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Mechanics of Materials

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46.1 Introduction

The subject of *mechanics of materials* involves analytical methods for determining the *strength*, *stiffness* (deformation characteristics), and *stability* of the various members in a structural system. Alternatively, the subject may be called the strength of materials, mechanics of solid deformable bodies, or simply mechanics of solids. The behavior of a member depends not only on the fundamental laws that govern the equilibrium of forces, but also on the mechanical characteristics of the material. These mechanical characteristics come from the laboratory, where materials are tested under accurately known forces and their behavior is carefully observed and measured. For this reason, mechanics of materials is a blended science of experiment and Newtonian postulates of analytical mechanics.

The advent of computer technology has made possible remarkable advances in the analytical methods, notably the *finite element method*, for solving problems of mechanics of materials. Prior to the computer, practical solutions were largely restricted to simple and idealized problems. Today, the technology is capable of analyzing complex three-dimensional structural systems with nonlinear material properties. Although this chapter will be limited to presenting the classical topics, the relatively simple methods employed are unusually useful as they apply to a vast number of technically important and practical problems of structural engineering.

46.2 Stress

Method of Sections

Engineering mechanics is in large part the study of the nature of forces within a body. To study these internal forces a uniform approach — the method of sections — is applied. In this approach an arbitrary plane cuts the original solid body into two distinct parts (see Fig. 46.1). If the body as a whole is in equilibrium, then each part must also be in equilibrium, which leads to the fundamental conclusion that the external forces are balanced by the internal forces.

Definition of Stress

Stress is defined as the intensity of forces acting on infinitesimal areas of a cut section (Fig. 46.1). The mathematical definition of stress *t* is

$$\tau = \lim_{\Delta A \to 0} \frac{\Delta P}{\Delta A} \tag{46.1}$$

It is advantageous to resolve these intensities perpendicular and parallel to the section (see Fig. 46.1). The perpendicular component is called the normal stress. The parallel component is called the shearing stress. Normal stresses that pull away from the section are tensile stresses, while those that push against it are compressive stresses. Normal stresses are alternatively designated by the letter σ.

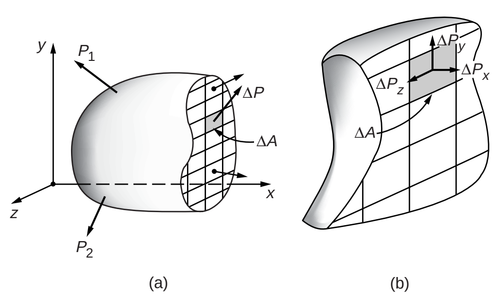


FIGURE 46.1 Sectioned body: (a) free body with some internal forces; (b) enlarged view with components of ΔP.

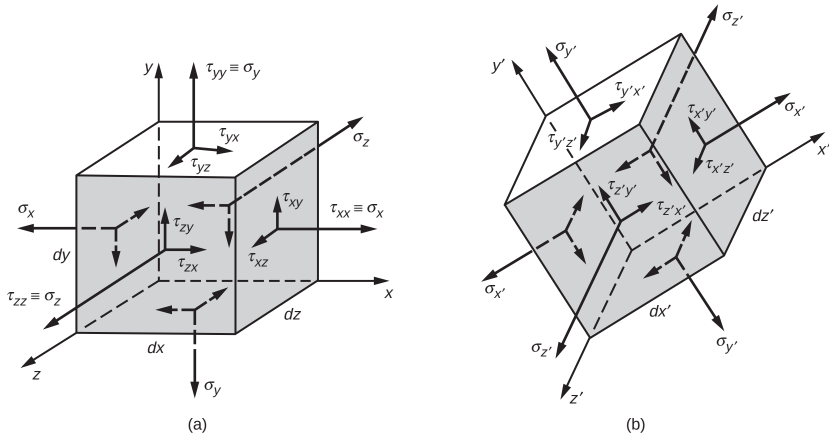


FIGURE 46.2 (a) General state of stress acting on an infinitesimal element in the initial coordinate system. (b) General state of stress acting on an infinitesimal element defined in a rotated system of coordinate axes. All stresses have positive sense.

Stress Tensor

A cube of infinitesimal dimensions is isolated from a body, as shown in Fig. 46.2. All stresses acting on this cube are identified on the figure. The first subscript of τ gives reference to the axis that is perpendicular to the plane on which the stress is acting; the second subscript gives the direction of the stress. Using this cube, the state of stress at any point can be defined by three components on each of the three mutually perpendicular axes. In mathematical terminology this is called a tensor, and the matrix representation of the stress tensor is

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \tag{46.2}$$

This is a second-rank tensor requiring two indices to identify its components. There are three normal stresses, $\tau_{xx} = \sigma_x$, $\tau_{yy} = \sigma_y$, $\tau_{zz} = \sigma_z$, and six shearing stresses, τ_{xy}, τ_{yx}, τ_{yz}, τ_{zy}, τ_{zx}, τ_{xz}. For brevity, a stress tensor can be written in indicial notation as τ_{ij}, where it is understood that i and j can assume designations x, y, and z. To satisfy the requirement of moment equilibrium, a stress tensor is symmetric, i.e., $\tau_{ij} = \tau_{ji}$. Thus subscripts of t are cummutative, and shear stresses on mutually perpendicular planes are numerically equal. Moment equilibrium cannot be satisfied by a single pair of shear stresses; two pairs are required, with their arrowheads meeting at diametrically opposite corners of an element.

When stresses on the z plane do not exist, the third column and third row of Eq. (46.2) are zeros, and this two-dimensional state is referred to as plane stress. If all shear stresses are absent, only the diagonal terms remain and the stresses are said to be triaxial; for plane stress $\tau_{zz} = 0$, and the state of stress is biaxial. If τ_{yy} is further eliminated, the state of stress is referred to as uniaxial.

Differential Equations for Equilibrium

For an infinitesimal element to be in equilibrium the following equation must hold in the x direction:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0 \tag{46.3}$$

where X is the inertial or body forces. Similar equations hold for the y and z directions. These equations are applicable whether a material is elastic, plastic, or viscoelastic. Note that there are not enough

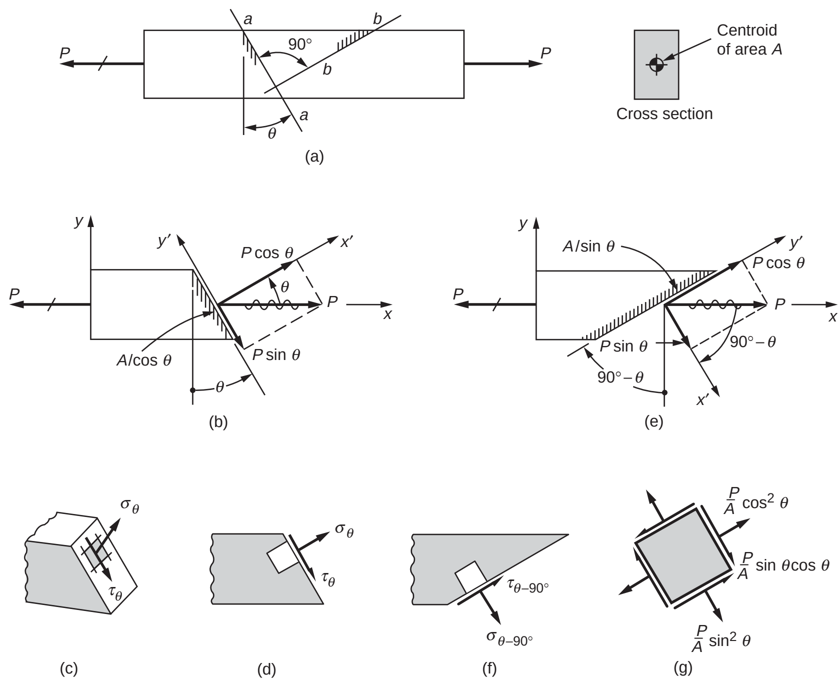


FIGURE 46.3 Sectioning of a prismatic bar loaded axially.

equations of equilibrium to solve for the unknown stresses. Thus all problems in stress analysis are internally intractable or indeterminate and can be solved only when supplemented by other equations given by kinematic requirements and the mechanical properties of the material.

Stress Analysis of Axially Loaded Bars

Figure 46.3 shows a bar loaded axially by a force P and a free body diagram isolated by the section *aa*, which is at an angle θ with the vertical. The equilibrium force on the inclined section is equal to P and is resolved into the two components: the normal force component, $P\cos\theta$, and the shear component, $P\sin\theta$. The area of the inclined section is $A/\cos\theta$. Therefore, given the definition of stress as force per unit area, the normal stress and the shear stress are given by the following equations:

$$\sigma_\theta = \frac{P}{A}\cos^2\theta \tag{46.4}$$

and

$$\tau_\theta = \frac{P}{A}\sin\theta\cos\theta \tag{46.5}$$

These equations show that the normal and shear stresses vary with the angle. Therefore the maximum normal stress is reached when $\theta = 0$ or when the section is perpendicular to the x axis, and $\sigma_{max} = \sigma_x = P/A$. By differentiating Eq. (46.5), the maximum shear stress occurs on planes either $\pm 45°$ with the axis of the bar, and $\tau_{max} = P/2A = \sigma_x/2$. It is important to note that the basic procedure of engineering mechanics of solids used here gives the average or mean stress at a section.

46.3 Strain

Normal Strain

A solid body deforms when subjected to an external load or a change of temperature. When an axial rod, shown in Fig. 46.4, is subjected to an increasing force *P*, a change in length occurs between any two points, such as *A* and *B*. The gage length L_0 is the initial distance between *A* and *B*. If after loading the observed length is *L*, the gage elongation $\Delta L = L - L_0$. The elongation ε per unit of initial gage length is then given as

$$\varepsilon = \frac{L - L_0}{L_0} = \frac{\Delta L}{L_0} \tag{46.6}$$

This expression defines the *extensional strain*, which is a dimensionless quantity. Since this strain is associated with the normal stress, it is usually called the *normal strain*.

In some applications, where strains are large, one defines the *natural* or *true strain* $\bar{\varepsilon}$. The strain increment *d* for this strain is defined as dL/L, where *L* is the instantaneous length of the specimen and *dL* is the incremental change in length. Analytically,

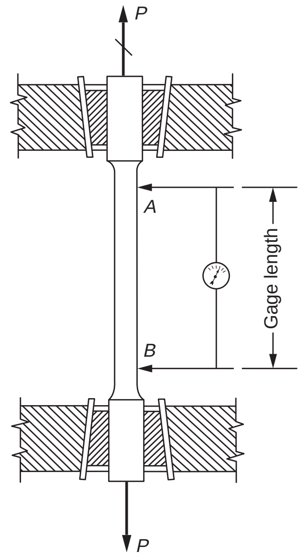


FIGURE 46.4 Diagram of a tension specimen in a testing machine.

$$\bar{\varepsilon} = \int_{L_0}^{L} dL/L = \ln L/L_0 = \ln(1+\varepsilon) \tag{46.7}$$

Stress–Strain Relationships

The mechanical behavior of real materials under loads is of primary importance. Information of this behavior is generally provided by plotting stress against strain from experiments, as shown in Fig. 46.5, for a variety of engineering materials. Each material has its own characteristics. Typical physical properties of some common materials are given in Table 46.1. It should be noted that even for the same material the stress–strain diagram will differ, depending on the temperature at which the test was conducted, the speed of the test, and a number of other variables. The vast majority of engineering materials are generally assumed to be completely *homogeneous* (sameness from point to point) and *isotropic* (having essentially the same physical properties in different directions). Materials capable of withstanding large strains without a significant increase in stress are referred to as *ductile* materials (see Fig. 46.6). The converse applies to *brittle* materials.

When stresses are computed on the basis of the original area of a specimen, they are referred to as *conventional* or *engineering stresses*. For some materials, such as mild steel and aluminum, significant transverse contraction or expansion takes place near the breaking point, referred to as *necking*. Dividing the applied load by the corresponding actual area of a specimen gives the so-called *true stress*; see Fig. 46.6(a).

Hooke's Law

As illustrated by Fig. 46.6(a), the experimental values of stress versus strain obtained for mild steel lie essentially on a straight line for a limited range from the origin. This idealization is known as Hooke's law and can be expressed by the equation

$$\sigma = E\varepsilon \tag{46.8}$$

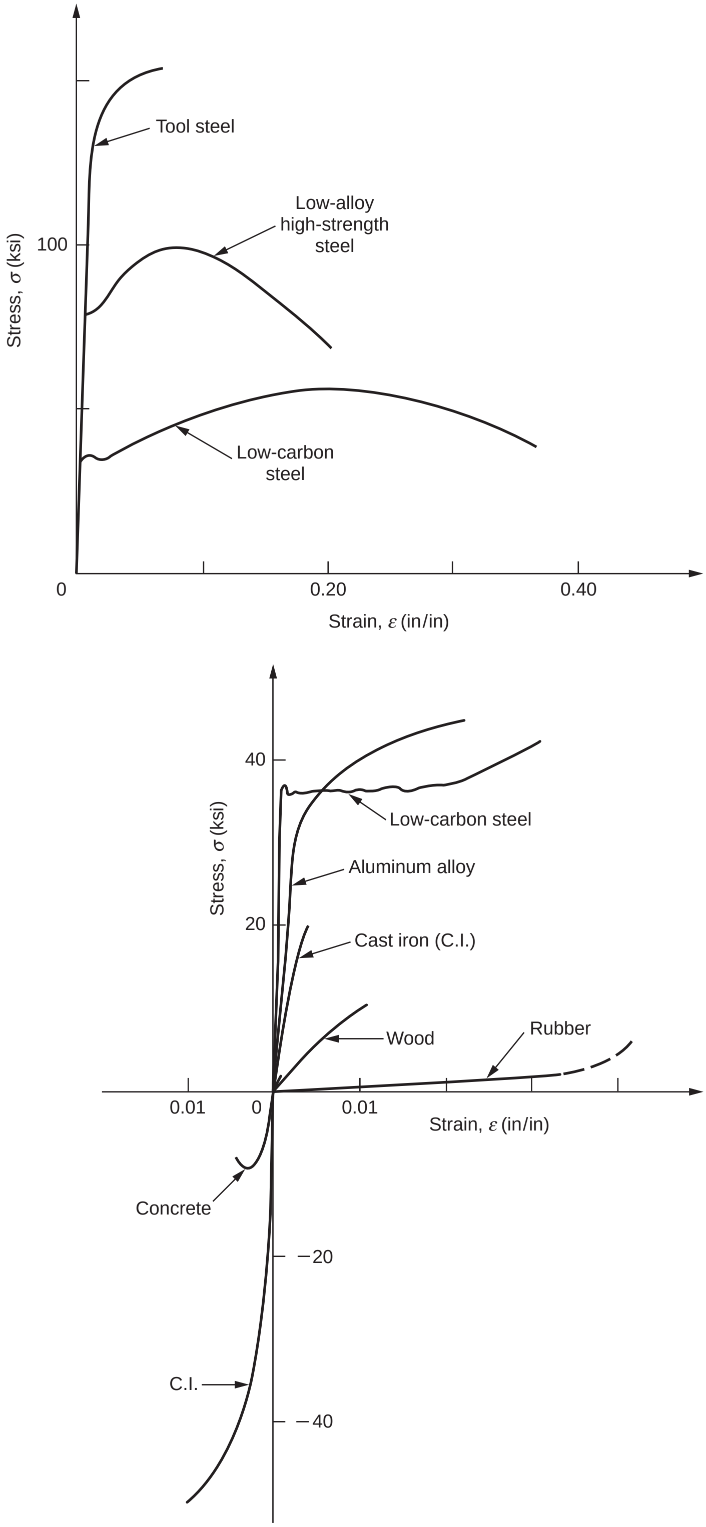


FIGURE 46.5 (a) Typical stress–strain diagrams for different steels. (b) Typical stress–strain diagrams for different materials.

TABLE 46.1A Typical Physical Properties of and Allowable Stresses for Some Common Materials (in U.S. Customary System Units)

Material	Unit Weight (lb/in.^{3})	Ultimate Strength (ksi)			Yield Strength (ksi)		Allow Stresses (psi)		Elastic Moduli ($\times 10^{-6}$ psi)		Coefficient of Thermal Expansion ($\times 10^6$/°F)
		Tension	Compression	Shear	Tension	Shear	Tension or Compression	Shear	Tension or Compression	Shear	
Aluminum alloy (extruded)											
2024-T4	0.100	60	—	32	44	25			10.6	4.00	12.9
6061-T6		38	—	24	35	20			10.0	3.75	13.0
Cast iron											
gray	0.276	30	120	—	—	—			13	6	5.8
malleable		54	—	48	36	24			25	12	6.7
Concrete											
8 gal/sack	0.087	—	3	—	—	—	–1350	66	3	—	6.0
6 gal/sack		—	5	—	—	—	–2250	86	5	—	—
Magnesium alloy, AM100A	0.065	40	—	21	22	—	—	—	6.5	2.4	14.0
Steel											
0.2% carbon (hot rolled)		65	—	48	36	24	±24,000	14,500			
0.6% carbon (hot rolled)	0.283	100	—	80	60	36			30	12	6.5
0.6% carbon (quenched)		120	—	100	75	45					
3½% Ni, 0.4% C		200	—	150	150	90					
Wood											
Douglas fir (coast)	0.018	—	7.4	1.1	—	—	±1900	120	1.76	—	—
Southern pine (longleaf)	0.021	—	8.4	1.5	—	—	±2250	135	1.76	—	—

TABLE 46.1B Typical Physical Properties of and Allowable Stresses for Some Common Materials (in SI System Units)

Material	Unit Mass ($\times 10^3$ kg/m^3)	Ultimate Strength (MPa)			Yield Strength (MPa)		Allow Stresses (MPa)		Elastic Moduli (GPa)		Coefficient of Thermal Expansion ($\times 10^{-6}$/°C)
		Tension	Compression	Shear	Tension	Shear	Tension or Compression	Shear	Tension or Compression	Shear	
Aluminum alloy (extruded)											
2014-T6	2.77	414	—	220	300	170			73	27.6	23.2
6061-T6		262	—	165	241	138			70	25.9	23.4
Cast iron											
gray	7.64	210	825	—	—	—			90	41	10.4
malleable		370	—	330	250	165			170	83	12.1
Concrete											
0.70 water–cement ratio	2.41	—	20	—	—	—	–9.31	0.455	20	—	10.8
0.53 water–cement ratio		—	35	—	—	—	–15.5	0.592	35	—	
Magnesium alloy, AM100A	1.80	275	—	145	150	—	—	—	45	17	25.2
Steel											
0.2% carbon (hot rolled)		450	—	330	250	165	±165.0	100			
0.6% carbon (hot rolled)	7.83	690	—	550	415	250			200	83	11.7
0.6% carbon (quenched)		825	—	690	515	310					
3½% Ni, 0.4% C		1380	—	1035	1035	620					
Wood											
Douglas fir (coast)	0.50	—	51	7	—	—	±13.1	0.825	12.1	—	—
Southern pine (longleaf)	0.58	—	58	10	—	—	±15.5	0.930	12.1	—	—

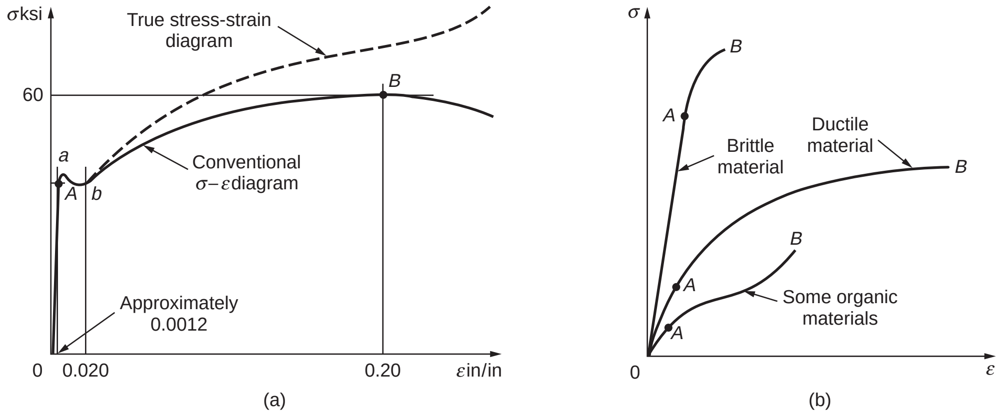


FIGURE 46.6 Stress–strain diagrams: (a) mild steel; (b) typical materials.

which simply means that stress is directly proportional to strain. The constant of proportionality E is called the *elastic modulus*, *modulus of elasticity*, or *Young's modulus*. Graphically, E is the slope of the straight line from the origin to point A, which is the *proportional* or *elastic limit* of the material. Physically, the elastic modulus represents the stiffness of the material to an imposed load, and it is a definite property of a material (Table 46.1). Almost all materials have an initial range where the relationship between stress and strain is linear and Hooke's law is applicable.

In Fig. 46.6(a), the highest point B corresponds to the *ultimate strength* of the material. Stress associated with the plateau ab in Fig. 46.6(a) is called the *yield strength* of the material. The yield strength is often so near the proportional limit that the two may be taken to be the same. For materials with no well-defined yield strength, the *offset method* is usually used where a line offset of an arbitrary amount — generally 0.2% of strain — is drawn parallel to the straight-line portion of the material; the yield strength is taken where the line offset intersects the stress–strain curve.

Constitutive Relations

The relationships between stress and strain are frequently referred to as *constitutive relations* or laws. A *linear elastic* material implies that stress is directly proportional to strain and Hooke's law is applicable. A *nonlinear elastic* material responds in a nonproportional manner, yet when unloaded returns back along the loading path to its initial stress-free state. If in stressing an *inelastic* or *plastic* material its elastic limit is exceeded, then on unloading it usually responds approximately in a linearly elastic manner, but a permanent deformation, or set, develops at no external load. Figure 46.7 shows three other types of

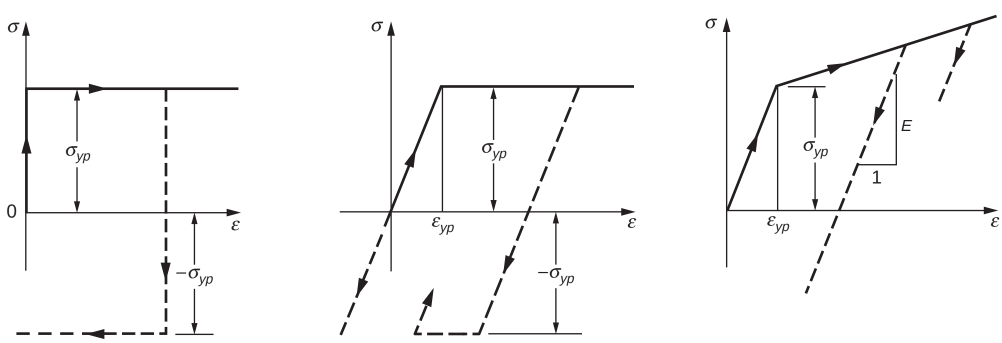


FIGURE 46.7 Idealized stress–strain diagrams: (a) rigid-perfectly plastic material; (b) elastic-perfectly plastic material; elastic-linearly hardening material.

idealized stress–strain diagrams. A rigid, perfectly plastic diagram of Fig. 46.7(a) is applicable to problems in which the elastic strains can be neglected in relation to the plastic ones. If both the elastic and perfectly plastic strains have to be included, Fig. 46.7(b) is more appropriate. Beyond the elastic range, many materials resist additional stress, a phenomenon referred to as *strain hardening*; then Fig. 46.7(c) would provide a reasonable approximation. For a more accurate idealization, equations capable of representing a wide range of stress–strain curves have been developed, for example, by Ramberg and Osgood [1943]. This equation is

$$\frac{\varepsilon}{\varepsilon_0} = \frac{\sigma}{\sigma_0} + \frac{3}{7}\left(\frac{\sigma}{\sigma_0}\right)^n \tag{46.9}$$

where the constants ε_0 and σ_0 correspond to the yield point and n is a characteristic constant of the material. This equation has the important advantage of being a continuous mathematical function by which an *instantaneous* or *tangent modulus* E_t, defined as

$$E_t = \frac{d\sigma}{d\varepsilon} \tag{46.10}$$

can be uniquely determined.

Deformation of Axially Loaded Bars

Consider the axially loaded bar shown in Fig. 46.8 and recast Eq. (46.6) for a differential element dx. Let du be the change in length; then the normal strain in the x direction is

$$\varepsilon_x = \frac{du}{dx} \tag{46.11}$$

Assuming the origin of x at B, and integrating, the change in length Δ between points D and B is

$$\Delta = \int_0^L \varepsilon_x dx \tag{46.12}$$

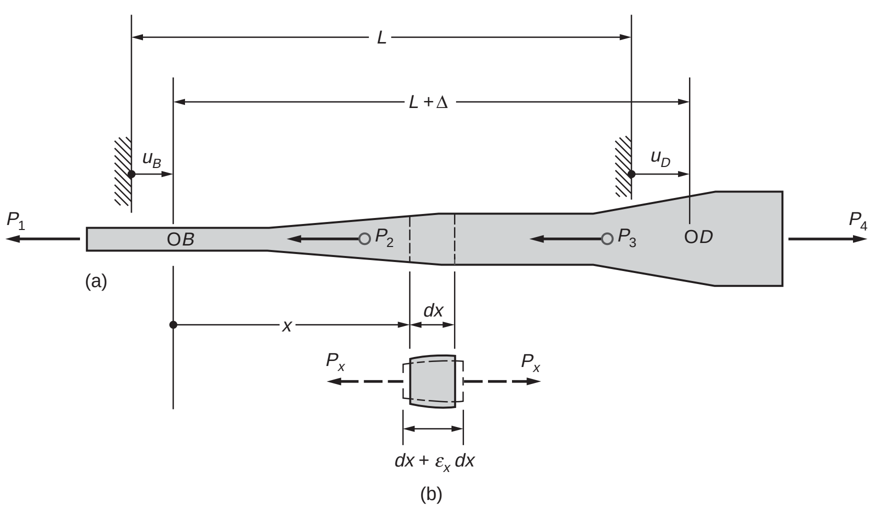


FIGURE 46.8 An axially loaded bar.

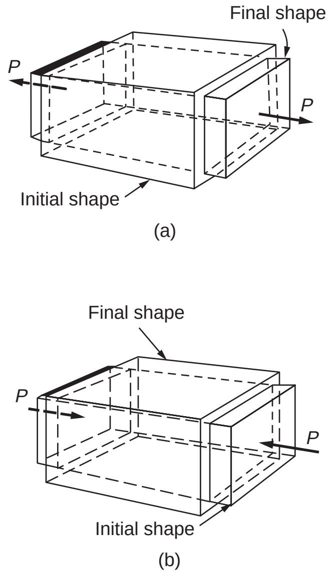


FIGURE 46.9 (a) Lateral contraction and (b) lateral expansion of solid bodies subjected to axial forces (Poisson's effect).

For linear elastic materials, according to Hooke's law, $\varepsilon_x = \sigma_x/E$, where $\sigma_x = P_x/A_x$. Thus,

$$\Delta = \int_0^L \frac{P_x dx}{A_x E_x} \tag{46.13}$$

where the force P_x, area A_x, and elastic modulus E_x can vary along the length of a bar. For a beam of length L with a constant cross-sectional area A and an elastic modulus E, $\Delta = PL/AE$.

Poisson's Ratio

As illustrated by Fig. 46.9, when a solid body is subjected to an axial tension, it contracts laterally; when it is compressed, the material "squashes" out sideways. These lateral deformations on a relative basis are termed *lateral strains* and bear a constant relationship to the axial strains. This constant is a definite property of a material and is called Poisson's ratio. It is denoted by ν and is defined as follows:

$$\nu = \left| \frac{\text{Lateral Strain}}{\text{Axial Strain}} \right| \tag{46.14}$$

The value of ν fluctuates for different materials over a relatively narrow range. Generally, it is on the order of 0.25 to 0.35. The largest possible value is 0.5 and is normally attained during plastic flow and signifies constancy of volume.

Thermal Strain and Deformation

With changes of temperature, solid bodies expand on an increase of temperature and contract on its decrease. The thermal strain ε_T caused by a change in temperature from T_0 to T can be expressed as

$$\varepsilon_T = \alpha(T - T_0) \tag{46.15}$$

where α is an experimentally determined coefficient of linear thermal expansion (see Table 46.1).

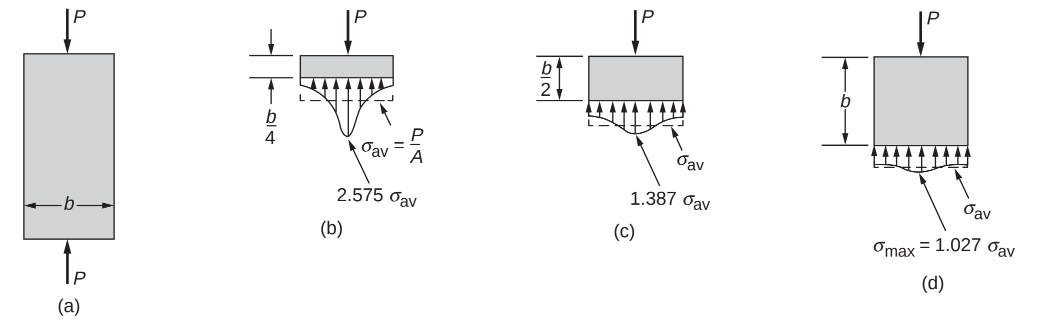


FIGURE 46.10 Stress distribution near a concentrated force in a rectangular elastic plate.

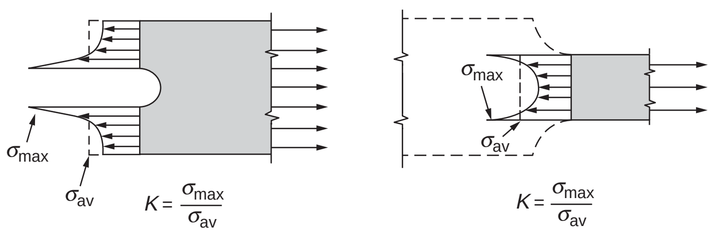


FIGURE 46.11 Meaning of the stress-concentration factor *K*.

For a body of length L subjected to a uniform temperature, the extensional deformation Δ_T due to a change in temperature of $\delta T = T - T_0$ is

$$\Delta_T = \alpha(\delta T)L \tag{46.16}$$

Saint-Venant's Principle and Stress Concentrations

Saint-Venant's principle states that the manner of force application on stresses is important only in the vicinity of the region where the force is applied. Figure 46.10 illustrates how normal stresses at a distance equal to the width of the member are essentially uniform. Only near the location where the concentrated force is applied is the stress nonuniform. Saint-Venant's principle also applies to changes in a cross section, as shown by Fig. 46.11. The ratio of the maximum stress to the average stress is called the *stress concentration factor* K, which depends on the geometrical proportions of the members. The maximum normal stress can then be expressed as

$$\sigma_{maz} = K\sigma_{av} = K\frac{P}{A} \tag{46.17}$$

Many stress concentration factors K can be found tabulated in the literature [Roark and Young, 1975].

Elastic Strain Energy for Uniaxial Stress

The product of force (stress multiplied by area) and deformation is the internal work done in a body by the externally applied forces. This internal work is stored in an elastic body as the internal elastic energy of deformation or the elastic strain energy. The internal elastic strain energy U for an infinitesimal element subjected to uniaxial stress is

$$dU = \tfrac{1}{2}\sigma_x \varepsilon_x dV \tag{46.18}$$

where dV is the volume of the element.

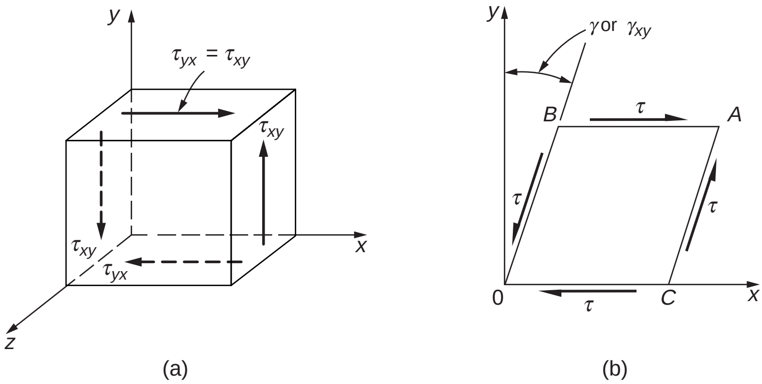


FIGURE 46.12 Element in pure shear.

In the elastic range, Hooke's law applies, $\sigma_x = E\varepsilon_x$. Then

$$U = \int_{vol} \frac{\sigma_x^2}{2E} dV \tag{46.19}$$

The strain energy stored in an elastic body per unit volume of the material (its *strain-energy density*), U_0, is equal to $\sigma^2/2E$. Substituting the value of the stress at the proportional limit gives the *modulus of resilience*, an index of a material's ability to store or absorb energy without permanent deformation. Analogously, the area under a complete stress–strain diagram gives a measure of a material's ability to absorb energy up to fracture and is called its *toughness*.

46.4 Generalized Hooke's Law

Stress–Strain Relationships for Shear

An element in pure shear is shown in Fig. 46.12. The change in the initial right angle between any two imaginary planes in a body defines *shear strain* γ. For infinitesimal elements, these small angles are measured in radians. Below the yield strength of most materials, a linear relationship exists between pure shear and the angle γ. Therefore, mathematically, extension of Hooke's law for shear stress and strain reads

$$\tau = G\gamma \tag{46.20}$$

where G is a constant of proportionality called the *shear modulus of elasticity*, or the *modulus of rigidity*.

Elastic Strain Energy for Shear Stress

An expression for the elastic strain energy for shear stresses may be established in a manner analogous to that for one in uniaxial stress, given in the previous section. Thus,

$$U_{shear} = \int_{vol} \frac{\tau^2}{2G} dV \tag{46.21}$$

Mathematical Definition of Strain

Since strains generally vary from point to point, the definitions of strain must relate to an infinitesimal element. As shown in Fig. 46.13(a), consider an extensional strain taking place in one direction. Points

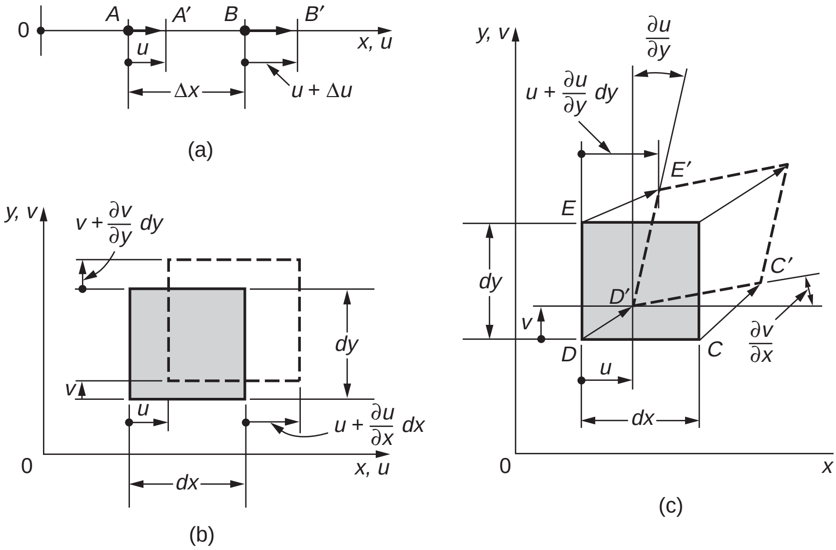


FIGURE 46.13 One- and two-dimensional strained elements in initial and final positions.

A and *B* move to *A′* and *B′*, respectively. During straining, point *A* experiences a displacement *u*. Point *B* experiences a displacement $u + \Delta u$, since in addition to the rigid-body displacement *u*, common to the whole element Δx, a stretch Δu takes place within the element. On this basis, the definition of the extensional or normal strain is

$$\varepsilon = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx} \tag{46.22}$$

For the two-dimensional case shown in Fig. 46.13(b), a body is strained in orthogonal directions and subscripts must be attached to differentiate between the directions of the strains, and it is also necessary to change the ordinary derivatives to partial ones. Therefore, if at a point of a body, *u*, *v*, and *w* are the three displacement components occurring in the *x*, *y*, and *z* directions, respectively, of the coordinate axes, the basic definitions of normal strain become

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \tag{46.23}$$

In addition to normal strain, an element can also experience shear strain as shown in Fig. 46.13(c) for the *xy* plane. This inclines the sides of the deformed element in relation to the *x* and the *y* axes. Since *v* is the displacement in the *y* direction, as one moves in the *x* direction, $\partial v/\partial x$ is the slope of the initially horizontal side of the infinitesimal element. Similarly, the vertical side tilts through an angle $\partial u/\partial y$. On this basis, the initially right angle CDE is reduced by the amount $\partial v/\partial x + \partial u/\partial y$. Analogous descriptions can be used for shear strains in the *xz* and *yz* planes. Therefore, for small angle changes, the definitions of the shear strain become

$$\begin{aligned}
\gamma_{xy} &= \gamma_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\
\gamma_{xz} &= \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \\
\gamma_{yz} &= \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}
\end{aligned} \tag{46.24}$$

It is assumed that tangents of small angles are equal to angles measured in radians, and a positive sign applies for the shear strain when the element is deformed, as depicted in Fig. 46.13(c).

Strain Tensor

In order to obtain the strain tensor, an entity that must obey certain laws of transformation, it is necessary to redefine the shear strain $\varepsilon_{xy} = \varepsilon_{yx}$ as one half of γ_{xy}. The strain tensor in matrix representation can then be assembled as follows:

$$\begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \varepsilon_z \end{bmatrix} \equiv \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{46.25}$$

The strain tensor is symmetric. Just as for the stress tensor, using indicial notation, one can write ε_{ij} for the strain tensor. For a two-dimensional problem, the third row and column are eliminated, and one has a case of *plane strain*.

Generalized Hooke's Law for Isotropic Materials

Six basic relationships between a general state of stress and strain can be synthesized using the principle of superposition. This set of equations is referred to as the generalized Hooke's law, and for isotropic linearly elastic materials it can be written for use with Cartesian coordinates as

$$\begin{aligned} \varepsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\ \varepsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \\ \varepsilon_z &= -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \\ \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \gamma_{yz} &= \frac{\tau_{yz}}{G} \\ \gamma_{zx} &= \frac{\tau_{zx}}{G} \end{aligned} \tag{46.26}$$

E, G, and Relationship

Using the relationship between shear and extensional strains and the fact that a pure shear stress at a point can be alternatively represented by the normal stresses at 45° with the directions of the shear stresses, one can obtain the following relationship among E, G, and ν:

$$G = \frac{E}{2(1+\nu)} \tag{46.27}$$

Dilatation and Bulk Modulus

For volumetric changes in elastic materials subjected to stress, change in volume per unit volume, often referred to as *dilatation*, is defined as

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z \tag{46.28}$$

The shear strains cause no change in volume.

If an elastic body is subjected to hydrostatic pressure of uniform intensity p, then, based on the generalized Hooke's law, it can be shown that

$$\frac{-p}{e} = k = \frac{E}{3(1-2\nu)} \tag{46.29}$$

The quantity k represents the ratio of the hydrostatic compressive stress to the decrease in volume and is called the *modulus of compression*, or *bulk modulus*.

The six equations of Eq. (46.26) have an inverse that can be solved to express stresses in terms of strain and may be written as

$$\begin{aligned}
\sigma_x &= \lambda e + 2G\varepsilon_x \\
\sigma_y &= \lambda e + 2G\varepsilon_y \\
\sigma_z &= \lambda e + 2G\varepsilon_z \\
\tau_{xy} &= G\gamma_{xy} \\
\tau_{yz} &= G\gamma_{yz} \\
\tau_{zx} &= G\gamma_{zx}
\end{aligned} \tag{46.30}$$

where

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{46.31}$$

46.5 Torsion

Torsion of Circular Elastic Bars

To establish a relation between the internal torque and the stresses it sets up in members with circular solid and tubular cross sections, it is necessary to make two assumptions (Fig. 46.14): (1) A plane section

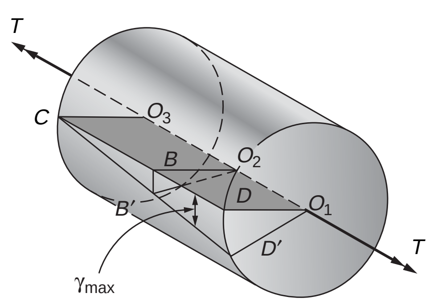


FIGURE 46.14 Variation of strain in circular member subjected to torque.

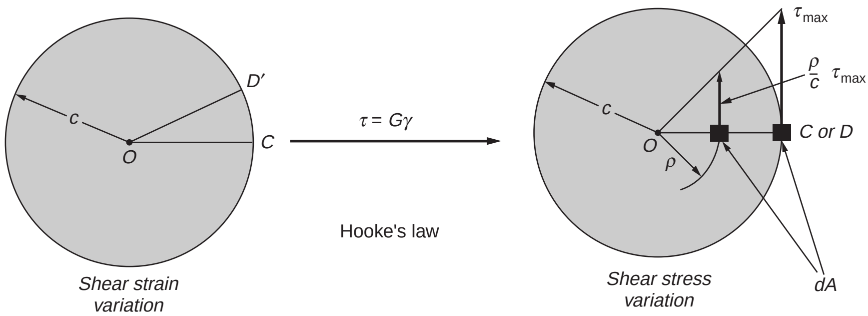


FIGURE 46.15 Shear strain assumption leading to elastic shear stress distribution in a circular member.

of material perpendicular to the axis of a circular member remains plane after the torque is applied, i.e., no warpage or distortion of parallel planes normal to the axis of member takes place; (2) shear strains γ vary linearly from the central axis, reaching γ_{max} at the periphery. This means that in Fig. 46.14 an imaginary plane such as DO_1O_3C moves to $D'O_1O_3C$ when the torque is applied, and the radii O_1D and O_2B remain straight. In the elastic case, shear stresses vary linearly from the central axis of a circular member, as shown in Fig. 46.15. Thus the maximum stress, τ_{max}, occurs at the radius c from the center, and at any distance ρ from O, the shear stress is $(\rho/c)\tau_{max}$. For equilibrium, the internal resisting torque must equal the externally applied torque T. Hence,

$$\frac{\tau_{max}}{c}\int_A \rho^2 dA = T \tag{46.32}$$

However, $\int_A \rho^2 dA$ is the *polar moment of inertia* of a cross-sectional area and is a constant. By using the symbol J for the polar moment of inertia of a circular area, that is, $J = \pi c^4/2$, Eq. (46.32) may be written more compactly as

$$\tau_{max} = \frac{Tc}{J} \tag{46.33}$$

For a circular tube with inner radius b, $J = (\pi c^4/2) - (\pi b^4/2)$.

Angle-of-Twist of Circular Members

The governing differential equation for the angle-of-twist for solid and tubular circular elastic shafts subjected to torsional loading can be determined by referring to Fig. 46.16, which shows a differential shaft of length dx under a differential twist $d\phi$. Arc DD' can be express as $\gamma_{max}\, dx = d\phi c$. Then substituting Eq. (46.33) and assuming Hooke's law is applicable, the governing differential equation for the angle-of-twist is obtained:

$$\frac{d\phi}{dx} = \frac{T}{JG} \quad \text{or} \quad d\phi = \frac{Tdx}{JG} \tag{46.34}$$

Hence, a general expression for the angle-of-twist between any two sections A and B on a shaft is

$$\phi = \phi_B - \phi_A = \int_A^B d\phi = \int_A^B \frac{T_x dx}{J_x G} \tag{46.35}$$

where ϕ_B and ϕ_A are the global shaft rotations at ends B and A, respectively. In this equation, the internal torque T_x and the polar moment of inertia J_x may vary along the length of a shaft. The direction of the

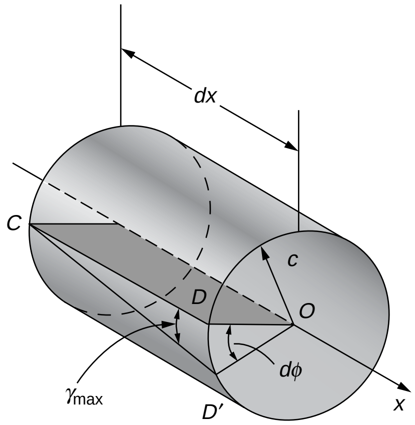


FIGURE 46.16 Deformation of a circular bar element due to torque.

angle of twist ϕ coincides with the direction of the applied torque. For an elastic shaft of length L and constant cross section $\phi = TL/JG$.

Torsion of Solid Noncircular Members

The two assumptions made for circular members do not apply for noncircular members. Sections perpendicular to the axis of a member warp when a torque is applied. The nature of the distortions that take place in a rectangular section can be surmised from Fig. 46.17. For a rectangular member, the corner elements do not distort at all. Therefore shear stresses at the corners are zero; they are maximum at the midpoints of the long sides. Analytical solutions for torsion of rectangular, elastic members have been obtain [Timoshenko and Goodier, 1970]. The methods used are mathematically complex and beyond

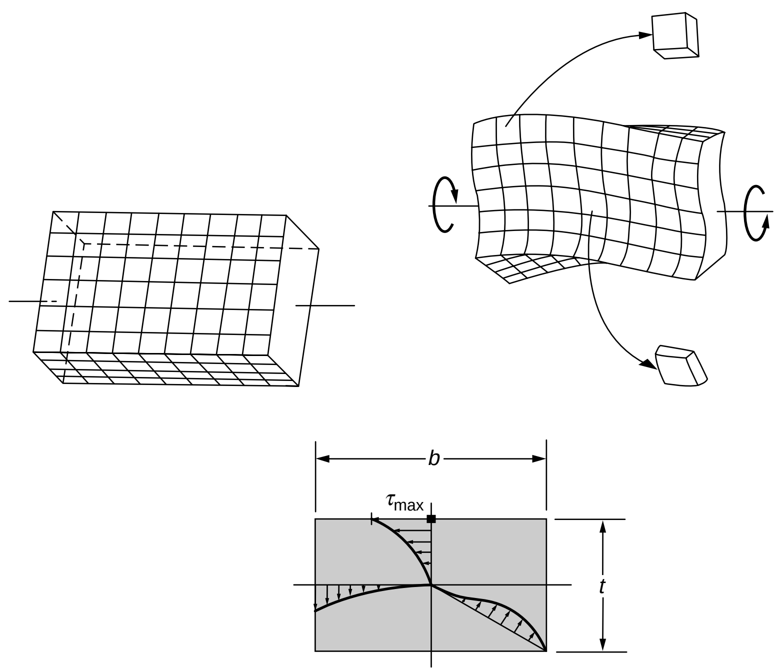


FIGURE 46.17 Rectangular bar (a) before and (b) after a torque is applied. (c) Shear stress distribution in a rectangular shaft subjected to a torque.

TABLE 46.2 Table of Torsional Coefficients for Rectangular Bars

b/t	1.00	1.50	2.00	3.00	6.00	10.0	8
α	0.208	0.231	0.246	0.267	0.299	0.312	0.333
β	0.141	0.196	0.229	0.263	0.299	0.312	0.333

the scope of the present discussion. However, the results for the maximum shear stresses and the angle-of-twist can be put into the following form:

$$\tau_{\max} = \frac{T}{\alpha b t^2} \quad \text{and} \quad \phi = \frac{TL}{\beta b t^3 G} \tag{46.36}$$

where T is the applied torque, b is the length of the long side, and t is the thickness or width of the short side of a rectangular section. The values of parameters α and β depend on the ratio b/t, given in Table 46.2. For thin sections, where b is much greater than t, the values of α and β approach 1/3. It is useful to recast the second Eq. (46.36) to express the torsional stiffness k_t for a rectangular section, giving

$$k_t = \frac{T}{\phi} = \beta b t^3 \frac{G}{L} \tag{46.37}$$

For cases that cannot be conveniently solved mathematically, the *membrane analogy* has been devised. It happens that the solution of the partial differential equation that must be solved in the elastic torsion problem is mathematically identical to that for a thin membrane, such as a soap film, lightly stretched over a hole. This hole must be geometrically similar to the cross section of the shaft being studied. Then the following can be shown to be true:

1. The shear stress at any point is proportional to the slope of the stretched membrane at the same point (Fig. 46.18).
2. The direction of a particular shear stress at a point is at right angles to the slope of the membrane at the same point.
3. Twice the volume enclosed by the membrane is proportional to the torque carried by the section.

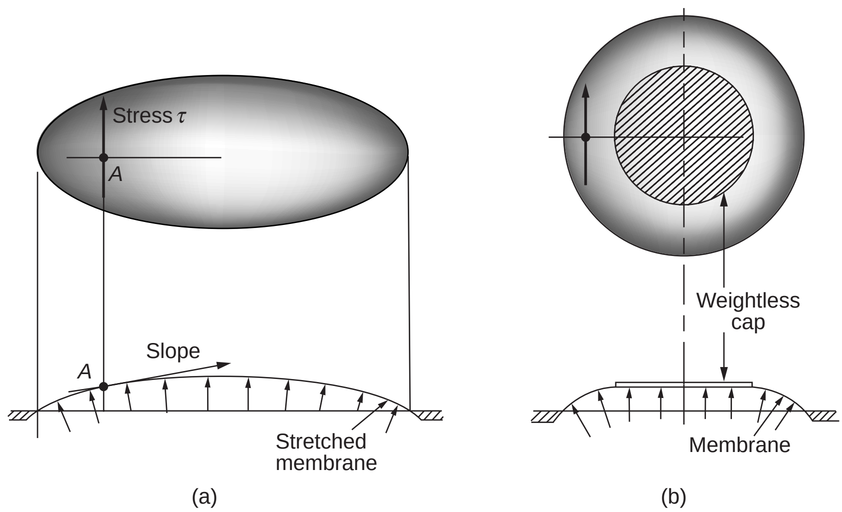


FIGURE 46.18 Membrane analogy: (a) simply connected region; (b) multiply connected (tubular) region.

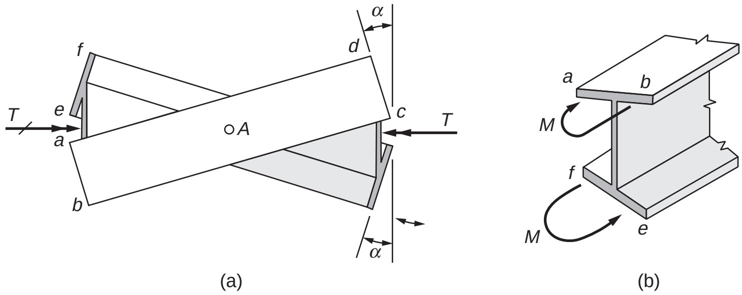


FIGURE 46.19 Cross-sectional warpage due to applied torque.

For plastic torsion, the *sand-heap analogy* has been developed, and it has similar interpretations as those in the membrane analogy [Nadai, 1950]. Dry sand is poured onto a raised flat surface having the shape of the cross section of the member. The surface of the sand heap so formed assumes a constant slope. The volume of the sand heap, hence its weight, is proportional to the fully plastic torque carried by a section.

Warpage of Thin-Walled Open Sections

For a narrow rectangular bar, no shear stresses develop along a line bisecting its thickness. This means that no in-plane deformation can take place along the entire width and length of the bar's middle surface. In this sense, an I section, shown in Fig. 46.19, consists of three flat bars, and during twisting, the three middle surfaces of these bars do not develop in-plane deformations. By virtue of symmetry, this I section twists around its centroidal axis, which in this case is also the center of twist. During twisting, as the beam flanges displace laterally, the undeformed middle surface *abcd* rotates about point A, Fig. 46.19(a). Similar behavior is exhibited by the middle surface of the other flange. In this manner, plane sections of an I beam warp, i.e., cease to be plane, during twisting.

Cross-sectional warpage, or its restraint, may have an important effect on member strength, particularly on its stiffness. Warpage of cross sections in torsion is restrained in many applications. For example, by welding an end of a steel I beam to a rigid support, the attached cross section cannot warp. To maintain required compatibility of deformations, in-plane flange moments M, shown in Fig. 46.19(b), must develop. Such an enforced restraint effectively stiffens a beam and reduces its twist. This effect is local in character and, at some distance from the support, becomes unimportant. Nevertheless, for short beams, cutouts, etc., the warpage-restraint effect is dominant. For further details, refer to the reference of Oden and Ripperger [1981].

Torsion of Thin-Walled Hollow Members

Unlike solid noncircular members, thin-walled tubes of any shape can be rather simply analyzed for the magnitude of the shear stresses and the angle-of-twist caused by a torque applied to the tube. Thus, consider a tube of an arbitrary shape with varying wall thickness, such as shown in Fig. 46.20(a), subjected to torque T. Isolate an element from this tube, as shown enlarged in Fig. 46.20(b). This element must be in equilibrium under the action of forces F_1, F_2, F_3, and F_4. These forces are equal to the shear stresses acting on the cut planes multiplied by the respective areas. From summation of forces, and since the longitudinal sections were taken an arbitrary distance apart, it follows that the product of the shear stress and the wall thickness is the same, i.e., constant, on any such planes. This constant will be denoted by q, which is measured in the units of force per unit distance along the perimeter, since shear stresses on mutually perpendicular planes are equal at a corner of an element. Hence, at a corner such as A in Fig. 46.20(b), $\tau_2 = \tau_3$; similarly, $\tau_1 = \tau_4$. Therefore, $\tau_4 t_1 = \tau_3 t_2$, or, in general, q is constant in the plane of a section perpendicular to the axis of a member. The quantity q has been termed the shear flow. Next consider the cross section of the tube as shown in Fig. 46.20(c). The force per unit distance of the

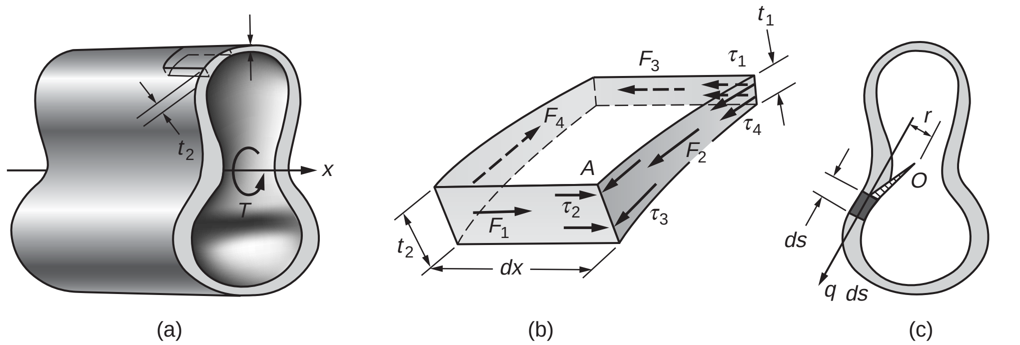


FIGURE 46.20 Thin-walled tubular member of variable thickness.

perimeter of this tube, by virtue of the previous argument, is constant and is the shear flow q. This shear flow multiplied by the length ds of the perimeter gives a force $q\ ds$ per differential length. The product of this infinitesimal force $q\ ds$ and r around some convenient point such as 0, Fig. 46.20(c), gives the contribution of an element to the resistance of applied torque T. Adding or integrating this,

$$T = q\oint r ds \tag{46.38}$$

where the integration process is carried around the tube along the center line of the perimeter. A simple interpretation of the integral is available. It can be seen from Fig. 46.20(c) that rds is twice the value of the shaded area of an infinitesimal triangle of altitude r and base ds. Hence, the complete integral is twice the whole area bounded by the center line of the perimeter of the tube. Defining this area by the special symbol Ⓐ, one obtains

$$T = 2Ⓐq \text{ or } q = \frac{T}{2Ⓐ} \tag{46.39}$$

This equation applies only to thin-walled tubes. The area Ⓐ is approximately an average of the two areas enclosed by the inside and the outside surfaces of a tube, or, as noted, it is an area enclosed by the center line of the wall's contour. It is not applicable at all if the tube is slit.

Since for any tube the shear flow q is constant, from the definition of shear flow, the shear stress at any point of a tube where the wall thickness is t is

$$\tau = \frac{q}{t} \tag{46.40}$$

In the elastic range, Eqs. (46.39) and (46.40) are applicable to any shape of tube. For inelastic behavior, Eq. (46.40) applies only if thickness t is constant. For linearly elastic materials, the angle of twist for a hollow tube can be found by applying the principle of conservation of energy. Equating the elastic strain energy to the external work per unit length of member, the following governing differential equation is obtained,

$$\theta = \frac{d\phi}{dx} = \frac{T}{4Ⓐ^2 G}\oint \frac{ds}{t} \tag{46.41}$$

It is useful to recast this equation to express the torsional stiffness k_t for a thin-walled hollow tube. Since for a prismatic tube subjected to a constant torque, $\phi = \theta L$,

$$k_t = \frac{T}{\phi} = \frac{4Ⓐ^2}{\oint ds/t}\frac{G}{L} \tag{46.42}$$

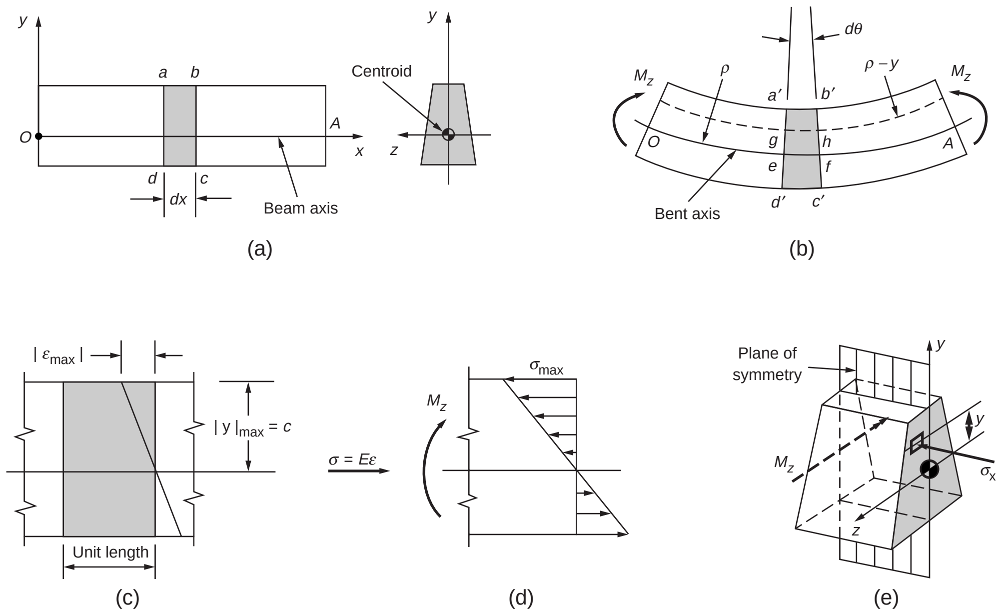


FIGURE 46.21 Assumed behavior of elastic beam in bending.

46.6 Bending

The Basic Kinematic Assumption

Consider a typical element of the beam between two planes perpendicular to the beam axis, as shown in Fig. 46.21. In side view, such an element is identified in the figure as *abcd*. When such a beam is subjected to equal end moments M_z acting around the z axis, Fig. 46.21(b), this beam bends in the plane of symmetry, and the planes initially perpendicular to the beam axis tilt slightly. Nevertheless, the lines such as *ad* and *bc* becoming *a′d′* and *b′c′* remain straight. This observation forms the basis for the fundamental hypothesis of the flexure theory. It may be stated thus: plane sections through a beam taken normal to its axis remain plane after the beam is subjected to bending.

In pure bending of a prismatic beam, the beam axis deforms into a part of a circle of radius ρ, as shown by Fig. 46.21(b). For an element defined by an infinitesimal angle $d\theta$, the fiber length *ef* of the beam axis is given as $ds = \rho d\theta$. Hence,

$$\frac{d\theta}{ds} = \frac{1}{\rho} = \kappa \tag{46.43}$$

where the reciprocal of ρ defines the axis *curvature* κ.

The fiber length *gh* located on a radius $(\rho - y)$ can be found similarly, and the difference between fiber lengths *gh* and *ef* can be expressed as $(-y d\theta)$, which is equal to du, since the deflection and rotations of the beam axis are very small. Then one obtains the normal strain $\varepsilon_x = du/dx$, as

$$\varepsilon_x = -\kappa y \tag{46.44}$$

This equation establishes the expression for the basic kinematic hypothesis for the flexure theory: the strain in a bent beam varies along the beam depth linearly with y.

The Elastic Flexure Formula

By using Hooke's law, the expression for the normal strain given by Eq. (46.44) can be recast into a relation for the normal longitudinal stress:

$$\sigma_x = E\varepsilon_x = -E\kappa y \tag{46.45}$$

To satisfy equilibrium, the sum of all forces at a section in pure bending must vanish,

$$\int_A \sigma_x dA = 0$$

which can be rewritten as

$$-E\kappa \int_A y dA = 0$$

By definition, the integral

$$\int_A y dA = \bar{y}A$$

where $\bar{y}$ is the distance from the origin to the centroid of an area A. Since the integral equals zero here and area A is not zero, distance $\bar{y}$ must be set equal to zero. Therefore, the z axis must pass through the centroid of a section. In bending theory, this axis is also referred to as the neutral axis of a beam. On this axis both the normal strain ε_x and the normal stress σ_x are zero. Based on this result, linear variation in strain is schematically shown in Fig. 46.21(c). The corresponding elastic stress distribution in accordance with Eq. (46.45) is shown in Fig. 46.21(d). Both the absolute maximum strain ε_{max} and the absolute maximum stress σ_{max} occur at the largest value of y.

Equilibrium requires the additional condition that the sum of the externally applied and the internal resisting moments must vanish. For the beam segment in Fig. 46.21(d), this yields

$$M_z = E\kappa \int_A y^2 dA$$

In mechanics, the last integral, depending only on the geometrical properties of a cross-sectional area, is called the rectangular moment of inertia or the second moment of inertia of the area A and is designated by I. Since I must always be determined with respect to a particular axis, it is often meaningful to identify it with a subscript corresponding to such an axis. For the case considered, this subscript is z, that is,

$$I_z = \int_A y^2 dA \tag{46.46}$$

With this notation, the basic relation giving the curvature of an elastic beam subjected to a specified moment is expressed as

$$\kappa = \frac{M_z}{EI_z} \tag{46.47}$$

By substituting Eq. (46.47) into Eq. (46.45), the elastic flexure formula for beams is obtained:

$$\sigma_x = -\frac{M_z}{I_z} y \tag{46.48}$$

It is customary to recast the flexure formula to give the maximum normal stress σ_{max} directly and to designate the value $|y|_{max}$ by c, as in Fig. 46.21(c). It is also common practice to dispense with the sign, as in Eq. (46.48), as well as with the subscripts on M and I. Since the normal stresses must develop a couple statically equivalent to the internal bending moment, their sense can be determined by inspection. On this basis, the flexure formula becomes

$$\sigma_{max} = \frac{Mc}{I} \tag{46.49}$$

Elastic Strain Energy in Pure Bending

Using the section "Elastic Strain Energy for Uniaxial Stress" as the basis, the elastic strain energy for beams in pure bending can be found. By substituting the flexure formula into Eq. (46.18) and integrating over the volume, V, of the beam, the expression for the elastic strain energy, U, in a beam in pure bending is obtained:

$$U = \int_0^L \frac{M^2}{2EI} dx \tag{46.50}$$

Unsymmetric Bending and Bending with Axial Loads

Consider the rectangular beam shown in Fig. 46.22, where the applied moments M act in the plane *abcd*. By using the vector representation for M shown in Fig. 46.22(b), this vector forms an angle α with the z axis and can be resolved into the two components, M_y and M_z. Since the cross section for this beam has symmetry about both axes, Eqs. (46.45) to (46.49) are directly applicable. By assuming elastic behavior of the material, a superposition of the stresses caused by M_y and M_z is the solution to the problem. Hence, using Eq. (46.48),

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{46.51}$$

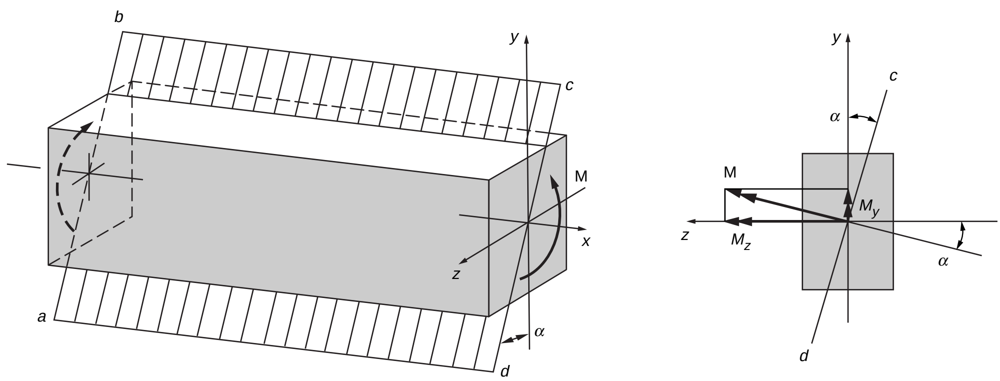


FIGURE 46.22 Unsymmetrical bending of a beam with doubly symmetric cross section.

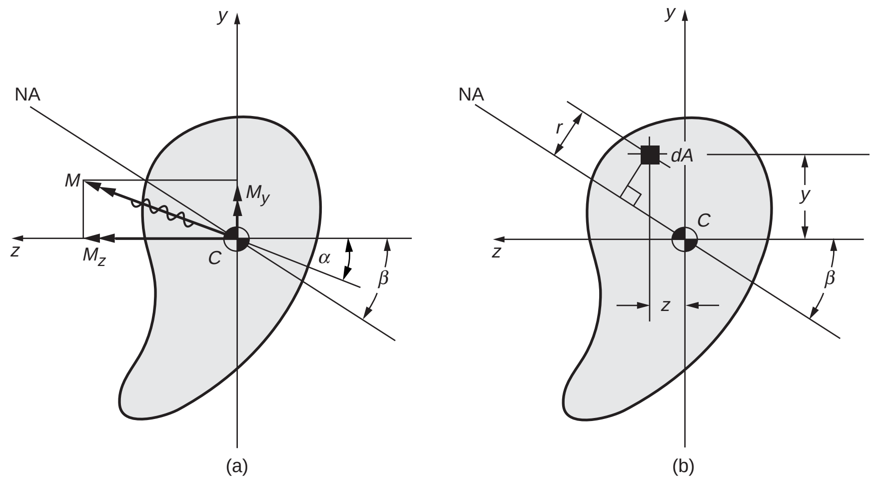


FIGURE 46.23 Bending of unsymmetric cross section.

A line of zero stress, i.e., a neutral axis, forms at an angle with the z axis and can be determined from the following equation:

$$\tan\beta = \frac{I_z}{I_y}\tan\alpha \qquad (46.52)$$

In general, the neutral axis does not coincide with the normal of the plane in which the applied moment acts.

Superposition can again be employed to include the effect of axial loads, leading Eq. (46.51) to be generalized into

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y} \qquad (46.53)$$

where P is taken positive for axial tensile forces and bending takes place around the two principal y and z axes. Further, if an applied axial force causes compression, a member must be stocky, lest a buckling problem of the type considered in a later section arises.

Bending of Beams with Unsymmetric Cross Section

A general equation for pure bending of elastic members of an arbitrary cross section (Fig. 46.23), whose reference axes are not the principal axes, can be formulated using the same approach as for the symmetrical cross sections. This *generalized flexure formula* is

$$\sigma_x = -\frac{M_z I_y + M_y I_{yz}}{I_y I_z - I_{yz}^2} y + \frac{M_y I_z + M_z I_{yz}}{I_y I_z - I_{yz}^2} z \qquad (46.54)$$

By setting this equation equal to zero, the angle β for locating the neutral axis in the arbitrary coordinate system is obtained, giving

$$\tan\beta = \frac{y}{z} = \frac{M_y I_z + M_z I_{yz}}{M_z I_y + M_y I_{yz}} \qquad (46.55)$$

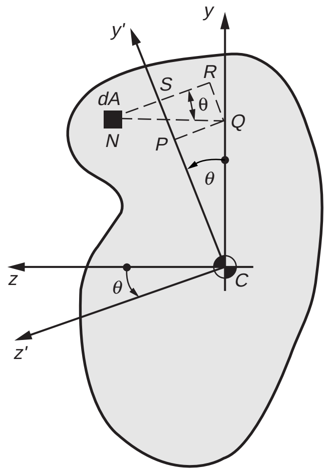


FIGURE 46.24 Rotation of coordinate axes.

Area Moments of Inertia

The concept of moments of inertia is generalized here for two orthogonal axes for any cross-sectional shape (Fig. 46.24). With the *yz* coordinates chosen as shown, by definition, the moments and product of inertia of an area are given as

$$I_z = \int y^2 dA \qquad I_y = \int z^2 dA \qquad I_{yz} = \int yz\, dA \tag{46.56}$$

Note that these axes are chosen to pass through the centroid *C* of the area, and the product of the inertia vanishes for either doubly or singly symmetric areas.

If the orthogonal axes are rotated by θ, forming a new set of $y'z'$ coordinates, it can be shown that the moments and product of inertia are transformed to the following quantities:

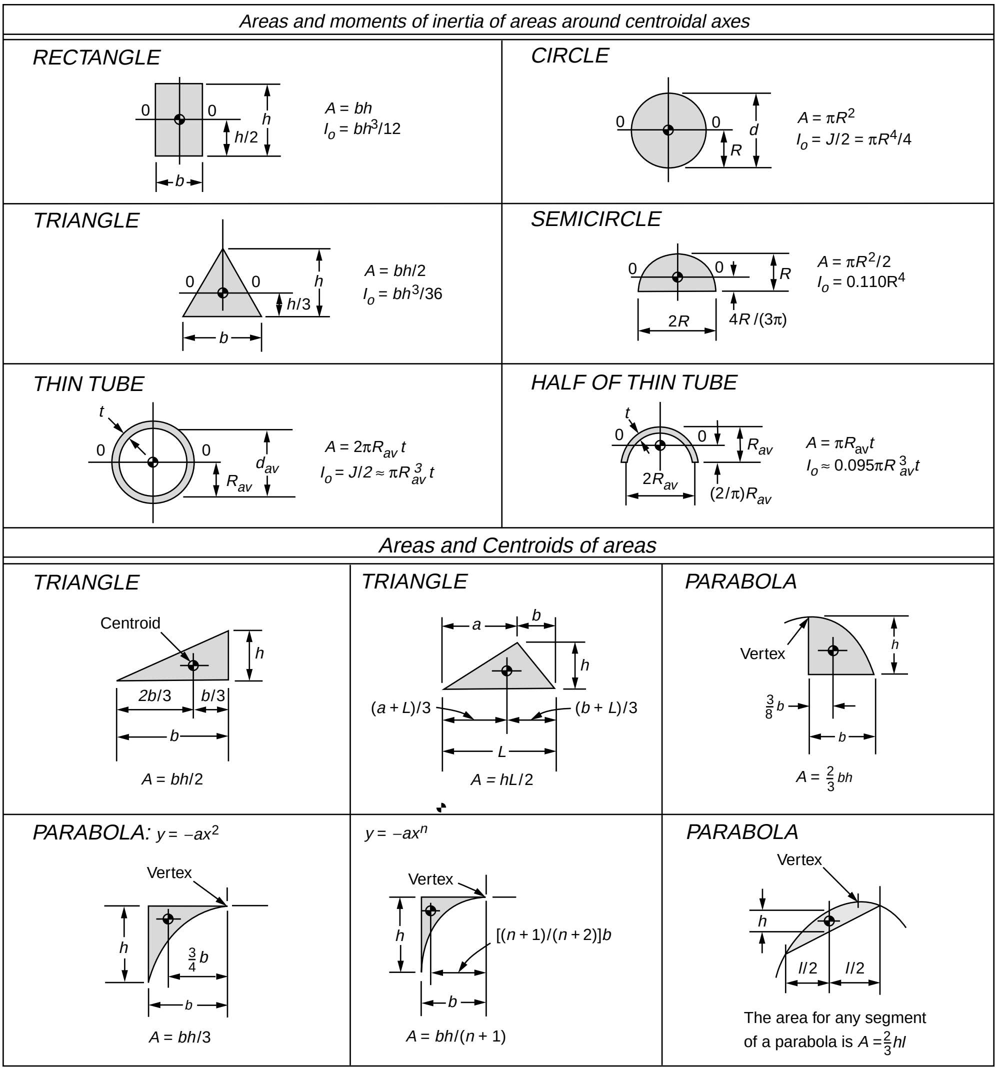
$$\begin{aligned} I_{z'} &= \frac{I_z + I_y}{2} + \frac{I_z - I_y}{2}\cos 2\theta + I_{yz}\sin 2\theta \\ I_{y'} &= \frac{I_z + I_y}{2} - \frac{I_z - I_y}{2}\cos 2\theta - I_{yz}\sin 2\theta \\ I_{z'y'} &= -\frac{I_z - I_y}{2}\sin 2\theta + I_{yz}\cos 2\theta \end{aligned} \tag{46.57}$$

Note that the sum of the moments of inertia around two mutually perpendicular axes is invariant, that is, $I_{y'} + I_{z'} = I_y + I_z$.

Table 46.3 provides formulas for the areas, centroids, and moments of inertia of some simple shapes. Most cross-sectional areas used may be divided into a combination of these simple shapes. To find *I* for an area composed of several simple shapes, the *parallel-axis theorem* (sometimes called the *transfer formula*) is necessary. It can be stated as follows: the moment of inertia of an area around any axis is equal to the moment of inertia of the same area around a parallel axis passing through the area's centroid, plus the product of the same area and the square of the distance between the two axes. Hence,

$$I_z = I_{zc} + Ad_z^2 \tag{46.58a}$$

TABLE 46.3 Some Properties of Areas



where d_z is the distance from the centroid of the subarea to the centroid of the whole area, as shown in Fig. 46.25. In calculation, Eq. (46.58a) must be applied to each subarea for which a cross-sectional area has been divided and the results summed to obtain I_z for the whole section:

$$I_z(\text{whole section}) = \sum \left(I_{zc} + Ad_z^2 \right) \tag{46.58b}$$

46.7 Shear Stresses in Beams

Shear Flow

Consider an elastic beam made from several continuous longitudinal planks whose cross section is shown in Fig. 46.26. To make this beam act as an integral member, it is assumed that the planks are fastened at intervals by vertical bolts. If an element of this beam, Fig. 46.26(b), is subjected to a bending moment

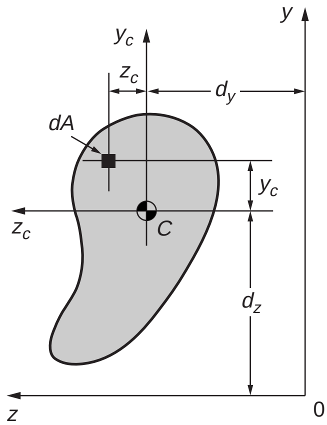


FIGURE 46.25 Area for deriving the parallel-axis theorem.

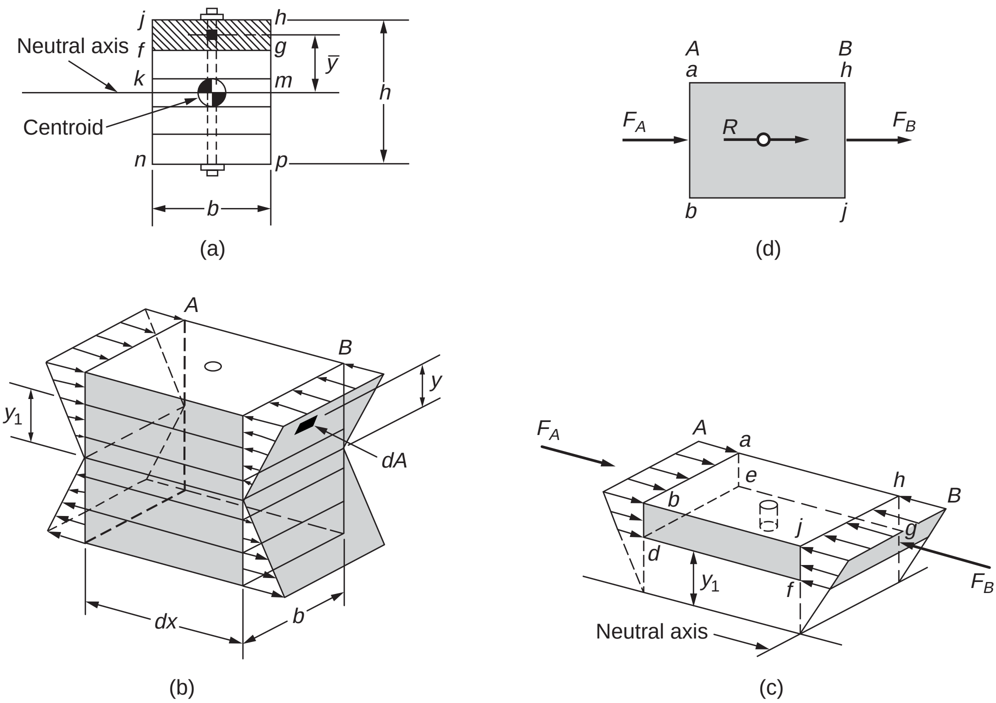


FIGURE 46.26 Elements for deriving shear flow in a beam.

$+M_A$ at end A and $+M_B$ at end B, bending stresses that act normal to the sections are developed. These bending stresses vary linearly from the neutral axis in accordance with the flexure formula My/I. The top plank of the beam element is isolated, as shown in Fig. 46.26(c). The forces acting perpendicular to the ends A and B of this plank may be determined by summing the bending stresses over their respective areas. Denoting the total force acting normal to the area *fghj* by F_B, and remembering that, at section B, M_B and I are constants, one obtains the following relation:

$$F_B = -\frac{M_B}{I} \int_{area=fghj} y\,dA = -\frac{M_B Q}{I} \tag{46.59}$$

where

$$Q = \int_{area=fghj} y\,dA = A_{fghj}\bar{y} \tag{46.60}$$

The integral defining Q is the *first*, or *statical*, *moment* of area *fghj* around the neutral axis. By definition, y is the distance from the neutral axis to the centroid of A_{fghi}. Similarly, one can express the total force acting normal to the area *abcd* as $F_A = -M_A Q/I$. If M_A is not equal to M_B, which is always the case when shears are present at the adjoining sections, F_A is not equal to F_B. Equilibrium of the horizontal forces in Fig. 46.26(c) may be attained only by developing a horizontal resisting force in the bolt R, as in Fig. 46.26(d). Taking a differential beam element of length dx, $M_B = M_A + dM$ and $dF = |F_B| - |F_A|$, and substituting these relations into the expression for F_A and F_B found above, one obtains $dF = dM\ Q/I$. It is more significant to obtain the force per unit length of beam length, dF/dx, which will be designated by q and referred to as the *shear flow*. Then, noting that $dM/dx = V$, one obtains the following expression for the shear flow in beams:

$$q = \frac{VQ}{I} \tag{46.61}$$

In this equation, I is the moment of inertia of the entire cross-sectional area around the neutral axis, and Q extends only over the cross-sectional area of the beam to one side, at which q is investigated.

Shear-Stress Formula for Beams

The shear-stress formula for beams may be obtained by modifying the shear flow formula. In a solid beam, the force resisting dF may be developed only in the plane of the longitudinal cut taken parallel to the axis of the beam, as shown in Fig. 46.27. Therefore, assuming that the shear stress τ is uniformly distributed across the section of width t, the shear stress in the longitudinal plane may be obtained by dividing dF by the area $t\,dx$. This yields the horizontal shear stress τ, which for an infinitesimal element is numerically equal to the shear stress acting on the vertical plane (see Fig. 46.27(b)). Since $q = dF/dx$, one obtains

$$\tau = \frac{VQ}{It} = \frac{q}{t} \tag{46.62}$$

where t is the width of the imaginary longitudinal cut, which is usually equal to the thickness or width of the member. The shear stress at different longitudinal cuts through the beam assumes different values as the values of Q and t for such sections differ.

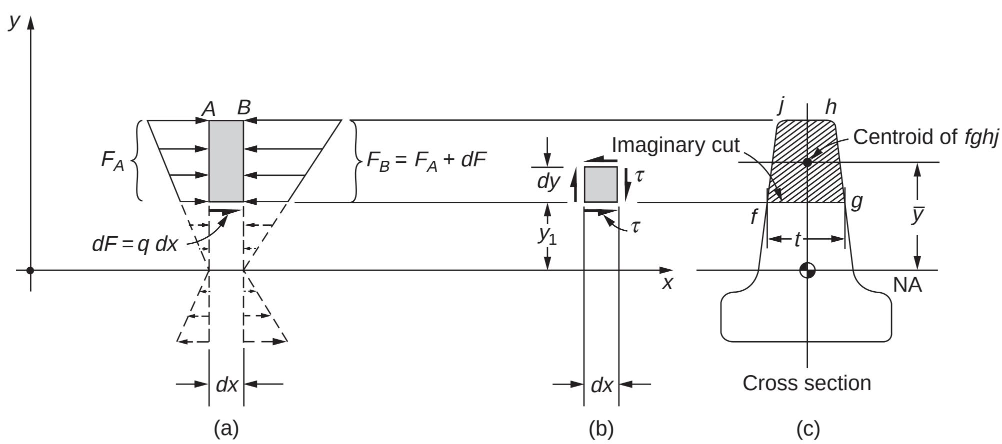


FIGURE 46.27 Derivation of shear stress in a beam.

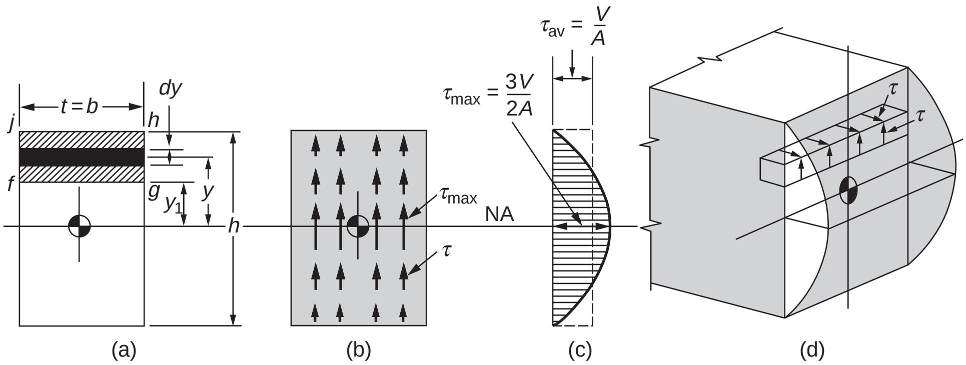


FIGURE 46.28 Shear stresses in a rectangular beam.

Since the shear-stress formula for beams is based on the flexure formula, all the limitations imposed on the flexure formula apply. The material is assumed to be elastic with the same elastic modulus in tension as in compression. The theory developed applies only to straight beams. Moreover, in certain cases such as a wide flange section, the shear-stress formula may not satisfy the requirement of a stress-free boundary condition. However, no appreciable error is involved by using Eq. (46.62) for thin-walled members, and the majority of beams belong to this group.

Shear Stresses in a Rectangular Beam

The cross-sectional area of a rectangular beam is shown in Fig. 46.28(a). A longitudinal cut through the beam at a distance y_1 from the neutral axis isolates the partial area *fghj* of the cross section. Here $t = b$ and the infinitesimal area of the cross section may be conveniently expressed as $b\,dy$. By applying Eq. (46.62), the horizontal shear stress is found at level y_1 of the beam. At the same cut, numerically equal vertical shear stresses act in the plane of the cross section ($\tau_{xy} = \tau_{yx}$):

$$\tau = \frac{VQ}{It} = \frac{V}{It}\int_{\substack{area\\ fghj}} y\,dA = \int_{y1}^{h/2} by\,dy = \frac{V}{2I}\left[\left(\frac{h}{c}\right)^2 - y_1^2\right] \tag{46.63}$$

This equation shows that in a beam of rectangular cross section, both the horizontal and the vertical shear stresses vary parabolically. The maximum value of the shear stress is obtained when y_1 is equal to zero. In the plane of the cross section, Fig. 46.28(b), this is diagrammatically represented by τ_{max} at the neutral axis of the beam. At increasing distances from the neutral axis, the shear stresses gradually diminish. At the upper and lower boundaries of the beam, the shear stresses cease to exist as $y_1 = \pm h/2$. These values of the shear stresses at the various levels of the beam may be represented by the parabola shown in Fig. 46.28(c). An isometric view of the beam with horizontal and vertical shear stresses is shown in Fig. 46.28(d).

The maximum shear stress in a rectangular beam occurs at the neutral axis, and for this case, the general expression for τ_{max} may be simplified by setting $y_1 = 0$.

$$\tau_{max} = \frac{Vh^2}{8I} = \frac{Vh^2}{8bh^3/12} = \frac{3}{2}\frac{V}{bh} = \frac{3}{2}\frac{V}{A} \tag{46.64}$$

where V is the total shear and A is the entire cross-sectional area. Since beams of rectangular cross-sectional area are used frequently in practice, this equation is very useful. It is widely used in the design of wooden beams, since the shear strength of wood on planes parallel to the grain is small. Thus, although equal shear stresses exist on mutually perpendicular planes, wooden beams have a tendency to split longitudinally along the neutral axis. Note that the maximum shear stress is 1½ times as great as the average shear stress V/A. Nevertheless, in the analysis of bolts and rivets, it is customary to determine their shear strengths by dividing the shear force V by the cross-sectional area A. Such practice is considered justified since the allowable and ultimate strengths are initially determined in this manner from tests.

Warpage of Plane Sections Due to Shear

A solution based on the mathematical theory of elasticity for a rectangular beam subjected simultaneously to bending and shear shows that plane sections perpendicular to the beam axis warp, i.e., they do not remain plane. This can also be concluded from Eq. (46.63). According to Hooke's law, shear strains must be associated with shear stresses. Therefore, the shear stresses given by Eq. (46.63) give rise to shear strains. According to this equation, the maximum shear stress, hence, maximum shear strain, occurs at $y = 0$; conversely, no shear strain takes place at $y = \pm h/2$. This behavior warps the initially plane sections through a beam, as shown qualitatively in Fig. 46.29, and contradicts the fundamental assumption of the simplified bending theory for pure flexure. However, based on rigorous analysis, warpage of the sections is known to be important only for very short members and is negligibly small for slender members. An examination of analytical results, as well as experimental measurements on beams, suggests that the assumption of "plane sections" is reasonable. It should also be noted that if shear force V along a beam is constant and the boundaries provide no restraint, the warping of all cross sections is the same. Therefore, the strain distribution caused by bending remains the same as in pure bending. Based on these considerations, a far-reaching conclusion can be made that the presence of shear at a section does not invalidate the expressions for bending stresses derived earlier.

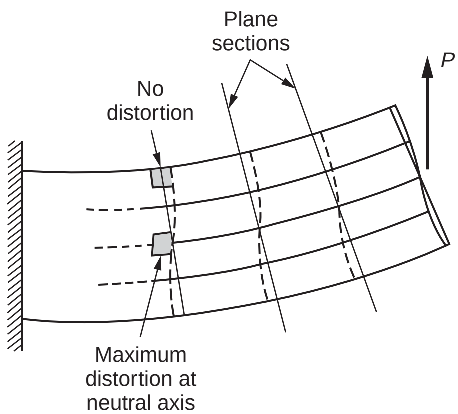


FIGURE 46.29 Shear distortions in a beam.

Shear Stresses in Beam Flanges

The shear-stress formula for beams is based on the flexure formula. Hence, all of the limitations imposed on the flexure formula apply. The material is assumed to be elastic with the same elastic modulus in tension as in compression. The theory developed applies only to straight beams. Moreover, there are additional limitations that are not present in the flexure formula.

Consider a section through the I beam shown in Fig. 46.30. The shear stresses computed for level 1-1 apply to the infinitesimal element a. The vertical shear stress is zero for this element. Likewise, no shear stresses exist on the top plane of the beam. This is as it should be, since the top surface of the beam is a free surface. In mathematical phraseology, this means that the conditions at the boundary are satisfied. For beams of rectangular cross section, the situation at the boundaries is correct. A different condition is found when the shear stresses determined for the I beam at level 2-2 are scrutinized. The shear stresses were found to be 570 psi for the elements, such as b or c shown in the figure. This requires matching horizontal shear stresses on the inner surfaces of the flanges. However, the latter surfaces must be free of the shear stresses because they are free boundaries of the beam. This leads to a contradiction that cannot be resolved by the methods of engineering mechanics of solids. The more advanced techniques

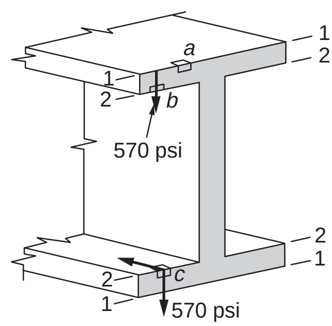


FIGURE 46.30 Boundary conditions are not satisfied at level 2-2.

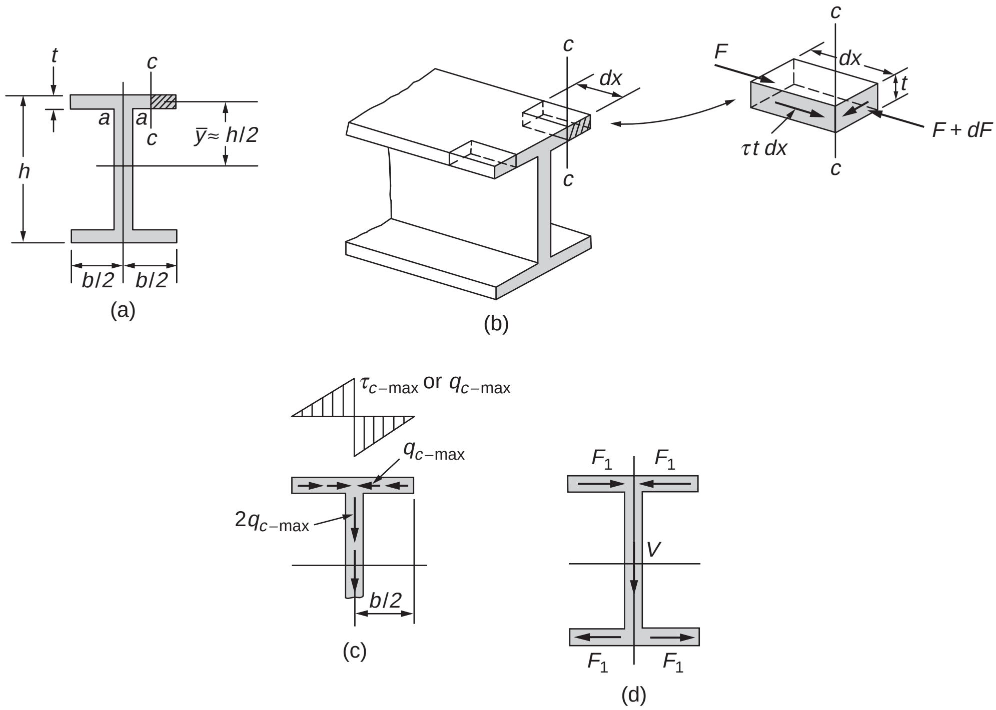


FIGURE 46.31 Shear forces in the flanges of an I beam.

of the mathematical theory of elasticity or three-dimensional finite-element analysis must be used to obtain an accurate solution.

Fortunately, the above defect of the shear-stress formula for beams is not serious. The vertical shear stresses in the flanges are small. The large shear stresses occur in the web and, for all practical purposes, are correctly given by Eq. (46.62). No appreciable error is involved by using the relations derived in this section for thin-walled members, and the majority of beams belong to this group. Moreover, as stated earlier, the solution for the shear stresses of a beam with a rectangular cross section is correct.

In an I beam, the shear stresses acting in a vertical longitudinal cut, as *c-c* in Fig. 46.31(a), act perpendicular to the plane of the paper. Their magnitude may be found by applying Eq. (46.62), and their sense follows by considering the bending moments at the adjoining sections through the beam. For example, if for the beam shown in Fig. 46.31(b) positive bending moments increase toward the reader, larger normal forces act on the near section. For the elements shown, $\tau t\, dx$ or $q\, dx$ must aid the smaller force acting on the partial area of the cross section. This fixes the sense of the shear stresses in the longitudinal cuts. However, numerically equal shear stresses act on the mutually perpendicular planes of an infinitesimal element, and the shear stresses on such planes either meet or part with their directional arrowheads at a corner. Hence, the sense of the shear stresses in the plane of the section also becomes known.

The magnitude of the shear stresses varies for the different vertical cuts. For example, if cut *c-c* in Fig. 46.31(a) is at the edge of the beam, the hatched area of the beam's cross section is zero. However, if the thickness of the flange is constant, and cut *c-c* is made progressively closer to the web, this area increases from zero at a linear rate. Moreover, as y remains constant for any such area, Q also increases linearly from zero toward the web. Therefore, since V and I are constant at any section through the beam, shear flow $q_c = VQ/I$ follows the same variation. If the thickness of the flange remains the same, the shear stress $\tau_c = VQ/It$ varies similarly. The same variation of q_c and τ_c applies on both sides of the axis of symmetry of the cross section. However, as may be seen from Fig. 46.31(b), these quantities in the plane of the cross section act in opposite directions on the two sides. The variation of these shear stresses or shear flows is represented in Fig. 46.31(c), where for simplicity, it is assumed that the web has zero thickness.

In common with all stresses, the shear stresses shown in Fig. 46.31(c), when integrated over the area on which they act, are equivalent to a force. The magnitude of the horizontal force F_1 for one half of the flange, Fig. 46.31(d), is equal to the average shear stress multiplied by one half of the whole area of the flange, i.e.,

$$F_1 = \left(\frac{\tau_{c-\max}}{2}\right)\left(\frac{bt}{2}\right) \quad \text{or} \quad F_1 = \left(\frac{q_{c-\max}}{2}\right)\left(\frac{b}{2}\right) \tag{46.65}$$

If an I beam transmits a vertical shear, these horizontal forces act in the upper and lower flanges. However, because of the symmetry of the cross section, these equal forces occur in pairs and oppose each other, and cause no apparent external effect.

To determine the shear flow at the juncture of the flange and the web, cut *a-a* in Fig. 46.31(a), the whole area of the flange times y must be used in computing the value of Q. However, since in finding $q_{c\text{-max}}$, one half the flange area times the same y has already been used, the sum of the two horizontal shear flows coming in from opposite sides gives the vertical shear flow at cut *a-a*. Hence, figuratively speaking, the horizontal shear flows "turn through 90° and merge to become the vertical shear flow." Thus, the shear flows at the various horizontal cuts through the web may be determined in the manner explained in the preceding sections. Moreover, the resistance to the vertical shear V in thin-walled I beams is developed mainly in the web, as shown in Fig. 46.31(d). The sense of the shear stresses and shear flows in the web coincides with the direction of the shear V. Note that the vertical shear flow "splits" upon reaching the lower flange. This is represented in Fig. 46.31(d) by the two forces F_1 that are the result of the horizontal shear flows in the flanges.

The shear forces that act at a section of an I beam are shown in Fig. 46.31(d), and for equilibrium, the applied vertical forces must act through the centroid of the cross-sectional area to be coincident with V. If the forces are so applied, no torsion of the member will occur. This is true for all sections having cross-sectional areas with an axis of symmetry. To avoid torsion of such members, the applied forces must act in the plane of symmetry of the cross section and the axis of the beam. A beam with an unsymmetrical section will be discussed next.

Shear Center

The channel section shown in Fig. 46.32 does not have a vertical axis of symmetry. Thus with bending around the horizontal axis, there is a tendency for the channel to twist around some longitudinal axis. To prevent twisting and thus maintain the applicability of the flexure formula, the externally applied force P shown in Fig. 46.32(c) must be applied in such a manner as to balance the internal couple F_1h. This location is called the *shear center* or *center of twist* and is designated by the letter S. The shear center for any cross section lies on a longitudinal line parallel to the axis of the beam. Any transverse force

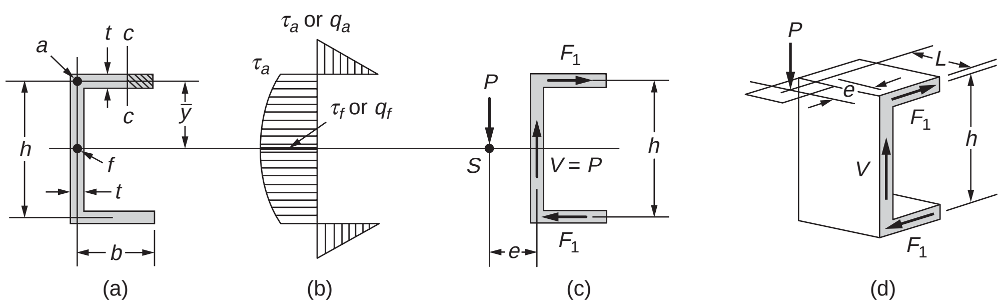


FIGURE 46.32 Deriving location of shear center for a channel.

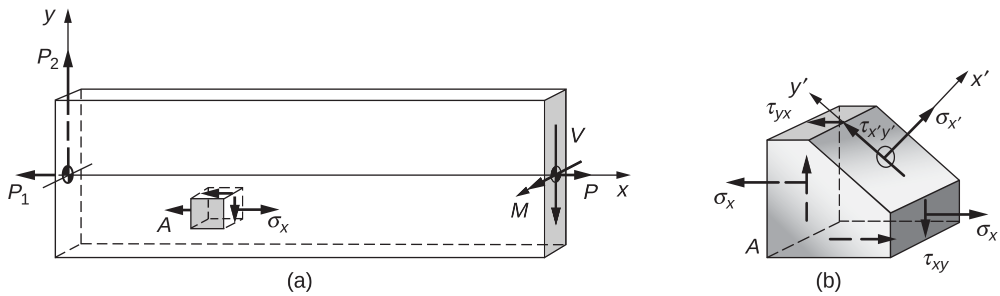


FIGURE 46.33 State of stress at a point on different planes.

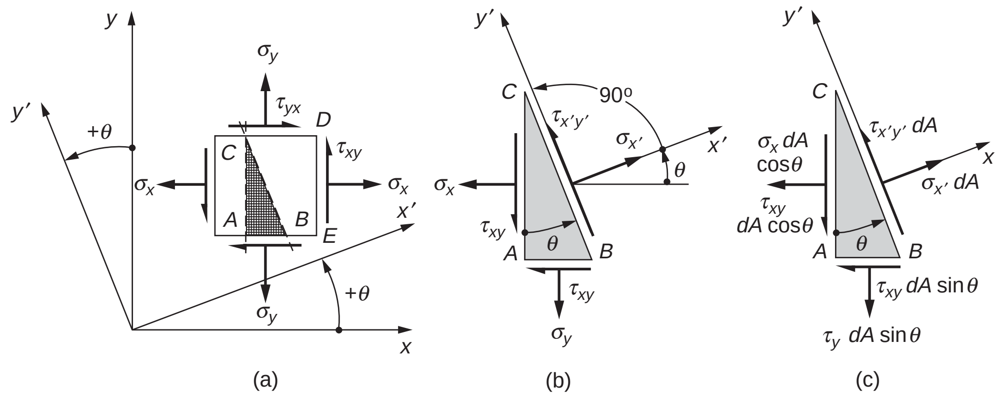


FIGURE 46.34 Derivation of stress transformation on an inclined plane.

applied through the shear center causes no torsion of the beam. For a channel section, the shear center location measured by e from the center of the web is equal to $b^2h^2t/4I$. For a symmetrical angle, the shear center is located at the intersection of the centerlines of its legs.

46.8 Transformation of Stress and Strain

Transformation of Stress

In stress analysis, a more general problem often arises, as illustrated in Fig. 46.33, in which element A is subjected to a normal stress σ_x, due to axial pull and bending, and simultaneously experiences a direct shear stress τ_{xy}. The combined normal stress with the shear stress requires a consideration of stresses on an inclined plane, such as shown by Fig. 46.33(b). Since an inclined plane may be chosen arbitrarily, the state of stress at a point can be described in an infinite number of ways, all of which are equivalent. The planes on which the normal or shear stresses reach their maximum intensity have a particularly significant effect on materials.

An infinitesimal element of unit thickness, as shown in Fig. 46.34(a), is used to describe the state of two-dimensional stress. To determine stresses on any inclined plane, the fundamental procedure involves isolating a wedge and using the equations of the equilibrium of forces. By multiplying the stresses shown in Fig. 46.34(b) by their respective areas, a diagram with the forces acting on the wedge is constructed, as in Fig. 46.34(c). Then, by applying the equations of static equilibrium to the forces acting on the wedge, the stresses $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$ are obtained:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{46.66}$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_z - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \tag{46.67}$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{46.68}$$

These equations are the equations of transformation of stress from one set of coordinates to another. The sign conventions assumed for positive stress and positive angle are depicted in Fig. 46.34(a).

By adding Eqs. (46.66) and (46.67), $\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$, meaning that the sum of the normal stresses on any two mutually perpendicular planes remain the same, that is, *invariant*, regardless of the angle θ.

Principal Stresses

Interest often centers on the determination of the largest possible stress, as given by Eqs. (46.66) to (46.68), and the planes on which such stresses occur. To find the plane for a maximum or a minimum normal stress, Eq. (46.66) is differentiated with respect to θ and the derivative is set equal to zero. Hence,

$$\tan 2\theta_1 = \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2} \tag{46.69}$$

where the subscript of the angle is used to designate the angle that defines the plane of the maximum or minimum normal stress. If the location of planes on which no shear stresses act is wanted, Eq. (46.68) must be set equal to zero. This yields the same relation as that in Eq. (46.69). Therefore, an important conclusion is reached: on planes on which maximum or minimum normal stresses occur there are no shear stresses. These planes are called the *principal planes* of stress, and the stresses acting on these planes — the maximum and minimum normal stresses — are called the *principal stresses*.

Equation (46.69) has two roots that are 90° apart. One of these roots locates a plane on which the maximum normal stress acts; the other root locates the corresponding plane for the minimum normal stress. The magnitude of the principal stresses is obtained by substituting the values of the double angle given by Eq. (46.69) into Eq. (46.66). The expression for the maximum normal stress (denoted by σ_1) and the minimum normal stress (denoted by II_σ) becomes

$$\sigma_{1\text{ or }2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{46.70}$$

where the positive sign in front of the square root must be used to obtain σ_1 and the negative sign to obtain σ_2.

Maximum Shear Stress

Similarly, to locate the planes on which the maximum or the minimum shear stresses act, Eq. (46.68) must be differentiated with respect to and the derivative set equal to zero. Hence,

$$\tan 2\theta_2 = -\frac{\left(\sigma_x - \sigma_y\right)/2}{\tau_{xy}} \tag{46.71}$$

where the subscript 2 designates the plane on which the shear stress is a maximum or minimum. Again, the two planes defined by this equation are mutually perpendicular. Moreover, Eq. (46.71) is a negative reciprocal of Eq. (46.69). This means that the angles that locate the planes of maximum or minimum shear stress form angles of 45° with the planes of principal stresses. A substitution of the results of Eq. (46.71) into Eq. (46.68) gives the maximum and the minimum values of the shear stresses:

$$\tau_{\text{max or min}} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{46.72}$$

Thus, the maximum shear stress differs from the minimum shear stress only in sign. From the physical point of view, these signs have no meaning, and for this reason, the largest shear stress regardless of sign will often be called the maximum shear stress. By substituting θ_2 into Eq. (46.66), the normal stresses σ' that act on the planes of the maximum shear stresses are

$$\sigma' = \frac{\sigma_x + \sigma_y}{2} \tag{46.73}$$

If σ_x and σ_y in Eq. (46.73) are the principal stresses, τ_{xy} is zero and Eq. (46.72) simplifies to

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} \tag{46.74}$$

For pure shear, with the absence of normal stresses, the principal stresses are numerically equal to the shear stress, as displayed by Fig. 46.35.

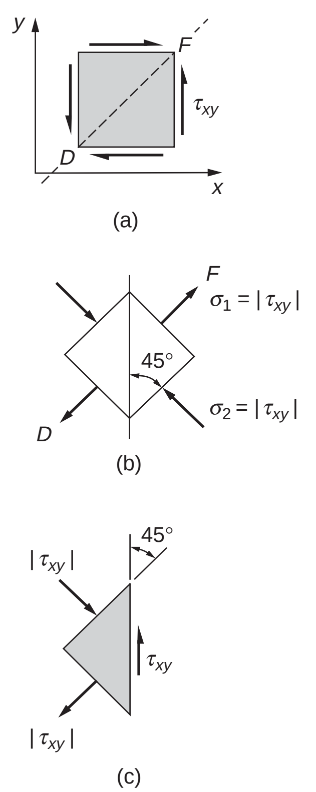


FIGURE 46.35 Equivalent representations for pure shear stress.

Mohr's Circle of Stress

The equations of stress transformation given by Eqs. (46.66), (46.67), and (46.68) may be presented graphically. They can be shown to represent a circle written in parametric form,

$$(\sigma_{x'} - a)^2 + \tau_{x'y'}^2 = b^2 \tag{46.75}$$

where the center of the circle C is at $(a, 0)$ and the circle radius is equal to b:

$$a = \frac{\sigma_x + \sigma_y}{2} \tag{46.76}$$

$$b = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{46.77}$$

In a given problem, σ_x, σ_y, and τ_{xy} are the three known stresses of the element. Hence, if a circle satisfying Eq. (46.75) is plotted, as shown in Fig. 46.36, the simultaneous values of a point (x, y) on this circle correspond to $\sigma_{x'}$ and $\tau_{x'y'}$ for a particular orientation of an inclined plane. The circle so constructed is called the *circle of stress* or *Mohr's circle of stress*.

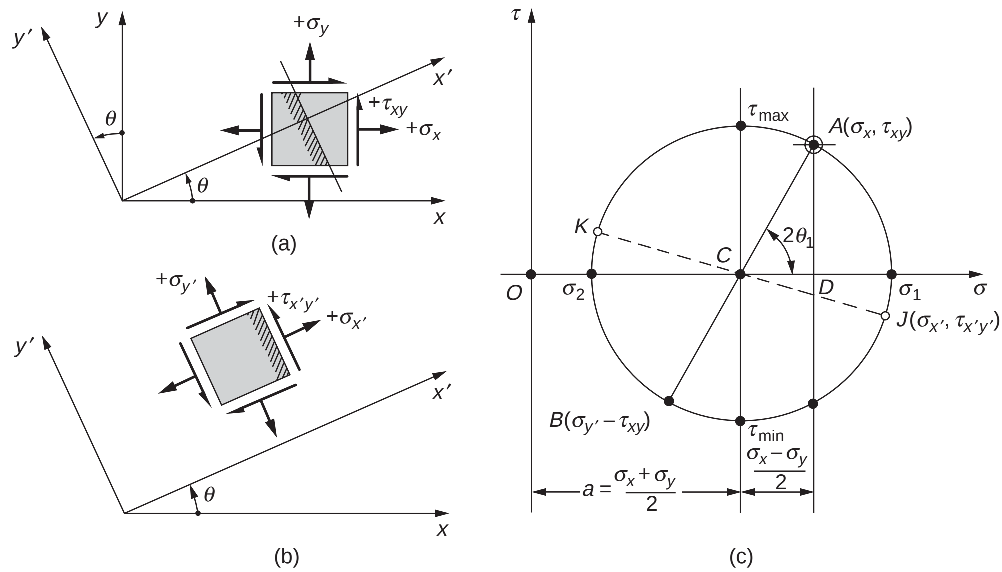


FIGURE 46.36 Mohr's circle of stress.

The coordinates for point A on the circle correspond to the stresses in Fig. 46.36(a) on the right face of the element. The coordinates for the conjugate point B correspond to the stresses on the upper face of the element. For any other orientation θ of an element, such as shown in Fig. 46.36(b), a pair of conjugate points J and K can always be found on the circle to give the corresponding stresses, as in Fig. 46.36(c). This may be easily accomplished by rotating the AB axis by a corresponding 2θ. The following important observations regarding the state of stress at a point can be made based on the Mohr's circle:

1. The largest possible normal stress is σ_1, and the smallest is σ_2. No shear stresses exist with these principal stresses.
2. The largest shear stress τ_{max} is numerically equal to the radius of the circle. A normal stress equal to $(\sigma_1 + \sigma_2)/2$ acts on each of the planes of maximum shear stress.
3. If $\sigma_1 = \sigma_2$, Mohr's circle degenerates into a point, and no shear stresses develop in the xy plane.
4. If $\sigma_x + \sigma_y = 0$, the center of Mohr's circle coincides with the origin of the $\sigma - t$ coordinates, and the state of pure shear exists.
5. The sum of the normal stresses on any two mutually perpendicular planes is invariant, that is, $\sigma_x + \sigma_y = \sigma_1 + \sigma_2 = \sigma_{x'} + \sigma_{y'} =$ constant.

Principal Stresses for a General State of Stress

Consider a general state of stress and define an infinitesimal tetrahedron, as shown in Fig. 46.37. The unknown stresses are sought on an arbitrary oblique plane ABC in the three-dimensional xyz coordinate system. A set of known stresses on the other three faces of the mutually perpendicular planes of the tetrahedron is given. These stresses are the same as those shown earlier in Fig. 46.2. A unit normal n to the oblique plane defines its orientation. This unit vector is identified by its direction cosines l, m, and n, as illustrated by Fig. 46.37(b). Further, if the infinitesimal area ABC is defined as dA, then the three areas of the tetrahedron along the coordinate axes are $dA\ l$, $dA\ m$, and $dA\ n$. Force equilibrium for the tetrahedron can now be written by multiplying the stresses given in Fig. 46.37 by the respective areas established. It will be assumed that only a normal stress σ_n (i.e., a principal stress) is acting on face ABC; then a system of linear homogeneous equations is obtained that has nontrivial solution if and only if the determinant of the coefficients of l, m, and n vanishes. Expansion of this determinant gives

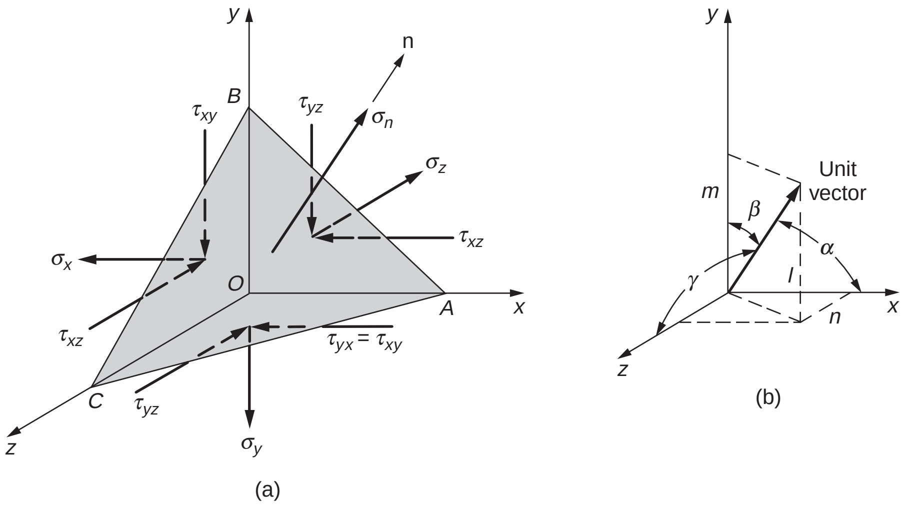


FIGURE 46.37 Tetrahedron for deriving a principal stress on an oblique plane.

$$\sigma_n^3 - I_\sigma \sigma_n^2 + II_\sigma \sigma_n - III_\sigma = 0 \tag{46.78}$$

where

$$I_\sigma = \sigma_x + \sigma_y + \sigma_z$$

$$II_\sigma = \left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right) - \left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)$$

$$III_\sigma = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{xz} - \sigma_x\tau_{yz}^2 + \sigma_y\tau_{xz}^2 + \sigma_z\tau_{xy}^2$$

The constants I_σ, II_σ, and III_σ are invariant, since if the initial coordinate system is changed, thereby changing the three mutually perpendicular planes of the tetrahedron, the σ_n on the inclined plane must remain the same. In general, Eq. (46.78) has three real roots. These roots are the eigenvalues of the determinant and are the principal stresses of the problem.

Transformation of Strain

The transformation of normal and shear strain from one set of rotated axes to another (Fig. 46.38) is completely analogous to the transformation of normal and shear stresses presented earlier.

Fundamentally, this is because both stresses and strains are second-rank tensors and mathematically obey the same laws of transformation. One may then obtain equations of strain transformation from the equations of stress transformation by simply substituting the normal stress σ with the normal strain ε and the shear stress τ with shear strain γ. Hence, the basic expressions for strain transformation in a plane in an arbitrary direction defined by the x' axis are

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \tag{46.79}$$

$$\gamma_{x'y'} = -\left(\varepsilon_x - \varepsilon_y\right)\sin 2\theta + \gamma_{xy}\cos 2\theta \tag{46.80}$$

The sign convention adopted corresponds to the element distortions shown in Fig. 46.38(a) for positive strain. Likewise, Mohr's circle of strain can be constructed where every point on the circle gives two values: one for the normal strain, and the other for the shear strain divided by 2.

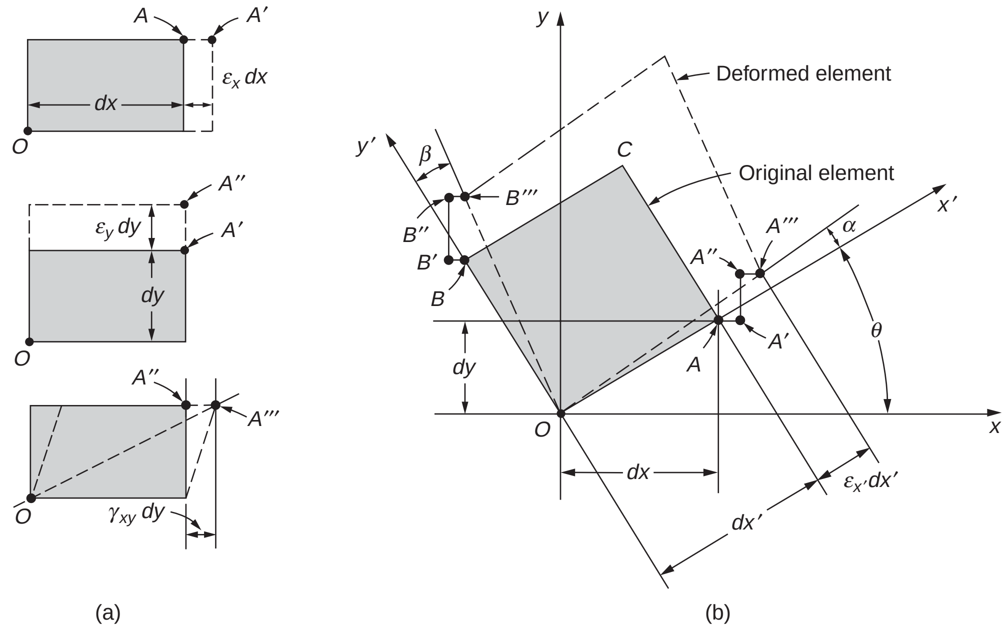


FIGURE 46.38 Exaggerated deformations of elements for deriving strains along new axes.

Yield and Fracture Criteria

Thus far the idealized mathematical procedures for determining the states of stress and strain, as well as their transformation to different coordinates, have been presented. However, it must be pointed out that the precise response of real materials to such stresses and strains defies accurate formulations. As yet, no comprehensive theory can provide accurate predictions of material behavior under the multitude of static, dynamic, impact, and cyclic loading, as well as temperature effects. Only the classical idealizations of yield and fracture criteria for materials are discussed in this section.

The *maximum shear-stress theory*, or simply the *maximum shear theory*, results from the observation that in a ductile material slip occurs during yielding along critically oriented planes. This suggests that the maximum shear stress plays the key role, and it is assumed that yielding of the material depends only on the maximum shear stress that is attained within an element. Therefore, whenever a certain critical value σ_n is reached, yielding in an element commences. For a given material, this value usually is set equal to the shear stress at yield σ_{yp} in simple tension or compression. Hence, according to Eq. (46.72), $t_{\max} = t_{cr} = \sigma_{yp}/2$. In applying this criterion to a biaxial plane stress problem, two different cases arise. In one case, if the signs of the principal stresses σ_1 and σ_2 are the same, the maximum shear stress is of the same magnitude, as would occur in a simple uniaxial stress. Therefore, the criteria corresponding to this case are

$$|\sigma_1| \le \sigma_{yp} \quad \text{and} \quad |\sigma_2| \le \sigma_{yp} \tag{46.81}$$

In the second case, the signs of σ_1 and σ_2 are opposite, and the maximum shear stress $t_{\max} = (|\sigma_1| + |\sigma_2|)/2$. Hence,

$$\left| \pm \frac{\sigma_1 - \sigma_2}{2} \right| \le \frac{\sigma_{yp}}{2} \tag{46.82}$$

A plot of Eqs. (46.81) and (46.82) is shown in Fig. 46.39. If a point defined by σ_2/σ_{yp} and σ_1/σ_{yp} falls on the hexagon shown, a material begins and continues to yield. No such stress points can lie outside

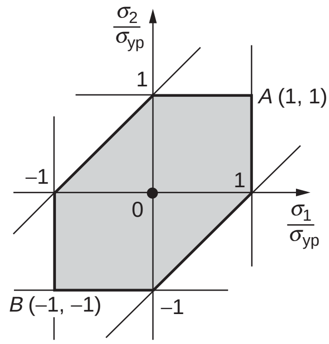


FIGURE 46.39 Yield criterion based on maximum shear stress.

the hexagon because one of the three yield criteria equations would be violated. The stress points falling within the hexagon indicate that a material behaves elastically. The derived yield criterion for perfectly plastic material is often referred to as the *Tresca yield condition* and is one of the widely used laws of plasticity.

Another widely accepted criterion of yielding for ductile isotropic materials is based on energy concepts. In this approach, the total elastic energy is divided into two parts: one associated with the volumetric changes of the material, and the other causing shear distortions. By equating the shear distortion energy at yield point in simple tension to that under combined stress, the yield criterion for combined stress is established.

Based on the concept of superposition, it is possible to consider the stress tensor of the three principal stresses — σ_1, σ_2, and σ_3 — to consist of two additive component tensors:

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \bar{\sigma} & 0 & 0 \\ 0 & \bar{\sigma} & 0 \\ 0 & 0 & \bar{\sigma} \end{bmatrix} + \begin{bmatrix} \sigma_1 - \bar{\sigma} & 0 & 0 \\ 0 & \sigma_2 - \bar{\sigma} & 0 \\ 0 & 0 & \sigma_3 - \bar{\sigma} \end{bmatrix} \tag{46.83}$$

where $\bar{\sigma} = (\sigma_1 + \sigma_2 + \sigma_3)/3$ is the "hydrostatic stress." The stresses associated with the first tensor component, which is called the *spherical stress tensor*, or alternatively the *dilatational stress tensor*, are the same in every possible direction. The second tensor component causes no volumetric changes, but instead distorts or deviates the element from its initial cubic shape. It is called the *deviatoric* or *distortional stress tensor.*

Extending the results from the section "Elastic Strain Energy for Uniaxial Stress," generalizing for three dimensions and expressing in terms of the principal stresses, the total strain energy per unit volume (i.e., strain density) can be found:

$$U_{total} = \frac{1}{2E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right) - \frac{\nu}{E}\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) \tag{46.84}$$

The strain energy per unit volume due to the dilatational stress can be determined from this equation and expressed in terms of the principal stress. Thus,

$$U_{dilatation} = \frac{1 - 2\nu}{6E}\left(\sigma_1 + \sigma_2 + \sigma_3\right)^2 \tag{46.85}$$

By subtracting Eq. (46.85) from Eq. (46.84), simplifying and noting that $G = E/2(1 + \nu)$, one finds the distortion strain energy for combined stress:

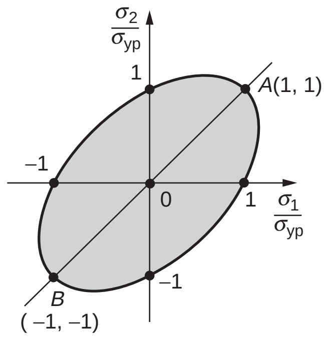


FIGURE 46.40 Yield criterion based on maximum distortion energy.

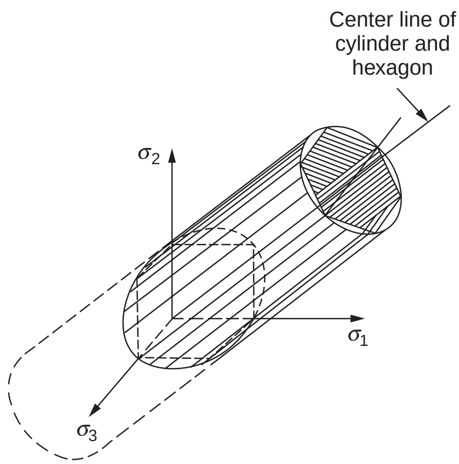


FIGURE 46.41 Yield surfaces for triaxial state of stress.

$$U_{distortion} = \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \tag{46.86}$$

According to the basic assumption of the distortion-energy theory, the expression of Eq. (46.86) must be equal to the maximum elastic distortion energy in simple tension, which is equal to $2\sigma_{yp}^2/12G$. After minor simplifications, one obtains the basic law for yielding of an ideally plastic material:

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 = 2\sigma_{yp}^2 \tag{46.87}$$

The fundamental relation given by Eq. (46.87) is widely used for perfectly plastic material and is often referred to as the *Huber–Hencky–Mises*, or simply the *von Mises yield condition*.

For plane stress, $\sigma_3 = 0$, Eq. (46.87) is an equation of an ellipse, a plot of which is shown in Fig. 46.40. Any stress falling within the ellipse indicates that the material behaves elastically. Points on the ellipse indicate that the material is yielding.

In three-dimensional stress space, the yield surface becomes a cylinder with an axis having all three direction cosines equal to $1/\sqrt{3}$. Such a cylinder is shown in Fig. 46.41. The ellipse in Fig. 46.40 is simply the intersection of this cylinder with the $\sigma_1 - \sigma_2$ plane. It can be shown that the yield surface for the maximum shear stress criterion is a hexagon that fits into the tube, Fig. 46.41(b).

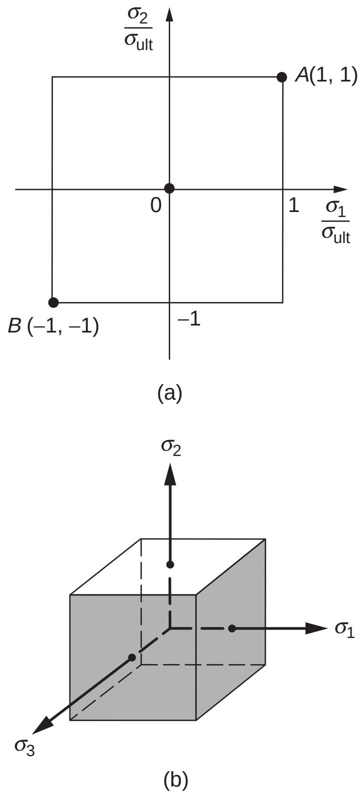


FIGURE 46.42 Fracture envelope based on maximum stress criterion.

The *maximum normal stress theory* or simply the *maximum stress theory* asserts that failure or fracture of a material occurs when the maximum normal stress at a point reaches a critical value regardless of the other stresses. Only the largest principal stress must be determined to apply this criterion. The critical value of stress $\sigma_{u/t}$ is usually determined in a tensile experiment. Experimental evidence indicates that this theory applies well to brittle materials in all ranges of stresses, providing that a tensile principal stress exists. Failure is characterized by separation fracture, or cleavage. The maximum stress theory can be interpreted on graphs, as shown in Fig. 46.42. Unlike the previous theories, the stress criterion gives a bounded surface of the stress space.

46.9 Stability of Equilibrium: Columns

Governing Differential Equation for Deflection

Figure 46.43 shows a deflected beam segment with point A' directly above its initial position A by a displacement v_A. The tangent to the elastic curve at the same point and a plane section with the centroid at A' rotate through an angle dv/dx. Therefore, assuming small angles, the displacement u of a material point at a distance y from the elastic curve is

$$u = -y\frac{dv}{dx} \tag{46.88}$$

Next, recall Eq. (46.22), which states that $\varepsilon_x = du/dx$. Therefore, from Eq. (46.88), $\varepsilon_x = -yd^2v/dx^2$. The same normal strain also can be found from Eqs. (46.26) and (46.48), yielding $\varepsilon_x = -My/EI$. On equating

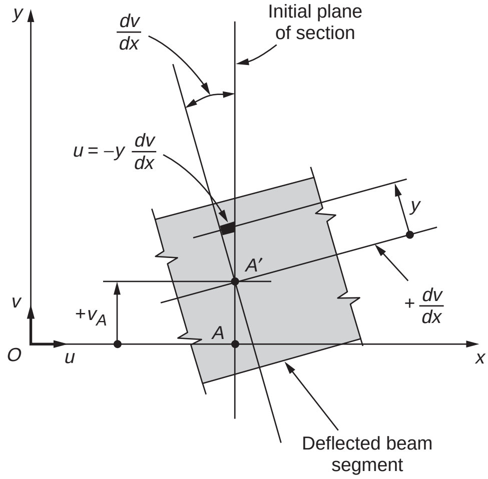


FIGURE 46.43 Longitudinal displacement in a beam due to rotation of a plane section.

the two alternative expressions for ε_x, one obtains the governing differential equation for small deflections of elastic beams as

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \tag{46.89}$$

Buckling Theory for Columns

The procedures of stress and deformation analysis in a state of stable equilibrium were discussed in the preceding sections. But not all structural systems are necessarily stable. The phenomenon of structural instability occurs in numerous situations where compressive stresses are present. The consideration of material strength alone is not sufficient to predict the behavior of such members. Stability considerations are primary in some structural systems.

Consider the ideal perfectly straight column with pinned supports at both ends; see Fig. 46.44. The least force at which a buckled mode is possible is the *critical* or *Euler buckling load*. In a general case

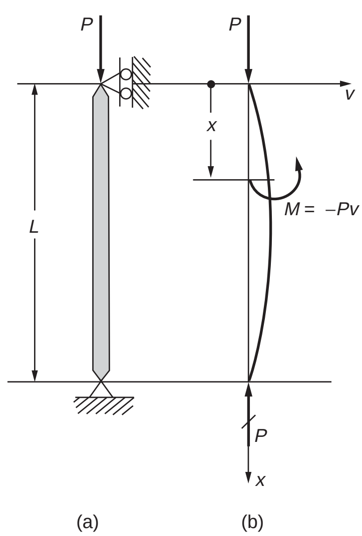


FIGURE 46.44 Column pinned at both ends.

where a compression member does not possess equal flexural rigidity in all directions, the significant flexural rigidity EI of a column depends on the minimum I, and at the critical load a column buckles either to one side or the other in the plane of the major axis.

In order to determine the critical load for this column, the compressed column is displaced as shown in Fig. 46.44. In this position, the bending moment is $-Pv$. By substituting this value of moment into Eq. (46.89), the differential equation for the elastic curve for the initially straight column becomes

$$\frac{d^2\nu}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}\nu \tag{46.90}$$

Letting $\lambda^2 P/EI$ and transposing

$$\frac{d^2\nu}{dx^2} + \lambda^2\nu = 0 \tag{46.91}$$

This is an equation of the same form as the one for simple harmonic motion, and its solution is

$$v = A\sin\lambda x + Bcox\lambda x \tag{46.92}$$

where A and B are arbitrary constants that must be determined from the boundary conditions.

For Fig. 46.44 these conditions are $v\ (0) = 0$ and $v\ (L) = 0$. Hence, $B = 0$ and

$$A\sin\lambda L = 0 \tag{46.93}$$

This equation can be satisfied by taking $A = 0$. However, with A and B each equal to zero, this is a solution for a straight column, and is usually referred to as a trivial solution. An alternative solution is obtained by requiring the sine term in Eq. (46.92) to vanish. This occurs when λL equals $n\pi$, where n is an integer. Therefore, since λ was defined as $\sqrt{P/(EI)}$, the nth critical P_n that makes the deflected shape of the column possible follows from solving $\sqrt{P/(EI)}l = n\pi$. Hence,

$$P_n = \frac{n^2\pi^2 EI}{L^2} \tag{46.94}$$

which are the eigenvalues for this problem. However, since in stability problems only the least value of P_n is of importance, n must be taken as unity, and the critical or Euler load P_{cr} for an initially perfectly straight elastic column with pinned ends becomes

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{46.95}$$

where E is the elastic modulus of the material, I is the least moment of inertia of the constant cross-sectional area of a column, and L is its length. This case of a column pinned at both ends is often referred to as the fundamental case.

According to Eq. (46.92), at the critical load, since $B = 0$, the equation of the buckled elastic curve is

$$v = A\sin\lambda x \tag{46.96}$$

This is the characteristic, or eigenfunction, of this problem, and since $\lambda = np/L$, n can assume any integer value. There are an infinite number of such functions. In this linearized solution, amplitude A of the buckling mode remains indeterminate. For the fundamental case $n = 1$, the elastic curve is a half-wave sine curve. This shape and the modes corresponding to $n = 2$ and 3 are shown in Fig. 46.45. The higher modes have no physical significance in buckling problems, since the least critical buckling load occurs at $n = 1$.

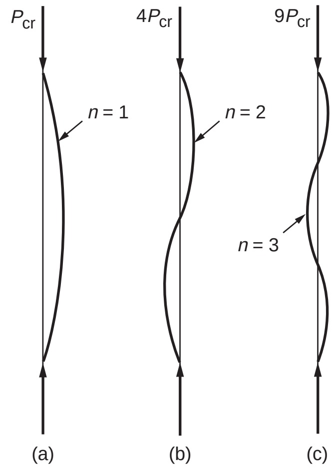


FIGURE 46.45 First three buck-lining modes for a column pinned at both ends.

It should be noted that the elastic modulus E was used for the derivation of the Euler formulas for columns; therefore, the formulas are applicable while the material behavior remains linearly elastic. To bring out this significant limitation, Eq. (46.94) is rewritten in a different form: by definition, $I = Ar^2$, where A is the cross-sectional area and r is its *radius of gyration*. Substitution of this relation into Eq. (46.94) gives

$$\sigma_{cr} = \frac{P'_{cr}}{A} = \frac{\pi^2 E}{(L/r)^2} \tag{46.97}$$

where the critical stress σ_{cr} for a column is the average stress over the cross-sectional area A of a column at the critical load P_{cr}. The ratio L/r of the column length to the least radius of gyration is called the column *slenderness ratio*. Note that σ_{cr} always decreases with increasing ratios of L/r. Since Eq. (46.97) is based on elastic behavior, σ_{cr} determined by this equation cannot exceed the proportional limit.

Euler Loads for Columns with Different End Restraints

The same procedure as that discussed before can be used to determine the critical axial loads for columns with different boundary conditions. The solutions of buckling problems are very sensitive to the end restraints. Some of these different cases of end restraint combinations are shown in Fig. 46.46. It can be shown that the buckling formulas for all these cases can be made to resemble the fundamental case, Eq. (46.95), provided that the effective column lengths are used, instead of the actual column length. The effective column length L_e for the fundamental case is L, but for a free-standing column it is $2L$. For a column fixed at both ends, the effective length is $L/2$, and for a column fixed at one end and pinned at the other, it is $0.7L$. For a general case, $L_e = KL$, where K is the *effective length factor*, which depends on the end restraints.

Generalized Euler Buckling Load Formulas

Beyond the proportional limit, it may be said that a column of different material has been created, since the stiffness of the material is no longer represented by the elastic modulus. At this point the material stiffness is given instantaneously by the tangent to the stress–strain curve, that is, by the tangent modulus E_t; see the section on constitutive relations. The column remains stable if its new flexural rigidity E_tI is sufficiently large and it can carry a higher load. Substitution of the tangent modulus E_t for the elastic

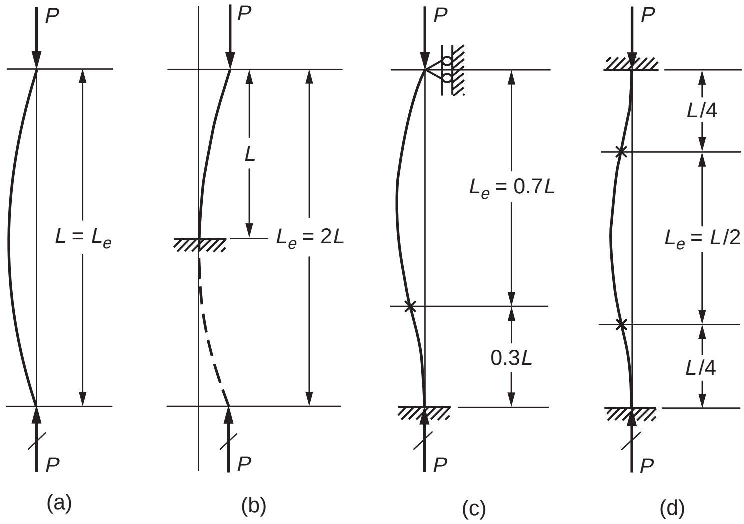


FIGURE 46.46 Effective lengths of columns with different restraints.

modulus E is the only modification necessary to make the elastic buckling formulas applicable in the inelastic range. Hence, the *generalized Euler buckling load formula*, or the *tangent modulus formula*, becomes

$$\sigma_{cr} = \frac{\pi^2 E_t}{(L/r)^2} \tag{46.98}$$

Eccentric Loads and the Secant Formula

Since no column is perfectly straight and the applied forces are not perfectly concentric, the behavior of real columns may be studied with some statistically determined imperfections or possible misalignments of the applied loads. To analyze the behavior of an eccentrically loaded column, consider the column shown in Fig. 46.47. The differential equation for the elastic curve is the same as that for a concentrically

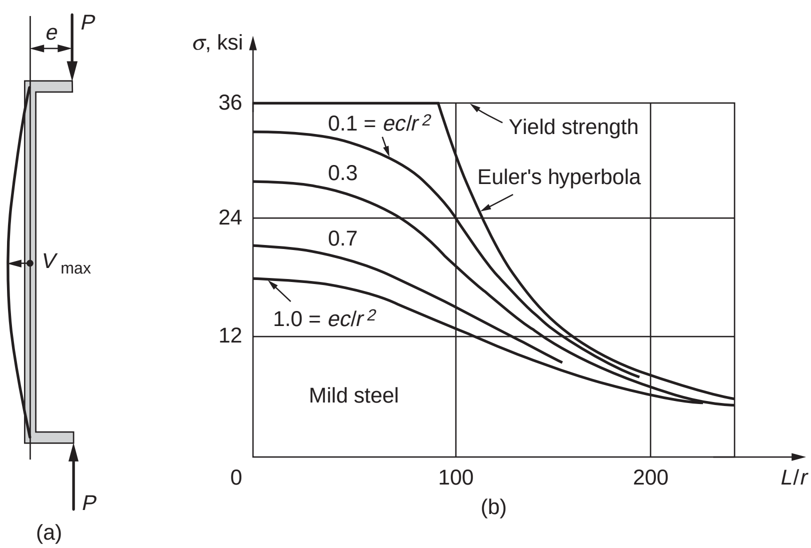


FIGURE 46.47 Results of analyses for different columns by the secant formula.

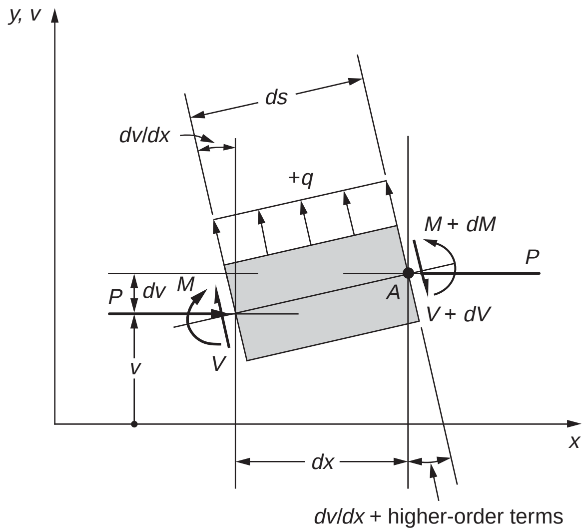


FIGURE 46.48 Beam-column element.

loaded column, derived earlier. However, the boundary conditions are now different. At the upper end, v is equal to the eccentricity e of the applied load. And given that the elastic curve has a vertical tangent at the midheight of the column, the equation for the elastic curve is obtained:

$$v = e\left(\frac{\sin \lambda L/2}{\cos \lambda L/2}\sin \lambda x + \cos \lambda x\right) \tag{46.99}$$

No indeterminacy of any constants appears in this equation, and the maximum deflection v_{max} can be found from it. This maximum occurs at $L/2$. Hence, $v_{max} = e \sec(\lambda L/2)$.

For the column shown in Fig. 46.47, the largest bending moment M is developed at the point of maximum deflection and numerically is equal to Pv_{max}. Therefore, the maximum compressive stress occurring in the column can be computed by superposition of the axial and bending stresses, that is, $(P/A) + (Mc/I)$, to give

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{ec}{r^2}\sec\frac{L}{r}\sqrt{\frac{P}{4EA}}\right) \tag{46.100}$$

This equation, because of the secant term, is known as the *secant formula* for columns. It applies to columns of any length, provided the maximum stress does not exceed the elastic limit.

Note that in Eq. (46.100), the relation between σ_{max} and P is not linear; σ_{max} increases faster than P. Therefore, the solutions for maximum stresses in columns caused by different axial forces cannot be superposed; instead, the forces must be superposed first, and then the stresses can be calculated. A plot of Eq. (46.100) is shown in Fig. 46.47. Note the large effect load eccentricity has on short columns and the negligible one it has on very slender columns. A condition of equal eccentricities of the applied forces in the same direction causes the largest deflection.

Differential Equations for Beam-Columns

As was the case in the preceding section, a member acted upon simultaneously by an axial force and transverse forces or moments causing bending is referred to as a *beam-column*. To obtain the governing differential equation, consider the beam-column element shown in Fig. 46.48. Applying the equilibrium equations and small deflection approximations, one obtains two equations:

$$\frac{dV}{dx} = q \qquad (46.101)$$

and

$$V = \frac{dM}{dx} + P\frac{dv}{dx} \qquad (46.102)$$

Therefore, the shear V, in addition to depending on the rate of change of moment M, as in beams, now also depends on the magnitude of the axial force and the slope of the elastic curve.

On substituting Eq. (46.102) into Eq. (46.101) and using the beam curvature–moment relation $d^2v/dx^2 = M/EI$, one obtains the two alternative governing differential equations for beam-columns:

$$\frac{d^2M}{dx^2} + \lambda^2 M = q \qquad (46.103)$$

or

$$\frac{d^4v}{dx^4} + \lambda^2\frac{d^2v}{dx^2} = \frac{q}{EI} \qquad (46.104)$$

where, as before, $\lambda^2 = P/EI$.

Defining Terms

Constitutive relation — The relationship between stress and strain.

Elastic strain energy — The internal work (product of force and deformation) stored in an elastic body.

Elasticity — When unloaded, material responds by returning back along the loading path to its initial stress-free state of deformation.

Euler or **critical buckling load —** The least force at which a buckled mode is possible for an elastic material.

Hooke's law — The idealization of stress as being directly proportional to strain, that is, the stress–strain relationship is a straight line. The constant of proportionality E is called the elastic modulus, modulus of elasticity, or Young's modulus.

Inelasticity or **plasticity —** Material response characterized by a stress–strain diagram that is nonlinear and that retains a permanent strain on complete unloading.

Mechanics of materials — A branch of applied mechanics that deals with the behavior of various load-carrying members. It involves analytical methods for determining strength, stiffness, and stability.

Mohr's circle of stress — A graphical representation of stress transformation.

Poisson's ratio — For a body subjected to axial tension, the constant ratio of lateral strain to axial strain.

Principal stresses — For any general state of stress there exist three mutually perpendicular planes on which the shear stresses vanish. The normal stress components remaining are called principal stresses.

Stability — Stability refers to the ability of a load-carrying member to resist buckling under compressive loads.

Strain — Deformation per unit length. Normal or extensional strain is the change in length per unit of initial gage length. Shear strain is the change in initial right angle between any two imaginary planes in a body.

Stress concentration — Large stresses due to discontinuities that develop in a small portion of a member.

Stress — Intensity of force per unit area. Normal stress is the intensity of force perpendicular to or normal to the section at a point. Shear stress is the intensity of force parallel to the plane of the elementary area.

Transformation of stress and strain — The mathematical process for changing the components of the state of stress or strain given in one set of coordinate axes to any other set of rotated axes.

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Further Information

For further information the reader may consult *Engineering Mechanics of Solids* by Egor P. Popov, which served as the main source of information for this section. Permission from Prentice-Hall to use figures from that textbook is gratefully acknowledged.

For a more advanced treatment of the subject matter, consult the series of books written by S.P. Timoshenko and his coauthors: *Theory of Elasticity, Theory of Elastic Stability, Theory of Plates and Shells,* and *Strength of Materials*. Other good sources include *Theory of Flow and Fracture of Solids* by A. Nadai, *Introduction to the Mechanics of Continuous Medium* by L.E. Malven, *Mathematical Theory of Elasticity* by A.E.H. Love, and *Advanced Mechanics of Materials* by A.P. Boresi, R.J. Schmidt, and O.M. Sidebottom. *Mechanics of Elastic Structures* by J.T. Oden and E.A. Ripperger includes a good presentation of the problem of torsion and warpage. Plasticity is covered in depth in *Plasticity for Structural Engineers* by W.F. Chen and D.J. Han. For further information regarding the mechanical characteristics of materials, see *Materials Science and Engineering* by W.D. Callister, *Introduction to Material Science for Engineers* by J.F. Shackelford, and *Material Science for Engineers* by L.H. Van Vlack. On the finite element method, refer to *The Finite Element Method* by O.C. Zienkiewicz and R.L. Taylor and *Finite Element Fundamentals* by R.H. Gallagher. Two early books on the topic of stability are *Buckling Strength of Metal Structures* by F. Bleich and *Principles of Structural Stability* by H. Ziegler. A more recent treatment can be found in *Structural Stability* by W.F. Chen and E.M. Lui. For problems in mechanics of interest to physical and biological engineers and scientists, there is *A First Course in Continuum Mechanics* by Y.C. Fung.

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