52

Structural Reliability

Ser-Tong Quek
National University of Singapore

52.1 Introduction

The principle aim of structural design is the assurance of satisfactory performance within the constraints of economy. A primary complication toward achieving this in practice is imperfect execution and the lack of complete information. The existence of uncertainties in structural engineering has long been recognized and quantitatively accounted for through the use of safety factors in design. Reliability analysis, using probability theory as a tool, provides a rational and consistent basis for determining the appropriate safety margins (Ang and Tang, 1984). Its success is exhibited by the numerous reliability-based provisions developed in recent code revisions to achieve a target reliability range in the design of structural elements (e.g., AISC, ACI, AASHTO). Over the last 20 years, research studies have been carried out to provide similar reliability provisions at the structural systems level, and perhaps they will have a more direct and substantial influence in design specifications over the next decade.

This chapter aims to provide the basic knowledge for structural engineers who have little exposure in this field and to serve as a platform for understanding the basic philosophy behind reliability-based design.

Definition of Reliability

Reliability can be defined as the probabilistic measure of assurance of performance with respect to some prescribed condition(s). A condition can refer to an ultimate limit state (such as collapse) or serviceability limit state (such as excessive deflection and/or vibration).

As a simple illustration, consider a bar with ultimate tensile capacity R (which can be viewed as the supply to the system) that has to resist a tensile load S (which can be viewed as the demand of the system).

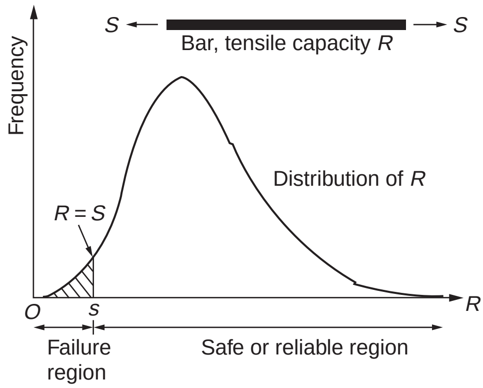

FIGURE 52.1 Distribution of R, failure, and safe regions.

Performance against failure is ensured if $R > S$ (i.e., supply exceeds demand). However, the capacity of this particular bar cannot be known exactly unless it is tested to failure. Nevertheless, some estimates can be obtained based on test results of similar bars, which can be summarized in the form of a distribution. The proportion of bars with strength equal to or above S (assumed deterministic) gives an indication of the reliability of this bar (see Fig. 52.1). Complementary to this, the proportion (shaded region) of bars below S indicates the probability of failure of the system. Hence, reliability can be viewed as a complementary to the *probability of failure*.

The simple example above can be extended to the case where S is not known with certainty. Similarly, one can consider a more complicated function for R, e.g., a reinforced concrete beam where the capacity is a function of many variables, such as the properties of the concrete and reinforcing bars used. One can also look at the reliability of a structure comprising more than one bar or element. An exposition to some basic probability concepts is prerequisite to understanding the complexity and solutions of such problems.

52.2 Basic Probability Concepts

Random Variables and Probability Distributions

For the case of the tensile capacity of the bar mentioned above, its strength can be modeled as a *continuous random variable* and denoted in general as X. Other engineering parameters may take on only discrete values, such as the number of significant earthquakes, and hence modeled as a *discrete random variable*. In either case, a histogram can be constructed once data are available and normalized such that the area under it for the continuous random variable case (or the summation of the ordinates for the discrete case) is unity.

A mathematical expression can be used to describe the distribution represented by the histogram, which for the discrete case is known as the *probability mass function* (PMF), denoted as $p_X(x)$, and for the continuous case as *probability density function* (PDF), denoted as $f_X(x)$.

The cumulative value of the mass or density from the smallest value of X can be described by its *cumulative distribution function* (CDF), commonly denoted as $F_X(x)$ (see Fig. 52.2). Hence, one can write the probability of X taking on values less than or equal to a as

$$P(X \le a) = F_X(a) = \int_{-\infty}^{a} f_X(x)dx \qquad \textit{for continuous } X \tag{52.1a}$$

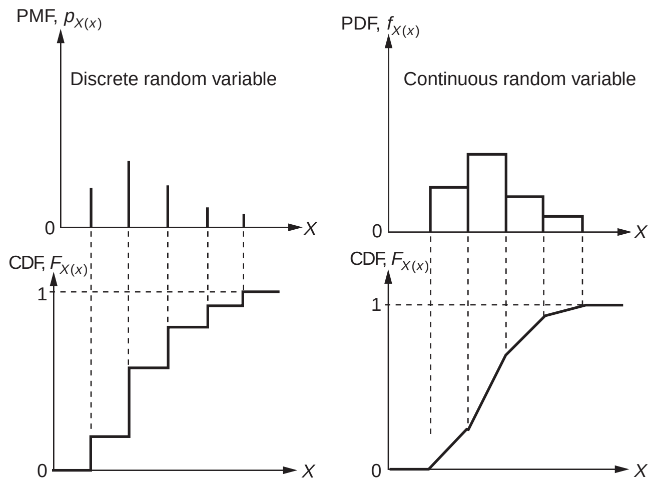

FIGURE 52.2 PMF (discrete random variable), PDF (continuous random variable), and corresponding CDFs.

$$= \sum_{x_i \le a} p_X(x_i) \qquad \textit{for discrete } X \tag{52.1b}$$

Commonly used probability functions related to engineering problems can be found in textbooks on probability and statistics (e.g., Ang and Tang, 1975). Table 52.1 is provided for convenience.

The most common distribution used is the *normal distribution*, with two parameters, namely, mean, μ_X, and standard deviation, σ_X. It cumulative function $F_X(x)$ cannot be expressed in closed form and is often denoted as

$$P(X \le a) = F_X(a) = \int_{-\infty}^{a} \frac{1}{\sigma_X \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right] = \Phi\left(\frac{a-\mu_X}{\sigma_X}\right) \tag{52.2}$$

where $\Phi(.)$ denotes the CDF of a normal distribution with a mean equal to 0 and standard deviation equal to 1. Its concise tabulated form is given in Table 52.2.

Another common distribution used, which spans over positive values of X, is the *lognormal distribution*, with parameters λ_X and ζ_X. Note that the transformation $Y = ln\ X$ produces a normal distribution for Y. The CDF of X can be conveniently evaluated as

$$P(X \le a) = F_X(a) = \Phi\left(\frac{\ln a - \lambda_X}{\zeta_X}\right) \tag{52.3}$$

Expectation and Moments

Consider a function $g(X)$ where X is a random variable with PMF $p_X(x)$ or PDF $f_X(x)$. The expected value (also known as *expectation*) of $g(X)$ is defined as

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \qquad \textit{for continuous } X \tag{52.4a}$$

$$= \sum_{all\, x_i} g(x_i) p_X(x_i) \qquad \textit{for discrete } X \tag{52.4b}$$

TABLE 52.1 Some Distribution Type

Distribution	PMF ($p_X(x)$) or PDF ($f_X(x)$)	Mean, $E[X]$	Variance, Var$[X]$
Binomial	$p_X(x)=\binom{n}{x}p^x(1-p)^{n-x}$ $x=0,1,2,\ldots,n$	np	$np(1-p)$
Poisson	$p_X(x)=\dfrac{(\nu t)^x}{x!}e^{-\nu t}$	νt	νt
Normal	$f_X(x)=\dfrac{1}{\sigma\sqrt{2\pi}}\exp\left[-\dfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^2\right]$ $-\infty<x<\infty$	μ	σ^2
Lognormal	$f_X(x)=\dfrac{1}{x\zeta\sqrt{2\pi}}\exp\left[-\dfrac{1}{2}\left(\dfrac{\ln x-\lambda}{\zeta}\right)^2\right]$ $0<x<\infty$	$e^{\left(\lambda+\frac{1}{2}\zeta^2\right)}$	$e^{\left(2\lambda+\zeta^2\right)}\left[e^{\zeta^2}-1\right]$
Rayleigh	$f_X(x)=\dfrac{x}{\alpha^2}\exp\left[-\dfrac{1}{2}\left(\dfrac{x}{\alpha}\right)^2\right]$ $0\le x<\infty$	$\alpha\sqrt{\dfrac{\pi}{2}}$	$\left(2-\dfrac{\pi}{2}\right)\alpha^2$
Exponential	$f_X(x)=\lambda\exp\left[-\lambda(x-\tau)\right]$ $\tau\le x<\infty$	$\tau+\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$
Gumbel type I maximum	$f_X(x)=\alpha\exp\left[-\alpha(x-u)-e^{-\alpha(x-u)}\right]$ $-\infty<x<\infty$	$u+\dfrac{0.5772}{\alpha}$	$\dfrac{\pi^2}{6\alpha^2}$
Fretchet type II maximum	$f_X(x)=\dfrac{k}{\nu-\tau}\left(\dfrac{\nu-\tau}{x-\tau}\right)^{k+1}\exp\left[-\left(\dfrac{\nu-\tau}{x-\tau}\right)^k\right]$ $\varepsilon<x<\infty$	$(\nu-\tau)\Gamma\left(1-\dfrac{1}{k}\right)+\tau$	$(\nu-\tau)^2\left[\Gamma\left(1-\dfrac{2}{k}\right)\right]-\Gamma^2\left(1-\dfrac{1}{k}\right)$
Weibull type III minimum	$f_X(x)=\dfrac{k}{w-\varepsilon}\left(\dfrac{x-\varepsilon}{w-\varepsilon}\right)^{k-1}\exp\left[-\left(\dfrac{x-\varepsilon}{w-\varepsilon}\right)^k\right]$ $\varepsilon<x<\infty$	$(w-\varepsilon)\Gamma\left(1-\dfrac{1}{k}\right)+\varepsilon$	$(w-\varepsilon)^2\left[\Gamma\left(1-\dfrac{2}{k}\right)+\Gamma^2\left(1-\dfrac{1}{k}\right)\right]$

If $g(X) = X$, then Eq. (52.4) gives the *population mean* (denoted as μ_X), which physically describes the central tendency of the distribution of X. The mean is also known as the *first moment* of X.

In general, if $g(X) = X^n$, evaluation of Eq. (52.4) gives the nth moment of X, denoted as $\mu_X^{(n)}$.

If $g(X) = (X - \mu_X)^2$, then Eq. (52.4) gives the *population variance* (denoted as σ_X^2), which measures the spread of data around the mean value. It is also known as the *second central moment* and is related to the second moment and the mean by

$$\sigma_X^2 = E\left[\left(X-\mu_X\right)^2\right] = E\left[X^2\right]-\mu_X^2 \qquad (52.5)$$

The square root of variance is the *population standard deviation*, and for $\mu_X \neq 0$, its normalized form is known as the *coefficient of variation*, that is,

$$COV = V_X = \sigma_X/\mu_X \qquad (52.6)$$

TABLE 52.2 Values for Φ(Z)

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
–0.0	5.0000E-1	4.9601E-1	4.9202E-1	4.8803E-1	4.8405E-1	4.8006E-1	4.7608E-1	4.7210E-1	4.6812E-1	4.6414E-1
–0.1	4.6017E-1	4.5620E-1	4.5224E-1	4.4828E-1	4.4433E-1	4.4038E-1	4.3644E-1	4.3251E-1	4.2858E-1	4.2465E-1
–0.2	4.2074E-1	4.1683E-1	4.1294E-1	4.0905E-1	4.0517E-1	4.0129E-1	3.9743E-1	3.9358E-1	3.8974E-1	3.8591E-1
–0.3	3.8209E-1	3.7828E-1	3.7448E-1	3.7070E-1	3.6993E-1	3.6317E-1	3.5942E-1	3.5569E-1	3.5197E-1	3.4827E-1
–0.4	3.4458E-1	3.4090E-1	3.3724E-1	3.3360E-1	3.2997E-1	3.2636E-1	3.2276E-1	3.1918E-1	3.1561E-1	3.1207E-1
–0.5	3.0854E-1	3.0503E-1	3.0153E-1	2.9806E-1	2.9460E-1	2.9116E-1	2.8774E-1	2.8434E-1	2.8096E-1	2.7760E-1
–0.6	2.7425E-1	2.7093E-1	2.6763E-1	2.6435E-1	2.6109E-1	2.5785E-1	2.5463E-1	2.5143E-1	2.4825E-1	2.4510E-1
–0.7	2.4196E-1	2.3885E-1	2.3576E-1	2.3270E-1	2.2965E-1	2.2663E-1	2.2363E-1	2.2065E-1	2.1770E-1	2.1476E-1
–0.8	2.1186E-1	2.0897E-1	2.0611E-1	2.0327E-1	2.0045E-1	1.9766E-1	1.9489E-1	1.9215E-1	1.8943E-1	1.8673E-1
–0.9	1.8406E-1	1.8141E-1	1.7879E-1	1.7619E-1	1.7361E-1	1.7106E-1	1.6853E-1	1.6602E-1	1.6354E-1	1.6109E-1
–1.0	1.5866E-1	1.5625E-1	1.5386E-1	1.5151E-1	1.4917E-1	1.4686E-1	1.4457E-1	1.4231E-1	1.4007E-1	1.3786E-1
–1.1	1.3567E-1	1.3350E-1	1.3136E-1	1.2924E-1	1.2714E-1	1.2507E-1	1.2302E-1	1.2100E-1	1.1900E-1	1.1702E-1
–1.2	1.1507E-1	1.1314E-1	1.1123E-1	1.0935E-1	1.0749E-1	1.0565E-1	1.0383E-1	1.0204E-1	1.0027E-1	9.8525E-2
–1.3	9.6800E-2	9.5098E-2	9.3418E-2	9.1759E-2	9.0123E-2	8.8508E-2	8.6915E-2	8.5343E-2	8.3793E-2	8.2264E-2
–1.4	8.0757E-2	7.9270E-2	7.7804E-2	7.6359E-2	7.4934E-2	7.3529E-2	7.2145E-2	7.0781E-2	6.9437E-2	6.8112E-2
–1.5	6.6807E-2	6.5522E-2	6.4255E-2	6.3008E-2	6.1780E-2	6.0571E-2	5.9380E-2	5.8208E-2	5.7053E-2	5.5917E-2
–1.6	5.4799E-2	5.3699E-2	5.2616E-2	5.1551E-2	5.0503E-2	4.9471E-2	4.8457E-2	4.7460E-2	4.6479E-2	4.5514E-2
–1.7	4.4565E-2	4.3633E-2	4.2716E-2	4.1815E-2	4.0930E-2	4.0059E-2	3.9204E-2	3.8364E-2	3.7538E-2	3.6727E-2
–1.8	3.5930E-2	3.5148E-2	3.4380E-2	3.3625E-2	3.2884E-2	3.2157E-2	3.1443E-2	3.0742E-2	3.0054E-2	2.9379E-2
–1.9	2.8717E-2	2.8067E-2	2.7429E-2	2.6803E-2	2.6190E-2	2.5588E-2	2.4998E-2	2.4419E-2	2.3852E-2	2.3295E-2
–2.0	2.2750E-2	2.2216E-2	2.1692E-2	2.1178E-2	2.0675E-2	2.0182E-2	1.9699E-2	1.9226E-2	1.8763E-2	1.8309E-2
–2.1	1.7864E-2	1.7429E-2	1.7003E-2	1.6586E-2	1.6177E-2	1.5778E-2	1.5386E-2	1.5003E-2	1.4629E-2	1.4262E-2
–2.2	1.3903E-2	1.3553E-2	1.3209E-2	1.2874E-2	1.2545E-2	1.2224E-2	1.1911E-2	1.1604E-2	1.1304E-2	1.1011E-2
–2.3	1.0724E-2	1.0444E-2	1.0170E-2	9.9031E-3	9.6419E-3	9.3867E-3	9.1375E-3	8.8940E-3	8.6563E-3	8.4242E-3
–2.4	8.1975E-3	7.9763E-3	7.7603E-3	7.5494E-3	7.3436E-3	7.1428E-3	6.9469E-3	6.7557E-3	6.5691E-3	6.3872E-3
–2.5	6.2097E-3	6.0366E-3	5.8677E-3	5.7031E-3	5.5426E-3	5.3861E-3	5.2336E-3	5.0849E-3	4.9400E-3	4.7988E-3
–2.6	4.6612E-3	4.5271E-3	4.3965E-3	4.2692E-3	4.1453E-3	4.0246E-3	3.9070E-3	3.7926E-3	3.6811E-3	3.5726E-3
–2.7	3.4670E-3	3.3642E-3	3.2641E-3	3.1667E-3	3.0720E-3	2.9798E-3	2.8901E-3	2.8028E-3	2.7179E-3	2.6354E-3
–2.8	2.5551E-3	2.4771E-3	2.4012E-3	2.3274E-3	2.2557E-3	2.1860E-3	2.1182E-3	2.0524E-3	1.9884E-3	1.9262E-3
–2.9	1.8658E-3	1.8071E-3	1.7502E-3	1.6948E-3	1.6411E-3	1.5889E-3	1.5382E-3	1.4890E-3	1.4412E-3	1.3949E-3
–3.0	1.3499E-3	1.3062E-3	1.2639E-3	1.2228E-3	1.1829E-3	1.1442E-3	1.1067E-3	1.0703E-3	1.0350E-3	1.0008E-3
–3.1	9.6760E-4	9.3544E-4	9.0426E-4	8.7403E-4	8.4474E-4	8.1635E-4	7.8885E-4	7.6219E-4	7.3638E-4	7.1136E-4
–3.2	6.8714E-4	6.6367E-4	6.4095E-4	6.1895E-4	5.9765E-4	5.7703E-4	5.5706E-4	5.3774E-4	5.1904E-4	5.0094E-4
–3.3	4.8342E-4	4.6648E-4	4.5009E-4	4.3423E-4	4.1889E-4	4.0406E-4	3.8971E-4	3.7584E-4	3.6243E-4	3.4946E-4
–3.4	3.3693E-4	3.2481E-4	3.1311E-4	3.0179E-4	2.9086E-4	2.8029E-4	2.7009E-4	2.6023E-4	2.5071E-4	2.4151E-4
–3.5	2.3263E-4	2.2405E-4	2.1577E-4	2.0778E-4	2.0006E-4	1.9262E-4	1.8543E-4	1.7849E-4	1.7180E-4	1.6534E-4
–3.6	1.5911E-4	1.5310E-4	1.4730E-4	1.4171E-4	1.3632E-4	1.3112E-4	1.2611E-4	1.2128E-4	1.1662E-4	1.1213E-4
–3.7	1.0780E-4	1.0363E-4	9.9611E-5	9.5740E-5	9.2010E-5	8.8417E-5	8.4957E-5	8.1624E-5	7.8414E-5	7.5324E-5
–3.8	7.2348E-5	6.9483E-5	6.6726E-5	6.4072E-5	6.1517E-5	5.9059E-5	5.6694E-5	5.4418E-5	5.2228E-5	5.0122E-5
–3.9	4.8096E-5	4.6148E-5	4.4274E-5	4.2473E-5	4.0741E-5	3.9076E-5	3.7475E-5	3.5936E-5	3.4458E-5	3.3037E-5
–4.0	3.1671E-5	3.0359E-5	2.9099E-5	2.7888E-5	2.6726E-5	2.5609E-5	2.4536E-5	2.3507E-5	2.2518E-5	2.1569E-5
–4.1	2.0658E-5	1.9783E-5	1.8944E-5	1.8138E-5	1.7365E-5	1.6624E-5	1.5912E-5	1.5230E-5	1.4575E-5	1.3948E-5
–4.2	1.3346E-5	1.2769E-5	1.2215E-5	1.1685E-5	1.1176E-5	1.0689E-5	1.0221E-5	9.7736E-6	9.3447E-6	8.9337E-6
–4.3	8.5399E-6	8.1627E-6	7.8015E-6	7.4555E-6	7.1241E-6	6.8069E-6	6.5031E-6	6.2123E-6	5.9340E-6	5.6675E-6
–4.4	5.4125E-6	5.1685E-6	4.9350E-6	4.7117E-6	4.4979E-6	4.2935E-6	4.0980E-6	3.9110E-6	3.7322E-6	3.5612E-6
–4.5	3.3977E-6	3.2414E-6	3.0920E-6	2.9492E-6	2.8127E-6	2.6823E-6	2.5577E-6	2.4386E-6	2.3249E-6	2.2162E-6
–4.6	2.1125E-6	2.0133E-6	1.9187E-6	1.8283E-6	1.7420E-6	1.6597E-6	1.5810E-6	1.5060E-6	1.4344E-6	1.3660E-6
–4.7	1.3008E-6	1.2386E-6	1.1792E-6	1.1226E-6	1.0686E-6	1.0171E-6	9.6796E-7	9.2113E-7	8.7648E-7	8.3391E-7
–4.8	7.9333E-7	7.5465E-7	7.1779E-7	6.8267E-7	6.4920E-7	6.1731E-7	5.8693E-7	5.5799E-7	5.3043E-7	5.0418E-7
–4.9	4.7918E-7	4.5538E-7	4.3272E-7	4.1115E-7	3.9061E-7	3.7107E-7	3.5247E-7	3.3476E-7	3.1792E-7	3.0190E-7

Note: $\Phi(-Z) = 1 - \Phi(Z)$, e.g., $\Phi(2) = 1 - \Phi(-2)$.

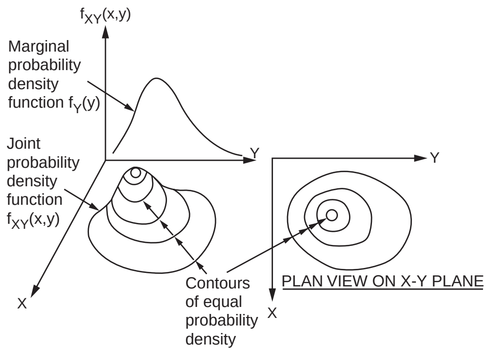

FIGURE 52.3 Joint probability density function.

Higher order central moments can be defined, where $g(X) = (X - \mu_X)^n$. For example, $n = 3$ gives the third central moment and is a measure of the skewness of the distribution, and $n = 4$ gives the fourth central moment and is a measure of the peakedness (or flatness) of the distribution. The third and fourth central moments for a normal distribution are 0 and $3\sigma_x^2$, respectively.

Joint Distribution and Correlation Coefficient

Invariably, all engineering problems involve more than one variable that are random and may be related to one another. To estimate the distribution of multiple random variables, data are jointly collected, from which a multidimensional histogram can be plotted. A suitable *joint probability mass function* or *density function* may be used to represent the spread of data. An example of a joint PDF $f_{XY}(x, y)$ for two random variables, X and Y, is illustrated in Fig. 52.3. The plan view is a two-dimensional contour representation of the three-dimensional plot. Lower probability information can be deduced from the joint probability function; for example, the marginal PDF of Y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx \tag{52.7}$$

as illustrated in Fig. 52.3.

Consider the random variable Z, which is the sum of two random variables X and Y, that is, $Z = X + Y$. The mean and variance of Z can be expressed as

$$\mu_Z = E[Z] = E[X + Y] = E[X] + E[Y] = \mu_X + \mu_Y \tag{52.8}$$

$$\begin{aligned}\sigma_Z^2 = Var[Z] &= E\left[(Z-\mu_Z)^2\right] = E\left[\left\{(X-\mu_X)+(Y-\mu_Y)\right\}^2\right] \\ &= E\left[(X-\mu_X)^2\right] + E\left[(Y-\mu_Y)^2\right] + 2E\left[(X-\mu_X)(Y-\mu_Y)\right] \\ &= \sigma_X^2 + \sigma_Y^2 + 2Cov[X,Y] = \sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y\end{aligned} \tag{52.9}$$

where $Cov[X, Y] = E[(X - \mu_X)(Y - \mu_Y)]$ is the *covariance* between X and Y. It is a measure of their linear interdependence, and its normalized form is known as the *correlation coefficient*, given by

$$\rho_{XY} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y} \tag{52.10}$$

When $\rho_{XY} = +1$, X and Y are said to be *perfectly positive correlated*, whereas $\rho_{XY} = -1$ implies *perfect negative correlation* (“negative” meaning that high values of X occur with low values of Y and vice versa). When $\rho_{XY} = 0$, X and Y are said to be uncorrelated.

It is appropriate to mention here that for the special case where X and Y are normally distributed, their sum (and also difference), Z, follows a normal distribution.

Statistical Independence

Practical problems involve more than one random variable with varying degrees of interdependence among them. An extreme case is when the random variables, say X and Y, are *statistically independent*, meaning that the event of one variable taking on some value does not affect the probability of the other variable taking on another value. That is,

$$P\left(X \le a \,\middle|\, Y \le b\right) = P\left(X \le a\right) \tag{52.11}$$

where the symbol | denotes “given that,” and the left-hand side of Eq. (52.11) is a *conditional probability*. As a consequence of Eq. (52.11), the probability of the two events, $X \le a$ and $Y \le b$, happening together (denoted by the intersection symbol $\cap$) can be simplified as

$$P\left(X \le a \cap Y \le b\right) = P\left(X \le a \,\middle|\, Y \le b\right) P\left(Y \le b\right) = P\left(X \le a\right) P\left(Y \le b\right) \tag{52.12}$$

Note that if two variables are statistically independent, they must also be uncorrelated, but the converse is not true in general.

The concept of statistical independence permits simplification in solving complex reliability problems approximately. In fact, a number of real quantities can be reasonably assumed as statistically independent. For example, one would expect dead load to be relatively independent of wind load, the occurrence of tornado to be independent of the occurrence of earthquake, and loads to be independent of structural capacity.

52.3 Assessment of Reliability

A brief exposure is provided here for engineers who wish to have some basic understanding of structural reliability theory. Those interested in a more complete treatment should refer to the many textbooks in this field, such as Ang and Tang (1984), Ditlevsen and Madsen (1996), Madsen et al. (1986), Melchers (1999), and Nowak and Collins (2000).

Fundamental Case

Consider the bar in tension shown in Fig. 52.1 and denote the PDF of R as $f_R(r)$ and the deterministic load as $S = s_1$. Then the probability of failure is given by

$$p_f = P\left(R \le S \,\middle|\, S = s_1\right) = \int_{-\infty}^{s_1} f_{R|S}\left(r \,\middle|\, S = s_1\right) dr \tag{52.13}$$

For the case where S is also random, described by the PDF $f_s(s)$, Eq. (52.13) becomes

$$\begin{aligned} p_f &= \int_{-\infty}^{\infty} P\left(R \le S \,\middle|\, S = s\right) f_S(s)\, ds = \int_{-\infty}^{\infty}\int_{-\infty}^{s} f_{R|S}\left(r \,\middle|\, S = s\right) f_S(s)\, dr\, ds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{s} f_{RS}(r,s)\, dr\, ds \end{aligned} \tag{52.14}$$

which is the volume of the joint PDF in the failure region defined by $R \le S$.

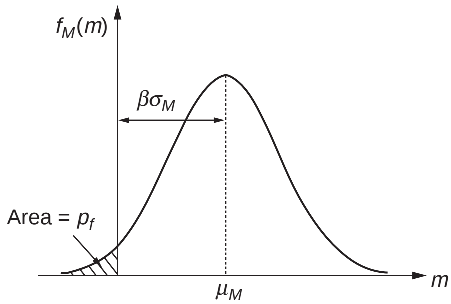

FIGURE 52.4 Probabilistic interpretation of safety margin.

TABLE 52.3 Reliability Indices and Corresponding Failure Probabilities

Reliability Index β	Failure Probability p_f
10^{-1}	1.28
10^{-2}	2.33
10^{-3}	3.10
10^{-4}	3.72
10^{-5}	4.25
10^{-6}	4.75

An equivalent formulation is to define a performance function

$$M = g(R,S) = R - S \tag{52.15}$$

In this case, M can be interpreted as the *margin of safety*, and $g(R, S)$ is a limit state function. The mean μ_M and standard deviation σ_M of M can be computed following Eqs. (52.8) and (52.9). If both R and S are normally distributed, then M is also normally distributed. Equation (52.15) can then be evaluated as

$$p_f = P(M \le 0) = \Phi\left(\frac{0 - \mu_M}{\sigma_M}\right) = \Phi\left(-\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2 + 2\rho_{RS}\sigma_R\sigma_S}}\right) \tag{52.16}$$

The quantity μ_M/σ_M is denoted as β and is known as the *reliability index*. Its relationship to the safety margin in the probabilistic sense is illustrated in Fig. 52.4. Some β values in the practical range and their corresponding failure probabilities with respect to the normal distribution are given in Table 52.3.

For the case where both R and S can be modeled more accurately by the lognormal distribution, the factor of safety format will result in an easier computation of p_f. That is, define the performance function as

$$M = g(R,S) = R/S \tag{52.17}$$

By taking the natural logarithm and in view of Eq. (52.3), Eq. (52.14) can be simplified as

$$p_f = P(M \le 0) = \Phi\left(-\frac{\lambda_R - \lambda_S}{\sqrt{\zeta_R^2 + \zeta_S^2 + 2\rho_{\ln R \ln S}\zeta_R\zeta_S}}\right) \tag{52.18}$$

where

$$\rho_{\ln R \ln S} = \frac{\ln\left(1 + \rho_{RS} V_R V_S\right)}{\zeta_R \zeta_S} \cong \rho_{RS}$$

for small V_R and V_S.

For problems involving n random variables, $\boldsymbol{X} = \{X_1, X_2, X_3, \ldots X_n\}$, with performance function

$$M = g(\boldsymbol{X}) = a_0 + \sum_{i=1}^{n} a_i X_i \tag{52.19}$$

The mean and variance of M in Eq. (52.16) is computed as

$$\mu_M = a_0 + \sum_{i=1}^{n} a_i \mu_{X_i}, \qquad \sigma_M^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \rho_{X_i X_j} \sigma_{X_i} \sigma_{X_j} \tag{52.20}$$

First-Order Second-Moment Index

It should be mentioned here that in the early stage of the development of structural reliability theory, β given in Eq. (52.16) is proposed as the first-order second-moment index, since only the first two moments of the random variables are involved. The failure probability is approximated by Eq. (52.16), which is exact if the random variables are normally distributed and $g(\boldsymbol{X})$ is a linear function.

Hasofer–Lind Reliability Index

Consider the case where R and S are uncorrelated. Define the standardized form of the random variable as $R' = (R - \mu_R)/\sigma_R$ and $S' = (S - \mu_S)/\sigma_S$. The limit state function of Eq. (52.15) can be rewritten as

$$M = g(R, S) = \sigma_R R' - \sigma_S S' + \mu_R - \mu_S = 0 \tag{52.21}$$

If the limit state function is plotted on the $R' - S'$ coordinate system (see Fig. 52.5), then β is the shortest distance from the origin to the limit state function $g(R', S') = 0$, also known as the *Hasofer–Lind reliability index*. The point x^* has the highest probability density value in the failure region and is hence termed the *most likely failure point*.

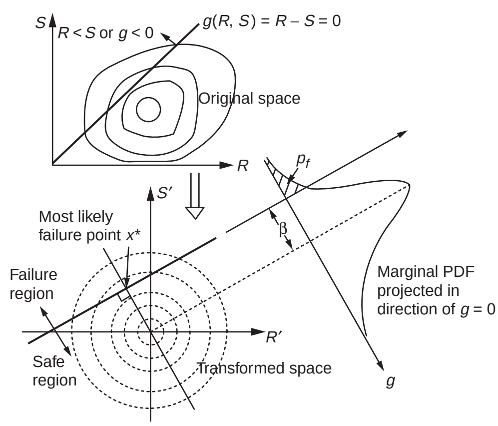

FIGURE 52.5 Hasofer–Lind reliability index and most likely failure point.

Another important set of information that can be extracted from this computation is the sensitivity of each random variable to the reliability index. This is given by the direction cosines, which for this example is

$$\alpha_R = \frac{\sigma_R}{\sqrt{\sigma_R^2 + \sigma_{S'}^2}}, \quad \alpha_S = \frac{-\sigma_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{52.22}$$

The most likely failure point in the transformed coordinate space is given by

$$\left(r'^{*}, s'^{*}\right) = \left(-\alpha_R\beta, -\alpha_S\beta\right) \quad \text{or} \quad \left(r^{*}, s^{*}\right) = \left(\mu_R - \alpha_R\beta\sigma_R, \mu_S - \alpha_S\beta\sigma_S\right) \tag{52.23}$$

Reliability Estimate by FORM

In general, the limit state function can be nonlinear and the distribution of the random variables different from normal or lognormal. The first-order reliability method (FORM) has been developed to evaluate the failure probability with reasonable accuracy and cost for realistic structures. The basic idea of the method can be summarized as follows:

1. For the general case of correlated nonnormal random variables, the *Rosenblatt transformation* can be employed and requires the complete joint PDF (see, e.g., Ditlevsen and Madsen, 1996). In many practical problems, only the marginal distribution functions and the covariance matrix are known. For such cases, approximations using the Nataf distribution can be employed (Der Kiureghian and Liu, 1986). A more approximate but simpler procedure and its variations have been developed. Essentially, the random variables are first transformed into uncorrelated random variables using the correlation matrix (basically finding its eigenvalues and eigenvectors). The *principle of normal tail approximation* (Rackwitz and Fiessler, 1978) is then employed to replace the nonnormal distributions by normal distributions. For the latter, the equivalent normal distribution parameters are determined such that the PDF as well as the CDF values at the most likely failure point corresponding to the actual and the normal distributions are equal.
2. The nonlinear performance function, $g_{NL}(\boldsymbol{X})$, is linearized at the most likely failure point (denoted as $\boldsymbol{X}^*$) in the standardized normal coordinate space. That is, $g_{NL}(\boldsymbol{X})$ is now approximated by Eq. (52.19) with a_i given by the partial differential $\partial g_{NL}(\boldsymbol{X})/\partial X_i$ evaluated at $\boldsymbol{X}^*$. This is basically a first-order Taylor series expansion of the nonlinear performance function. Since $\boldsymbol{X}^*$ is not known apriori, an iterative solution to obtain β is inevitable.

Based on the principle of normal tail approximation and the first-order approximation of $g_{NL}(\boldsymbol{X})$ described above, a simple algorithm to compute β is given by the following steps:

1. Find the eigenvalues λ and corresponding eigenvectors $\boldsymbol{T}$ corresponding to the correlation matrix ρ_{XX}.
2. Assume an initial guess $\boldsymbol{x}^*$ for $\boldsymbol{X}$, satisfying $g_{NL}(\boldsymbol{x}^*) = 0$. This can be achieved by using the mean values for n-1 random variables, with the value for the remaining variable obtained by enforcing the condition $g_{NL}(\boldsymbol{x}^*) = 0$. Next, compute $\partial g_{NL}(\boldsymbol{X})/\partial X_i$ at $\boldsymbol{x}^*$.
3. For each of the nonnormal random variables, say with CDF $F_{X_i}(x)$ and PDF $f_{X_i}(x)$, compute the equivalent normal parameters as follows:

$$\begin{aligned} \sigma_{X_i}^N &= \frac{1}{f_{X_i}\left(x_i^*\right)} \phi\left[\Phi^{-1}\left(F_{X_i}\left(x_i^*\right)\right)\right] \\ \mu_{X_i}^N &= x_i^* - \sigma_{X_i}^N\left[\Phi^{-1}\left(F_{X_i}\left(x_i^*\right)\right)\right] \end{aligned} \tag{52.24}$$

The values for the vector of reduced variates z^* at the failure point can be obtained where

$$\beta = \frac{\mathbf{G^t T}\lambda^{1/2}\{\mathbf{z}\}}{\sqrt{\mathbf{G^t}\rho_{\mathbf{XX}}\mathbf{G}}} \quad \text{where} \quad G_i = -\sigma_{X_i}^N \left.\frac{\partial g_{NL}(\mathbf{X})}{\partial X_i}\right|_{x^*}$$

4. The reliability index, corresponding to the most likely failure point, can be computed using the formula

$$\beta = \frac{\mathbf{G^t T}\lambda^{1/2}\{\mathbf{z}\}}{\sqrt{\mathbf{G^t}\rho_{\mathbf{XX}}\mathbf{G}}} \quad \text{where} \quad G_i = -\sigma_{X_i}^N \left.\frac{\partial g_{NL}(\mathbf{X})}{\partial X_i}\right|_{x^*} \tag{52.25}$$

   where the superscript **t** denotes transpose.
5. The sensitivity factors can be estimated as

$$\alpha = \frac{\lambda^{1/2}\mathbf{T^t G}}{\sqrt{\mathbf{G^t}\rho_{\mathbf{XX}}\mathbf{G}}} \tag{52.26}$$

6. A new set of n-1 random variables can be formulated as

$$x_i^* = \mu_{X_i}^N - \alpha_i \beta \sigma_{X_i}^N \tag{52.27}$$

7. and x_n^* obtained by enforcing the condition $g_{NL}(\boldsymbol{x}^*) = 0$.
8. Steps 3 to 6 are repeated until β and $\boldsymbol{x}^*$ converge. The failure probability is then approximated by $p_f = \Phi(-\beta)$.

Reliability Estimate by Monte Carlo Simulation

The computation of failure probability is equivalent to evaluating the integral

$$p_f = \int \cdots \int_{g(\mathbf{X}) \le 0} f_{\mathbf{X}}(\mathbf{X})dx = \int \cdots \int I\left[g(\mathbf{X}) \le 0\right] f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{52.28}$$

where $I[.]$ is a function taking a value of 1 if the condition in the bracket is true, and is assigned zero otherwise. Analytical solution or even numerical integration using quadrature-based techniques is only possible for limited cases. Hence, approximate numerical techniques such as FORM or higher order methods (e.g., second-order reliability method (SORM)) have been developed. Alternatively, numerical integration can be performed using Monte Carlo simulation (MCS) techniques. A brief introduction will be given here. The generation of random numbers can be found in standard textbooks and will not be discussed here.

In essence, MCS involves the random generation of many realizations of a set of random variables, say N realizations, and to count how many of these result in the condition $g(\boldsymbol{X}) \le 0$. The failure probability is estimated as

$$p_f = \frac{1}{N}\sum_{i=1}^{N} I\left[g(\boldsymbol{x}_i) \le 0\right] \tag{52.29}$$

The variance of this estimate can be approximated by (Melchers, 1999)

$$s_{p_f}^2 = \frac{1}{(N-1)}\left(\frac{1}{N}\left[\sum_{i=1}^{N} I^2\left[g(\mathbf{x}_i) \le 0\right]\right] - \left[\frac{1}{N}\sum_{i=1}^{N} I\left[g(\mathbf{x}_i) \le 0\right]\right]^2\right) \tag{52.30}$$

From Eq. (52.29), it is obvious that when p_f is small, N has to be very large to get a reasonable estimate, which makes MCS unattractive. This limitation can be further compounded by cases where the dimension of $\boldsymbol{X}$ is large or $g(\boldsymbol{X})$ is not easy to evaluate (such as the need to perform a finite element computation). In addition, the variance decreases slowly with N. By using additional information to focus the simulation on a more fruitful region, N can be made small and the variance can be significantly reduced. Among the many variance reduction techniques, the *importance sampling technique* is currently one of the most popular in structural reliability and is briefly described below.

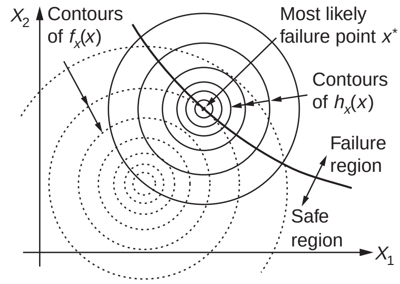

FIGURE 52.6 Concept of importance sampling in MCS.

The region that contributes most to p_f is around the most likely failure point $\boldsymbol{x}^*$. Hence, one can selectively generate the realizations around this vicinity. For example, one can sample from distributions that follow $f_X(\boldsymbol{x})$, but with their means shifted to $\boldsymbol{x}^*$, denoted as $h_X(\boldsymbol{x})$, as proposed by Harbitz (1983). This will result in having an order of $N/2$ points in the failure region (see Fig. 52.6) and should logically reduce the size of N needed, subjected to some conditions being satisfied, such as the nature of the performance function and the suitable choice of $h_X(\boldsymbol{x})$. Equation (52.29), in view of the modified sampling space, becomes

$$\begin{aligned} p_f &= \int \cdots \int I\left[g(\boldsymbol{X}) \le 0\right] \frac{f_X(\boldsymbol{x})}{h_X(\boldsymbol{x})} h_X(\boldsymbol{x})\, dx \\ &\cong \frac{1}{N} \sum_{i=1}^{N} I\left[g(\boldsymbol{x}_i) \le 0\right] \frac{f_X(\boldsymbol{x}_i)}{h_X(\boldsymbol{x}_i)} \end{aligned} \tag{52.31}$$

The optimal choice of $h_X(\boldsymbol{x})$ is by no means simple and is the subject of many research papers that cannot be adequately discussed in this brief introduction. Nevertheless, the following points should be noted (Melchers, 1991):

1. $h_X(\boldsymbol{x})$ should not be too flat or skewed. As such, the use of normal distribution has been suggested.
2. $\boldsymbol{x}^*$ may not be unique. The use of multiple $h_X(\boldsymbol{x})$ functions with corresponding weights may be necessary.
3. A highly concave limit state function gives rise to low efficiency, and N may need to be large for such cases. This may be overcome by using multiple $h_X(\boldsymbol{x})$ functions.

52.4 Systems Reliability

Structural engineering design, for the sake of simplicity, is invariably based on satisfying various individual limit state functions. Similarly, a structure is usually designed on member basis, although its optimal performance as an entire structure is desired. Codified optimal design at the structural systems level has been the subject of research for many decades. Classical systems reliability concepts have been employed in various applications, such as nuclear power plants, offshore installations, and bridges. In view of space limitations, only issues closely related to structures will be briefly discussed in this section. A general treatment of systems reliability pertaining to civil engineering can be found in Ang and Tang (1984), whereas that pertaining to structures is fairly well treated by Melchers (1999).

Systems in Structural Reliability Context

Structural reliability problems involving more than one limit state are solved using systems reliability concepts. Hence, in a general sense, a *structural system* can comprise only one element, such as a beam

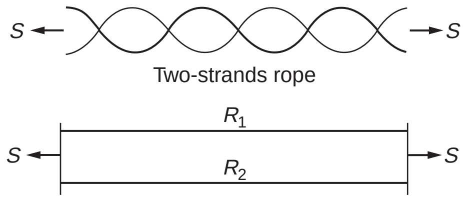

FIGURE 52.7 Model of a two-strand parallel structure.

that involves combined stresses and deflection limit states, or can comprise many elements, as in a truss or frame structure.

As an illustration, consider a simply supported beam of length l and stiffness EI subjected to a uniform load w per unit length and designed with flexural capacity M_u and shear capacity V_u that must satisfy a deflection limit Δ. The limit state equations can be written as follows:

$$\begin{aligned} g_1(\boldsymbol{X}) &= M_u - \frac{1}{8}wl^2, & g_2(\boldsymbol{X}) &= V_u - \frac{1}{2}wl, \\ g_3(\boldsymbol{X}) &= 1 - \left(\frac{wl^2}{8M_u} + \frac{wM_u}{2V_u^2}\right), & g_4(\boldsymbol{X}) &= \Delta - \frac{5wl^4}{384EI} \end{aligned} \tag{52.32}$$

The nonperformance or failure of this system happens when any one of the limit state equations is violated. The failure probability can be formulated as

$$p_f = P\left[\left(g_1(\boldsymbol{X}) \le 0\right) \cup \left(g_2(\boldsymbol{X}) \le 0\right) \cup \left(g_e(\boldsymbol{X}) \le 0\right) \cup \left(g_4(\boldsymbol{X}) \le 0\right)\right] \tag{52.33}$$

where the symbol ∪ is the Boolean or operator to denote the *union* of events. For example, $A \cup B$ denotes the occurrence of event A, event B, or both events A and B. Such a system is known as a *series system*.

Another class of systems is the *parallel system*. Consider a simple redundant system comprising a bundle of two steel strands with capacity R_1 and R_2 under a load S, schematically shown in Fig. 52.7. Assume the capacities to be random but correlated with a coefficient denoted as $\rho_{R_1R_2}$. The failure probability of this system can be written as

$$\begin{aligned} p_f &= P\left[\left(g_1(\boldsymbol{X}) \le 0\right) \cap \left(g_2(\boldsymbol{X}) \le 0\right)\right] \\ g_1(\boldsymbol{X}) &= R_1 - S, \qquad g_2(\boldsymbol{X}) = R_2 - S \end{aligned} \tag{52.34}$$

where the symbol ∩ is the Boolean and operator to denote the *intersection* of events. For example, $A \cap B$ denotes the occurrence both events A and B. Such a system is known as a parallel system.

The failure region corresponding to the two different systems described by Eqs. (52.33) and (52.34) is best illustrated assuming that there are only two random variables shown in Fig. 52.8. It should be noted that even for cases where the individual $g_i(\boldsymbol{X})$ are linear, the failure region is bounded by piecewise linear boundaries. Hence, the evaluation of Eqs. (52.33) and (52.34) is nonlinear and can be quite formidable.

Note that an equivalent form of Eq. (52.34) can be written by considering the complementary events, such as $g_1(\boldsymbol{X}) > 0$, which is also denoted as $\overline{g_1(\boldsymbol{X}) \le 0}$, where the overbar indicates the complementary of the event under the bar. Since the nonfailure of the two-strand structure implies the nonfailure of at least one strand, one can write

$$p_f = 1 - P\left[\overline{\left(g_1(\mathbf{X}) \le 0\right)} \cup \overline{\left(g_2(\mathbf{X}) \le 0\right)}\right] \tag{52.35}$$

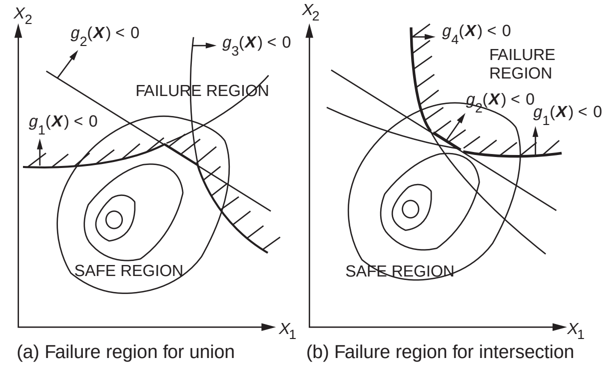

FIGURE 52.8 Failure region for system involving (a) union and (b) intersection of events.

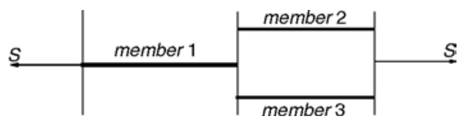

FIGURE 52.9 Model of a simple series–parallel three-bar system.

Based on the same consideration, the equivalence of Eq. (52.33) is

$$p_f = 1 - P\left[\overline{\left(g_1(\mathbf{X}) \le 0\right)} \cap \overline{\left(g_2(\mathbf{X}) \le 0\right)} \cap \overline{\left(g_1(\mathbf{X}) \le 0\right)} \cap \overline{\left(g_2(\mathbf{X}) \le 0\right)}\right] \tag{52.36}$$

The equivalence of the two sets of formulation is basically an application of *De Morgan's rule* in set theory.

In general, a complete system may be cast as a combination of parallel and series subsystems. For example, consider the three-bar system shown in Fig. 52.9. Denote F_i, $i = 1$, 2, and 3, as the failure of members 1, 2, and 3, respectively. The system cannot withstand the load if member 1 fails or if both members 2 and 3 fail. Thus, the probability of failure of the system can be formulated as

$$p_f = P\left[F_1 \cup \left(F_2 \cap F_3\right)\right] \tag{52.37}$$

The evaluation of equations such as Eq. (52.37) can be a formidable task, and numerical techniques need to be used. As an alternative, first-order and second-order bounds are often computed instead. The bounds for the series and the parallel systems will be considered next, namely,

$$p_{\cup} = P\left[G_1 \cup G_2 \cup G_3 \cup \ldots \cup G_n\right] \tag{52.38}$$

$$p_{\cap} = P\left[G_1 \cap G_2 \cap G_3 \cap \ldots \cap G_n\right] \tag{52.39}$$

First-Order Probability Bounds

One extreme case is to consider all the events in Eq. (52.39) to be mutually independent. Hence, by virtue of Eq. (52.12), Eq. (52.39) becomes

$$p_{\cap indp} = P\left(G_1 \cap G_2 \cap \ldots \cap G_n\right) = P\left(G_1\right)P\left(G_2\right)\ldots P\left(G_n\right) = \prod_{i=1}^{n} P\left(G_i\right) \tag{52.40}$$

Similarly, Eq. (52.39) for mutually independent events is simplified as

$$
\begin{aligned}
p_{\cup indp} &= 1 - P\left(\overline{G}_1 \cap \overline{G}_2 \cap \ldots \cap \overline{G}_n\right) = 1 - \prod_{i=1}^{n} P\left(\overline{G}_i\right) \\
&= 1 - \prod_{i=1}^{n}\left[1 - P\left(G_i\right)\right] \le \sum_{i=1}^{n} P\left(G_i\right)
\end{aligned}
\tag{52.41}
$$

Note that if $P(G_i)$ is small, as in most practical structural systems, then

$$
p_{\cup indp} \approx \sum_{i=1}^{n} P\left(G_i\right).
$$

The other extreme is when the G_i are perfect-positively correlated. The corresponding probabilities are

$$
p_{\cap ppc} = \min_{i=1}^{n} P\left(G_i\right) \quad \text{and} \quad p_{\cup ppc} = \max_{i=1}^{n} P\left(G_i\right) \tag{52.42}
$$

Based on the above, the first-order probability bounds can be summarized as:

$$
\begin{aligned}
&\max_{i=1}^{n} P\left(G_i\right) \le P\left(\bigcup_{i-1}^{n} G_i\right) \le 1 - \prod_{i=1}^{n}\left[1 - P\left(\overline{G}_i\right)\right] && \text{for } \rho \ge 0 \\
&1 - \prod_{i=1}^{n}\left[1 - P\left(G_i\right)\right] \le P\left(\bigcup_{i-1}^{n} G_i\right) \le \min\left[1, \sum_{i=1}^{n} P\left(G_i\right)\right] && \text{for } \rho \le 0
\end{aligned}
\tag{52.43}
$$

$$
\begin{aligned}
&\prod_{i=1}^{n} P\left(G_i\right) \le P\left(\bigcap_{i-1}^{n} G_i\right) \le \min_{i=1}^{n} P\left(G_i\right) && \text{for } \rho \ge 0 \\
&0 \le P\left(\bigcap_{i-1}^{n} G_i\right) \le \prod_{i=1}^{n} P\left(G_i\right) && \text{for } \rho \le 0
\end{aligned}
\tag{52.44}
$$

Second-Order Probability Bounds

The probability bounds given by Eqs. (52.43) and (52.44) for some applications can be wide and have limited use. Hence, second-order bounds have been proposed where the joint probability of two events $P(G_iG_j)$ are used in the computation. The probability bounds for the union of events is given by

$$
\begin{aligned}
&P\left(\bigcup_{i-1}^{n} G_i\right) \le \min\left[1, \sum_{i=1}^{n} P\left(G_i\right) - \sum_{i=2}^{n} \max_{j<i} P\left(G_iG_j\right)\right] \\
&P\left(\bigcup_{i-1}^{n} G_i\right) \ge P\left(G_1\right) + \sum_{i=2}^{n} \max\left[0, P\left(G_i\right) - \sum_{j=1}^{i-1} P\left(G_iG_j\right)\right]
\end{aligned}
\tag{52.45}
$$

where the events G_i are ordered in terms of decreasing probability, as a rule of thumb, for optimal results. The joint probability of two events can be estimated either by numerical integration or by the following approximate bounds:

$$\max\left[P(A), P(B)\right] \le P\left(G_i G_j\right) \le P(A) + P(B) \qquad \text{for } \rho \ge 0$$
$$0 \le P\left(G_i G_j\right) \le \min\left[P(A), P(B)\right] \qquad \text{for } \rho \le 0 \tag{52.46}$$

where

$$P(A) = \Phi(-\beta_i)\Phi\left(-\frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}}\right), \quad P(B) = \Phi(-\beta_j)\Phi\left(-\frac{\beta_i - \rho_{ij}\beta_j}{\sqrt{1-\rho_{ij}^2}}\right) \tag{52.47}$$

in which β_i is the reliability index corresponding to $P(G_i \le 0)$ and ρ_{ij} is the correlation coefficient between G_i and G_j. An estimate of the latter is given by

$$\rho_{ij} = \left(\sum_{k=1}^{n} \alpha_{ik}\alpha_{jk}\right) \Bigg/ \sqrt{\left(\sum_{k=1}^{n} \alpha_{ik}^2 \sum_{k=1}^{n} \alpha_{jk}^2\right)} \tag{52.48}$$

where α_{ik} are the components of the direction cosines $\boldsymbol{\alpha}$ at the most likely failure point corresponding to G_i, given in Eq. (52.26), with n as the number of basic random variables.

For tighter bounds, the use of higher order probabilities has been proposed (e.g., Greig, 1992). This will not be treated here.

Monte Carlo Solution

The concept of MCS in estimating the failure probability for the case of a single limit state function described earlier can be extended directly to that for series and parallel systems. For example, the indicator function in Eqs. (52.28) and (52.31) for series system becomes

$$I\left[\bigcup_{i=1}^{n} g_i(x) < 0\right] = 1 \qquad \text{if } I[.] \text{ is true}$$
$$= 0 \qquad \text{otherwise} \tag{52.49}$$

The difference in the application of importance sampling technique in this case is that the presence of numerous limit state functions (see Fig. 52.8) complicates the choice of $h_x(\boldsymbol{x})$. A simple solution is to consider a multimodal sampling function given by

$$h_X(x) = \sum_{i=1}^{n} w_i h_{Xi}(x) \quad \text{where} \quad \sum_{i=1}^{n} w_i = 1 \tag{52.50}$$

and $h_{xi}(\boldsymbol{x})$ is the sampling distribution determined based on the ith limit state function and w_i is the weight, which is inversely proportional to β_i.

Applications to Structural Systems

Design codes generally try to ensure that actual structural systems fail in ductile modes rather than brittle ones. It is therefore reasonable to assume that the commonly used rigid-plastic model provides a reasonable approximation to structural system behavior (Bjerager, 1984). The dependence of the probability of failure on the load path for such a case is not a significant issue. On the other extreme, there are cases where the actual member behavior can be better idealized as elastic-brittle, where within a structural

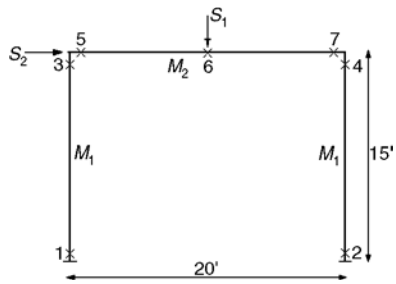

FIGURE 52.10 Simple frame structure with potential plastic hinges (numbered 1 to 7).

TABLE 52.4 Mechanisms, Performance Functions, and Failure Probabilities

Mechanism i	Location of Hinges	Performance Functions G_i	Reliability Index β_i	Failure Probability p_{fi}
1	1, 2, 3, 6	$4M_1 + 2M_2 - 10S_1 - 15S_2$	1.8167	0.03463
2	1, 2, 3, 4	$4M_1 - 15S_2$	2.2123	0.01347
3	1, 2, 5, 6	$2M_1 + 4M_2 - 10S_1 - 15S_2$	2.2589	0.01195
4	5, 6, 7	$4M_2 - 10S_1$	3.0177	0.00127
5	1, 2, 5, 7	$2M_1 + 2M_2 - 15S_2$	3.2276	0.00062
6	3, 4, 6	$2M_1 + 2M_2 - 10S_1$	3.3024	0.00048

system, deformation at zero capacity is possible after the peak capacity has been reached (Quek and Ang, 1986). Real structures will behave somewhat in between these two idealized models and can be too complex for accurate evaluation of the system failure probability. Consequently, one can consider probabilistic upper and lower bound solutions.

For structures that ultimately fail in a ductile manner, the collapse mechanism approach can be employed. As an illustration, consider a simple one-story one-bay frame under concentrated vertical and lateral loads, as shown in Fig. 52.10, with the seven critical sections marked and numbered (Ma and Ang, 1981). The member plastic moment capacities, M_1 and M_2, have mean values of 360 and 480 ft-kips with standard deviations of 54 and 72 ft-kips, respectively. The loads, S_1 and S_2, have mean values of 100 and 50 kips with standard deviations of 10 and 15 kips, respectively. All four random variables are assumed to be independent normal variates. Six physically admissible mechanisms can be generated, with the limit state function corresponding to each given in Table 52.4.

The reliability index and failure probability of each individual performance function can be obtained using Eqs. (52.16) and (52.20) and Table 52.2. For example,

$$P(G_1 \le 0) = \Phi\left(-\frac{4\mu_{M_1} + 2\mu_{M_2} - 10\mu_{S_1} - 15\mu_{S_2}}{\sqrt{16\sigma_{M_1}^2 + 4\sigma_{M_2}^2 + 100\sigma_{S_1}^2 + 225\sigma_{S_2}^2}}\right)$$

$$= \Phi\left(-\frac{650}{\sqrt{128{,}017}}\right) = \Phi(-1.8167) = 0.03463$$

To find the first-order probability bounds, Eq. (52.43) gives $0.03463 \le p_f \le 0.06124$.

To obtain the second-order bounds, the joint probability of pairs of events is needed. It is illustrated using G_1 and G_2. First, the correlation coefficient is computed using Eq. (52.48). The α values for G_1 and G_2 are (0.6037, 0.4025, –0.2795, and –0.6289) and (0.6925, 0, 0, and –0.7214), respectively. Hence, ρ_2= 0.8718. From Eq. (52.47),

$$P(A) = \Phi(-1.8167)\Phi\left(-\frac{2.2123 - 0.8718 * 1.8167}{\sqrt{1 - 0.8718^2}}\right) = 0.003422,$$

$$P(B) = 0.007953$$

Hence, from Eq. (52.46), $0.007953 \le P(G_1 G_2) \le 0.011375$. Performing similar computations for other joint probabilities and applying Eq. (52.45) yields the second-order bounds as $0.0367 \le p_f \le 0.0447$.

52.5 Reliability-Based Design

There are two principal considerations in the design of structures: the first is the optimization of the total expected utility of the structure by the designer, and the second is the optimization of the design code by the controlling authority. In the latter, the optimization covers as much as possible the range of practical structures and includes issues such safety, serviceability, and overall cost. A complete practical design code will have to be simple, yet consider all the above factors for the whole family of structures and account for many details, some of which may not have been fully tested. As such, it is natural that code formats and requirements evolved over a long period of time and collective wisdom and consensus of the profession remain significant factors. Such a process is seen to be less formal and has obvious drawbacks when new materials, structural principles, and technological developments are introduced. Nevertheless, previous generation codes still play a major role as guides to new generation codes in the process known as *code calibration*.

Reliability concepts presented in the earlier sections serve as a plausible vehicle for code calibration, where various limit state conditions such as ultimate and serviceability limit states need to be addressed. Based on the calibration results, it is possible in principle when formulating the new code to incorporate the type of failure and the associated consequences of failure through a cost component. To enforce such requirements explicitly will make the codified design procedure for general use unnecessarily too complex a process in the present-day context. Nevertheless, it can be incorporated in the code in a less obvious manner through the numerical factors specified in the code. Reliability analysis is being accepted currently as a practical, consistent, and formal tool for which partial factors can be derived given the safety-checking formats in code formulation.

Load and Resistance Factor Design Format

Before outlining the code calibration and formulation procedure, it is appropriate to briefly mention safety-checking formats. The format for codes of practice differs between countries, and there are attempts to unify such format. For example, all European structural design codes follow the general form specified by the Comite Europeen du Beton (CEB, 1976). Numerous partial factors for both materials and loads are imposed to account for various uncertain components. The Canadian building codes adopted a less complicated format by using only one partial factor to account for material uncertainties (NRCC, 1977). The AISC code in the U.S. uses the load and resistance factor design (LRFD) format, expressed as (Ravindra and Galambos, 1978)

$$\phi R_n = \sum_{k=1}^{i} \gamma_k S_{km} \tag{52.51}$$

where ϕ and γ_k are the resistance and load factors, respectively, R_n the nominal resistance, and S_{km} the mean load effects.

Code Calibration Procedure

Changes or improvements to existing codes can result for reasons of harmonization of different codes or development of a simpler code. The first comprehensive change toward reliability-based design code

TABLE 52.5 Summary of Statistical Data on Resistance

Designation	R/R_n	V_R	Probability Distribution
Reinforced concrete, flexure			
Grade 60	1.05	0.11	Normal
Grade 40	1.14	0.14	Normal
Reinforced concrete, short-tied columns	0.95	0.14	Normal
Reinforced concrete beams, shear, minimum stirrups	1.00	0.19	Normal
Structural steel			
Tension members, yield	1.05	0.11	Lognormal
Compact beam, uniform moment (plastic design)	1.07	0.13	Lognormal
Beam-column (plastic design)	1.07	0.15	Lognormal
Cold-formed steel, braced beams	1.17	0.17	Lognormal
Aluminum, laterally braced beams	1.10	0.08	Lognormal
Unreinforced masonry walls in compression, inspected workmanship	5.3	0.18	Lognormal
Glulam beams			
Live load	1.97	0.18	Weibull
Snow load	1.62	0.18	Weibull

Source: From Table 1 of Galambos, T.V. et al., *J. Struct. Div. ASCE,* 108, 959, 1982.

formulation in the United States was the American National Standard A58 on minimum design loads in buildings (Ellingwood et al., 1980). Calibration has been defined as the process of assigning values to the parameters in a design code to achieve a desired level of reliability accounting for practical constraints, resulting in a specific design code. It involves the combination of judgment, fitting to existing design practice, and optimization.

The first step in the code calibration procedure is to define the *scope* of applicability, namely, the class of structures (e.g., bridges), materials (e.g., concrete), failure modes, geographical domain of validity, and geometrical properties. This will result in selecting the range of values for the design variables, such as length, cross-sectional areas, permitted yield stress, and applied loads. For practical computation, discrete zones of such values are used and the corresponding frequency of occurrence corresponding to each zone in actual practice estimated. For example, the dead-to-live load ratios in building structures are usually confined within the range of 0.5 to 2.

In the second step, for each zone the existing structural design code is used to *design* the various elements, such as beams and columns, with specified geometry and load conditions.

The third step involves the specification of *performance functions*. The limit state conditions to be specified in the new code must be defined, such as those related to flexure, shear, local buckling, and deflection. The performance function for each limit state is expressed in terms of the basic variables.

In order to compute the reliability index corresponding to each performance function, in the fourth step, the *statistics of the basic variables* are assumed to be available. These are often compiled from extensive survey data. An example of statistical data on resistance can be found in Galambos et al. (1982) and is reproduced in Table 52.5.

The fifth step involves computing the *reliability indices* of each member designed under the old code using techniques such as FORM. An example of the reliability indices obtained from such a process is shown in Fig. 52.11, reproduced directly from Galambos et al. (1982). It shows the range of reliability indices implied in existing code for gravity loads. If the range is unacceptably wide, further work is needed to obtain a new set of factors satisfying the intended goals.

The specification of the *goal* of the revised or new code is a major step of the process of calibration. The overall goal could be to maximize the expected utility or to achieve a specified failure probability or reliability index taking into consideration the failure consequences. The target reliability index to be used for a revised code can be based on the values obtained in the previous step. For example, a statistically determined value can be chosen, such as the weighted average or some specified percentile value of β

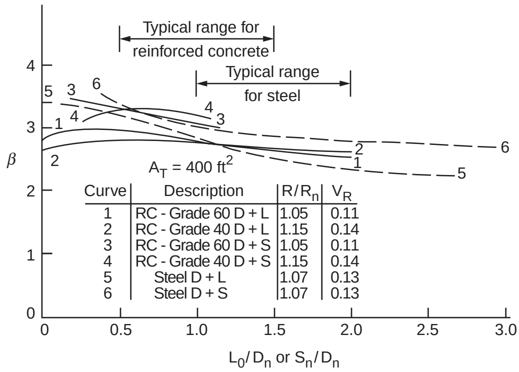

FIGURE 52.11 Reliability index for steel and reinforced concrete beams. (From Galambos, T.V. et al., *J. Struct. Div. ASCE,* 108, 959, 1982.)

based on the occurrence frequency of the various designs encountered in practice. For example, Siu et al. (1975) obtained an average β value of 4.67 for hot-rolled steel columns under compression failure for the 1975 NBCC code. It is unlikely that a single value will be selected as the target for an entire code. For example, failure consequence is a major design consideration, and higher values of β are often imposed for brittle-type failure. To achieve the desired goal, a quantitative measure or objective function must be formulated for example (Ditlevsen and Madsen, 1996),

$$\Delta = \sum_{materials} \sum_{\substack{failure \\ modes}} \sum_{parameters} w_i M(\beta_i) \tag{52.52}$$

where w_i are the relative frequencies of occurrences (determined in the first step). The function $M(.)$ can be simply in terms of reliability indices, such as $(\beta_i^* - \beta_i)^2$ where β_i^* is the target reliability indices. This function can take on a more complex form by introducing cost variables (including cost of initial construction and cost of failure consequences).

For a given code safety-checking format, the set of partial factors is determined based on the objective function. First, trial values of partial factors are used for the new format and the respective β_i computed. The objective function is computed using Eq. (52.52). By repeating for different sets of values for the partial factors, the set of values that minimize Eq. (52.52) are adopted. The same procedure is repeated for the various load combinations to be implemented in the new (or revised) code.

Evaluation of Load and Resistance Factors

As an illustration of the computation of the load and resistance factors, a simple example involving only the dead and live loads is considered. The new code format is specified as

$$\phi R_n \geq \gamma_D D_n + \gamma_L L_n \tag{52.53}$$

where R_n, D_n , and L_n are the nominal resistance, dead load, and live load, respectively, and ϕ, γ_D, and γ_L are the corresponding partial factors. Consider the case of a steel member under tension with the performance function expressed as

$$g(\mathbf{X}) = R - D - L \tag{52.54}$$

The statistics of the basic variables are given in Table 52.6. For the specific class of loads corresponding to a mean live-to-dead load ratio $\bar{L}/\bar{D}$ of 1.5, the partial factors are to be computed for a target reliability index of 3.0.

TABLE 52.6 Statistics of Basic Variables

Variables	$\overline{R}/R_n$	Coefficient of Variation, V	Probability Distribution
Resistance, R	1.05	0.11	Lognormal
Dead load, D	1.05	0.10	Normal
Live load, L	1.00	0.25	Type I extreme

The computation can be summarized as follows:

1. To commence the computation, the most likely failure point is assumed, say $r^* = 0.866\overline{R}$, $d^* = 1.03\overline{D}$ and $l^* = 1.99\overline{L}$.
2. The equivalent normal parameters for R and L are next computed. For R, which is lognormal distributed,

$$\sigma_R^N = r^* \varsigma_R = r^* \sqrt{\ln\left(1+V_R^2\right)} = 0.09497\overline{R}$$

$$\mu_R^N = r^*\left(1-\ln\frac{r^*}{\overline{R}} - \frac{1}{2}\ln\left(1+V_R^2\right)\right) = 0.9854\overline{R}$$

For L, which follows the type I extreme distribution,

$$\alpha_n = \frac{\pi}{\sigma_L\sqrt{6}} = \frac{5.1302}{\overline{L}} = \frac{3.4201}{\overline{D}}$$

$$u_n = L - -\frac{0.5772}{\alpha_n} = 1.3312\overline{D}$$

$$F_L\left(l^*\right) = \exp\left[-\exp\left\{-\alpha_n\left(l^* - u_n\right)\right\}\right] = 0.9965$$

$$f_L\left(l^*\right) = \alpha_n \exp\left[-\alpha_n\left(l^* - u_n\right) - \exp\left\{-\alpha_n\left(l^* - u_n\right)\right\}\right] = \frac{0.01191}{\overline{D}}$$

$$\sigma_L^N = \frac{\phi\left(\Phi^{-1}\left[F_L\left(l^*\right)\right]\right)}{f_L\left(l^*\right)} = 0.8821\overline{D}$$

$$\mu_L^N = l^* - \mu_L^N\,\Phi^{-1}\left[F_L\left(l^*\right)\right] = 0.606\overline{D}s$$

3. Based on the performance function and the target reliability index,

$$\beta = \frac{\mu_R^N - \mu_D^N - \mu_L^N}{\sqrt{\sigma_R^{N^2} + \sigma_D^{N^2} + \sigma_L^{N^2}}} = 3.0$$

$$0.8898\overline{R}^2 - 3.9491\overline{R}\overline{D} - 4.5137\overline{D}^2 = 0 \quad \text{or} \quad \overline{R} = 4.6482\overline{D}$$

4. The sensitivity factors at the most likely failure point is given by the direction cosines and computed as

$$\alpha_R^* = \frac{\sigma_R^N}{\sqrt{\sigma_R^{N^2} + \sigma_D^{N^2} + \sigma_L^{N^2}}} = 0.4452$$

$$\alpha_D^* = \frac{-\sigma_D^N}{\sqrt{\sigma_R^{N^2} + \sigma_D^{N^2} + \sigma_L^{N^2}}} = -0.1009$$

$$\alpha_L^* = -0.8897$$

5. The most likely failure point is then updated,

$$r^* = \mu_R^N - \alpha_R^* \sigma_R^N \beta = 0.9854\overline{R} - 0.4452 * 0.09497\overline{R} * 3.0 = 0.8586\overline{R}$$

$$d^* = 1.030\overline{D}$$

$$l^* = 2.960\overline{D} = 1.9736\overline{L}$$

6. If the results in step 5 are significantly different from the values assumed in step 1, then the values in step 5 are used for the next iteration. Steps 2 to 5 are then repeated until convergence. The final solution of the most likely failure point is used to estimate the partial factors.
7. Assuming the values given in step 5 are correct, the mean partial factors of safety are given by

$$\overline{\phi} = r^* / \overline{R} = 0.859$$

$$\overline{\gamma}_D = d^* / \overline{D} = 1.030$$

$$\overline{\gamma}_L = l^* / \overline{L} = 1.974$$

The nominal partial factors of safety are

$$\phi = \frac{r^* * \overline{R}}{\overline{R} * R_n} = 0.859 * 1.05 = 0.902$$

$$\gamma_D = 1.030 * 1.05 = 1.082$$

$$\gamma_L = 1.974 * 1.0 = 1.974$$

Hence the code checking equation can be written as

$$0.90R_n \geq 1.08D_n + 1.97L_n$$

The implied overall mean factor of safety can be estimated as

$$\frac{\overline{R}}{\overline{D} + \overline{L}} \geq \frac{\left(1.03\overline{D} + 1.974\overline{L}\right) / 0.859}{\overline{D} + \overline{L}} = 1.86$$

whereas the overall nominal factor of safety is approximately 1.80.

Acknowledgment

The author expresses his gratitude to Dr. Kok-Kwang Phoon for reading through the manuscript and providing constructive suggestions.

References

Ang, A.H.-S. and Tang, W.H., *Probability Concepts in Engineering Planning and Design, Vol. I, Basic Principles*, John Wiley & Sons, New York, 1975.

Ang, A.H.-S. and Tang, W.H., *Probability Concepts in Engineering Planning and Design, Vol. II, Decision, Risk, and Reliability*, John Wiley & Sons, New York, 1984.

Bjerager, P., *Reliability Analysis of Structural Systems,* Series R, No. 183, Department of Structural Engineering, Technical University of Denmark, Lyngby, 1984.
CEB, Common Unified Rules for Different Types of Construction and Material, Bulletin d'Information 116-E, Comite Europeen du Beton, Paris, 1976.
Der Kiureghian, A. and Liu, P.L., Structural reliability under incomplete information, *J.Eng. Mech. ASCE*, 112, 579, 1986.
Ditlevsen, O. and Madsen, H.O., *Structural Reliability Methods*, John Wiley & Sons, New York, 1996.
Ellingwood, B. et al., Development of a Probability Based Load Criterion for American National Standard A58, NBS Special Publication SP577, National Bureau of Standards, Washington, D.C., 1980.
Galambos, T.V. et al., Probability based load criteria: assessment of current design practice, *J. Struct. Div. ASCE,* 108, 959, 1982.
Greig, G.L., An assessment of high-order bounds for structural reliability, *Struct. Safety,* 11, 213, 1992.
Harbitz, A., Efficient and accurate probability of failure calculation by the use of importance sampling technique, in *Applications of Statistics and Probability in Soil and Structural Engineering*, Augusti, G., Borri, A., and Vannuchi, O., Eds., Pitagora Editrice, Bologna, 1983, p. 825.
Ma, H.F. and Ang, A.H.-S., Reliability Analysis of Redundant Ductile Structural Systems, Civil Engineering Studies, SRS 494, University of Illinois at Urbana-Champaign, 1981.
Madsen, H.O., Krenk, S., and Lind, N.C., *Methods of Structural Safety,* Prentice Hall, Englewood Cliffs, NJ, 1986.
Melchers, R.E., Simulation in time-invariant and time-variant reliability problems, in *Reliability and Optimization of Structural Systems*, Rackwitz, R. and Thoft-Christensen, P., Eds., Springer, Berlin, 1991, p. 39.
Melchers, R.E., *Structural Reliability: Analysis and Prediction,* John Wiley & Sons, Chichester, U.K., 1999.
Nowak, A.S. and Collins, K.R., *Reliability of Structures,* McGraw-Hill, New York, 2000.
NRCC, *National Building Code of Canada,* National Research Council of Canada, Ottawa, 1977.
Quek, S.T. and Ang, A.H.-S., Structural System Reliability by the Method of Stable Configuration, Civil Engineering Series, SRS 529, University of Illinois at Urbana-Champaign, 1986.
Rackwitz, R. and Fiessler, B., Structural reliability under combined random load sequences, *Comput. Struct.,* 9, 489, 1978.
Ravindra, M.K. and Galambos, T.V., Load and resistance factor design for steel, *J. Struct. Div. ASCE,* 104, 1337, 1978.
Siu, W.W., Parimi, S.R., and Lind, N.C., Practical approach to code calibration, *J. Struct. Div. ASCE,* 101, 489, 1975.

Related Journals and Conferences

Although papers on structural reliability can be found in numerous journals and conference proceedings, the author wishes to draw attention to two civil engineering-related journals and two conferences. These are the *Journal of Structural Safety*, the *Journal of Probabilistic Engineering Mechanics*, the International Conference on Structural Safety and Reliability, and the International Conference on the Applications of Statistics and Probability. In addition, there are numerous specialty conferences organized in this field.