordinates *x*, *y*, *z*; we represent the position of a point *A* through three distances *x*, *y*, *z* (coordinates) to three perpendicular planes, the *yz*-, *xz*-, *xy*-planes, respectively. The intersecting lines between the three planes are the perpendicular coordinate axes. The position of point *A* is thought to be represented by the vector x with elements $\{x, y, z\}$:

$$\mathrm{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{55.3}$$

Three-Dimensional Polar Coordinates: Spherical

We may want to represent the position of these points with respect to a sphere with radius *R*. We make use of so-called spherical coordinates. The earth's radius is about $R = 6371.0$ km.

The sphere is intersected by two perpendicular planes, both of which pass through the center *O* of the sphere: a reference *equatorial plane* (perpendicular to the rotation axis) and a reference *meridian plane* (through the rotation axis). The angle between the vector and the reference equatorial plane is called *latitude,* ψ. The angle between the reference meridian plane (through Greenwich) and the local meridian plane (through *A*) is called *longitude,* λ (positive east). The distance to the surface of the sphere we call *height, h*. Consequently, the position of a point *A* is represented by $\{\lambda, \psi, h\}$ or $\{\lambda, \psi, r\}$ or $\{\lambda, \psi, R + h\}$; see Fig. 55.1.

Three-Dimensional Polar Coordinates: Ellipsoidal

The earth is flattened at the poles, and the average ocean surface has about the shape of an ellipsoid. For this reason, ellipsoidal coordinates are more often used in geodesy than spherical coordinates. We express the coordinates with respect to an ellipsoid of revolution with an equatorial semimajor axis *a* and a polar semiminor axis *b*. The semimajor axis thus represents the equatorial radius, and *b* is the distance between the ellipsoidal origin and the poles. The equation of such an ellipsoid of revolution is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{55.4}$$

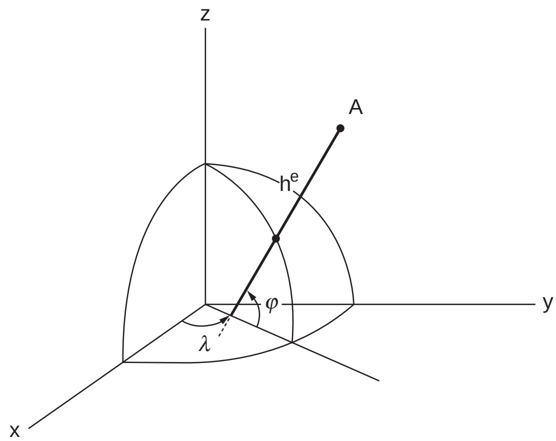

FIGURE 55.2 (Geodetic) ellipsoidal coordinates: longitude λ, latitude ϕ, and height *h*.

For the earth we have a semimajor axis a = 6378.137 km and a semiminor axis b = 6356.752 km. This means that the poles are about 21.4 km closer to the center of the earth than the equator.

The flattening of the earth is expressed by f and the (first) eccentricity by e:

$$f = \frac{a-b}{a} \left(\approx 1/298.257\right) \tag{55.5}$$

$$e^2 = \frac{a^2 - b^2}{a^2} \left(\approx 0.00669438\right) \tag{55.6}$$

See also Fig. 55.2.

Coordinate Transformations

We have to distinguish between two classes of transformations:

- Transformations between dissimilar coordinate representations. An example would be the transformation between Cartesian coordinates and curvilinear coordinates, such as the ellipsoidal (geodetic) coordinates.
- Transformations between similar coordinate frames. An example is the relation between geocentric Cartesian coordinates and topocentric Cartesian coordinates.

The latter group is to be discussed subsequently in this section, after we consider transformations between dissimilar coordinate representations.

Transformations of Different Kind

If the xy-plane coincides with the equator plane and the xz-plane with the reference meridian plane, then we have the following:

From spherical to Cartesian:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(R + h\right) \begin{pmatrix} \cos\psi \cos\lambda \\ \cos\psi \sin\lambda \\ \sin\psi \end{pmatrix} \tag{55.7}$$

From Cartesian to spherical:

$$\begin{pmatrix} \lambda \\ \psi \\ h \end{pmatrix} = \begin{pmatrix} \arctan(y/x) \\ \arctan\left(z/\sqrt{x^2+y^2}\right) \\ \sqrt{x^2+y^2+z^2} - R \end{pmatrix} \tag{55.8}$$

From ellipsoidal to Cartesian:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} [N + h]\cos\phi\cos\lambda \\ [N + h]\cos\phi\sin\lambda \\ [N(1-e^2)+h]\sin\phi \end{pmatrix} \tag{55.9}$$

with

$$N = \frac{a}{W} \tag{55.10}$$

and

$$W = \sqrt{1 - e^2 \sin^2\phi} \tag{55.11}$$

In these equations the variable N has a distinct geometric significance: it is the radius of curvature in the prime vertical plane. This plane goes through the local normal and is perpendicular to the meridian plane. In other words, N describes the curvature of the curve obtained through the intersection of the prime vertical plane and the ellipsoid. The curve formed through the intersection of the meridian plane and the ellipsoid is given by M; see Fig. 55.3.

The varying radius of curvature M of the elliptic meridian is given by

$$M = \frac{a(1-e^2)}{W^3} \tag{55.12}$$

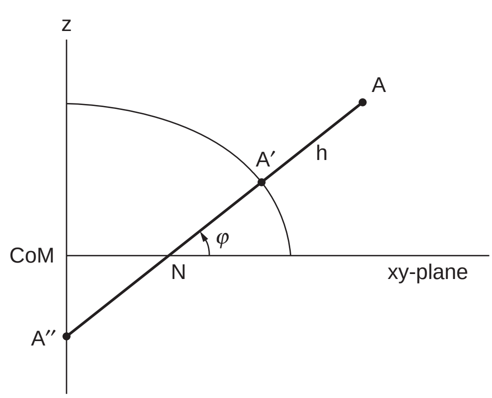

FIGURE 55.3 Meridian plane through point *A*.

From Cartesian to ellipsoidal:

$$\lambda = \arctan(y/x) \tag{55.13}$$

The geodetic latitude ϕ can be obtained by the following iteration scheme, starting with an approximate value for the geodetic latitude ϕ_0:

$$\begin{aligned} \phi_0 &= \arctan\left[z\Big/\sqrt{x^2+y^2}\right] \\ N_0 &= a\Big/\sqrt{1-e^2\sin^2\phi_0} \\ \phi &= \arctan\left[\left(z+N_0e^2\sin\phi_0\right)\Big/\sqrt{x^2+y^2}\right] \end{aligned} \tag{55.14}$$

If $|\phi - \phi_0| > \varepsilon$, then set ϕ_0 equal to ϕ and go back to the computation of N. After the iteration, h can be computed directly:

$$h = \sqrt{[x^2 + y^2 + \left(z + Ne^2\sin\phi\right)^2]} - N \tag{55.15}$$

Through more cumbersome expressions an analytical solution for the geodetic latitude ϕ as a function of the Cartesian coordinates $\{x, y, z\}$ is possible.

Transformations of Same Kind

Orthogonal Transformations: Translation and Rotation

When groups of points are known in their relative position with respect to each other, the use of Cartesian coordinates ($3n$ in total) becomes superfluous. As a matter of fact there are 6° of freedom, since the position of the origin with respect to the group is arbitrary, as is the orientation of the frame axes. Two groups of identical points or, for that matter, one and the same group of points expressed in two arbitrary but different coordinate frames may be represented by the following orthogonal transformation:

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}' \tag{55.16}$$

or

$$\mathbf{x}' = \mathbf{R}(\mathbf{x} - \mathbf{t}) \tag{55.17}$$

The vector $\mathbf{t}$ represents a translation. In Eq. (55.16) $\mathbf{t}'$ represents the vector of the old origin in the new frame ($\mathbf{x}' - \mathbf{t}' = \mathbf{R}\mathbf{x}$); in Eq. (55.17) $\mathbf{t}$ represents the coordinates of the origin of the new frame in the old coordinate frame. The relation between the two translation vectors is represented by

$$\mathbf{t}' = -\mathbf{R}\mathbf{t} \tag{55.18}$$

The rotation matrix describes the rotations around the frame axes. In Eq. (55.16) $\mathbf{R}$ describes a rotation around axes through the origin of the $\mathbf{x}$ frame; in Eq. (55.17) $\mathbf{R}$ describes a rotation around axes through the origin of the $\mathbf{x}'$ frame. We define the sense of rotations as follows: the argument angle of the rotation matrix is taken positive if one views the rotation as counterclockwise from the positive end of the rotation axis looking back to the origin. For an application relating coordinates in a local frame to coordinates in a global frame, see Section 55.3.

In the above equations we assume three consecutive rotations, first around the z axis with an argument angle γ, then around the y axis around an argument angle β, and finally around the x axis with the argument angle α. So we have

$$\mathbf{R} = \mathbf{R}_1(\alpha)\mathbf{R}_2(\beta)\mathbf{R}_3(\gamma) \tag{55.19}$$

$$R = \begin{pmatrix} \cos\beta & \cos\beta\sin\gamma & -\sin\beta \\ -\cos\alpha\ \sin\gamma + \sin\alpha\sin\beta\cos\gamma & \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & \sin\alpha\cos\beta \\ \sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma & -\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{pmatrix} \quad (55.20)$$

One outcome of these orthogonal transformations is an inventory of variables that are invariant under these transformations. Without any proof, these include lengths, angles, sizes and shapes of figures, and volumes — important quantities for the civil or survey engineer.

Similarity Transformations: Translation, Rotation, Scale

In the previous section we saw that the relative location of n points can be described by fewer than $3n$ coordinates: $3n - 6$ quantities (for instance, an appropriate choice of distances and angles) are necessary but also sufficient. Exceptions have to be made for so-called critical configurations such as four points in a plane. The 6 is nothing else than the 6° of freedom supplied by the orthogonal transformation: three translations and three rotations.

A simple but different reasoning leads to the same result. Imagine a tetrahedron in a three-dimensional frame. The four corner points are connected by six distances. These are exactly the six necessary but sufficient quantities to describe the form and shape of the tetrahedron. These six sides determine this figure completely in size and shape. A fifth point will be positioned by another three distances to any three of the four previously mentioned points. Consequently, a field of n points (in three-dimensional) will be necessarily but sufficiently described by $3n - 6$ quantities. We need these types of reasoning in three-dimensional geometric satellite geodesy.

If we just consider the *shape* of a figure spanned by n points (we are not concerned any more about the *size* of the figure), then we need even one quantity fewer than $3n - 6$ (i.e., $3n - 7$); we are now ignoring the scale, in addition to the position and orientation of the figure. This constitutes just the addition of a seventh parameter to the six-parameter orthogonal transformation: the scale parameter σ. So we have

$$\mathbf{x}' = \sigma\mathbf{R}\mathbf{x} + \mathbf{t}' \quad (55.21)$$

or

$$\mathbf{x}' = \sigma\mathbf{R}(\mathbf{x} - \mathbf{t}) \quad (55.22)$$

Here also the vector $\mathbf{t}$ represents a translation. In Eq. (55.21) $\mathbf{t}'$ represents the vector of the old origin in the new scaled and rotated frame ($\mathbf{x}' - \mathbf{t}' = \sigma\mathbf{R}\mathbf{x}$); in Eq. (55.22) $\mathbf{t}$ represents the coordinates of the origin of the new frame in the old coordinate frame. The relation between the two translation vectors is represented by

$$\mathbf{t}' = -\sigma\mathbf{R}\mathbf{t} \quad (55.23)$$

One outcome of these similarity transformations is an inventory of invariant variables under these transformations. Without any proof, these include length ratios, angles, shapes of figures, and volume ratios, which are important quantities for the civil or survey engineer. The reader is referred to Leick and van Gelder [1975] for other important properties.

Curvilinear Coordinates and Transformations

One usually prefers to express coordinate differences in terms of the curvilinear coordinates on the sphere or ellipsoid or even locally, rather than in terms of the Cartesian coordinates. This approach also facilitates the study of effects due to changes in the adopted values for the reference ellipsoid (so-called *datum transformations*).

Curvilinear Coordinate Changes in Terms of Cartesian Coordinate Changes

Differentiating the transformation formulas in which the Cartesian coordinates are expressed in terms of the ellipsoidal coordinates (see Eq. (55.9)), we obtain a differential formula relating the Cartesian total differentials $\{dx, dy, dz\}$ as a function of the ellipsoidal total differentials $\{d\lambda, d\phi, dh\}$:

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \mathbf{J} \begin{pmatrix} d\lambda \\ d\phi \\ dh \end{pmatrix} \tag{55.24}$$

The projecting matrix **J** is nothing else than the Jacobian of partial derivatives:

$$\mathbf{J} = \frac{\partial(x, y, z)}{\partial(\lambda, \phi, h)} \tag{55.25}$$

Carrying out the differentiation, one finds

$$\mathbf{J} = \begin{bmatrix} -(N+h)\cos\phi\sin\lambda & -(M+h)\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ (N+h)\cos\phi\cos\lambda & -(M+h)\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & (M+h)\cos\phi & \sin\phi \end{bmatrix} \tag{55.26}$$

On inspection, this Jacobian **J** is simply a product of a rotation matrix $\mathbf{R}(\lambda, \phi)$ and a metric matrix $\mathbf{H}(\phi, h)$ [Soler, 1976]:

$$\mathbf{J} = \mathbf{RH} \tag{55.27}$$

or, in full,

$$\mathbf{R} = \begin{bmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \tag{55.28}$$

$$\mathbf{H} = \begin{bmatrix} (N+h)\cos\phi & 0 & 0 \\ 0 & (M+h) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{55.29}$$

It turns out that the rotation matrix $\mathbf{R}(\lambda, \phi)$ relates the local $\{e, n, u\}$ frame to the geocentric $\{x, y, z\}$ frame; see further the discussion of earth-fixed coordinates in Section 55.3. The metric matrix $\mathbf{H}(\phi, h)$ relates the curvilinear coordinates' longitude, latitude, and height in radians and meters to the curvilinear coordinates, all expressed in meters.

The formulas just given are the simple expressions relating a small arc distance ds to the corresponding small angle $d\alpha$ through the radius of curvature. The radius of curvature for the longitude component is equal to the radius of the local parallel circle, which in turn

$$\begin{pmatrix} d\lambda_m \\ d\phi_m \\ dh_m \end{pmatrix} = \mathbf{H}(\phi, h) \begin{pmatrix} d\lambda_{\text{rad}} \\ d\phi_{\text{rad}} \\ dh_m \end{pmatrix} \tag{55.30}$$

equals the radius of curvature in the prime vertical plane times the cosine of the latitude.

The power of this evaluation is more apparent if one realizes that the inverse Jacobian, expressing the ellipsoidal total differentials $\{d\lambda, d\phi, dh\}$ as a function of the Cartesian total differentials $\{dx, dy, dz\}$, is easily obtained, whereas an analytic solution expressing the geodetic ellipsoidal coordinates in terms of the Cartesian coordinates is extremely difficult to obtain. So, we have

$$\begin{pmatrix} d\lambda \\ d\phi \\ dh \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{55.31}$$

With the relationship in Eq. (55.27) $\mathbf{J}^{-1}$ becomes simply

$$\mathbf{J}^{-1} = (\mathbf{RH})^{-1} = \mathbf{H}^{-1}\mathbf{R}^{\mathbf{T}} \tag{55.32}$$

or, in full,

$$\mathbf{J}^{-1} = \begin{bmatrix} \frac{\sin\lambda}{(N+h)\cos\phi} & \frac{\cos\lambda}{(N+h)\cos\phi} & 0 \\ \frac{-\sin\phi\cos\lambda}{M+h} & \frac{-\sin\phi\sin\lambda}{M+h} & \frac{\cos\phi}{M+h} \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix} \tag{55.33}$$

This equation gives a simple analytic expression for the inverse Jacobian, whereas the analytic expression for the original function is virtually impossible.

Curvilinear Coordinate Changes Due to a Similarity Transformation

Differentiating Eq. (55.21) with respect to the similarity transformation parameters α, β, γ, t'_x, t'_y, t'_z, and σ, one obtains

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}_7 = \mathbf{J}_7 \begin{pmatrix} d\alpha \\ d\beta \\ d\gamma \\ dt'_x \\ dt'_y \\ dt'_z \\ d\sigma \end{pmatrix} \tag{55.34}$$

with

$$\mathbf{J}_7 = \frac{\partial(x, y, z)}{\partial(\alpha, \beta, \gamma, t'_x, t'_y, t'_z, \sigma)} \tag{55.35}$$

The Jacobian $\mathbf{J}_7$ is a matrix that consists of seven column vectors

$$\mathbf{J}_7 = [\mathbf{j}_1 | \mathbf{j}_2 | \mathbf{j}_3 | \mathbf{j}_4 | \mathbf{j}_5 | \mathbf{j}_6 | \mathbf{j}_7 |] \tag{55.36}$$

with

$$\begin{aligned}\mathbf{j}_1 &= \sigma \mathbf{L}_1 \mathbf{R}_1(\alpha)\mathbf{R}_2(\beta)\mathbf{R}_3(\gamma)\mathbf{x} \\ &= \sigma \mathbf{L}_1 \mathbf{R}\mathbf{x} \\ &= \mathbf{L}_1 \mathbf{x}'\end{aligned} \tag{55.37}$$

$$\mathbf{j}_2 = \sigma \mathbf{R}_1(\alpha)\mathbf{L}_2\mathbf{R}_2(\beta)\mathbf{R}_3(\gamma)\mathbf{x} \tag{55.38}$$

$$\begin{aligned}\mathbf{j}_3 &= \sigma \mathbf{R}_1(\alpha)\mathbf{R}_2(\beta)\mathbf{R}_3(\gamma)\mathbf{L}_3\mathbf{x} \\ &= \sigma \mathbf{R}\mathbf{L}_3\mathbf{x}\end{aligned} \tag{55.39}$$

$$\left[\mathbf{j}_4 \middle| \mathbf{j}_5 \middle| \mathbf{j}_6\right] = \mathbf{I} \qquad (3 \times 3) \text{ identity matrix} \tag{55.40}$$

$$\begin{aligned}\mathbf{j}_7 &= \mathbf{R}_1(\alpha)\mathbf{R}_2(\beta)\mathbf{R}_3(\gamma)\mathbf{x} \\ &= \mathbf{R}\mathbf{x} \\ &= (\mathbf{x}' - \mathbf{t}')/\sigma\end{aligned} \tag{55.41}$$

since

$$\partial \mathbf{R}_1/\partial\alpha = \mathbf{L}_1\mathbf{R}_1(\alpha) = \mathbf{R}_1(\alpha)\mathbf{L}_1 \tag{55.42}$$

$$\partial \mathbf{R}_2/\partial\beta = \mathbf{L}_2\mathbf{R}_2(\beta) = \mathbf{R}_2(\beta)\mathbf{L}_2 \tag{55.43}$$

$$\partial \mathbf{R}_3/\partial\gamma = \mathbf{L}_3\mathbf{R}_3(\gamma) = \mathbf{R}_3(\gamma)\mathbf{L}_3 \tag{55.44}$$

and

$$\mathbf{L}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{55.45}$$

$$\mathbf{L}_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{55.46}$$

$$\mathbf{L}_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{55.47}$$

The advantage of these **L** matrices is that in many instances the derivative matrix (product) can be written as the original matrix pre- or postmultiplied by the corresponding **L** matrix [Lucas, 1963].

Curvilinear Coordinate Changes Due to a Datum Transformation

Differentiating Eq. (55.9) with respect to the semimajor axis *a* and the flattening *f*, one obtains

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}_{a,f} = \mathbf{J}_{a,f} \begin{pmatrix} da \\ df \end{pmatrix} \tag{55.48}$$

with (see Soler and van Gelder [1987])

$$\mathbf{J}_{a,f} = \frac{\partial(x,y,z)}{\partial(a,f)} \tag{55.49}$$

and

$$\mathbf{J}_{a,f} = \begin{bmatrix} \cos\phi\cos\lambda/W & a(1-f)\sin^2\phi\cos\phi\cos\lambda/W^3 \\ \cos\phi\sin\lambda/W & a(1-f)\sin^2\phi\cos\phi\sin\lambda/W^3 \\ (1-e^2)\sin\phi/W & (M\sin^2\phi-2N)(1-f)\sin\phi \end{bmatrix} \tag{55.50}$$

Also see Soler and van Gelder [1987] for the second-order derivatives.

Curvilinear Coordinate Changes Due to a Similarity and a Datum Transformation

The curvilinear effects of a redefinition of the coordinate frame due to a similarity transformation and a datum transformation are computed by adding Eqs. (55.34) and (55.48) and substituting them into

$$\begin{pmatrix} d\lambda \\ d\phi \\ dh \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{55.51}$$

with

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}_7 + \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}_{a,f} \tag{55.52}$$

55.3 Coordinate Frames Used in Geodesy and Some Additional Relationships

Earth-Fixed

Earth-Fixed Geocentric

From the moment satellites were used to study geodetic aspects of the earth, one had to deal with modeling the motion of the satellite (a point mass) around the earth's center of mass (CoM). The formulation of the equations of motion is easiest when referred to the CoM. This point became almost naturally the origin of the coordinate frame in which the earthbound observers were situated. For the orientation of

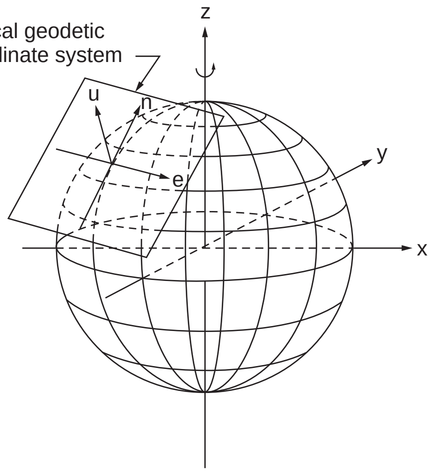

FIGURE 55.4 A geocentric and a local Cartesian coordinate frame.

the x and z axes, see the discussion of spherical three-dimensional polar coordinates in Section 55.2 and the discussion of polar motion in this section.

Earth-Fixed Topocentric Cartesian

An often used local frame is the earth-fixed topocentric coordinate frame. The origin resides at the position of the observer's instrument. Although in principle arbitrary, one often chooses the x axis pointing east, the y axis pointing north, and the z axis pointing up. This e, n, u frame is again a right-handed frame. With respect to the direction of the local z or u axis, various choices are possible: the u axis coincides with the negative direction of the local gravity vector (the first axis of a leveled theodolite) or along the normal perpendicular to the surface of the ellipsoid.

An Important Relationship Using an Orthogonal Transformation

The transformation formulas between a geocentric coordinate frame and a local coordinate frame are (see Fig. 55.4):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{R}_3\left(-\lambda_a - \frac{\pi}{2}\right)\mathbf{R}_1\left(+\phi_a - \frac{\pi}{2}\right)\begin{pmatrix} e \\ n \\ u \end{pmatrix} + \begin{pmatrix} \left[N_a + h_a\right]\cos\phi_a \cos\lambda_a \\ \left[N_a + h_a\right]\cos\phi_a \sin\lambda_a \\ \left[N_a\left(1-e^2\right)+h_a\right]\sin\phi_a \end{pmatrix} \tag{55.53}$$

One should realize that this transformation formula is of the orthogonal type shown in Eq. (55.16):

$$\mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{t}' \tag{55.54}$$

whereby $\mathbf{x}'$ = the geocentric Cartesian vector
$\mathbf{R} = \mathbf{R}_3(-\lambda_a - \pi/2)\mathbf{R}_1(\phi_a - \pi/2)$
$\mathbf{t}'_a$ = the location of a in the (new) geocentric frame and is equal to:

$$\mathbf{t}'_a = \begin{pmatrix} \left[N_a + h_a\right]\cos\phi_a \cos\lambda_a \\ \left[N_a + h_a\right]\cos\phi_a \sin\lambda_a \\ \left[N_a\left(1-e^2\right)+h_a\right]\sin\phi_a \end{pmatrix} \tag{55.55}$$

The rotation matrix $\mathbf{R} = \mathbf{R}_3(-\lambda_a - \pi/2)\mathbf{R}_1(\phi_a - \pi/2)$ is given in Section 55.2 as Eq. (55.28).

Given the geocentric coordinates $\{x, y, z\}$ of an arbitrary point (e.g., a satellite), if one wants to compute the local coordinates $\{e, n, u\}$ of that point, the local frame being centered at a, then one obtains for the inverse relationship

$$\begin{pmatrix} e \\ n \\ u \end{pmatrix} = \mathbf{R}_1\left(-\phi_a + \frac{\pi}{2}\right)\mathbf{R}_3\left(+\lambda_a + \frac{\pi}{2}\right)\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} \left[N_a + h_a\right]\cos\phi_a \cos\lambda_a \\ \left[N_a + h_a\right]\cos\phi_a \sin\lambda_a \\ \left[N_a\,(1-e^2) + h_a\right]\sin\phi_a \end{pmatrix}\right] \tag{55.56}$$

The rotation matrix $\mathbf{R}_1 = (-\phi_a + \pi/2)\mathbf{R}_3(+\lambda_a + \pi/2)$ is the transpose of the matrix given in Section 55.2 as Eq. (55.28).

Earth-Fixed Topocentric Spherical

Satellites orbit the earth at finite distances. For such purposes as visibility calculations, one relates the local e, n, u coordinates to local spherical coordinates El (elevation or altitude angle), Az (azimuth, clockwise positive from the north), and Sr (slant range to the object):

$$\begin{pmatrix} e \\ n \\ u \end{pmatrix} = \mathrm{Sr}\begin{pmatrix} \cos \mathrm{El} \sin \mathrm{Az} \\ \cos \mathrm{El} \cos \mathrm{Az} \\ \sin \mathrm{El} \end{pmatrix} \tag{55.57}$$

The inverse relationships are

$$\begin{pmatrix} \mathrm{El} \\ \mathrm{Az} \\ \mathrm{Sr} \end{pmatrix} = \begin{pmatrix} \arctan\left[u/\sqrt{e^2 + n^2}\right] \\ \arctan(e/n) \\ \sqrt{e^2 + n^2 + u^2} \end{pmatrix} \tag{55.58}$$

Note again that El, Az, and Sr form themselves a right-handed (curvilinear) frame.

Some Important Relationships Using Similarity and Datum Transformations

Increasing measurement accuracies and improved insights in the physics of the earth often cause reference frames to be reviewed. For instance, if coordinates of a station are given in an old frame, then with current knowledge of similarity transformation parameters relating the old $\mathbf{x}$ frame to the new $\mathbf{x}'$ frame, the new coordinates can be computed according to

$$\mathbf{x}' = \sigma\mathbf{R}\mathbf{x} + \mathbf{t}' \tag{55.59}$$

In many instances the translation and rotation transformation parameters are small, and the scale parameter σ deviates little from 1, so we introduce the following new symbols:

$$\begin{aligned} \sigma &= 1 + \delta\sigma \\ \alpha &= \delta\in \\ \beta &= \delta\psi \end{aligned} \tag{55.60}$$

$$\gamma = \delta\omega$$

$$t'_x = \Delta x$$

$$t'_y = \Delta y$$

$$t'_z = \Delta z$$

Neglecting second-order effects, the rotation matrix **R** can be written as the sum of an identity matrix **I** and a skew-symmetric matrix δ**R**:

$$\mathbf{R} = \mathbf{R}_1(\delta\in)\mathbf{R}_2(\delta\psi)\mathbf{R}_3(\delta\omega) \tag{55.61}$$

$$= \mathbf{I} + \delta\mathbf{R} \tag{55.62}$$

with

$$\delta\mathbf{R} = \begin{pmatrix} 0 & \delta\omega & -\delta\psi \\ -\delta\omega & 0 & \delta\varepsilon \\ \delta\psi & -\delta\in & 0 \end{pmatrix} \tag{55.63}$$

Equation (55.59) becomes

$$\mathbf{x}' = (1 + \delta\sigma)(\mathbf{I} + \delta\mathbf{R})\mathbf{x} + \Delta\mathbf{x} \tag{55.64}$$

or, neglecting second-order effects,

$$\mathbf{x}' = \mathbf{x} + \mathbf{dx} \tag{55.65}$$

with

$$\mathbf{dx} = \delta\sigma\mathbf{x} + \delta\mathbf{R}\mathbf{x} + \Delta\mathbf{x} \tag{55.66}$$

See Section 55.8 for a variety of parameter sets relating the various reference frames and datum values.

Inertial and Quasi-Inertial

Inertial Geocentric Coordinate Frame

For the derivation of the equations of motion of point masses in space we need so-called inertial frames. These are frames where Newton's laws apply. These frames are nonrotating, where point masses either have uniform velocity or are at rest. Popularly speaking, in these frames the stars or, better, extragalactic points or quasars, are "fixed" (i.e., not moving in a rotational sense). Since the stars are at such large distances from the earth, it is often sufficient in geodetic astronomy to consider the inertial directions. Instead of the inertial coordinates of the stars we consider the vector **d**, consisting of the three direction cosines. One has to realize that these direction cosines are dependent on only two angles. Consequently, only two direction cosines contain independent information because the three direction cosines squared sum up to 1.

The two angles are (see Fig. 55.5)

α right ascention

δ declination

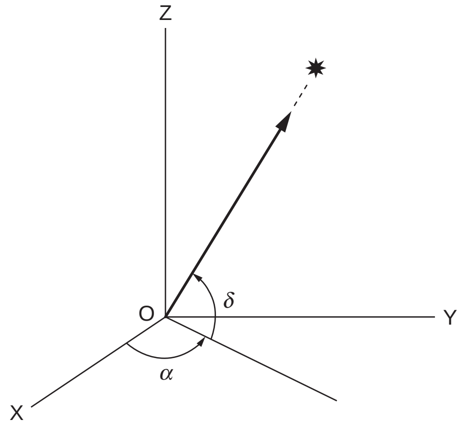

FIGURE 55.5 Direction to a satellite or star: right ascension, α, and declination, δ.

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = l\mathbf{d} \tag{55.67}$$

with

$$\mathbf{d} = \begin{pmatrix} \cos\delta & \cos\alpha \\ \cos\delta & \sin\alpha \\ \sin\delta & \end{pmatrix} \tag{55.68}$$

The right ascension is counted counterclockwise positive from the X axis and is defined as the intersection of the earth's equatorial plane and the plane of the earth's orbit around the sun. One of the points of intersection is called the *vernal equinox:* it is that point in the sky among the stars where the sun appears as viewed from earth at the beginning of spring in the northern hemisphere. The declination is counted from the equatorial plane in the same manner as the latitude; see Fig. 55.6.

Quasi-Inertial Coordinate Frame

In the previous section the position of the origin was not defined yet: the origin of the inertial frame is not to coincide with the center of mass of the earth; since the earth itself orbits around the sun, the center of mass of the earth is subject to accelerations. Similarly, the center of mass of the sun and all its planets is rotating around the center of our galaxy, and the galaxy experiences gravitational forces from other galaxies. A continuation of this reasoning will improve the quality of "inertiality" of the coordinate frame, but the practical application for the description of the motion of earth-orbiting satellites has been completely lost.

In Section 55.5 a practical solution is presented: in orientation the frame is as inertial as possible, but the origin has been chosen to coincide with the earth's center of mass. Such frames are called *quasi-inertial* frames. The apparent forces caused by the (small) accelerations of the origin have to be accounted for later.

Relation between Earth-Fixed and Inertial

Satellite equations of motion are easily dealt with in an inertial frame, but we observers are likely to model our positions and relatively slow velocities in an earth-fixed frame. The relationship between these two frames has to be dealt with.

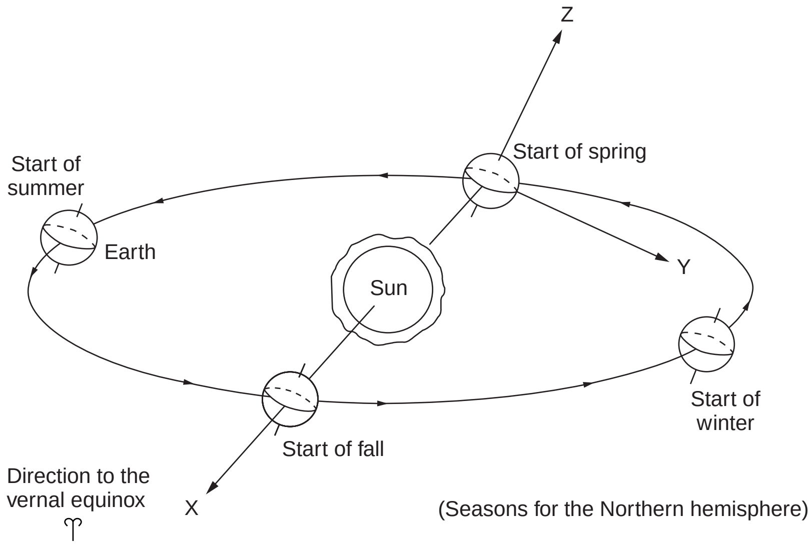

FIGURE 55.6 The (quasi-)inertial reference frame with respect to sun and earth.

Time and Sidereal Time

The diurnal rotation of the earth is given by its average angular velocity:

$$\omega_e = 7.292115 \times 10^{-5} \text{ rad/s} \tag{55.69}$$

This inertial angular velocity results in an average day length of

$$T = \frac{2\pi}{\omega_e} = 86164.1 \text{sec} \tag{55.70}$$

This *sidereal day*, based on the earth's spin rotation with respect to the fixed stars, deviates from our 24 x 60 x 60 = 86,400-sec day by 3 min, 55.9 sec. That is why we see an arbitrary star constellation in the same position in the sky each day about 4 min earlier. Our daily lives are based on the earth's spin with respect to the sun, the *solar day*. Since the earth advances about 1° per day in its orbit around the sun, the earth has not completed a full spin with respect to the sun when it completes one full turn with respect to the stars; see Fig. 55.7.

For practical purposes the angular velocity must include the effect of precession; see the next section. We have

$$\omega_c^* = 7.2921158553 * 10^{-5} \text{ rad/s} \tag{55.71}$$

The angle between the vernal equinox and the Greenwich meridian as measured along the equator is *Greenwich apparent sidereal time* (GAST). This angle increases in time by ω_c^* per second. With the help of the formula of Newcomb, we are able to compute GAST [IERS, 1992]:

$$\text{GAST}(t) = a + bT_u + cT_u^2 + dT_u^3 + \text{ee} \tag{55.72}$$

with

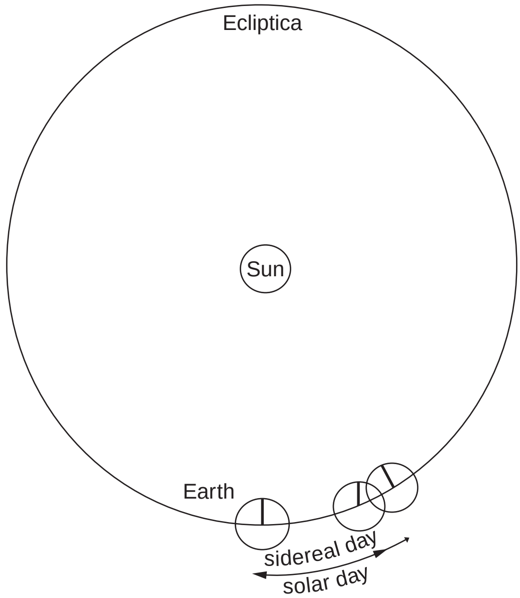

FIGURE 55.7 The earth's orbit around the sun: sidereal day and solar day.

$$\begin{aligned} a &= 18\,h\ 41\ \text{min}\ 50.54841\ \text{s} \\ b &= 8{,}640{,}184.812866\ \text{s}/\text{century}^3 \\ c &= 0.093104\ \text{s}/\text{century}^3 \\ d &= -0.0000062\ \text{s}/\text{century}^3 \\ \text{ee} &= \text{the equation of the equinoxes} \end{aligned} \tag{55.73}$$

T_u is measured in Julian centuries of 36,525 universal days, since 1.5 January 2000 (JD_0 = 2,451,545.0). This means that T_u, until the year 2000, is negative. T_u can be computed from

$$T_u = \frac{(\text{JD} - 2{,}451{,}545.0)}{36{,}525} \tag{55.74}$$

when the Julian day number, *JD*, is given.

Polar Motion

Polar motion, or on a geological time scale "polar wandering," represents the motion of the earth's spin axis with respect to an earth-fixed frame. Polar motion changes our latitudes, since if the *z* axis were chosen to coincide with the instantaneous position of the spin axis, our latitudes would change continuously.

Despite the earth's nonelastic characteristics, excitation forces keep polar motion alive. Polar motion is the motion of the instantaneous rotation axis, or celestial ephemeris pole (CEP), with respect to an adopted reference position, the conventional terrestrial pole (CTP) of the conventional terrestrial reference frame (CTRF). The adopted reference position, or conventional international origin (CIO), was the main position of the CEP between 1900 and 1905. Since then the mean CEP has drifted about 10 m away from the CIO in a direction of longitude 280°.

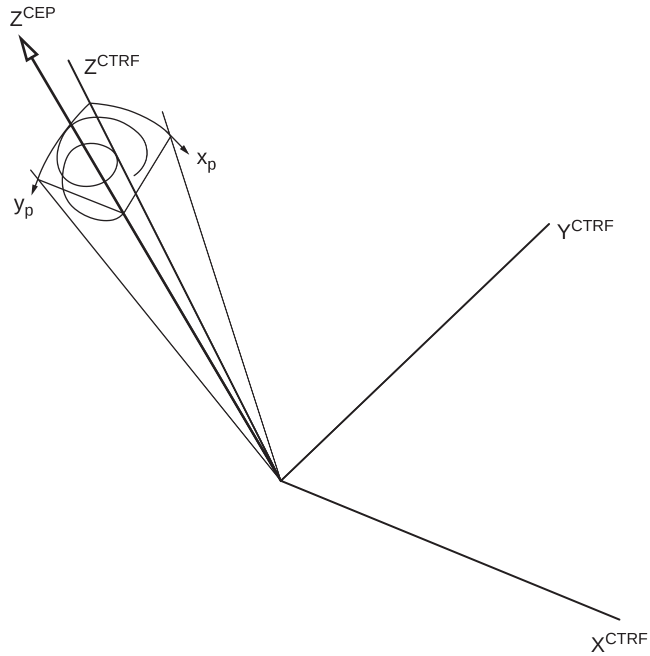

FIGURE 55.8 Local curvilinear pole coordinates x_p, y_p with respect to the conventional terrestrial reference frame.

The transformation from the somewhat earth-fixed frame $\mathbf{x}_{\mathrm{CEP}}$ to $\mathbf{x}_{\mathrm{CTRF}}$ is

$$\mathbf{x_{CTRF}} = \mathbf{R}_2\left(-x_p\right)\mathbf{R}_1\left(-y_p\right)\mathbf{x_{CEP}} \tag{55.75}$$

The position of the pole is expressed in local curvilinear coordinates that have their origin in the CIO. The angle x_p (radians) increases along the Greenwich meridian south, whereas the angle y_p increases along the meridian $\lambda = 270°$ south; see Fig. 55.8. As we saw in the previous subsection, the earth rotates daily around its (moving) CEP axis. Expanding the transformation of Eq. (55.75) but now also including the sidereal rotation of the earth, we obtain the following relationship between an inertial reference frame and the CTRF:

$$\mathbf{x_{CEP}} = \mathbf{R}_3(\mathrm{GAST})\mathbf{x_{in}} \tag{55.76}$$

This relationship is shown in Fig. 55.9. Combining Eqs. (55.75) and (55.76), we have

$$\mathbf{x_{CTRF}} = \mathbf{R}_2\left(-x_p\right)\mathbf{R}_1\left(-y_p\right)\mathbf{R}_3(\mathrm{GAST})\mathbf{x_{in}} \tag{55.77}$$

or, in short,

$$\mathbf{x_{CTRF}} = \mathbf{R}_S\mathbf{x_{in}} \tag{55.78}$$

where $\mathbf{R}_S$ represents the combined earth rotation due to polar motion and diurnal rotation (length of day).

Two main frequencies make up the polar motion: the Chandler wobble of 435 days (14 months, more or less) and the annual wobble of 365.25 days. A prediction model for polar motion is

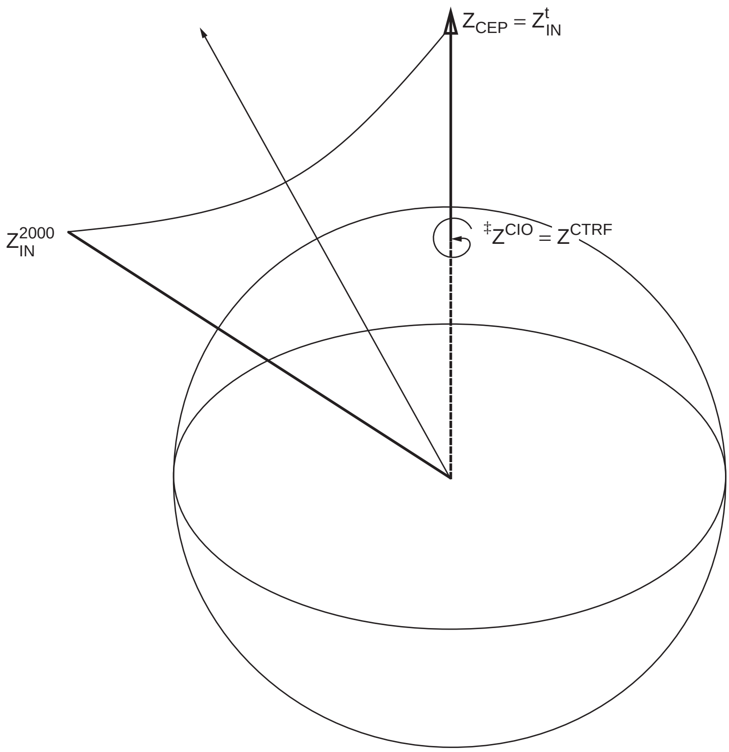

FIGURE 55.9 Relationship between the quasi-earth-fixed frame $\mathbf{X}_{CEP}$ and the inertial frame $\mathbf{X}_{in}$.

$$\begin{aligned} x_p(t) = x_p(t_0) + \dot{x}_p(t-t_0) + A_s^x \sin\left[\frac{2\pi}{T_1}(t-t_0)\right] + A_c^x \cos\left[\frac{2\pi}{T_1}(t-t_0)\right] \\ + C_s^x \sin\left[\frac{2\pi}{T_2}(t-t_0)\right] + C_c^x \cos\left[\frac{2\pi}{T_2}(t-t_0)\right] \end{aligned} \tag{55.79}$$

$$\begin{aligned} y_p(t) = y_p(t_0) + \dot{y}_p(t-t_0) + A_s^y \sin\left[\frac{2\pi}{T_1}(t-t_0)\right] + A_c^y \cos\left[\frac{2\pi}{T_1}(t-t_0)\right] \\ + C_s^y \sin\left[\frac{2\pi}{T_2}(t-t_0)\right] + C_c^y \cos\left[\frac{2\pi}{T_2}(t-t_0)\right] \end{aligned} \tag{55.80}$$

where T_1 = 365.25 days
T_2 = 435 days

Similarly, the ever-increasing angle GAST has to be corrected for seasonal variations in ω. These variations are in the order of ±1 msec. A prediction model for length-of-day variations is

$$\delta\mathrm{UT1} = a \sin 2\pi t + b \cos 2\pi t + c \sin 4\pi t + d \cos 4\pi t \tag{55.81}$$

with a = +0.0220 sec
b = –0.0120 sec
c = –0.0060 sec
d = +0.0070 sec
t = 2000.000 + [(MJD – 51544.03)/365.2422].

With Eq. (55.72), we have

$$\text{GAST} = \text{GAST}_{(t)} + \delta\text{UT1} \tag{55.82}$$

Due to the flattening, f, $\mathbf{x}_{\text{in}}$ in Eq. (55.77) is not star-fixed but would be considered a proper (quasi-) inertial frame if we froze the frame at epoch t. So, Eq. (55.78) is more properly expressed as

$$\mathbf{x}_{\mathbf{CTRF}} = \mathbf{R}_S \mathbf{x}_{\mathbf{in}}^t \tag{55.83}$$

The motions of the CEP with respect to the stars are called *precession* and *nutation*.

Precession and Nutation

Precession and nutation represent the motion of the earth's spin axis with respect to an inertial frame.

To facilitate comparisons between observations of spatial objects (satellites, quasars, stars) that may be made at different epochs, a transformation is carried out between the inertial frame at epoch t (CEP at t) and the mean position of the inertial frame at an agreed-upon reference epoch t_0. The reference epoch is again 1.5 January 2000, for which JD_0 = 2,451,545.0; see the discussion on time and sidereal time earlier in this section. At this epoch we define the conventional inertial reference frame (CIRF). So we have

$$\mathbf{x}_{\mathbf{in}}^t = \mathbf{R}_{\text{NP}} \mathbf{x}_{\mathbf{in}}^{\mathbf{2000}} \tag{55.84}$$

All masses in the ecliptic plane (earth and sun) and close to it (planets) exert a torque on the tilted equatorial bulge of the earth. The result is that the CEP describes a cone with its half top angle equal to the obliquity ε: precession. The period is about 25,800 years. The individual orbits of the planets and moon cause deviations with respect to this cone: nutation. The largest effect is about 9 sec of arc, with a period of 18.6 years, caused by the inclined orbit of the moon.

The transformation $\mathbf{R}_{\text{NP}}$ is carried out in two steps. The mean position of the CEP is first updated for precession to a mean position at epoch t (now):

$$\mathbf{x}_{\mathbf{in}}^{t,\mathbf{mean}} = \mathbf{R}_P \mathbf{x}_{\mathbf{in}}^{\mathbf{2000,mean}} \tag{55.85}$$

Subsequently, the mean position of the CEP at epoch t is transformed to the true position of the CEP at epoch t due to nutation:

$$\mathbf{x}_{\mathbf{in}}^{t,\mathbf{true}} = \mathbf{R}_N \mathbf{x}_{\mathbf{in}}^{t,\mathbf{mean}} \tag{55.86}$$

Combining all transformations, we have

$$\mathbf{x}_{\mathbf{CTRF}} = \mathbf{R}_S \mathbf{R}_N \mathbf{R}_P \mathbf{x}_{\mathbf{CIRF}} \tag{55.87}$$

where $\mathbf{x}_{\text{CTFR}}$ is identical to $\mathbf{x}_{\text{earth-fixed}}^{\text{CIO}}$ and $\mathbf{x}_{\text{CIRF}}$ is identical to $\mathbf{x}_{\text{star-fixed}}^{2000}$

The rotation matrices $\mathbf{R}_P$ and $\mathbf{R}_N$ depend on the obliquity ε, longitude of sun, moon, etc. We have

$$\mathbf{R}_P = \mathbf{R}_3(-z)\mathbf{R}_2(\theta)\mathbf{R}_3(-\zeta) \tag{55.88}$$

with (see, e.g., IERS [1992])

$\zeta = 2\ 306''.218\ 1T_u + 0''.301\ 88T_u^2 + 0''.0.17\ 998T_u^3$
$\theta = 2\ 004''.310\ 9T_u - 0''.426\ 65T_u^2 - 0''.0.41\ 833T_u^3$
$z = 2\ 306''.218\ 1T_u + 1''.094\ 68T_u^2 + 0''.0.18\ 203T_u^3$

and

$$\mathbf{R}_N = \mathbf{R}_1(-\varepsilon - \Delta\varepsilon)\mathbf{R}_3(-\Delta\psi)\mathbf{R}_1(\varepsilon) \tag{55.89}$$

with $\varepsilon = 84381''.448 - 46''.8150T_u - 0''.018203T_u^2 + 0''.001813T_u^3$.

For the nutation in longitude, $\Delta\psi$, and the nutation in obliquity, $\Delta\varepsilon$, a trigonometric series expansion is available consisting of 106 × 2 × 2 parameters and five arguments: mean anomaly of the moon, the mean anomaly of the sun, mean elongation of the moon from the sun, the mean longitude of the ascending node of the moon, and the difference between the mean longitude of the moon and the mean longitude of the ascending node of the moon. There are 2 × 2 constants in each term: a sine coefficient, a cosine coefficient, a time-invariant coefficient, and a time-variant coefficient.

For more detail on these transformations, refer to earth orientation literature, such as Mueller [1969], Moritz and Mueller [1988], and IERS [1992].

55.4 Mapping

The art of mapping is referred to a technique that maps information from an n-dimensional space $\mathbf{R}_n$ to an m-dimensional space $\mathbf{R}_m$. Often information belonging to a high-dimensional space is mapped to a low-dimensional space. In other words, one has

$$n > m \tag{55.90}$$

In geodesy and surveying one may want to map a three-dimensional world onto a two-dimensional world. In photogrammetry, an aerial photograph can be viewed as a mapping procedure as well: a two-dimensional photo of the three-dimensional terrain. In least-squares adjustment we map an n-dimensional observation space onto a u-dimensional parameter space. In this section we restrict our discussion to the mapping:

$$\mathbf{R}_n \to \mathbf{R}_m \tag{55.91}$$

with

$$n = 3 \tag{55.92}$$

$$m = 2 \tag{55.93}$$

A three-dimensional earth, approximated by a sphere or, better, by an ellipsoid of revolution, cannot be mapped onto a two-dimensional surface, which is flat at the start (plane) or can be made flat (the surface of a cylinder or a cone), without distorting the original relative positions in $\mathbf{R}_3$. Any figure or, better, the relative positions between an arbitrary number of points on a sphere or ellipsoid, will also be distorted when mapped onto a plane, cylinder, or cone. The distortions of the figure (or part thereof) will increase with the area. Likewise, the mapping will introduce distortions that will become larger as the extent of the area to be mapped increases.

If one approximates (maps) a sphere of the size of the earth, the radius R being

$$R \approx 6371.000 \text{ km} \tag{55.94}$$

onto a plane tangent in the center of one's engineering project of diameter D km, one finds increased errors in lengths, angles, and heights the further one gets away from the center of the project. Table 55.1 lists these errors in distance dS, angle $d\alpha$, and height dh, if one assumes the following case: one measures in the center of the project one angle of 60° and two equal distances of S km. In the plane assumption we would find in the two terminal points of both lines two equal angles of 60° and a distance between them of exactly S km. Basically, we have an equilateral triangle. In reality, on the curved, spherical earth we would measure angles larger than 60° and a distance between them shorter than S km. The error dh shows how the earth curves away from underneath the tangent plane in the center of the project.

TABLE 55.1 Errors in Length *dS* (km), Angle *dα* (arcseconds), and Height *dh* (km)[a]

Diameter, *D* (km)	Length, *S* (km)	*dS* (m)	*dS* (ppm)	*dα* (arcseconds)	*dα* (ppm)	*dh* (m)
0.100	0.050	–0.000	–0.000	0.000	0.000	0.000
0.200	0.100	–0.000	–0.000	0.000	0.000	0.001
0.500	0.250	–0.000	–0.000	0.000	0.000	0.005
2.000	1.000	–0.000	–0.003	0.001	0.005	0.078
5.000	2.500	–0.000	–0.019	0.007	0.032	0.491
20.000	10.000	–0.003	–0.308	0.110	0.509	7.848
50.000	25.000	–0.048	–1.925	0.688	3.184	49.050
200.000	100.000	–3.080	–30.797	11.002	50.937	784.790
500.000	250.000	–48.123	–192.494	68.768	318.372	4904.409
2000.000	1000.000	–3084.329	–3084.329	1101.340	5098.796	78,319.621

[a] Depending on the diameter *D* (km) of a project or distance *S* (=*D* 12) from the center of the project for an equilateral triangle with sides of *S* km.

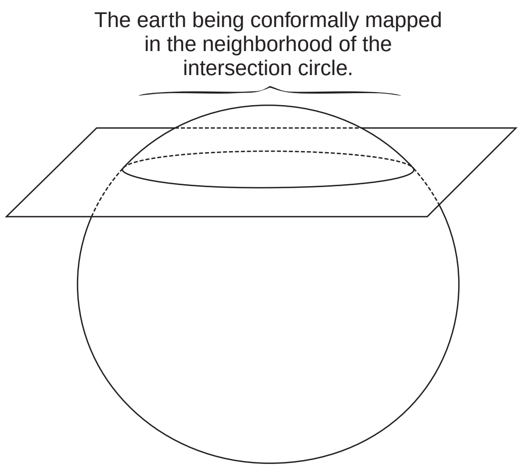

FIGURE 55.10 Mapping a sphere to a plane or lowered plane.

The table shows that errors in length and angle of larger than 1 ppm start to occur for project diameters larger than 20 km. Height differences obtained through leveling would be accurate enough, but vertical angles would start deviating from 90° by 1 arcsecond per 30 m. Within an area of 20 km fancy mapping procedures would not be needed to avoid errors of 1 ppm. The trouble starts if one wants to map an area of the size of the state of Indiana. A uniform strict mapping procedure has to be adhered to if one wants to work in one consistent system of mapping coordinates. In practice, a state of the size of Indiana is actually divided into two regions to keep the distortions within bounds. One may easily reduce the errors by a factor of 2 by making the plane not tangent to the sphere, but by lowering the plane from the center of the project by such an amount that the errors of *dS* in the center of the area are equal, but of opposite sign, to the errors *dS* at the border of the area (see Fig. 55.10). The U.S. State Plane Coordinate Systems are based on this practice.

As an alternative one may choose the mapping plane to be cylindrical or conical so that the mapping plane "follows" the earth's curvature at least in one direction (see Figs. 55.11 and 55.12). Also, with these alternatives distortions are reduced even further by having the cylindrical or conical surface not tangent to the sphere or ellipsoid, but intersecting the surface to be mapped just below the tangent point. After mapping, the cylinder or cone can be "cut" and made into a two-dimensional map. A cylinder and a cone are called *developable surfaces*.

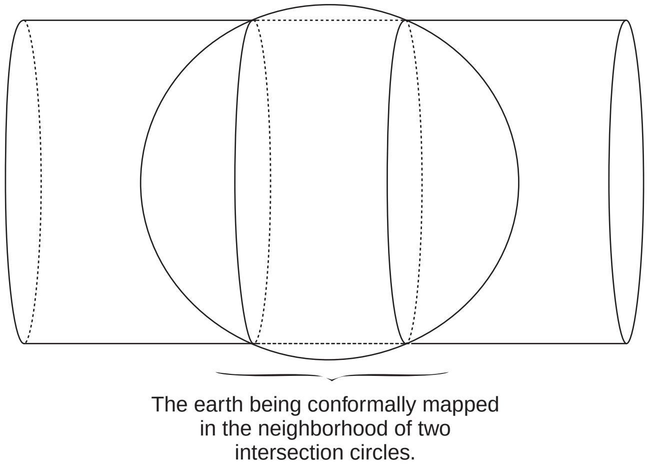

FIGURE 55.11 A cylinder trying to follow the earth's curvature.

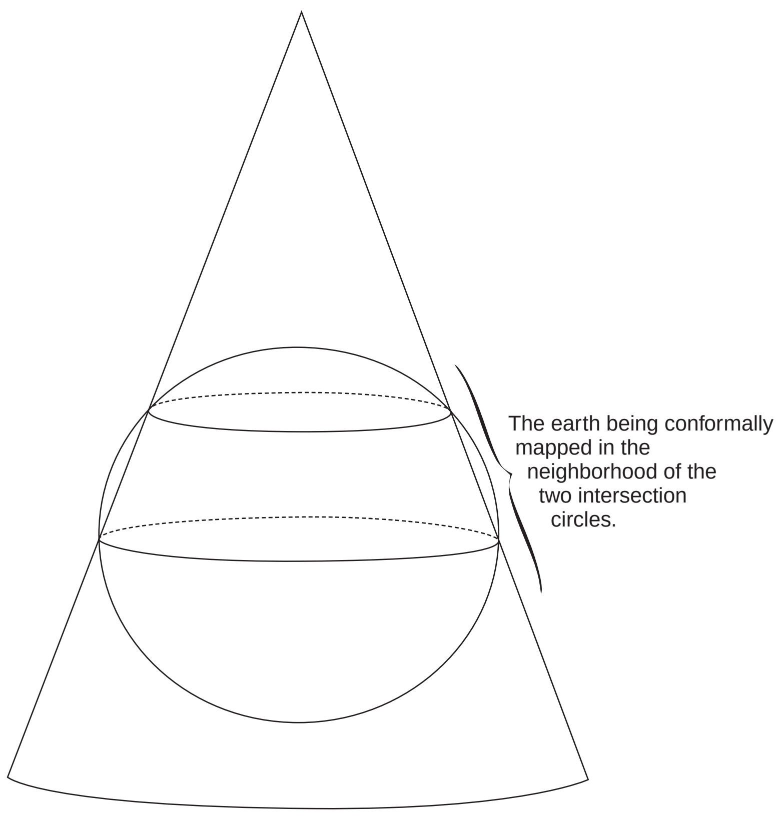

FIGURE 55.12 A cone trying to follow the earth's curvature.

The notion of distortion of a figure is applied to different elements of a figure. If a (spherical) triangle is mapped, one may investigate how the length of a side or the angle between two sides is distorted in the mapping plane.

Two Worlds

We live in R_3, which is to be mapped into R_2. In R_3 we need three quantities to position ourselves: $\{x, y, z\}$, $\{\lambda, \psi, h\}_R$, or $\{\lambda, \phi, h\}_{a,f}$; see Section 55.2. In R_2 we need only two quantities, such as two Cartesian mapping coordinates $\{X, Y\}$ or two polar mapping coordinates $\{r, \alpha\}$.

Real world → Mapped world

$$\left.\begin{array}{c}\{x, y, z\}\\ \text{or}\\ \{\lambda, \psi, h\}_R\\ \text{or}\\ \{\lambda, \phi, h\}_{a,f}\end{array}\right\} \rightarrow \left\{\begin{array}{c}\{X, Y\}\\ \text{or}\\ \{r, \alpha\}\end{array}\right.$$

The mapping M may be written symbolically as

$$\{r, \alpha\} = M\{x, y, z\}\,\} \tag{55.95}$$

or

$$\{X, Y\} = M'\{\lambda, \phi, h\}_{a,f} \tag{55.96}$$

M represents a mere mapping "prescription" of how the R_3 world is condensed into R_2 information. Note that the computer era made it possible to "store" the R_3 world digitally in R_3 (a file with three numbers per point). Only at the very end, if that information is to be presented on a map or computer screen, do we map to R_2.

The analysis of distortion is, in this view, the mere comparison of corresponding geometrical elements in the real world and in the mapped world. A distance $s\ (x, y, z)$ between points i and j in the real world is compared to a distance $S = S(X, Y)$ in the mapped world. A scale distortion may be defined as the ratio

$$\sigma = \frac{S\left(X_i, Y_i, X_j, Y_j\right)}{s\left(x_i, y_i, z_i, x_j, y_j, z_j\right)} \tag{55.97}$$

or, for infinitely small distances,

$$\sigma = \frac{dS\left(dX_{ij}, dY_{ij}\right)}{ds\left(dx_{ij}, dy_{ij}, dz_{ij}\right)} \tag{55.98}$$

where

$$x_{ij} = x_j - x_i \quad \text{and so forth} \tag{55.99}$$

Similarly, an angle θ_{jik} in point i to points j and k in the real world is being compared to an angle Θ_{jik} in the mapped world, the angular distortion $d\theta$:

$$d\theta = \Theta_{jik} - \theta_{jik} \tag{55.100}$$

Other features may be investigated to assess the distortion of a certain mapping; for example, one may want to compare the area in the real world to the area in the mapped world. Various mapping prescriptions (mapping equations) exist that minimize scale distortion, angular distortion, area distortion, or combinations of these. In surveying engineering, and for that matter in civil engineering, the most widely applied map projection is the one that minimizes angular distortion. Moreover, there exists a class of map projections that do not show any angular distortion throughout the map; in other words,

$$d\theta = 0 \tag{55.101}$$

This class of map projections is known as *conformal map projections*. One word of caution is needed: the angles are only preserved in an infinitely small area. In other words, the points *i*, *j*, *k* have to be infinitely close together.

The surveyor or civil engineer often works in a relatively small area (compared to the dimensions of the earth). Therefore, it is extremely handy for his or her angular measurements from theodolite or total station, made in the real world, to be preserved in the mapped world. As a matter of fact, any of a variety of conformal map projections are used throughout the world by national mapping agencies. The most widely used map projection, also used by the military, is a conformal map projection, the so-called *Universal Transverse Mercator* (UTM) projection. In the U.S. all states have adopted some sort of a state plane coordinate system. This system is basically a local reference frame based on a certain type of conformal mapping. See the following three subsections.

Without proof, the necessary and sufficient conditions for conformality are that the real-world coordinates p and q and the mapping coordinates X and Y fulfill the Cauchy–Riemann equations or conditions:

$$\begin{aligned} \frac{\partial X}{\partial p} &= +\frac{\partial Y}{\partial q} \\ \frac{\partial X}{\partial q} &= -\frac{\partial Y}{\partial p} \end{aligned} \tag{55.102}$$

Purposely, the real-world variables p and q have not been identified. First we want to enforce the natural restriction on p, q, X, Y: they have to be isometric coordinates. The mapping coordinates are often isometric by definition; however, the real-world coordinates, as the longitude λ and the spherical latitude ψ (or geodetic latitude ϕ), are not isometric.

For instance, one arcsecond in longitude expressed in meters is very latitude dependent and, moreover, is not equal to one arcsecond in latitude in the very same point; see Table 55.2.

Conformal Mapping Using Cartesian Differential Coordinates

In principle, we have four choices to map from R_3 to R_2:

A1: three-dimensional Cartesian → two-dimensional Cartesian
A2: three-dimensional Cartesian → two-dimensional curvilinear
B1: three-dimensional curvilinear → two-dimensional Cartesian
B2: three-dimensional curvilinear → two-dimensional curvilinear

Although all four modes have known applications, we treat an example in this section with the B1 mode of mapping, whereas the following subsection deals with an example from the B2 mode.

From Eqs. (55.29) and (55.30) we have

$$\begin{aligned} d\lambda_m &= (N+h)\cos\phi \, d\lambda_{\text{rad}} \\ d\phi_m &= (M+h)\, d\phi_{\text{rad}} \end{aligned} \tag{55.103}$$

TABLE 55.2 Radius of Curvature in the Meridian M, Radius of Curvature in the Prime Vertical N, and Metric Equivalence of 1 Arcsecond in Ellipsoidal or Spherical Longitude λ (m) and in Ellipsoidal or Spherical Latitude ϕ/ψ (m) as a Function of Geodetic Latitude ϕ and Spherical Latitude ψ

			Ellipsoid		Sphere	
ϕ/ψ	M	N	1 in. λ	1 in. ϕ	1 in. λ	1 in. ψ
00.0	6,335,439	6,378,137	30.922	30.715	30.887	30.887
10.0	6,337,358	6,378,781	30.455	30.724	30.418	30.887
20.0	6,342,888	6,380,636	29.069	30.751	29.025	30.887
30.0	6,351,377	6,383,481	26.802	30.792	26.749	30.887
40.0	6,361,816	6,386,976	23.721	30.843	23.661	30.887
50.0	6,372,956	6,390,702	19.915	30.897	19.854	30.887
60.0	6,383,454	6,394,209	15.500	30.948	15.444	30.887
70.0	6,392,033	6,397,072	10.607	30.989	10.564	30.887
80.0	6,397,643	6,398,943	05.387	31.017	05.364	30.887
90.0	6,399,594	6,399,594	00.000	31.026	00.000	30.887

Note: Ellipsoidal values for WGS84: a = 6,378,137 m, $1/f$ = 298.257 223 563. Spherical values: R = 6,371,000 m.

A line element (small distance) in the real world (on the ellipsoid, $h = 0$) is

$$ds^2 = d\lambda_m^2 + d\phi_m^2 \tag{55.104}$$

A line element (small distance) in the mapped world (on paper) is

$$dS^2 = dX^2 + dY^2 \tag{55.105}$$

Equation (55.104) leads to

$$ds^2 = N^2 \cos^2 \phi d\lambda_{\text{rad}}^2 + M^2 d\phi_{\text{rad}}^2 \tag{55.106}$$

or

$$ds^2 = N^2 \cos^2 \phi \left(d\lambda_{\text{rad}}^2 = \frac{M^2}{N^2 \cos^2 \phi} d\phi_{\text{rad}}^2 \right) \tag{55.107}$$

Since we want to work with isometric coordinates in the real world (note that the mapping coordinates $\{X, Y\}$ are already isometric), we introduce the new variable dq. In other words, Eq. (55.107) becomes

$$ds^2 = N^2 \cos^2 \phi \left(d\lambda^2 + dq^2 \right) \tag{55.108}$$

So, we have

$$dq = \frac{M}{N \cos\phi} d\phi_{\text{rad}} \tag{55.109}$$

Upon integration of Eq. (55.109) we obtain the isometric latitude q:

$$q = \ln \left[\tan\left(\frac{\pi}{4} + \frac{\phi}{2} \right) \left(\frac{1 - e \sin\phi}{1 + e \sin\phi} \right)^{e/2} \right] \tag{55.110}$$

The isometric latitude for a sphere ($e = 0$) becomes simply

$$q = \ln \tan\left(\frac{\pi}{4} + \frac{\psi}{2} \right) \tag{55.111}$$

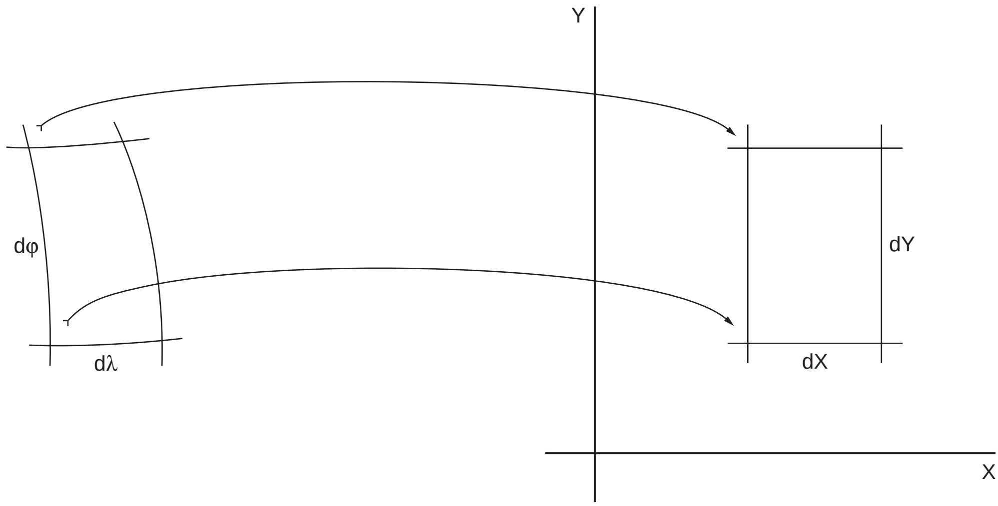

FIGURE 55.13 Spherical/ellipsoidal quadrangle mapped onto a planar Cartesian quadrangle.

Equating the variables from Eqs. (55.105) and (55.108) we have the mapping M between Cartesian mapping coordinates $\{X, Y\}$ and curvilinear coordinates $\{\lambda, q\}$ for both the sphere and the ellipsoid. Note that $\{\lambda, q\}$ play the role of the isometric coordinates $\{p, q\}$ in the Cauchy–Riemann equations.

So the mapping equations are simply

$$
\begin{aligned}
X &= \sigma\lambda \\
Y &= \sigma q
\end{aligned}
\tag{55.112}
$$

or, for the sphere,

$$
\begin{aligned}
X &= \sigma\lambda \\
Y &= \sigma \ln \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)
\end{aligned}
\tag{55.113}
$$

Equation (55.113) represents the conformal mapping equations from the sphere to R_2. These are the formulas of the well-known Mercator projection (cylindrical type). Inspection of the linear scale σ reveals that this factor depends on the term $M/(N \cos \phi)$ for the ellipsoid or $1/\cos \phi$ for the sphere. This means that the linear distortion is only latitude dependent. In order to minimize this distortion, we simply apply this mapping to regions that are elongated in the longitudinal direction, where the linear distortion is constant. In case we want to map an arbitrarily oriented elongated region, we simply apply a coordinate transformation. In the subsection on coordinate transformations and conformal mapping we will perform such a coordinate transformation on these mapping equations.

So far, we have mapped a spherical quadrangle $d\lambda$, $d\psi$ or ellipsoidal "quadrangle" $d\lambda$, $d\phi$ to a planar quadrangle dX, dY; see Fig. 55.13.

Conformal Mapping Using Polar Differential Coordinates

The alternative is to map an ellipsoidal or spherical quadrangle onto a polar quadrangle. Figure 55.14 shows that one option is to look for a mapping between the polar mapping coordinates $\{r, \alpha\}$ and the real-world coordinates $\{\lambda, \phi\}$ or $\{\lambda, \psi\}$.

The similarity of roles played by the radius r and the latitude ϕ or ψ is more apparent if we view the *colatitude* θ, since this real-world coordinate radiates from one point as the radius r does:

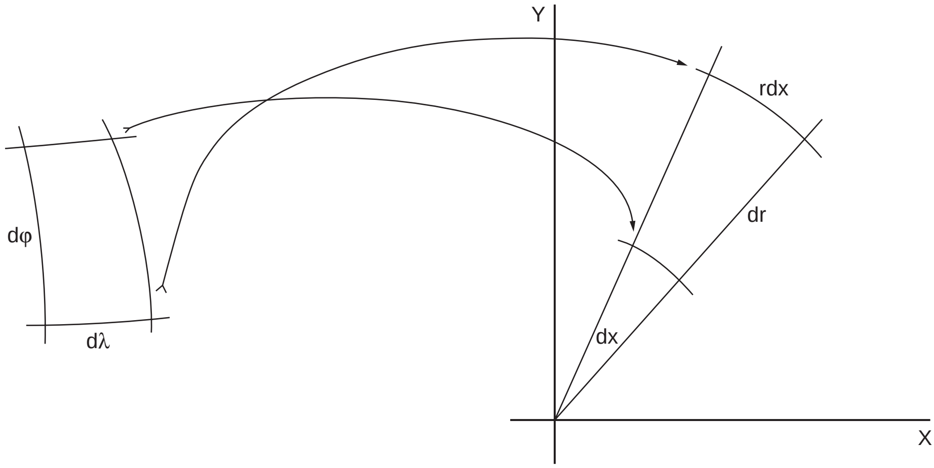

FIGURE 55.14 Spherical/ellipsoidal quadrangle mapped onto a planar polar quadrangle.

$$\theta = \frac{\pi}{2} - \psi \qquad (\text{or } \phi) \tag{55.114}$$

Considering now the line element ds on a sphere, we have

$$ds^2 = R^2 \sin^2 \theta d\lambda_{\text{rad}}^2 + R^2 d\theta_{\text{rad}}^2 \tag{55.115}$$

or

$$ds^2 = R^2 \sin^2 \theta \left(d\lambda_{\text{rad}}^2 + \frac{d\theta_{\text{rad}}^2}{\sin^2 \theta} \right) \tag{55.116}$$

Introducing the new variable dq', we have

$$dq' = \frac{d\theta_{\text{rad}}}{\sin \theta} \tag{55.117}$$

Integration of Eq. (55.117) gives the isometric colatitude q':

$$q' = \int dq = \int \frac{d\theta}{\sin \theta} \tag{55.118}$$

or

$$q' = -\ln \left(\cot \frac{\theta}{2} \right) \tag{55.119}$$

A line element on the sphere in terms of isometric coordinates is

$$ds^2 = R^2 \sin^2 \theta \left(d\lambda^2 + dq'^2 \right) \tag{55.120}$$

A line element dS (small distance) in the mapped world (on paper) is

$$dS^2 = r^2 d\alpha^2 + dr^2 \tag{55.121}$$

Now we want to derive isometric coordinates in the mapped world as well (we do not have a Cartesian, but a polar, representation). Among various options we choose

$$dS^2 = r^2\left(d\alpha^2 + \frac{dr^2}{r^2}\right) \tag{55.122}$$

Introducing the new variable $d\rho$,

$$d\rho = \frac{dr}{r} \tag{55.123}$$

Upon integration of Eq. (55.123), we obtain the isometric radius ρ:

$$\rho = \int d\rho = \int \frac{dr}{r} = \ln r + c_r \tag{55.124}$$

or

$$r = e^{\rho} \qquad r^2 = e^{2\rho} \tag{55.125}$$

The line element dS becomes with the new variable

$$dS^2 = e^{2\rho}\left(d\alpha^2 + d\rho^2\right) \tag{55.126}$$

The line element ds was

$$ds^2 = \mathbf{R}^2 \sin^2\theta\left(d\lambda^2 + dq'^2\right) \tag{55.127}$$

So the mapping equations are simply

$$\begin{aligned} \alpha &= \sigma\lambda \\ \rho &= \sigma q' \end{aligned} \tag{55.128}$$

or

$$\begin{aligned} \alpha &= \sigma\lambda \\ \ln r &= \sigma\left[-\ln\left(\cot\frac{\theta}{2}\right)\right] \end{aligned} \tag{55.129}$$

If one appropriately chooses the integration constant c_r, Eq. (55.126), to be

$$c_r = \ln 2 \tag{55.130}$$

the mapping Eq. (55.129) becomes

$$\begin{aligned} \alpha &= \sigma\lambda \\ \ln r &= \sigma\left[\ln 2 - \ln\left(\cot\frac{\theta}{2}\right)\right] = \ln\left(2\tan\frac{\theta}{2}\right) \end{aligned} \tag{55.131}$$

or

$$\begin{aligned} \alpha &= \sigma\lambda \\ r &= 2\sigma\tan\frac{\theta}{2} \end{aligned} \tag{55.132}$$

Equation (55.132) also represents conformal mapping equations from the sphere to R_2. They are the formulas of the well-known stereographic projection (planar type). Other choices of integration constants and integration interval would have led to the Lambert conformal projection (conical type).

The approaches laid out in this subsection and the preceding one are the theoretical basis of the U.S. State Plane Coordinate Systems. Refer to Stem [1991] for the formulas of the ellipsoidal equivalents.

Coordinate Transformations and Conformal Mapping

The two examples treated in the preceding two subsections can be treated for any arbitrary curvilinear coordinates. The widely used Transverse Mercator projection for the sphere is easily derived using a simple coordinate transformation. Rather than having the origin of the (co)latitude variable at the pole, we define a similar pole at the equator, and the new equator will be perpendicular to the old equator.

Having mapping poles at the equator leads to transverse types of conformal mapping. If the pole is neither at the North Pole nor at the equator, we obtain oblique variants of conformal mapping.

For the Transverse Mercator, the new equator may pass through a certain (old) longitude λ_0. For a UTM projection this λ_0 has specified values; for a state plane coordinate system the longitude λ may define the central meridian in a particular (part of the) state.

Two successive rotations will bring the old $\mathbf{x}$ frame to the new $\mathbf{x}'$ frame (see Section 55.2):

$$\mathbf{x}' = \mathbf{R}_1(\pi/2)\mathbf{R}_3(\lambda_0)\mathbf{x} \tag{55.133}$$

The original $\mathbf{x}$ frame expressed in curvilinear coordinates is

$$\mathbf{x} = R\begin{pmatrix} \cos\psi\cos\lambda \\ \cos\psi\sin\lambda \\ \sin\psi \end{pmatrix} \tag{55.134}$$

The new $\mathbf{x}'$-frame expressed in curvilinear coordinates is

$$\mathbf{x}' = R\begin{pmatrix} \cos\psi'\cos\lambda' \\ \cos\psi'\sin\lambda' \\ \sin\psi' \end{pmatrix} \tag{55.135}$$

Multiplying out the rotations in Eq. (55.135) we get

$$\mathbf{x}' = R\begin{pmatrix} \cos\psi'\cos\lambda' \\ \cos\psi'\sin\lambda' \\ \sin\psi' \end{pmatrix} = R\begin{pmatrix} x\cos\lambda_0 + y\sin\lambda_0 \\ z \\ x\sin\lambda_0 + y\cos\lambda_0 \end{pmatrix} \tag{55.136}$$

Substituting Eq. (55.134) into Eq. (55.136) and dividing by R we obtain

$$\begin{pmatrix} \cos\psi'\cos\lambda' \\ \cos\psi'\sin\lambda' \\ \sin\psi' \end{pmatrix} = \begin{pmatrix} \cos\psi\cos\lambda\cos\lambda_0 + \cos\psi\sin\lambda\sin\lambda_0 \\ \sin\psi \\ \cos\psi\cos\lambda\sin\lambda_0 - \cos\psi\sin\lambda\cos\lambda_0 \end{pmatrix} \tag{55.137}$$

which directly leads to

$$\tan\lambda' = \frac{\sin\psi}{\cos\psi\cos(\lambda-\lambda_0)} = \frac{\tan\psi}{\cos(\lambda-\lambda_0)}$$
$$\tan\psi' = \frac{\cos\psi\sin(\lambda-\lambda_0)}{\sqrt{\cos^2\psi\cos^2(\lambda-\lambda_0)+\sin^2\psi}} = \frac{\sin(\lambda-\lambda_0)}{\sqrt{\cos^2(\lambda-\lambda_0)+\tan^2\psi}} \tag{55.138}$$

The new longitudes λ' and latitudes ψ' are subjected to a (normal) Mercator projection according to

$$X' = \sigma\lambda'$$
$$Y' = \sigma q' = \sigma\ln\left[\tan\left(\frac{\pi}{4}+\frac{\psi'}{2}\right)\right] \tag{55.139}$$

The Transverse Mercator projection with respect to the central meridian λ_0 is obtained by a simple rotation, about –90°. The final mapping equations are

$$\begin{pmatrix} X \\ Y \\ 0 \end{pmatrix} = \mathbf{R}_3\left(-\frac{\pi}{2}\right)\begin{pmatrix} X' \\ Y' \\ 0 \end{pmatrix} = \begin{pmatrix} -Y' \\ X' \\ 0 \end{pmatrix} \tag{55.140}$$

When we start with ellipsoidal curvilinear coordinates, we cannot apply this procedure directly. However, when we follow a two-step procedure — mapping the ellipsoid conformal to the sphere, and then using the "rotated conformal" mapping procedure as just described — the treatise in this section will have a more general validity.

For more details on conformal projections using ellipsoidal coordinates, consult Bugayevskiy and Snyder [1998], Maling [1993], Stem [1991], and others.

55.5 Basic Concepts in Mechanics

Equations of Motion of a Point Mass in an Inertial Frame

To understand the motion of a satellite around the earth, we resort to two fundamental laws of physics: Isaac Newton's second law (the law of inertia) and Newton's law of gravitation.

Law of Inertia

The second law of Newton (the law of inertia) is as follows:

$$F = ma \tag{55.141}$$

The mass of a point mass m is the constant ratio that experimentally exists between the force **F**, acting on that point mass, and the acceleration that is the result of that force.

The acceleration **a**, the velocity **v**, and the distance **s** are related as follows:

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} \tag{55.142}$$

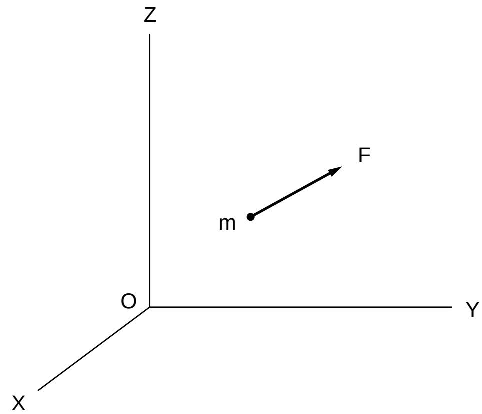

FIGURE 55.15 The acceleration of a point mass *m*.

Equation (55.141) can be written in vector form (see Fig. 55.15):

$$\mathbf{F} = m\mathbf{a} \quad \text{or} \quad \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = m \begin{pmatrix} \mathbf{a}_x \\ \mathbf{a}_y \\ \mathbf{a}_z \end{pmatrix} = m \begin{pmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{pmatrix} = m\ddot{\mathbf{X}} \tag{55.143}$$

with

$$\ddot{X} = \frac{d^2X}{dt^2} \text{ and similarly for } \ddot{Y} \text{ and } \ddot{Z} \tag{55.144}$$

Law of Gravitation

Until now we did not mention the cause of the force **F** acting on point mass *m*. If this force is caused by the presence of a second point mass, then Newton's law of gravitation says

$$F = G\frac{m_1 m_2}{|\mathbf{X}_{12}|^2} \tag{55.145}$$

Two point masses m_1 and m_2 attract each other with a force that is proportional to the masses of each point mass and inversely proportional to the square of the distance between them, $|\mathbf{X}_{12}|$.

$$G = 6.67259 \pm 0.00085 \times 10^{-11}\ \mathrm{m^3 kg^{-1} s^{-2}} \tag{55.146}$$

is the gravitation constant (e.g., Cohen and Taylor [1988]), and

$$|\mathbf{X}_{12}|^2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2 \tag{55.147}$$

The force **F** will be written in vector form (see Fig. 55.16):

$$\mathbf{F}_{12} = \begin{pmatrix} F_{12X} \\ F_{12Y} \\ F_{12Z} \end{pmatrix} = |\mathbf{F}_{12}| \begin{pmatrix} \sin\alpha \\ \sin\beta \\ \sin\gamma \end{pmatrix} = |\mathbf{F}_{12}| \begin{pmatrix} X_{12}/|\mathbf{X}_{12}| \\ Y_{12}/|\mathbf{X}_{12}| \\ Z_{12}/|\mathbf{X}_{12}| \end{pmatrix} \tag{55.148}$$

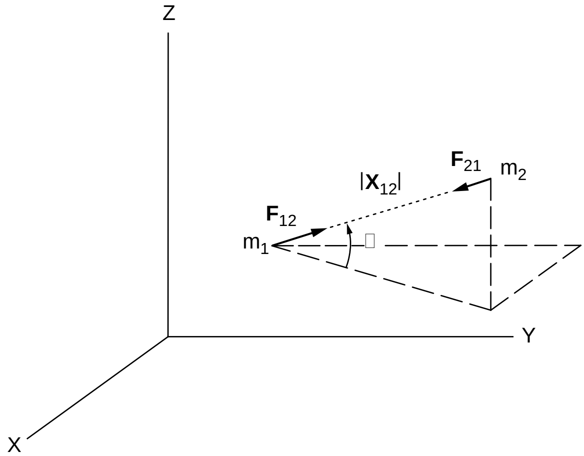

FIGURE 55.16 The attracting force $\mathbf{F}_{12}$ between two point masses m_1 and m_2.

with

$$X_{12} = X_2 - X_1 \qquad \text{and so forth} \tag{55.149}$$

Substituting Eq. (55.148) into Eq. (55.145),

$$\mathbf{F}_{12} = \frac{Gm_1m_2}{|\mathbf{X}_{12}|^2}\begin{pmatrix} X_{12}/|\mathbf{X}_{12}| \\ Y_{12}/|\mathbf{X}_{12}| \\ Z_{12}/|\mathbf{X}_{12}| \end{pmatrix} = \frac{Gm_1m_2}{|\mathbf{X}_{12}|^3}\begin{pmatrix} X_{12} \\ Y_{12} \\ Z_{12} \end{pmatrix} \tag{55.150}$$

or

$$\mathbf{F}_{12} = \frac{Gm_1m_2}{|\mathbf{X}_{12}|^3}\mathbf{X}_{12} \tag{55.151}$$

Equations (55.143) and (55.151) applied subsequently to two point masses m_1 and m_2 yield:

For m_1,

$$\mathbf{F}_1 = m_1\ddot{\mathbf{X}}_\mathbf{1} = \mathbf{F}_{12} = \frac{Gm_1m_2}{|\mathbf{X}_{12}|^3}\mathbf{X}_{12} \tag{55.152}$$

or

$$\ddot{\mathbf{X}}_\mathbf{1} = \frac{Gm_2}{|\mathbf{X}_{12}|^3}\mathbf{X}_{12} \tag{55.153}$$

For m_2,

$$\mathbf{F}_2 = m_2\ddot{\mathbf{X}}_\mathbf{2} = \mathbf{F}_{21} = -\mathbf{F}_{12} = \frac{-Gm_1m_2}{|\mathbf{X}_{12}|^3}\mathbf{X}_{12} \tag{55.154}$$

or

$$\ddot{\mathbf{X}}_\mathbf{2} = \frac{-Gm_1}{|\mathbf{X}_{12}|^3}\mathbf{X}_{12} \tag{55.155}$$

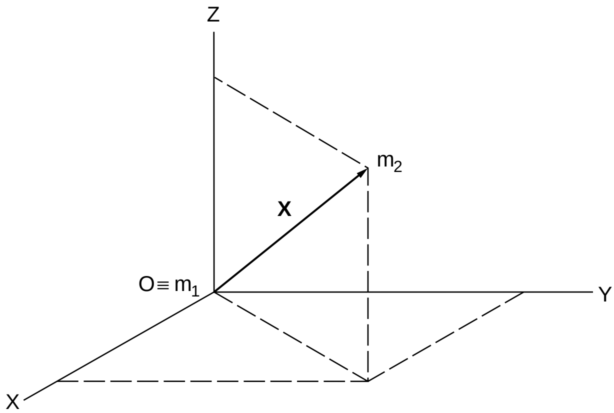

FIGURE 55.17 The inertial frame centered in m_1.

By subtracting Eq. (55.153) from Eq. (55.155) we get

$$\ddot{\mathbf{X}}_{12} = \ddot{\mathbf{X}}_2 - \ddot{\mathbf{X}}_1 = \frac{-G(m_1 + m_2)}{|\mathbf{X}_{12}|^3} \mathbf{X}_{12} \tag{55.156}$$

If m_1 resides in the origin of the inertial frame, then the subindices may be omitted:

$$\ddot{\mathbf{X}} = \frac{-G(m_1 + m_2)}{|\mathbf{X}|^3} \mathbf{X} \tag{55.157}$$

Equation (55.157) represents the equations of motion of m_2 in an inertial frame centered at m_1; see Fig. 55.17.

It will be clear that in m_1 the origin of an inertial frame can be defined if and only if $m_1 + m_2$. For the equations of motion of a satellite $m = m_2$ orbiting the earth $M = m_1$, Eq. (55.157) simplifies to

$$\ddot{\mathbf{X}} = \frac{-GM}{|\mathbf{X}|^3} \mathbf{X} \tag{55.158}$$

In Eq. (55.158) the product of the gravitational constant G and the mass of the earth M appears. For this product, the *geocentric gravitational constant*, a new equation is introduced:

$$\mu = GM \tag{55.159}$$

Potential

The equations of motion around a mass M,

$$\ddot{\mathbf{X}} = \frac{-\mu}{|\mathbf{X}|^3} \mathbf{X} \tag{55.160}$$

can be obtained through the definition of a scalar function V:

$$V = V(X, Y, Z) = \frac{\mu}{|\mathbf{X}|} = \frac{\mu}{(X^2 + Y^2 + Z^2)^{1/2}} \tag{55.161}$$

The partial derivatives of V with respect to $\mathbf{X}$ are

$$\frac{\partial V}{\partial X} = -\frac{\mu}{|\mathbf{X}|^2}\frac{\partial|\mathbf{X}|}{\partial X} = -\frac{\mu}{|\mathbf{X}|^3}X \tag{55.162}$$

$$\frac{\partial V}{\partial Y} = -\frac{\mu}{|\mathbf{X}|^2}\frac{\partial|\mathbf{X}|}{\partial Y} = -\frac{\mu}{|\mathbf{X}|^3}Y \tag{55.163}$$

$$\frac{\partial V}{\partial Z} = -\frac{\mu}{|\mathbf{X}|^2}\frac{\partial|\mathbf{X}|}{\partial Z} = -\frac{\mu}{|\mathbf{X}|^3}Z \tag{55.164}$$

Consequently, the equations of motion may be written as

$$\ddot{\mathbf{X}} = \begin{pmatrix} \dfrac{\partial V}{\partial X} \\ \dfrac{\partial V}{\partial Y} \\ \dfrac{\partial V}{\partial Z} \end{pmatrix} \equiv \operatorname{grad} V \equiv \nabla V \tag{55.165}$$

V has physical significance: it is the potential of a point mass of negligible mass in a gravity field of a point mass with sizable mass M at a distance $|\mathbf{X}|$.

55.6 Satellite Surveying

The Global Positioning System has become a tool used in a variety of fields within and outside engineering.

Positioning has become possible with accuracies from the subcentimeter level for high-accuracy geodetic applications — as used in state, national, and global geodetic networks and for deformation analysis in engineering and geophysics — and to the hectometer level in navigation applications. As in the space domain, a variety of accuracy classes may be assigned to the time domain: GPS provides a means to obtain position and velocity determinations averaged over time spans from subseconds (instantaneous) to 1 or 2 days. Stationary applications of the observatory type are used in GPS tracking for orbit improvement.

First the physics and mathematics of the space segment will be given (without derivations).

Numerical Solution of Three Second-Order Differential Equations

Equation (55.158) represents the equations of motion of a satellite expressed in vector form. Written in the three Cartesian components, we have

$$\begin{pmatrix} \dfrac{d^2X}{dt^2} \\ \dfrac{d^2Y}{dt^2} \\ \dfrac{d^2Z}{dt^2} \end{pmatrix} = \frac{-GM}{\left(X^2 + Y^2 + Z^2\right)^{3/2}} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{55.166}$$

or

$$\ddot{X} = C\left[X \Big/ \left(X^2 + Y^2 + Z^2\right)^{3/2}\right]$$
$$\ddot{Y} = C\left[Y \Big/ \left(X^2 + Y^2 + Z^2\right)^{3/2}\right] \qquad (55.167)$$
$$\ddot{Z} = C\left[Z \Big/ \left(X^2 + Y^2 + Z^2\right)^{3/2}\right]$$

with $C = GM$. Rather than solving for three second-order differential equations (DEs), we make a transformation to six first-order DEs. Introduce three new variables U, V, W:

$$U = \dot{X} = \frac{dX}{dt}; \quad V = \dot{Y} = \frac{dY}{dt}; \quad W = \dot{Z} = \frac{dZ}{dt} \qquad (55.168)$$

Equations (55.162) through (55.165) yield the following six DEs:

$$U = \dot{X}$$
$$V = \dot{Y}$$
$$W = \dot{Z}$$
$$\dot{U} = C\left[X \Big/ \left(X^2 + Y^2 + Z^2\right)^{3/2}\right] \qquad (55.169)$$
$$\dot{V} = C\left[Y \Big/ \left(X^2 + Y^2 + Z^2\right)^{3/2}\right]$$
$$\dot{W} = C\left[Z \Big/ \left(X^2 + Y^2 + Z^2\right)^{3/2}\right]$$

Integration of Eq. (55.169) results in six constants of integration. One is free to choose at an epoch t_0 six variables, $\{X_0, Y_0, Z_0, U_0, V_0, W_0\}$ or $\{X_0, Y_0, Z_0, \dot{X}_0, \dot{Y}_0, \dot{Z}_0\}$. These six starting values determine uniquely the orbit of m around M. In other words, if we know the position $\{X_0, Y_0, Z_0\}$ and its velocity $\{\dot{X}_0, \dot{Y}_0, \dot{Z}_0\}$ at an epoch t_0, then we are able to determine the position and velocity of m at any other epoch t by numerical integration of Eq. (55.169).

Analytical Solution of Three Second-Order Differential Equations

The differential equations of an earth-orbiting satellite can also be solved analytically. Without derivation, the solution is presented in computational steps in terms of transformation formulas.

In history the solution to the motion of planets around the sun was found before its explanation. Through the analysis of his own observations and those made by Tycho Brahe, Johannes Kepler discovered certain regularities in the motions of planets around the sun and formulated the following three laws:

1. Formulated 1609: The orbit of each planet around the sun is an ellipse. The sun is in one of the two focal points.
2. Formulated 1609: The sun–planet line sweeps out equal areas in equal time periods.
3. Formulated 1611: The ratio between the square of a planet's orbital period and the third power of its average distance from the sun is constant.

Kepler's third law leads to the famous equation

$$n^2 a^3 = GM \qquad (55.170)$$

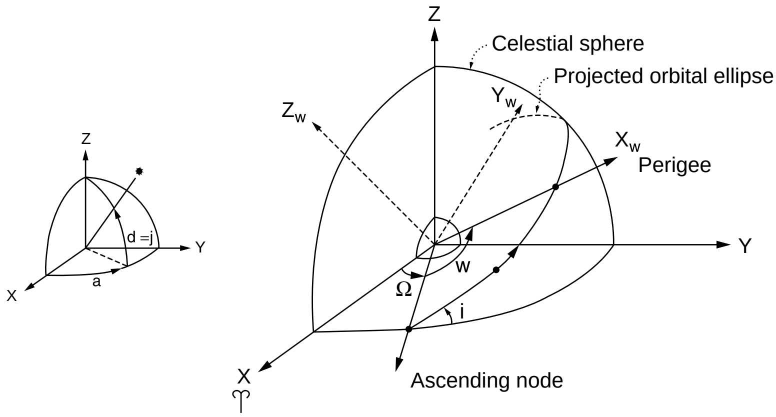

FIGURE 55.18 Celestial sphere with projected orbit ellipse and equator.

in which n is the average angular rate and a the semimajor axis of the orbital ellipse.

In 1665–1666 Newton formulated his more fundamental laws of nature (which were only published after 1687) and showed that Kepler's laws follow from them.

Orientation of the Orbital Ellipse

In a (quasi-)inertial frame the ellipse of an earth-orbiting satellite has to be positioned: the focal point will coincide with the center of mass of the earth. Instead of picturing the ellipse itself we project the ellipse on a celestial sphere centered at the CoM. On the celestial sphere we also project the earth's equator (see Fig. 55.18).

The orientation of the orbit ellipse requires three orientation angles with respect to the inertial frame *XYZ*: two for the orientation of the plane of the orbit (Ω and *I*) and one for the orientation of the ellipse in the orbital plane in terms of the point of closest approach, the perigee (ω).

Ω represents the right ascension (α) of the ascending node. The ascending node is the (projected) point where the satellite rises above the equator plane. *I* represents the inclination of the orbital plane with respect to the equator plane. ω represents the *argument of perigee*: the angle from the ascending node (in the plane of the orbit) to the perigee (for planets, the perihelion), which is that point where the satellite (planet) approaches closest to the earth (sun) or, more precisely, the CoM of the earth (sun).

We define now another reference frame $\mathbf{X}_\omega$, of which the $X_\omega Y_\omega$ plane coincides with the orbit plane. The X_ω axis points to the perigee, and the origin coincides with the earth's CoM (≡ focal point ellipse ≡ origin of **X** frame). The relationship between the inertial frames $\mathbf{X}_I$ and $\mathbf{X}_\omega$ is

$$\mathbf{X}_I = \mathbf{R}_{I?}\mathbf{X}_? \tag{55.171}$$

in which

$$\mathbf{R}_{I?} = \mathbf{R}_3(-\Omega)\mathbf{R}_1(-I)\mathbf{R}_3(-\omega) \tag{55.172}$$

and

$$\mathbf{X}_\omega = \begin{pmatrix} X_\omega \\ Y_\omega \\ Z_\omega = 0 \end{pmatrix} \tag{55.173}$$

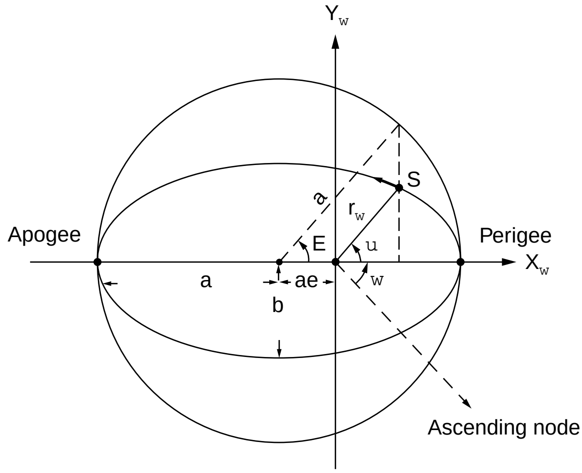

FIGURE 55.19 The position of the satellite S in the orbital plane.

Reference Frame in the Plane of the Orbit

Now that we know the orientation of the orbital ellipse we have to define the size and shape of the ellipse and the position of the satellite along the ellipse at a certain epoch t_0.

Similarly to the earth's ellipsoid, discussed in Section 55.2, we define the ellipse by a semimajor axis a and eccentricity e. In orbital mechanics it is unusual to describe the shape of the orbital ellipse by its flattening.

The position of the satellite in the orbital $X_\omega Y_\omega$ plane is depicted in Fig. 55.19. In the figure the auxiliary circle enclosing the orbital ellipse reveals the following relationships:

$$X_\omega = r_\omega \cos \nu = a(\cos E - e) \tag{55.174}$$

$$Y_\omega = r_\omega \sin \nu = a\sqrt{1-e^2}\, \sin E \tag{55.175}$$

in which

$$r_\omega = a(1 - e\cos E) \tag{55.176}$$

In Fig. 55.19 and the Eqs. (55.174) through (55.176), a is the the semimajor axis of the orbital ellipse, b is the semiminor axis of the orbital ellipse, e is the eccentricity of the orbital ellipse, with

$$e^2 = \frac{a^2 - b^2}{a^2} \tag{55.177}$$

ν is the the true anomaly, sometimes denoted by an f, and E is the eccentric anomaly.

The relation between the true and the eccentric anomalies can be derived to be

$$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\nu}{2}\right) \tag{55.178}$$

Substitution of Eqs. (55.174) through (55.176) into (55.171) gives

$$\mathbf{X_I} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathbf{R}_3(-\Omega)\mathbf{R}_1(-I)\mathbf{R}_3(-\omega)\begin{pmatrix} a(\cos E - e) \\ a\sqrt{1-e^2}\sin E \\ 0 \end{pmatrix} \tag{55.179}$$

In Eq. (55.179) the Cartesian coordinates are expressed in the six so-called Keplerian elements: a, e, I, Ω, ω, and E. Paraphrasing an earlier remark: if we know the position of the satellite at an epoch t_0 through $\{a, e, I, \Omega, \omega$, and $E_0\}$, we are capable of computing the position of the satellite at an arbitrary epoch t through the Eq. (55.179) if we know the relationship in time between E and E_0. In other words, how does the angle E increase with time?

We define an auxiliary variable (angle) M, which increases linearly in time with the mean motion n (= $(GM/a^3)^{1/2}$) according to Kepler's third law. The angle M, the *mean anomaly*, may be expressed as function of time by

$$M = M_0 + n(t - t_0) \tag{55.180}$$

Through Kepler's equation

$$M = E - e\sin E \tag{55.181}$$

the (time) relationship between M and E is given. Kepler's equation is the direct result of the enforcement of Kepler's second law (equal area law).

Combining Eqs. (55.180) and (55.181) gives an equation that expresses the relationship between a given eccentric anomaly E_0 (or M_0 or ν_0) at an epoch t_0 and the eccentric anomaly E at an arbitrary epoch t:

$$E - E_0 = e(\sin E - \sin E_0) + n(t - t_0) \tag{55.182}$$

Transformation from Keplerian to Cartesian Orbital Elements

So far, the position vector $\{X, Y, Z\}$ of the satellite has been expressed in terms of the Keplerian elements. The transformation is complete when we express the velocity vector $\{\dot{X}_0, \dot{Y}_0, \dot{Z}_0\}$ in terms of those Keplerian elements. Differentiating Eq. (55.171) with respect to time we get

$$\dot{\mathbf{X}}_I = \mathbf{R}_{\mathbf{I}?}\dot{\mathbf{X}}_? + \dot{\mathbf{R}}_{I?}\mathbf{X}_? \tag{55.183}$$

Since we consider the two-body problem with $m_1 = M$? $m = m_2$, the orientation of the orbital ellipse is time independent in the inertial frame. This means that the orientation angles I, Ω, ω are time independent as well:

$$\dot{\mathbf{R}}_{\mathbf{I}\omega} = [0] \tag{55.184}$$

Equation (55.183) simplifies to

$$\dot{\mathbf{X}}_\mathbf{I} = \mathbf{R}_{I?}\dot{\mathbf{X}}_? \tag{55.185}$$

Differentiating Eqs. (55.174) through (55.176) with respect to time we have

$$\dot{X}_\omega = -a\dot{E}\sin E \tag{55.186}$$

$$\dot{Y}_\omega = -a\dot{E}\sqrt{1-e^2}\cos E \tag{55.187}$$

$$\dot{r}_\omega = a\dot{E}e\sin E \tag{55.188}$$

The remaining variable $\dot{E}$ is obtained through differentiation of Eq. (55.181):

$$\dot{E} = \frac{n}{1-e\cos E} \tag{55.189}$$

Now all transformation formulas express the Cartesian orbital elements (state vector elements) in terms of the six Keplerian elements:

$$\left[\mathbf{X}_I \cdot \dot{\mathbf{X}}_I\right] = \mathbf{R}_{I\omega}\left[\mathbf{X}_\omega \cdot \dot{\mathbf{X}}_\omega\right] \tag{55.190}$$

or

$$\left[\mathbf{X}_I \cdot \dot{\mathbf{X}}_I\right] = \begin{bmatrix} X & \cdot & \dot{X} \\ Y & \cdot & \dot{Y} \\ Z & \cdot & \dot{Z} \end{bmatrix} = \tag{55.191}$$

$$= \mathbf{R}_3(-\Omega)\mathbf{R}_1(-I)\mathbf{R}_3(-\omega)\begin{bmatrix} a(\cos E - e) & \cdot & -a\dot{E}\sin E \\ a\sqrt{1-e^2}\sin E & \cdot & a\dot{E}\sqrt{1-e^2}\cos E \\ 0 & \cdot & 0 \end{bmatrix} \tag{55.192}$$

Transformation from Cartesian to Keplerian Orbital Elements

To compute the inertial position of a satellite in a central force field, it is simpler to perform a time update in the Keplerian elements than in the Cartesian elements. The time update takes place in Eqs. (55.180) through (55.182).

Schematically the following procedure is to be followed:

$t_0:\{X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}\}$

$\downarrow$ Conversion to Keplerian elements, this subsection

$t_0:\{a, e, I, \Omega, \omega, E_0\}$

$\downarrow$ Conversion to Keplerian, Eq. (55.182)

$t_1:\{a, e, I, \Omega, \omega, E_1\}$

$\downarrow$ Conversion to Cartesian elements, previous subsection

$t_1:\{X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}\}$

The conversion from Keplerian elements to state vector elements has been treated in the previous subsection. In this section the somewhat more complicated conversion from position and velocity vector to Keplerian representation will be described. Basically we "invert" Eq. (55.192) by solving for the six elements $\{a, e, I, \Omega, \omega, E_0\}$ in terms of the six state vector elements.

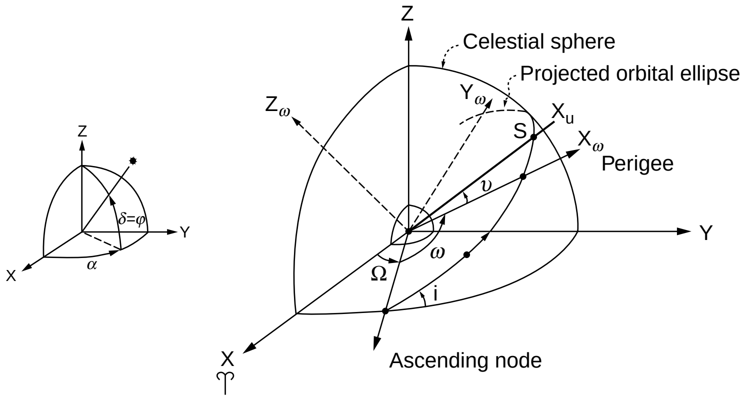

FIGURE 55.20 The orbital reference frame $\mathbf{X}_u$.

First, we introduce another reference frame $\mathbf{X}_u$:

X_I, Y_I, Z_I: inertial reference frame (X_I axis → vernal equinox)

X_ω, Y_ω, Z_ω: orbital reference frame (X_ω axis → perigee)

X_u, Y_u, Z_u: orbital reference frame (X_u axis → satellite)

The $\mathbf{X}_u$ frame is defined similarly to the $\mathbf{X}_\omega$ frame, except that the $\mathbf{X}_u$ axis continuously points to the satellites (see Fig. 55.20). Thus,

$$\mathbf{X_u} = \begin{pmatrix} \mathbf{X_u} \\ \mathbf{Y_u} \\ \mathbf{Z_u} \end{pmatrix} = \begin{pmatrix} \mathbf{X_u} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{55.193}$$

The angle in the orbital plane enclosed by the X_u axis and the direction to the ascending node is called u, the *argument of latitude*, with

$$u = \omega + \nu \tag{55.194}$$

As in the discussion of the orientation of the orbit ellipse, the following relationships hold:

$$\mathbf{X}_\omega = \mathbf{R}_{\omega I}\mathbf{X}_I \tag{55.195}$$

$$\mathbf{X}_u = \mathbf{R}_{uI}\mathbf{X}_I \tag{55.196}$$

$$\mathbf{X}_u = \mathbf{R}_{u?}\mathbf{X}_? = \mathbf{R}_{u?}\mathbf{R}_{?I}\mathbf{X}_I \tag{55.197}$$

Figure 55.20 reveals that

$$\mathbf{R}_{\omega I} = \mathbf{R}_3(\omega)\mathbf{R}_I(I)\mathbf{R}_3(\Omega) \tag{55.198}$$

$$\mathbf{R}_{uI} = \mathbf{R}_3(u)\mathbf{R}_I(I)\mathbf{R}_3(\Omega) \tag{55.199}$$

$$= \mathbf{R}_3(\nu + \omega)\mathbf{R}_I(I)\mathbf{R}_3(\Omega) \tag{55.200}$$

$$=\mathbf{R}_3(v)\mathbf{R}_3(\omega)\mathbf{R}_I(I)\mathbf{R}_3(\Omega) \tag{55.201}$$

$$=\mathbf{R}_3(v)\mathbf{R}_{\omega I}=\mathbf{R}_{u\omega}\mathbf{R}_{\omega I} \tag{55.202}$$

We define a vector **h** perpendicular to the orbital plane according to

$$\mathbf{h} \equiv \mathbf{X} \times \dot{\mathbf{X}} = |\mathbf{h}|\mathbf{w} = |\mathbf{h}| \begin{pmatrix} \sin\Omega \sin I \\ -\cos\Omega \sin I \\ \cos \mathbf{I} \end{pmatrix} \tag{55.203}$$

in which **w** is the unit vector along **h**. Consequently,

$$\mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} Y\dot{Z} - Z\dot{Y} \\ Z\dot{X} - X\dot{Z} \\ X\dot{Y} - Y\dot{X} \end{pmatrix} \tag{55.204}$$

h represents the angular momentum vector (vector product of position vector and velocity vector). The Keplerian elements Ω and I follow directly from Eqs. (55.203) and (55.204):

$$\tan\Omega = \frac{h_1}{-h_2} \tag{55.205}$$

$$\tan I = \frac{\sqrt{h_1^2 + h_2^2}}{h_3} \tag{55.206}$$

From

$$\mathbf{R}_3(-u)\mathbf{X}_u = \mathbf{R}_\mathrm{I}(I)\mathbf{R}_3(\Omega)\mathbf{X}_\mathrm{I} \tag{55.207}$$

it follows that

$$X_u \cos u = X\cos\Omega + Y\sin\Omega \tag{55.208}$$

$$X_u \sin u = -X\cos I\sin\Omega + Y\cos I\cos\Omega + Z\sin I \tag{55.209}$$

and

$$\tan u = \frac{-X\cos I\sin\Omega + Y\cos I\cos\Omega + Z\sin I}{X\cos\Omega + Y\sin\Omega} \tag{55.210}$$

Before determining the third Keplerian element defining the orientation of the orbit (ω) from the argument of latitude (u), we define the following quantities:

- Length r of the radius vector **X**:

$$r \equiv |\mathbf{X}| = \left(X^2 + Y^2 + Z^2\right)^{1/2} \tag{55.211}$$

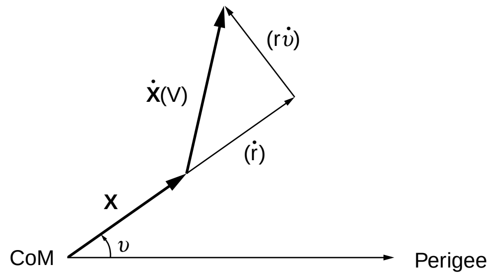

FIGURE 55.21 Illustration of the Vis-Viva equation.

- Length $\dot{r}$ of the radial component of the velocity vector:

$$\dot{r} \equiv \left|\mathbf{X}\cdot\dot{\mathbf{X}}\right| \big/ r = \left|X\dot{X}+Y\dot{Y}+Z\dot{Z}\right| \big/ r \tag{55.212}$$

The *Vis-Viva equation* gives the relationship between the length (magnitude) V of the velocity vector $\dot{\mathbf{X}}$ and the length r of the position vector $\mathbf{X}$ through the radial component ($\dot{r}$) and the tangential component ($r\dot{v}$) (see Fig. 55.21):

$$V^2 \equiv \left|\dot{\mathbf{X}}\right|^2 = \dot{r}^2 + (r\dot{v})^2 = \dot{r}^2 + \left|\mathbf{h}\right|^2 \big/ r^2 = GM\left(\frac{2}{r}-\frac{1}{a}\right) \tag{55.213}$$

From Eq. (55.213) it follows that

$$a = \frac{\left|\mathbf{X}\right|GM}{2GM-\left|\mathbf{X}\right|V^2} = \frac{rGM}{2GM-rV^2} \tag{55.214}$$

In a similar manner one arrives at

$$1-e^2 = \frac{\left|\mathbf{h}\right|^2}{aGM} \tag{55.215}$$

From Eqs. (55.188), (55.189), (55.176), and (55.170) the eccentric anomaly E may be computed:

$$\tan E = \frac{\sin E}{\cos E} = \left(\frac{r\dot{r}}{e\sqrt{GMa}}\right) \Big/ \left(\frac{a-r}{ae}\right) \tag{55.216}$$

The mean anomaly M is given by

$$M = E - e\sin E \tag{55.217}$$

The true anomaly v follows from Eqs. (55.174) and (55.175):

$$\tan v = \frac{\sqrt{1-e^2}\,\sin E}{\cos E - e} \tag{55.218}$$

after which, finally, the last Keplerian element, ω, is determined through Eq. (55.194):

$$\omega = u - v \tag{55.219}$$

Orbit of a Satellite in a Noncentral Force Field

The equations of motion for a real satellite are more difficult than reflected by Eq. (55.165). First of all, we do not deal with a central force field: the earth is not a sphere, and it does not have a radial symmetric density. Secondly, we deal with other forces, chiefly the gravity of the moon and the sun, atmospheric drag, and solar radiation pressure. Equation (55.165) gets a more general meaning if we suppose that the potential function is generated by the sum of the forces acting on the satellite:

$$V = V_c + V^t_{\text{nc}} + V^t_{\text{sun}} + V^t_{\text{moon}} + \dots \tag{55.220}$$

with the central part of the earth's gravitational potential

$$V_c = \mu / |\mathbf{X}| \tag{55.221}$$

and the noncentral and time-dependent part of the earth's gravitational field

$$V^t_{\text{nc}} = \left[\text{see Eq. } (52.224)\right] \tag{55.222}$$

and so forth.

The superscript t has been added to various potentials to reflect their time variance with respect to the inertial frame.

The equations of motion to be solved are

$$\begin{aligned} \ddot{\mathbf{X}} &= \nabla\left(V_c + V^t_{\text{nc}} + V^t_{\text{sun}} + V^t_{\text{moon}} + \dots\right) \\ &= \nabla V_c + \nabla V^t_{\text{nc}} + \nabla V^t_{\text{sun}} + \nabla V^t_{\text{moon}} + \dots \end{aligned} \tag{55.223}$$

For the earth's gravitational field we have (in an earth-fixed frame)

$$V_c + V_{\text{nc}} = \frac{\mu}{r}\left[1 + \sum_{l=1}^{\infty}\sum_{m=0}^{l}\left(\frac{a_e}{r}\right)^l \left(C_{lm}\cos m\lambda + S_{lm}\sin m\lambda\right) P_{lm}(\sin\phi)\right] \tag{55.224}$$

With Eq. (55.224) one is able to compute the potential at each point $\{\lambda, \phi, r\}$ necessary for the integration of the satellite's orbit. The coefficients C_{lm} and S_{lm} of the spherical harmonic expansion are in the order of 10^{-6}, except for C_{20} ($l = 2$, $m = 0$), which is about 10^{-3}. This has to do with the fact that the earth's equipotential surface at mean sea level can be best approximated by an ellipsoid of revolution. One has to realize that the coefficients C_{lm}, S_{lm} describe the shape of the potential field and not the shape of the physical earth, despite a high correlation between the two. P_{lm} sin ϕ are the associated Legendre functions of the first kind, of degree l and order m; a_e is some adopted value for the semimajor axis (equatorial radius) of the earth. See Section 55.8 for values of a_e, μ (= GM), and C_{20} (= $-J_2$).

The equatorial radius a_e, the geocentric gravitational constant GM, and the dynamic form factor J_2 characterize the earth by an ellipsoid of revolution of which the surface is an equipotential surface.

Restricting ourselves to the central part ($\mu = GM$) and the dynamic flattening ($C_{20} = -J_2$), Eq. (55.224) becomes

$$V_c + V_{nc} = \frac{\mu}{r}\left[1 + \frac{J_2 a_e^2}{2r^2}\left(1 - 3\sin^2\phi\right)\right] \tag{55.225}$$

with

$$\sin\phi = (\sin\delta =)\frac{z}{r} \tag{55.226}$$

in which ϕ is the latitude and δ the declination; see Fig. 55.20.

A solution of the DE (by substitution of Eq. (55.225) in Eq. (55.165))

$$\ddot{\mathbf{X}} = \nabla\left(V_c + V_{20}\right) \tag{55.227}$$

in a closed analytical expression is not possible. The solution expressed in Keplerian elements shows periodic perturbations and some dominant secular effects. An approximate solution using only the latter effects is (position only)

$$\mathbf{X}_I = \mathbf{R_3}[-\left(\Omega_0 + \dot{\Omega}\Delta t\right)]\mathbf{R_1}\left(-I\right)\mathbf{R_3}\left[-\left(\omega_0 + \dot{\omega}\Delta t\right)\right]\mathbf{X}_\omega \tag{55.228}$$

with

$$\Delta t = t - t_0 \tag{55.229}$$

with

$$\dot{\Omega} = -\frac{3}{2}\frac{J_2 a_e^2}{a^2\left(1-e^2\right)^2} n \cos I \tag{55.230}$$

$$\dot{\omega} = \frac{3}{2}\frac{J_2 a_e^2}{a^2\left(1-e^2\right)^2} n \left(2 - \frac{5}{2}\sin^2 I\right) \tag{55.231}$$

$$n = n_0\left[1 + \frac{3}{2}\frac{J_2 a_e^2 \sqrt{1-e^2}}{a^2\left(1-e^2\right)^2}\left(1 - \frac{3}{2}\sin^2 I\right)\right] \tag{55.232}$$

$$n_0 = \sqrt{\frac{GM}{a^3}} \tag{55.233}$$

whenever we have:

$I = 0°$	equatorial orbit
0°	$< I <$ 90°direct orbit
$I = 90°$	polar orbit
$90° < I < 180°$	retrograde orbit
$I = 180°$	retrograde equatorial orbit

Equation (55.230) shows that the ascending node of a direct orbit slowly drifts to the west. For a satellite at a height of about 150 km above the earth's surface the right ascension of the ascending node decreases about 9º per day. For satellites used in geodesy and geodynamics, such as STARLETTE (a = 7340 km, I = 50°) and LAGEOS-I (a = 12,270 km, I = 110°), these values are –4° per day and +1/3° per day, respectively.

The satellites belonging to the Global Positioning System have an inclination of about 55°. Their nodal regression rate is about –0.04187° per day.

The Global Positioning System

Introduction

The Navstar GPS space segment consists of 24 satellites, plus a few spare ones. This means that the full satellite constellation, in six orbital planes at a height of about 20,000 km, is completed. With this number of satellites, three-dimensional positioning is possible every hour of the day. However, care must be exercised, since an optimum configuration for three-dimensional positioning is not available on a full day's basis.

In the meantime, GPS receivers, ranging in cost between \$300 and \$20,000, are readily available. Over 100 manufacturers are marketing receivers, and the prices are still dropping. Magazines such as *G.I.M.*, *GPS World, P.O.B.,* and *Professional Surveyor* publish regularly on the latest models; see, for example, the recent GPS equipment surveys in *GPS World* [2002].

GPS consumer markets have been rapidly expanded. In the areas of land, marine, and aviation navigation, of precise surveying, of electronic charting, and of time transfer the deployment of GPS equipment seems to have become indispensable. This holds for military as well as civilian users.

Positioning

Two classes of positioning are recognized; standard positioning service (SPS) and precise positioning service (PPS):

In terms of positional accuracies one has to distinguish between SPS, with and without selective availability (SA), on the one hand and PPS on the other. Selective availability deliberately introduces clock errors and ephemeris errors in the data broadcast by the satellite. On May 1, 2000, selective availability was suspended. The current positional accuracy using the (civil) signal without SA is in the order of 10 m or better. With SA implemented, SPS accuracy is degraded to 50 to 100 m.

In several applications, GPS receivers are interfaced with other positioning systems, such as inertial navigation systems (INSs), hyperbolic systems, or even automatic braking systems (ABSs) in cars.

GPS receivers in combination with various equipment are able to answer such general questions as [Wells and Kleusberg, 1990]

Absolute positioning:	Where am I?
	Where are you?
Relative positioning:	Where am I with respect to you?
	Where are you with respect to me?
Orientation:	Which way am I heading?
Timing:	What time is it?

All of these questions may refer to either an observer at rest (*static* positioning) or one in motion (*kinematic* positioning). The questions may be answered immediately (*real-time* processing, often misnamed *DGPS*, for differential GPS) or after the fact (*batch* processing). The various options are summarized in Table 55.3 [Wells and Kleusberg, 1990]. Similarly, the accuracies for time dissemination are summarized in Table 55.4 [Wells and Kleusberg, 1990].

TABLE 55.3 Accuracy of Various GPS Positioning Modes

Absolute Positioning:	
SPS with SA	100 m
SPS without SA	40 m
PPS	20 m
Relative/Differential Positioning:	
Differential SPS	10 m
Carrier-smoothed code	2 m
Ambiguity-resolved carrier	10 cm
Surveying between fixed points	1 mm to 10 cm

Note: Accuracy of differential models depends on inter-receiver distance.

TABLE 55.4 Accuracy of Various GPS Time Dissemination Modes

Time and Time Interval:	
With SA	500 ns
Without SA, correct position	100 ns
Common mode, common view	<25 ns

Limiting Factors

Physics of the environment, instruments, broadcast ephemeris, and the relative geometry between orbits and network all form limiting factors on the final accuracy of the results. *Dilution of precision* (DOP) is used as a scaling factor between the observational accuracy and positioning accuracy.

The atmosphere of the earth changes the speed and the geometrical path of the electromagnetic signals broadcast by the GPS satellites. In the uppermost part of the atmosphere (the ionosphere) charged particles vary in number spatially as well as temporally. The so-called ionospheric refraction errors may amount to several tens of meters. Since this effect is frequency dependent, the first-order effect can be largely eliminated by the use of dual-frequency receivers. The lower part of the atmosphere (the troposphere) causes refraction errors of several meters. Fortunately, the effect can be modeled rather well by measuring the atmospheric conditions at the measuring site.

GPS instruments are capable of measuring one or a combination of the following signals:

C/A code:	with an accuracy of a few meters
P code:	with an accuracy of a few decimeters
Carrier phase:	with an accuracy of a few millimeters

In addition to this measurement noise, receiver clock errors are present and have to be modeled as to-be-solved-for parameters. It is this synchronization parameter between satellite time and receiver time that makes it necessary to have at least four satellites in view in order to get a three-dimensional fix.

Because of the high frequency of the GPS signals, multipath effects may hamper the final accuracy; the signal arriving at the receiver through a reflected path may be stronger than the direct signal. By careful antenna design and positioning of the antenna, multipath effects are reduced. The phase center of the antenna needs to be carefully calibrated with respect to a geometric reference point on the antenna assembly. However, because of the varying inclination angle of the incoming electromagnetic signals, effects of a moving phase center may be present at all times.

Information on the orbit of the satellite, as well as the orbital geometry relative to the network and receiver geometry, influences the overall positioning accuracy. The information the satellite broadcasts on its position and velocity is necessarily the result of a process of prediction. This causes the broadcast ephemeris to be contaminated with extrapolation errors. Typical values are

Radial error:	about 5 m
Across-track error:	about 10 m
Along-track error:	about 15 m

Also, the onboard satellite clock is not free of errors. Orbital and satellite clock errors can be largely taken care of by careful design of the functional model.

As mentioned before, deliberate contamination of the broadcast ephemeris and satellite time degrades the system to an accuracy of several tens of meters. PPS users are able to use the P code on two frequencies and have access to the SA code. Consequently, they are capable of eliminating the ionospheric effects and of removing deliberately introduced orbital and satellite clock errors. The resulting measurement error will be about 5 m. SPS users are able to use the C/A code (on one frequency only). They do not have access to the SA code. Consequently, the ionospheric error can only be roughly modeled, and these users are stuck with the deliberate errors. The resulting measurement error may be as large as 50 m.

Translocation techniques, such as having a stationary receiver continuously supporting the other (roving) receivers, will reduce the measurement error to well below the 10 m for SPS users.

Differencing techniques applied to the carrier phase measurements are successfully used to eliminate a wide variety of errors, provided the receivers are not too far apart. In essence, two close-by receivers are influenced almost equally by (deliberate) orbital errors and by part of the atmosphere error. Differencing of the measurements of both receivers will cancel a large portion of the first-order effects of these errors.

Modeling and the GPS Observables

Developing well-chosen functional models **F**, relating the GPS measurements **L** to the modeled parameters **X**, enables users to fit GPS perfectly to their needs. A wide class of applications, from monitoring the subsidence of oil rigs in the open sea to real-time navigating vehicles collecting geoinformation, belong to the range of possibilities that have been opened up by the introduction of GPS.

In satellite geodesy, one traditionally modeled the state of the satellite: a vector combining the positional ($\mathbf{X}$) and velocity ($\dot{\mathbf{X}}$) information. Although a known fact in the area of navigation, nowadays GPS provides geodesists, or geoscientists in general, with a tool by which the state of the observer, also in terms of position ($\mathbf{x}$) and velocity ($\dot{\mathbf{x}}$) can be determined with high accuracy and often in real time.

The GPS satellite geodetic model has evolved to

$$\mathbf{L} = \mathbf{F}\left(\mathbf{X}, \dot{\mathbf{X}}, \mathbf{x},\ \dot{\mathbf{x}},\ \mathbf{p},\ t\right) \tag{55.234}$$

where L = the C/A code, P code, or carrier phase observations
$i = 1, \ldots, n$
X = the three-dimensional position of the satellite at epoch t
$\dot{X}$ = the three-dimensional velocity of the satellite at epoch t
x = the one-dimensional, two-dimensional, or three-dimensional position of the observer at epoch t
$\dot{x}$ = the one-dimensional, two-dimensional, or three-dimensional velocity of the observer at epoch t
p = the vector of modeled (known or unknown) parameters
$j = 1, \ldots, u$
t = the epoch of measurement taking

Various differencing operators $\mathbf{D^k}$, up to order 3, are applied to the original observations in order to take full advantage of the GPS measurements. The difference operator $\mathbf{D^k}$ may be applied in the observation space spanned by the vector **L**, Eq. (55.235), or the $\mathbf{D^k}$ operator may be applied in the parameter space **x**, Eq. (55.236):

$$\mathbf{D^k}[\mathbf{L}] = \mathbf{D^k}\left[\mathbf{F}\left(\mathbf{X}, \dot{\mathbf{X}}, \mathbf{x},\ \dot{\mathbf{x}},\ \mathbf{p},\ t\right)\right] \tag{55.235}$$

$$\mathbf{L} = \mathbf{F}\left[\mathbf{X}, \dot{\mathbf{X}}, \mathbf{D^1}(\mathbf{x},\ \dot{\mathbf{x}}), \mathbf{p},\ t\right] \tag{55.236}$$

The latter method is sometimes referred to as *delta positioning*.

This is a difficult way of saying that one may either construct so-called derived observations from the original observations by differencing techniques or model the original observations, compute parameters (e.g., coordinates) in this way, and subsequently start a differencing technique on the results obtained from the roving receiver and the base receiver.

Pseudoranging

We restrict the discussion to the C/A-based *pseudorange observables*. The ranges are called *pseudo* because this technique is basically a one-way ranging technique with two independent clocks: the offset δt_E^S between the satellite clock S and the receiver clock E, which yields one additional parameter to solve for. Writing the observation equation in the earth-fixed reference frame, we have

$$\mathrm{pr} = \sqrt{(x^S - x_E)^2 + (y^S - y_E)^2 + (z^S - z_E)^2} - c\delta t_E^S \tag{55.237}$$

Inspection of the partials

$$\frac{\partial \mathrm{pr}}{\partial x^S} = \frac{x^S - x_E}{\mathrm{pr}} = -\frac{\partial \mathrm{pr}}{\partial x_E} \tag{55.238}$$

$$\frac{\partial \text{pr}}{\partial y^S} = \frac{y^S - y_E}{\text{pr}} = -\frac{\partial \text{pr}}{\partial y_E} \tag{55.239}$$

$$\frac{\partial \text{pr}}{\partial z^S} = \frac{z^S - z_E}{\text{pr}} = -\frac{\partial \text{pr}}{\partial z_E} \tag{55.240}$$

$$\frac{\partial \text{pr}}{\partial \left(\delta t_E^S\right)} = -c \tag{55.241}$$

reveals that:

- The coordinates of the stations are primarily obtained in a frame determined by the satellites or, better, by their broadcast ephemeris.
- Partials evaluated for neighboring stations are practically identical, so the coordinates of one station need to be adopted.

Phase (Carrier Wave) Differencing

For precise engineering applications the phase of the carrier wave is measured. Two wavelengths are available in principle:

$$\begin{aligned} L_1: \quad \lambda_1 &= \frac{c}{f_1} \quad \text{with} \quad f_1 = 1.57542 \text{ GHz} \\ &= 19.0 \text{ cm} \end{aligned} \tag{55.242}$$

and

$$\begin{aligned} L_2: \quad \lambda_2 &= \frac{c}{f_2} \quad \text{with} \quad f_2 = 1.22760 \text{ GHz} \\ &= 24.4 \text{ cm} \end{aligned} \tag{55.243}$$

For phase measurements the following observation equation can be set up:

$$\text{Range} = \Phi + N\lambda_l; \quad l = 1,\ldots,2 \tag{55.244}$$

or

$$\Phi_E^S = \sqrt{\left(x^S - x_E\right)^2 + \left(y^S - y_E\right)^2 + \left(z^S - z_E\right)^2} - N_E^S \lambda_l \tag{55.245}$$

where Φ_E^S is the phase observable in a particular *S–E* combination and N_E^S is the integer multiple of wavelengths in the range: the *ambiguity*. Phase measurements can be done with probably 1% accuracy. This yields an observational accuracy — in case the ambiguity *N* can be properly determined — in the millimeter range.

Using the various differencing operators on the phase measurements:

- $\mathbf{D}^{\mathbf{k}=\mathbf{1}}$ yields single differences:
  - Between receiver differences, $\Delta\Phi$, eliminating or reducing satellite-related errors
  - Between satellite differences, $\nabla\Phi$ eliminating or reducing receiver-related errors
  - Between epoch differences, δF, eliminating phase ambiguities per satellite and receiver combination

- $\mathbf{D}^{k=2}$ yields double differences:
  - Between receiver and satellite differences, $\nabla\Delta\Phi$ eliminating or reducing satellite- and receiver-related errors, and so forth
- $\mathbf{D}^{k=3}$ yields triple differences:
  - Between epoch, receiver, and satellite differences, $\delta\nabla\Delta\Phi$ eliminating or reducing satellite- and receiver-related errors and ambiguities

Receivers that use carrier wave observations have, in addition to the electronic components that do the phase measurements, a counter, which counts the complete cycles between selected epochs. GPS analysis software uses the triple differences to detect and possibly repair cycle slips occurring during loss of lock.

Design specifications and receiver selection depend on the specific project accuracy requirements. In the United States the Federal Geodetic Control Committee has adopted the specifications [FGCC, 1989] given in Tables 55.5 and 55.6.

GPS Receivers

A variety of receivers are on the market. Basically, they can be grouped in four classes, listed in Table 55.7. The types of observations of the first three types of receivers are subjected to models that can be characterized as *geometric* models. The position of the satellite is considered to be known, mostly the information taken from the broadcast ephemeris. The known positions are of course not errorless. First of all, the positions are predicted; thus they contain errors because of an extrapolation process in time. Second, the positions being broadcast may be corrupted by intentional errors (due to SA, see previous paragraphs). Differencing techniques are capable of eliminating most of the error if the separation between base station and roving receiver is not too large.

Millimeter-accurate observations from geodesy-grade receivers are often subjected to analysis through models of the *dynamic* type. Software packages containing dynamic models are very elaborate and allow for some kind of orbit improvement estimation process.

GPS World [2002] lists an overview of recent GPS receivers.

GPS Base Station

GPS, like most other classical survey techniques, has to be applied in a differential mode if one wants to obtain reliable relative positional information. This implies that for most applications of GPS in geodesy — surveying and mapping, photogrammetry, Geographic Information Systems (GIS), and so forth — one has to have at least two GPS receivers at one's disposal. If one of the receivers occupies a known location during an acceptable minimum period, than one may obtain accurate coordinates for the second receiver *in the same frame*. In surveying and geodesy applications one should preferably include three stations with known horizontal coordinates and at least four with known vertical (orthometric) heights. In most GIS applications one receiver is left at one particular site. This station serves as a so-called *base station*.

GIS, Heights, and High-Accuracy Reference Networks

In order to reduce influences from satellite-related errors and atmospheric conditions in geodesy, surveying, and Geographic Information Systems applications, GPS receivers are operated in a differential mode. Whenever the roving receiver is not too far from the base station receiver, "errors at high altitudes (satellite and atmosphere)" are more or less canceled if the "fix from the field" is differenced with the "fix from the base."

Washington editor of *GPS World*, Hale Montgomery, writes, "As a peripheral industry, the reference station business has grown almost into an embarrassment of riches, with stations proliferating nationwide and sometimes duplicating services." William Strange, chief geodesist at the National Geodetic Survey (NGS), is quoted as saying, "Only about 25 full-service, fixed stations would be needed to cover the entire United States" [Montgomery, 1993]. A group of the interagency Federal Geodetic Control subcommittee has compiled a list of about 90 operating base stations on a more or less permanent basis. If all GIS and GPS base stations being planned or in operation are included, the feared proliferation will be even larger.

From the point of view of the U.S. taxpaying citizen, "duplication of services" of work may be wasteful, although decentralization of services may often be more cost-effective than all-encompassing projects run by even more all-encompassing agencies. At the time of this writing (January 2002) the website of the National Geodetic Survey (http://www.ngs.noaa.gov/) lists a network of almost 300 continuous operating reference stations (CORS) in the U.S.

From the geodetic point of view, the duplication of services (base station-generated fixes in the field) will proliferate the coordinate fields and the reference frames they supposedly are tied to. The loss of money and effort in the years to come while trying to make sense out of these most likely not-matching point fields may be far larger than the money lost in "service duplicating" base stations. If we are not careful, the "coordinate duplicating" base stations will create chaos among GIS-applying agencies.

Everyone is convinced about the necessity to collect GIS data in one *common frame*. Formerly, the GIS community was satisfied with positions of the 3- to 5-m accuracy. Manufacturers are now aggressively marketing GPS and GIS equipment with 0.5-m accuracy ($10,000 per receiver, and remember, you need at least two). The increased demand for accuracy requires that a reference frame be in place that lasts at least two decades. This calls for a consistent reference of one, probably two orders of magnitude more accurate than presently available.

The accuracy of the classical horizontal control was in the order of 1 part in 100,000 (1 cm over 1 km). GPS is a survey tool with an accuracy of 1 part per 1,000,000 (1 cm over 10 km). All U.S. states put new High-Accuracy Reference Networks (HARNs) in place to accommodate the accuracy of GPS surveys. Even for GIS applications where 0.5-m accuracies are claimed for the roving receivers, one may speak of 1-ppm surveys whenever those rovers operate at a distance of 500 km from their base station.

It should not be forgotten that GPS is a geometric survey tool yielding results in terms of earth-fixed coordinate differences x_{ij}. From these coordinate differences expressed in curvilinear coordinates we obtain, at best, somewhat reproducible ellipsoidal height differences. These height differences are *not* easily converted to orthometric height differences of equal accuracy. The latter height differences are of interest in engineering and GIS applications; see Section 55.7.

55.7 Gravity Field and Related Issues

One-Dimensional Positioning: Heights and Vertical Control

One of the most accurate measurements surveyors are able to make are the determinations of height differences by spirit leveling. Since a leveling instrument's line of sight is tangent to the potential surface, one may say that leveling actually determines the height differences with respect to equipotential surfaces. If one singles out one particular equipotential surface at mean sea level (the so-called *geoid*), then the heights a surveyor determines are actually orthometric heights; see Fig. 55.22. Leveling in a closed loop is not a check on the actual height differences in a metrical sense, but in a potential sense: the distance between equipotential surfaces varies due to local gravity variations.

In spherical approximation the potential at a point A is

$$V = -\frac{GM}{r} = -\frac{GM}{R+h} \tag{55.246}$$

The gravity is locally dependent on the change in potential per height unit, or

$$\frac{dV}{dr} = g = \frac{GM}{r^2} \tag{55.247}$$

The potential dV difference between two equipotential surfaces is

$$dV = g\, dr \tag{55.248}$$

Consequently, if one levels in a loop, one has

$$\sum dV = \sum g\, dr = 0 \tag{55.249}$$

$$\oint dV = \oint g\ dr \tag{55.250}$$

This implies that for each metrically leveled height difference *dr*, one has to multiply this difference by the local gravity. Depending on the behavior of the potential surfaces in a certain area and the diameter of one's project, one has to "carry along a gravimeter" while leveling. The variations of local gravity vary depending on the geology of the area. Variations in the order of $10^{-7}\,g$ may yield errors as large as 10 mm for height differences in the order of several hundred meters. For precise leveling surveys (≤0.1 mm/km) gravity observations must be made with an interval of:

2 to 3 km in relatively flat areas
1 to 2 km in hilly terrain
0.5 to 1.5 km in mountainous regions

For more design criteria on leveling and gravity surveys, see FGCC [1989] and Table 55.8.

GPS surveys yield, at best, ellipsoidal height differences. These are rather meaningless from the engineering point of view. Therefore, extreme caution should be exercised when GPS height information, even after correction for geoidal undulations, is to be merged with height information from leveling. For two different points *i* and *j*,

$$h_i = H_i + N_i \tag{55.251}$$

$$h_j = H_j + N_j \tag{55.252}$$

Subtracting Eq. (55.251) from Eq. (55.252), we find the ellipsoidal height differences h_{ij} (from GPS) in terms of the orthometric height differences H_{ij} (from leveling) and the geoidal height differences N_{ij} (from gravity surveys):

$$h_{ij} = H_{ij} + N_{ij\ j} \tag{55.253}$$

where

$$\begin{aligned} h_{ij} &= h_j - h_i \\ H_{ij} &= H_j - H_i \\ N_{ij} &= N_j - N_i \end{aligned} \tag{55.254}$$

For instance, with the National Geodetic Survey's software program geoid99 package, geoidal height differences are as accurate as 5 cm over 100 km for the conterminous U.S. For GPS leveling this means that GPS competes with third-order leveling as long as the stations are more than 5 km apart.

In principle any equipotential surface can act as a vertical datum such as the National Geodetic Vertical Datum of 1929 (NGVD29). Problems may arise merging GPS heights, gravity surveys, and orthometric heights referring to NGVD29. Heights referring to the NGVD88 datum will be more suitable for use with GPS surveys. In the United States about 600,000 vertical control stations are in existence.

Two-Dimensional Positioning: East–North and Horizontal Control

In classical geodesy the measurements in height had to be separated (leveling) from the horizontal measurements (directions, angles, azimuths, and distances). To allow for the curvature of the earth and the varying gravity field, the horizontal observations were reduced first to the geoid, taking into account the orthometric heights. Subsequently, if it was desired to take advantage of geometrical properties between the once-reduced horizontal observations, the observations had to be reduced once more, but now from the geoid to the ellipsoid. An ellipsoid approximates the geoid up to 0.01%; the variations of the geoid are nowhere larger than 150 m. On the ellipsoid, which is a precise mathematical figure, one could check, for instance, whether the sum of the three angles equaled a prescribed value.

So far, geodesists have relied on a biaxial ellipsoid of revolution. A semimajor axis a and a semiminor axis b define the dimensions of the ellipsoid. Rather than this semiminor axis, one specifies the *flattening* of the ellipsoid

$$f = \frac{a-b}{a} \approx \frac{1}{298.257} \tag{55.255}$$

For a semimajor axis of about 6378.137 km, it implies that the semiminor axis is 6378.137/298.257 ≈ 22 km shorter than a.

Distance measurements need to be reduced to the ellipsoid. Angular measurements made with theodolites, total stations, or other instruments need to be corrected for several effects:

- The direction of local gravity does not coincide with the normal to the ellipsoid. The direction of the first axis of the instrument coincides with the direction of the local gravity vector. Notwithstanding this effect, the earth's curvature causes nonparallelism of first axes of 1 arcsecond for each 30 m.
- The targets aimed at generally do not reside on the ellipsoid.

The noncoincidence of the gravity vector and the normal is called *deflection of the vertical*. Proper knowledge of the behavior of the local geopotential surfaces is needed for proper distance and angle reductions. Consult Vaniček and Krakiwsky [1982], for example, for the mathematical background of these reductions. The FGCC adopted accuracy standards for horizontal control using classical geodetic measurement techniques; see Table 55.9. In the U.S. over 270,000 horizontal control stations exist.

Three-Dimensional Positioning: Geocentric Positions and Full Three-dimensional Control

Modern three-dimensional survey techniques, most noticeably GPS, allow for immediate three-dimensional relative positioning. Three-dimensional coordinates are equally accurate expressed in ellipsoidal, spherical, or Cartesian coordinates. Care should be exercised in properly labeling curvilinear coordinates as spherical (*geographic*) or ellipsoidal (*geodetic*). Table 55.10 shows the large discrepancies between the two. At the midlatitudes they may differ by more than 11′. This could result in a north–south error of 20 km. While merging GIS data sets, one should be aware of the meaning LAT/LON in any instance.

Despite their three-dimensional characteristics, networks generated by GPS are the weakest in the vertical component, not only because of the lack of physical significance of GPS determined heights, as described in the preceding subsection, but also because of the geometrical distribution of satellites with respect to the vertical: no satellite signals are received from "below the network." This lopsidedness makes the vertical the worst determined component in three-dimensional characteristics.

Because of the inclination of the GPS satellites, there are places on earth, most notoriously the midlatitudes, where there is not an even distribution of satellites in the azimuth sense. For instance, in the northern midlatitudes we never have as many satellites to the north as we have to the south; see, for example, Santerre [1991]. This makes the latitude the second-best-determined curvilinear coordinate. For space techniques the FGCC has proposed the classification in Table 55.11.

55.8 Reference Systems and Datum Transformations

Geodetic Reference Frames

There has always been a necessity, in a variety of fields but especially in engineering, to rely on a set of adopted parameter values that describe various quantities related to the earth to the current state of the art of accuracy. Questions asked of geodesists include:

- What is the best estimate of the equatorial radius of the earth?
- What is the mass of the earth?
- How fast does the earth rotate?
- How flattened is the earth?

The values adopted at the General Assemblies of the International Association of Geodesy (IAG) of Madrid (1924) and Stockholm (1930) were valid for a long time:

$$a = 6{,}378.388 \text{ km} \quad \text{and} \quad f = 1/297.0$$

This was the so-called international ellipsoid of the American geodesist J.F. Hayford (1909). The appropriate international gravity formula was

$$\gamma = 978.0490\left(1 + 0.005\,288\,4 \sin^2 \phi - 0.000\,0059 \sin 2\phi\right) \text{cm sec}^{-2} \text{ (gal)}$$

This formula for γ describes the gravity on the surface of the international ellipsoid as function of the geodetic (ellipsoidal) latitude *c*. After some years these values turned out to be not accurate enough, and the Soviet geodesist F.N. Krassovsky proposed a new ellipsoid (1943):

$$a = 6378.245 \text{ km} \quad \text{and} \quad f = 1/298.3$$

Geodetic Reference System 1967

In Hamburg (1964) the International Astronomical Union (IAU), after the advice of the International Union of Geodesy and Geophysics (IUGG), adopted values of three variables:

a the equatorial radius of the earth
GM the geocentric gravitational constant of the earth, including the atmosphere
J_2 the dynamical form factor of the earth

The General Assembly of the IAG in Lucerne (1967) followed this proposal, resulting in the so-called Geodetic Reference System (GRS) 1967:

$$a = 6{,}378{,}160 \text{ m}$$

$$GM = 398{,}603 \times 10^9 \text{ m}^3 \text{ sec}^{-2}$$

$$J_2 = 10{,}827 \times 10^{-7}$$

It was agreed upon that the semiminor axis of the reference ellipsoid would be parallel to the direction of the rotation axis as defined by the conventional international origin. In addition, the reference meridian would be parallel to the zero meridian, as resulted from the adopted values for the longitudes of various observatories by the Bureau International de l'Heure (BIH) in Paris.

TABLE 55.5 Guidelines for GPS Field Survey Procedures

Geometric Relative Positioning Standards	AA AA 0.01	A A 0.1	B B 1.0	C 1, 2-I&II, 3 10, 20, 50, 100
Two-frequency observations (I and L2) required[a]: daylight observations[b]	Y	Y	Y	op
Recommended number of receivers observing simultaneously, not less than	5	5	4	3
Satellite observations: GDOP values during observing session (m/cycle)[d]				
Period of observing session (observing span), not less than (min) (4 or more simultaneous satellite observations)[e]				
Triple difference processing[f]	na	na	240	60–120
Other processing techniques[g]				
General requirements[h]	240	240	120	30–60
Continuous and simultaneous between all receivers, period not less than[i]	180	120	60	20–30
Data sampling rate — maximum time interval between observations (sec)	15	30	30	15–30
Minimum number of quadrants from which satellite signals are observed	4	4	3	3 or 2[j]
Maximum angle above horizon for obstructions[u] (degrees)	10	15	20	20–40
Independent occupations per station[k]				
Three or more (percent of all stations, not less than)	80	40	20	10
Two or more (percent of stations, not less than)				
New stations	100	80	50	30
Vertical control stations	100	100	100	100
Horizontal control stations	100	75	50	25
Two or more for each station of "station-pairs"[l]	Y	Y	Y	Y
Master or fiducial stations[m]				
Required, yes or no[n]	Y	Y	Y	op
If yes, minimum number	4	3	2	—
Repeat baseline measurements, about equal number in N–S and E–W directions, minimum not less than (percent of total independently (nontrivial) determined base lines)	25	15	5	5
Loop closure, requirements when forming loops for postanalyses				
Baselines from independent observing sessions, not less than	3	3	2	2
Baselines in each loop, total not more than	6	8	10	10
Loop length, generally not more than (km)	2000	300	100	100
Baselines not meeting criteria for inclusion in any loop, not more than (percent of all independent nontrivial lines[o])	0	5	20	30
Stations not meeting criteria for inclusion in any loop, not more than (percent of all stations)	0	5	10	15
Direct connections are required; between any adjacent (NGRS or GPS) stations (new or old, GPS or non-GPS) located near or within project area, when spacing is less than (km)	30	10	5	3
Antenna setup				
Number of antenna phase center height measurements per session, not less than	3[p]	3[p]	2	2
Independent plumb point check required[q]	Y	Y	Y	op
Photograph (closeup) or pencil rubbing required for each mark occupied	Y	Y	Y	Y
Meteorological observations				
Per observing session, not less than	3[r]	3[r]	2[s]	2[s] or op
Sampling rate (measurement interval), not more than (min)	30	30	60	60
Water vapor radiometer measurements required at selected stations?	op	op	N	N
Frequency standard warm-up time (h)[t]				
Crystal	12	12	*u*	*u*
Atomic	—	1	*t*	*t*

Note: na = not applicable; op = optional.

[a] If two-frequency observations cannot be obtained, it is possible that an alternative method for estimating the ionospheric refraction correction would be acceptable, such as modeling the ionosphere using two-frequency data obtained from other sources. Or, if observations are during darkness, single-frequency observations may be acceptable, depending on the expected magnitude of the ionospheric refraction error.

[b] When spacing between any two stations occupied during an observing session is more than 50 km, two-frequency observations may need to be considered for accuracy standards of order 2 or higher.

[c] Multiple baseline processing techniques.

TABLE 55.5 (continued) Guidelines for GPS Field Survey Procedures

[d] Studies are under way to investigate the relationship of geometric dilution of precision (GDOP) values to the accuracy of the baseline determinations. Initial results of these studies indicate a possible correlation. It appears that the best results may be achieved when the GDOP values are changing during the observing session.
[e] The number of satellites that are observed simultaneously cannot be less than the number specified for more than 25% of the specified period for each observing session.
[f] Absolute minimum criterion is 100% of specified period.
[g] "Other" includes processing carrier phase data using single, double, nondifferencing, or other comparable precise relative positioning processing techniques.
[h] The times for the observing span are conservative estimates to ensure that the data quantity and quality will give results that will meet the desired accuracy standard.
[i] Absolute minimum criterion for the data collection observing span is that period specified for an observing session that includes continuous and simultaneous observations. Continuous observations are data collected that do not have any breaks involving all satellites; occasional breaks for individual satellites caused by obstructions are acceptable, but must be minimized. A set of observations for each measurement epoch is considered simultaneous when it includes data from at least 75% of the receivers participating in the observing session.
[j] Satellites should pass through quadrants diagonally opposite each other.
[k] Two or more independent occupations for the stations of a network are specified to help detect instrument and operator errors. Operator errors include those caused by antenna centering and height offset blunders. When a station is occupied during two or more sessions, back-to-back, the antenna/tripod will be reset and replumbed between sessions to meet the criteria for an independent occupation. To separate biases caused by receiver and/or antenna equipment problems from operator-induced blunders, a calibration test may need to be performed.
[l] Redundant occupations are required when pairs of intervisible stations are established to meet azimuth requirements, when the distance between the station pair is less than 2 km, and when the order is 2 or higher.
[m] Master or fiducial stations are those that are continuously monitored during a sequence of sessions, perhaps for the complete project. These could be sites with permanently tracking equipment in operation where the data are available for use in processing with data collected with the mobile units.
[n] If simultaneous observations are to be processed in the session or network for baseline determinations while adjusting one or more components of the orbit, then two or more master stations shall be established.
[o] For each observing session there are $r - 1$ independent baselines, where r is the number of receivers collecting data simultaneously during a session; e.g., if there were 10 sessions and 4 receivers used in each session, 30 independent baselines would be observed.
[p] A measurement will be made both in meters and feet, at the beginning, midpoint, and end of each station occupation.
[q] To ensure that the antenna was centered accurately with the optical plummet over the reference point on the marker, when specified, a heavyweight plumb bob will be used to check that the plumb point is within specifications.
[r] Measurements of station pressure (in millibars), relative humidity, and air temperature (in °C) will be recorded at the beginning, midpoint, and end, depending on the period of the observing session.
[s] Report only unusual weather conditions, such as major storm fronts passing over the sites during the data collection period. This report will include station pressure, relative humidity, and air temperature.
[t] The amount of warm-up time required is very instrument dependent. It is important to follow the manufacturer's specifications.
[u] An obstruction is any object that would effectively block the signal arriving from the satellite. Included are buildings, trees, fences, humans, vehicles, etc.

Source: FGCC, *Geometric Geodetic Accuracy Standards and Specifications for Using GPS Relative Positioning Techniques*, Version 5.0, Federal Geodetic Control Committee, Rockville, MD, 1989.

A fourth parameter completed the Geodetic Reference System 1967: the (inertial or sidereal) angular velocity of the earth:

$$\omega = 0.000\,072\,921151\,467 \text{ rad/sec}$$

The duration of one turn of the earth around its spin axis is then

$$\text{Length of day} = \frac{2\pi}{\omega} \approx 86{,}164.1 \text{ sec} \tag{55.256}$$

This is not equal to 24 × 60 × 60 = 86,400 sec, as explained in Section 55.3.

Officially, the XVth General Assembly of the IAG in Moscow (1971) approved and adopted these values [IAG, 1971].

TABLE 55.6 Office Procedures for Classifying GPS Relative Positioning Networks Independent of Connections to Existing Control

Geometric Relative Positioning Standards	AA 0.01	A 0.1	B 1.0	1 10	2–I 20	2–II 50	3 100
Ephemerides							
Orbit accuracy, minimum (ppm)	0.008	0.05	0.5	5	10	25	50
Precise ephemerides required?	Y[a]	Y[a]	Y	op	op	N	N
Loop Closure Analyses[b] — when forming loops, the following are minimum criteria:							
Baselines in loop from independent observations not less than	4	3	2	2	2	2	2
Baselines in each loop, total not more than	6	8	10	10	10	15	15
Loop length, not more than (km)	2000	300	100	100	100	100	100
Baselines not meeting criteria for inclusion in any loop, not more than (percent of all independent lines)	0	0	5	20	30	30	30
In any component (X, Y, Z), maximum misclosure not to exceed (cm)	10	10	15	25	30	50	100
In any component (X, Y, Z), maximum misclosure, in terms of loop length, not to exceed (ppm)	0.2	0.2	1.25	12.5	25	60	125
In any component (X, Y, Z), average misclosure, in terms of loop length, not to exceed (ppm)	0.09	0.09	0.9	8	16	40	80
Repeat Baseline Differences							
Baseline length, not more than (km)	2000	2000	500	250	250	100	50
In any component (X, Y, Z), maximum not to exceed (ppm)	0.01	0.1	1.0	10	20	50	100

Note: op = optional.

[a] The precise ephemerides is presently limited to an accuracy of about 1 ppm. The accuracy has been improved to about 0.1 ppm. It is unlikely orbital coordinate accuracies of 0.01 ppm will be achieved in the near future. Thus to achieve precisions approaching 0.01 ppm, it will be necessary to collect data simultaneously with continuous trackers or fiducial stations. Then all the data are processed in a session or network solution mode where the initial orbital coordinates are adjusted while solving for the baselines. In this method of processing the carrier phase data, the coordinates at the continuous trackers, are held fixed.

[b] Between any combination of stations, it must be possible to form a loop through three or more stations that never passes through the same station more than once.

Source: FGCC, *Geometric Geodetic Accuracy Standards and Specifications for Using GPS Relative Positioning Techniques*, Version 5.0, Federal Geodetic Control Committee, Rockville, MD, 1989.

TABLE 55.7 Accuracy Grades of (Civilian/Commercial) GPS Receivers

Grade	Accuracy	Code
Navigation grade	40–100 m	C/A code in stand-alone mode
Mapping (GIS) grade	2–5 m	C/A code in differenced mode
Surveying grade	1–2 cm (within 10 km)	C/A code plus phase, differenced
Geodesy grade	5–15 mm over any distance	C/A plus P codes plus phase, differenced

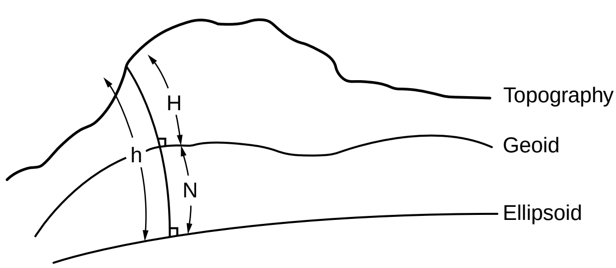

FIGURE 55.22 Orthometric heights.

TABLE 55.8 FGCC Vertical Control Accuracy Standards (Differential Leveling)

Class	b[a]
First Order	
I	< 0.5
II	0.7
Second Order	
I	1.0
II	1.3
Third Order	2.0

[a] $b = S/\sqrt{d}$ [mm km$^{-1/2}$], where S = standard deviation of elevation difference between control points (in mm) and d = approximate horizontal distance along leveled route (in km).

TABLE 55.9 FGCC Horizontal Control Accuracy Standards (Classical Techniques)

Class	Accuracy
First Order	1:100,000 (10 mm/km)
Second Order	
Class I	1:50,000 (20 mm/km)
Class II	1:20,000 (50 mm/km)
Third Order	
Class I	1:10,000 (100 mm/km)
Class II	1:5,000 (200 mm/km)

TABLE 55.10 Geographic (Spherical) Latitude as a Function of Geodetic Latitude

Geodetic Latitude			Geographic Latitude			Geodetic Minus Geographic Latitude		
°	′	″	°	′	″	°	′	″
00	0	0.000	00	00	00.000	00	00	00.000
10	0	0.000	09	56	03.819	00	03	56.181
20	0	0.000	19	52	35.868	00	07	24.132
30	0	0.000	29	50	01.089	00	09	58.911
40	0	0.000	39	48	38.198	00	11	21.802
50	0	0.000	49	48	37.402	00	11	22.598
60	0	0.000	59	49	59.074	00	10	00.926
70	0	0.000	69	52	33.576	00	07	26.424
80	0	0.000	79	56	02.324	00	03	57.676
90	0	0.000	90	00	00.000	00	00	00.000

TABLE 55.11 FGCC Three-Dimensional Accuracy Standards (Space System Techniques)

Order	Accuracy
AA order (global)	3 mm + 1:100,000,000 (1 mm/100 km)
A order (primary)	5 mm + 1:10,000,000 (1 mm/10 km)
B order (secondary)	8 mm + 1:1,000,000 (1 mm/km)
C order (dependent)	10 mm + 1:100,000 (10 mm/km)

Geodetic Reference System 1980

Accurate results of modern geodetic and astronomic measurement techniques required the replacement of GRS 1967 within 13 years by the Geodetic Reference System 1980:

$$a = 6378\,137 \text{ m}$$

$$GM = 3986\,05 \times 10^{8} \text{ m}^{3} \text{ sec}$$

$$J_2 = 108\,263 \times 10^{-8}$$

$$\omega = 7292\,115 \times 10^{-11} \text{ rad sec}^{-1}$$

GRS 1980 was adopted at the XVIIth General Assembly of the IUGG in Canberra, December 1979 [IAG, 1980]. Agreements with respect to the orientation of the reference ellipsoid were not altered.

1983 Best Values

During the XVIIIth General Assembly of the IUGG in Hamburg, August 1983, the IAG adopted a resolution that stated that the GRS 1980 did not need to be replaced, but that the following values were (at the time) the most representative [IAG, 1984]:

1983 Best Values

Velocity of light in vacuum	$c = (299{,}792{,}458 \pm 1.2)$ m sec^{-1}
Gravitational constant (Newton)	$G = (6{,}673 \pm 1) \times 10^{-14}$ m^3 sec^{-2} kg^{-1}
Angular velocity of the earth	$\omega = 7{,}292{,}115 \pm 10^{-11}$ rad sec^{-1}
Geocentric gravitational constant of the earth, including the atmosphere	$GM = (39{,}860{,}044 \pm 1) \times 10^{7}$ m^3 sec^{-2}
Geocentric gravitational constant of atmosphere	$GM_A = (35 \pm 0.3) \times 10^{7}$ m^3 sec^{-2}
Second-degree harmonic coefficient without permanent deformation due to tides	$J_2 = (1{,}082{,}629 \pm 1) \times 10^{9}$
Equatorial radius of the earth	$a = (6{,}378{,}136 \pm 1)$ m
Equatorial gravity	$\gamma_e = (978{,}032 \pm 1) \times 10^{-5}$ m sec^{-2}
Flattening	$1/f = (298{,}257 \pm 1) \times 10^{-3}$
Potential of the geoid	$W_0 = (6{,}263{,}686 \pm 2) \times 10$ m^2 sec^{-2}
Triaxial ellipsoid (rounded values):	
Equatorial flattening	$1/f_1 = 90{,}000$
Longitude semimajor axis	$\lambda_1 = 345°$ (east)

1987 Best Values and Secular Changes

In Hamburg a special study group (SSG 5.100), whose chairman was B.H. Chovitz, got the task to evaluate the status of the GRS 1980 and to prepare possible recommendations to the XIXth General Assembly of the IUGG in Vancouver, August, 1987. The result of the four-year study was that the GRS 1980 still did not need replacement [IAG, 1988a], but that a series of 1987 best values and secular changes could be forwarded. The interesting conclusion was that, from this moment on, two values had to be recognized for many parameters: a time-invariant component and the first derivative with respect to time [IAG,1988b].

In the meantime the velocity of light had been raised to a physical constant, adopting the value as listed in the previous section [Cohen and Taylor, 1988]:

$$c = 299{,}792{,}458 \text{ m sec}^{-1} \text{ exactly}$$

This means that the meter is now a derived parameter through the adopted length of a second.

1987 Best Values

Angular velocity of the earth (rounded)	$\omega = 7.292115 \times 10^{-5}$ rad sec^{-1}
Geocentric gravitational constant of the earth, including the atmosphere	$GM = (3{,}986{,}004.40 \pm 0.03) \times 10^{8}$ m^3 sec^{-2}
Equatorial radius of the earth	$a = (6{,}378{,}136 \pm 1)$ m
Second-degree spherical harmonic coefficient without permanent effect due to tides	$J_2 = (1{,}082{,}626 \pm 2) \times 10^{-9}$

1987 Best Secular Changes

Decrease of angular velocity due to tides	$d\omega_T/dt = (-6.0 \pm 0.3) \times 10^{-22}$ rad/sec^{-2}
Decrease of angular velocity due to other causes	$d\omega_{NT}/dt = (+1.4 \pm 0.3) \times 10^{-22}$ rad/sec^{-2}
Total decrease in angular velocity	$d\omega/dt = (-4.6 \pm 0.4) \times 10^{-22}$ rad/sec^{-2}
Relative change of the gravitational constant	$(dG/dt)/G = (0 \pm 1) \times 10^{-11}$ year^{-1}
Change of the earth's mass	$dM/dt = 0$ kg year^{-1}
Change of the earth's radius	$d\alpha/dt = 0$ mm year^{-1}
Decrease in the second-degree harmonic coefficient	$dJ_2/dt = (-2.8 \pm 0.3) \times 10^{-11}$ year^{-1}

It is interesting to note that we do not zero in on the earth's semimajor axis in an arbitrary manner. The earth seems to "shrink" under our increasingly accurate estimates of its equatorial radius, as shown in Table 55.12.

World Geodetic System 1984

The World Geodetic System 1984 (WGS 84) and its three predecessors (WGS 60, WGS 66, and WGS 72), developed by the American Department of Defense/Defense Mapping Agency (DoD/DMA), are also reference *systems* in the real sense of the word. The WGS 84 consists of a reference coordinate frame, an ellipsoidal gravity formula, an ellipsoid, an earth gravity model, a geoid, and a series of transformation parameters and formulas between various geodetic reference frames and datum values [DMA, 1988].

WGS 84 Coordinate Frame

The WGS 84 coordinate frame is based on the frames that were originally developed for the U.S. Navy's Doppler/Transit System. One of its most recent reference frames, known as NSWC 9Z-2, had the following known problems:

- Its geocentricity or, better, nongeocentricity
- The misorientation of the zero meridian, compared to the zero meridian adopted by the BIH
- A scale error of almost 1 ppm

These problems can be expressed in terms of specific values:

- The origin with respect to the center of mass of the earth, as accurately determined by satellite laser ranging (SLR), was 4.5 m too high.
- The *x* axis of the NSWC frame was 0.814 arcseconds to the east of the BIH zero meridian — an error of about 24 m at the equator.
- The scale of NSWC 9Z-2 was 0.6×10^6 too large.

The WGS 84 corrected these problems. The transformation formulas (see the discussion of earth-fixed frames in Section 55.3) from $\mathbf{X}_{\text{NSWC 9Z-2}}$ to $\mathbf{X}_{\text{WGS 84}}$ become

$$\mathbf{x}_{\text{WGS 84}} = \sigma \mathbf{R}_3(\gamma)\mathbf{x}_{\text{NSWC9Z-2}} + \mathbf{t} \tag{55.257}$$

TABLE 55.12 Decrease of the Semimajor Axis in Recent Decades

Reference	Estimate
International ellipsoid 1924/Hayford 1909	6,378,388 m
Krassovsky 1943	6,378,245 m
GRS 1967	6,378,160 m
GRS 1980	6,378,137 m
Best values 1983/1987	6,378,136 m

with

$$\sigma = 1. + \delta\sigma = 1. - 0.000\,000\,6 \tag{55.258}$$

$$\gamma = 0''.814 \tag{55.259}$$

$$\mathbf{t} = \begin{pmatrix} 0 \\ 0 \\ 4.5 \text{ m} \end{pmatrix} \tag{55.260}$$

WGS 84 Ellipsoid

The origin of the WGS 84 coordinate frame is also the center of the WGS 84 ellipsoid, so that the symmetry axis of the ellipsoid of revolution coincides with the *z* axis of the WGS 84 frame. The values for the standard WGS 84 earth are as follows:

WGS 84 Ellipsoid

Semimajor axis	$a = (6{,}378{,}137 \pm 2)$ m
Geocentric gravitational constant of the earth, including the atmosphere	$GM = (3{,}986{,}005 \pm 0.6) \times 10^8$ m^3 sec^{-2}
Geocentric gravitational constant of atmosphere	$GM_A = (3.5 \pm 0.1) \times 10^8$ m^3 sec^{-2}
Geocentric gravitational constant of the earth without atmosphere	$GM_{-A} = (3{,}986{,}001.5 \pm 0.6) \times 10^8$ m^3 sec^{-2}
Normalized second-degree zonal gravitational coefficient without permanent deformation due to tides	$\bar{C}_{2,0}^{-T} = (-484.166\ 85 \pm 0.00130) \times 10^{-6}$
Normalized second-degree zonal gravitational coefficient with permanent deformation due to tides ($J_2 = -\sqrt{5}\,\bar{C}_{2,0}$)	$\bar{C}_{2,0}^{+T} = (-484.171\ 01 \pm 0.00130) \times 10^{-6}$
Angular velocity of the earth	$\omega = (7{,}292{,}115 \pm 0.1500) \times 10^{-11}$ rad sec^{-1}
Angular velocity of the earth in a precessing frame	$\omega^* = (7{,}292{,}115.8553 \times 10^{-11} + 4.3 \times 10^{-15}\, T_u)$ rad sec^{-1} with T_u in Julian centuries, since January 1.5, 2000; see Eq. (55.74)

The velocity of light is the same as in GRS 1980; see earlier in this section.

Since the WGS 84 ellipsoid is a geocentric equipotential ellipsoid, the flattening *f* of this reference ellipsoid is a derived constant. It means that *f* can be computed from the listed "fundamental" parameters; see Heiskanen and Moritz [1967, sections 2-7 through 2-10] or IAG [1971].

The flattening *f* has the value:

$$\text{flattening (ellipticity)} \qquad f = 1/298.257\ 223\ 563$$

WGS 84 Ellipsoidal Gravity Formula

The normal gravity γ along the surface of the WGS 84 ellipsoid is given by

$$\gamma = 978{,}032.677\ 14 \times \left(1. + 0.001\ 931\ 851\ 386\ 39 \times \sin^2\phi\right) \Big/ \left(1. - 0.006\ 694\ 379\ 990\ 13 \times \sin^2\phi\right)^{1/2} \text{ milligal}$$

where 1 milligal = 0.001 cm sec^{-2}.

WGS 84 Earth Gravitational Model

The WGS 84 Earth Gravitational Model (EGM) is a gravity model with harmonic coefficients up to degree and order 180. For civilian users, the coefficients above degree and order 18 are classified.

WGS 84 Geoid

As with the EGM, for civilian users the geoid based on coefficients above degree and order 18 is classified.

In general, the geoidal undulations (geoidal heights) are accurate to about 2 to 6 m. For 55% of the earth's coverage the accuracy is 2 to 3 m; for 93%, 2 to 4 m.

WGS 84 Refinements since 1994

Regarding the WGS 84 Reference Frame, two adjustments have been implemented since January 1, 1994; the reference frames WGS 84 (G730), between January 2, 1994 and September 28, 1996, and the WGS 84 (G873), since September 29, 1996. Also a modified WGS 84 Earth Gravitational Model as well as a modified WGS 94 Geoid Model have been adopted since October 1, 1996: the WGS 84 EGM 96, see NIMA [1997]. These modifications led to a refined value for the WGS 84 GM parameter:

Geocentric gravitational constant of the earth including the atmosphere

$$GM = (3{,}986{,}004.418 \pm 0.008) \times 10^8 m^3 s^{-2}$$

With the geocentric gravitational constant of the atmosphere unchanged, a consistent value for GM excluding the atmosphere has become:

Geocentric gravitational constant of the earth without atmosphere

$$GM_{-A} = (3{,}986{,}000.9 \pm 0.6) \times 10^8 m^3 s^{-2}$$

The WGS 84 EGM 96 is a gravity model with harmonic coefficients up to degree and order 360, consisting of 130,317 coefficients. The WGS 84 EGM96 Geoid has an error range between 0.5 and 1.0 m (1 sigma) anywhere on Earth. Geoid undulations are available on a 15 ft × 15 ft (arcminutes) grid.

IERS Standards 1992

The International Earth Rotation Service (IERS) adopted a standard reference system to be used in earth rotation analysis. The system includes not only numerical values for a set of variables, but also sets of adopted equations reflecting certain physical models. The reader is referred to IERS [1992] for more details.

Datum and Reference Frame Transformations

In Section 55.3 the transformation formulas are described relating coordinates given in two different reference frames. The adoption of new ellipsoidal parameters (so-called datum transformations) causes the geodetic coordinates to change. Tables 55.13 to 55.15, from such sources as Soler and Hothem [1988], DMA [1988], and NIMA [1997], list the values for various ellipsoids and reference frame transformations.

TABLE 55.13 Transformation Parameters among Several Global Reference Frames

Coordinate System (Datum) (1)	Δx (m) (2)	Δy (m) (3)	Δr (m) (4)	$\delta\varepsilon$ (arc sec) (5)	$\delta\psi$ (arc sec) (6)	$\delta\omega$ (arc sec) (7)	δs (ppm) (8)
NWL-9D→WGS-72	0	0	0	0	0	–0.26	–0.827
NWL-9D→WGS-84	0	0	+4.5	0	0	–0.814	–0.6
WGS-72→WGS-84	0	0	+4.5	0	0	–0.554	+0.227
BTS87→NWL-9D	+0.071	–0.509	–4.666	–0.0179	+0.005	+0.8073	+0.583
BTS87→WGS-84	+0.071	–0.509	–0.166	–0.0179	+0.005	–0.0067	–0.107
BTS87→VLBI (NGS)	–0.089	+0.143	–0.016	+0.0043	–0.0093	+0.0033	+0.009
BTS87→SLR (GSFC)	0.000[a]	0.000[a]	0.000[a]	+0.0018	–0.0062	+0.0075	0.000[a]
WGS-84→WGS-84 (GPS)	+0.026	–0.006	+0.093	+0.001	0.000	+0.002	–0.128

[a] These values were held fixed (i.e., the BTS87 frame origin and scale are assumed to be defined through satellite laser ranging (GSFC)).

Source: Solar, T. and Hothem, L.D., *J. Surveying Eng.*, 114, 84, 1988.

TABLE 55.14 Parameters of Some Adopted Reference Ellipsoids

Coordinate System (Datum) (1)	Reference Ellipsoid Used (2)	*a* (m) (3)	1/*f* (4)
AGD	AN (or SA-69)	6,378,160	298.25
ED-79	International	6,378,388	297
GEM-8	GEM-8	6,378,145	298.255
GEM-9 (or GEM-10)	GEM-9 (or GEM-10)	6,378,140	298.255
GEM-10B	GEM-10B	6,378,138	298.257
GEM-T1	GEM-T1	6,378,137	298.257
NAD-27	Clarke 1866	6,378,206.4	294.9786982
NAD-83	GRS-80	6,378,137	298.257222101
NWL-9D = NSWC-9Z2	WGS-66	6,378,145	298.25
SA-69	SA-69 (or AN)	6,378,160	298.25
WGS-72	WGS-72	6,378,135	298.26
WGS-84	WGS-84	6,378,137	298.257223563

Note: AGD = Australian geodetic datum; AN = Australian national; ED = European datum; GEM = Goddard earth model; GRS = Geodetic Reference System; NAD = North American datum; NSWC = Naval Surface Warfare Center; NWL = Naval Weapons Laboratory; SA = South American; WGS = World Geodetic System.

Source: Solar, T. and Hothem, L.D., *J. Surveying Eng.*, 114, 84, 1988.

TABLE 55.15 Transformation Parameters, Local Geodetic Systems to WGS 84

	Reference Ellipsoids and Parameter Differences			Transformation Parameters		
Local Geodetic Systems	Name	Δa(m)	$\Delta f \times 10^4$	ΔX(m)	ΔY(m)	ΔZ(m)
Adindan	Clark 1880	–112.145	–0.54750714	–162	–12	206
Afgooye	Krassovsky	–108	0.00480795	–43	–163	45
Ain El ABD 1970	International	–251	–0.14192702	–150	–251	–2
Anna 1 Astro 1965	Australian National	–23	–0.00081204	–491	–22	435
ARC 1950	Clarke 1880	–112.145	–0.54750714	–143	–90	–294
ARC 1960	Clarke 1880	–112.145	–0.54750714	–160	–8	–300
Ascension Island 1958	International	–251	–0.14192702	–207	107	52
Astro Beacon “E”	International	–251	–0.14192702	145	75	–272
Astro B4 Sorol Atoll	International	–251	–0.14192702	114	–116	–333
Astro Dos 71/4	International	–251	–0.14192702	–320	550	–494
Astronomic Station 1952	International	–251	–0.14192702	124	–234	–25
Australian Geodetic 1966	Australian National	–23	–0.00081204	–133	–48	148
Australian Geodetic 1984	Australian National	–23	–0.00081204	–134	–48	149
Bellevue (IGN)	International	–251	–0.14192702	–127	–769	472
Bermuda 1957	Clarke 1866	–69.4	–0.37264639	–73	213	296
Bogota Observatory	International	–251	–0.14192702	307	304	–318
Campo Inchauspe	International	–251	–0.14192702	–148	136	90
Canton Astro 1966	International	–251	–0.14192702	298	–304	–375
Cape	Clarke 1880	–112.45	–0.54750714	–136	–108	–292
Cape Canaveral	Clarke 1866	–69.4	–0.37264639	–2	150	181
Carthage	Clarke 1880	–112.145	–0.54750714	–263	6	431
Chatham 1971	International	–251	–0.14192702	175	–38	113
Chua Astro	International	–251	–0.14192702	–134	229	–29
Corrego Alegre	International	–251	–0.14192702	–206	172	–6
Djakarta (Batavia)	Bessel 1841	739.845	0.10037483	–377	681	–50
DOS 1968	International	–251	–0.14192702	230	–199	–752
Easter Island 1967	International	–251	–0.14192702	211	147	111
European 1950	International	–251	–0.14192702	–87	–98	–121
European 1979	International	–251	–0.14192702	–86	–98	–119
Gandajika Base	International	–251	–0.14192702	–133	–321	50

TABLE 55.15 Transformation Parameters, Local Geodetic Systems to WGS 84

Local Geodetic Systems	Reference Ellipsoids and Parameter Differences			Transformation Parameters		
	Name	Δa(m)	$\Delta f \times 10^4$	ΔX(m)	ΔY(m)	ΔZ(m)
Geodetic Datum 1949	International	–251	–0.14192702	84	–22	209
Guam 1963	Clarke 1866	–69.4	–0.37264639	–100	248	259
GUX 1 Astro	International	–251	–0.14192702	252	–209	–751
Hjorsey 1955	International	–251	–0.14192702	–73	46	-86
Hong Kong 1963	International	–251	–0.14192702	–156	–271	–189
India	Everest	860.655	0.28361368	214	836	303
				289	734	257
Ireland 1965	Modified Airy	796.811	0.11960023	506	–122	611
ISTS 073 Astro 1969	International	–251	–0.14192702	208	–435	–229
Johnson Island 1961	International	–251	–0.14192702	191	–77	–204
Kandawala	Everest	860.655	0.28361368	–97	787	86
Kerguelen Island	International	–251	–0.14192702	145	–187	103
Kertau 1948	Modified Everest	832.937	0.2861368	–11	851	5
L.C. 5 Astro	Clarke 1866	–69.4	–0.37264639	42	124	147
Liberia 1964	Clarke 1880	–112.145	–0.54750714	–90	40	88
Luzon	Clarke 1866	–69.4	–0.37264639			
Philippines				–133	–77	–51
Mindanao Island				–133	–79	–72
Mahe 1971	Clarke 1880	–112.145	–0.54750714	41	–220	–134
Macro Astro	International	–251	–0.14192702	–289	–124	60
Massawa	Bessel 1841	739.845	0.10037483	639	405	60
Merchich	Clarke 1880	–112.145	–0.54750714	31	146	47
Midway Astro 1961	International	–251	–0.14192702	912	–58	1227
Minna	Clarke 1880	–112.145	–0.54750714	–92	–93	122
Nahrwan	Clarke 1880	–112.145	–0.54750714			
Masirah Island				–247	–148	369
United Arab Emirates				–249	–156	381
Saudi Arabia				–231	–196	482
Naparima, BWI	International	–251	–0.14192702	–2	374	172
North American 1927	Clarke 1866	–69.4	–0.37264639			
Mean Value (Conus)				–8	160	176
Alaska				–5	135	172
Bahamas				–4	154	178
San Salvador Island				1	140	165
Canada				–10	158	187
Canal Zone				0	125	201
Caribbean				–7	152	178
Central America				0	125	194
Cuba				–9	152	178
Greenland				11	114	195
Mexico				–12	130	190
North American 1983	GRS	0	–0.00000016	0	0	0
Observatorio 1966	International	–251	–0.14192702	–425	–169	81
Old Egyptian	Helmert 1906	–63	0.00480795	–130	110	–13
Old Hawaiian	Clarke 1866	–69.4	–0.37264639	61	–285	–181
Oman	Clarke 1880	–112.145	–0.54750714	–346	–1	224
Ordnance Survey of Great Britain 1936	Airy	573.604	0.11960023	375	–111	431
Pico De Las Nieves	International	–251	–0.14192702	–307	–92	127
Pitcairn Astro 1967	International	–251	–0.14192702	185	165	42
Provisional South Chilean 1963	International	–251	–0.14192702	16	196	93
Provisional South American 1956	International	–251	–0.14192702	–288	175	–376
Puerto Rico	Clarke 1866	–69.4	–0.37264639	11	72	–101
Qatar National	International	–251	–0.14192702	–128	–283	22
Qornoq	International	–251	–0.14192702	164	138	–189
Reunion	International	–251	–0.14192702	94	–948	–1262

TABLE 55.15 Transformation Parameters, Local Geodetic Systems to WGS 84

Local Geodetic Systems	Reference Ellipsoids and Parameter Differences			Transformation Parameters		
	Name	Δa(m)	$\Delta f \times 10^4$	ΔX(m)	ΔY(m)	ΔZ(m)
Rome 1940	International	–251	–0.14192702	–225	–65	9
Santo (DOS)	International	–251	–0.14192702	170	42	84
Sao Braz	International	–251	–0.14192702	–203	141	53
Sapper Hill 1943	International	–251	–0.14192702	–355	16	74
Schwarazeck	Bessel 1841	653.135	0.10037483	616	97	–251
South American 1969	South American 1969	–23	–0.00081204	–57	1	–41
South Asia	Modified Fischer 1960	–18	0.00480795	7	–10	–26
Southeast Base	International	–251	–0.14192702	–499	–249	314
Southwest Base	International	–251	–0.14192702	–104	167	–38
Timbalai 1948	Everest	860.655	0.28361368	–689	691	–46
Tokyo	Bessel 1841	739.845	0.10037483	–128	481	664
Tristan Astro 1968	International	–251	–0.14192702	–632	438	–609
Viti Levu 1916	Clarke 1880	–112.145	–0.54750714	51	391	–36
Wake–Eniwetok 1960	Hough	–133	–0.14192702	101	52	–39
Zanderij	International	–251	–0.14192702	–265	120	–358

Source: DMA, DMA Technical Report 8350.2, DMA, March 1, 1988.

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Further Information

Textbooks and Reference Books

For additional reading and more background — from the very basic to the advanced level — in geodesy, satellite geodesy, physical geodesy, mechanics, orbital mechanics, and relativity, refer to the following textbooks (in English):

Bomford, G., *Geodesy*, Clarendon Press, Oxford, 1980.
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Torge, W., *Geodesy,* Walter de Gruyter, Berlin, 2001.
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Journals and Organizations

The latest results from research in geodesy is published in:

- *Journal of Geodesy:* The *Journal of Geodesy* (http://link.springer.de/link/service/journals/00190/) is the recent merger of two international magazines, all published by Springer-Verlag, Berlin, under the auspices of the International Association of Geodesy:
  - Bulletin Géodésique
  - Manuscripta Geodaetica

Geodesy- and geophysics-related articles can be found in:

- American Geophysical Union: EOS and *Journal of Geophysical Research*, Washington, D.C.
- Royal Astronomical Society: *Geophysical Journal International*, London

GPS-related articles can be found in:

- *GPS Solutions,* published by Wiley (http://www.interscience.wiley.com/jpages/1080–5370/info.html)
- Institute of Navigation: *Navigation*
- American Society of Photogrammetry and Remote Sensing: *Photogrammetric Engineering & Remote Sensing*

Many national mapping organizations publish journals in which recent results in geodesy, surveying, and mapping are documented:

- American Congress of Surveying and Mapping: *Surveying and Land Information Systems*, *Cartography and Geographic Information Systems* (http://www.acsm.net/publist.html)
- American Society of Civil Engineers: *Journal of Surveying Engineering* (http://www.pubs.asce.org/journals/su.html)
- Deutscher Verein für Vermessungswesen: *Zeitschrift für Vermessungswesen*, Konrad Wittwer Verlag, Stuttgart
- The Canadian Institute of Geomatics: *Geomatica* (http://www.cig-acsg.ca/page.asp?intNodeID=15)
- The Royal Society of Chartered Surveyors: *Survey Review* (http://www.surveyreview.org.uk/)
- Institution of Surveyors of Australia: *Australian Surveyor*

Worth special mention are the following trade magazines:

- *GPS World*, published by Advanstar Communications, Eugene, OR (http://www.gpsworld.com)
- *P.O.B.* (Point of Beginning), P.O.B. Publishing Company, Canton, MI (http://www.pobonline.com)
- *Professional Surveyor*, published by American Surveyors Publishing Company, Inc., Arlington, VA (http://www.profsurv.com)
- *G.I.M. International (Geomatics Info Magazine)*, published by Geodetical Information & Trading Centre B.V., Lemmer, the Netherlands (http://www.gim-international.com)

National mapping organizations, such as the U.S. National Geodetic Survey (http://www.ngs.noaa.gov/), regularly make geodetic software available (free and at cost). Information can be obtained from:

National Geodetic Survey
Geodetic Services Branch
National Ocean Service, NOAA
1315 East–West Highway, Station 8620
Silver Spring, MD 20910–3282