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Open Channel Hydraulics

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Open channel hydraulics, a subject of great importance to civil engineers, deals with flows having a free surface in channels constructed for water supply, irrigation, drainage, and hydroelectric power generation; in sewers, culverts, and tunnels flowing partially full; and in natural streams and rivers. Open channel hydraulics includes steady flows that are unchanging in time, varied flows that have changes in depth and velocity along the channel, and transient flows that are time dependent. This chapter deals only with rigid-boundary channels without sediment deposition or erosion. In addition, this chapter assumes that wind and surface tension stresses exerted on the free surface are negligible, and that velocities are low enough that air is not entrained. The emphasis is on the one-dimensional treatment of uniform and nonuniform flows which are common in civil engineering practice. Design aspects of structures involving free surface flows are discussed in Chapter 35. Sediment transport in open channels is covered in Chapter 33.

30.1 Definitions and Principles

Open channel flow is the flow of a single phase liquid with a free surface in a gravitational field when the effects of surface tension and of the overlying gas can be neglected. Because laminar open channel flows

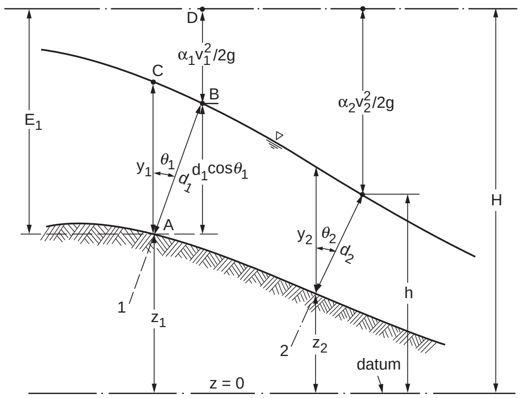

FIGURE 30.1 Definitions: y, depth of stream; d, thickness of stream; z, bottom elevation; θ, angle between channel bottom and horizontal; E, specific energy; h, piezometric head and water surface elevation; H, total head; $\alpha V^2/2g$, velocity head. Subscripts denote flow area.

are seldom encountered in civil engineering practice, only turbulent flows will be considered in this chapter. The analysis of open channel flows is largely based on the approximation that the mean streamlines are nearly parallel. As shown below, this implies that the piezometric head is nearly constant on planes normal to the flow, and allows a one-dimensional analysis. Regions of nonparallel streamlines are considered by using control volume arguments. In some cases, these assumptions are inadequate, and a much more complicated two- or three-dimensional analysis must be used.

For any given cross section, the following terminology and notation are used:

- The region of the cross section occupied by the liquid is called the *flow area*, A.
- The part of the cross section perimeter which is below the water surface is called the *wetted perimeter*, P.
- The length of the free surface is called the *top width*, T. This is normally assumed to be horizontal.
- The hydraulic depth is $D = A/T$.
- The hydraulic radius is $R = A/P$.
- The *water surface elevation*, h, is the vertical distance of the free surface above a reference elevation or datum. In Fig. 30.1, h is the water surface elevation at cross section 2.
- The *invert* is the lowest point of the cross section.
- The vertical distance to the free surface from the lowest point of the cross section is called the *depth of flow*, y, or *depth*. Referring to Fig. 30.1, y_1 is the depth corresponding to the invert at point A.
- The perpendicular distance from the invert to the free surface is called the *thickness of the stream*, d. Referring to Fig. 30.1, d_1 is the thickness of the stream at cross section 1. If the free surface is nearly parallel to the bottom of the channel, $d = y \cos \theta$, where θ is the angle between the bottom of the channel and the horizontal. For the small slopes normally encountered in rivers and canals $d \cong y$. The pressure head on the channel bottom is $y \cos^2\theta = d \cos \theta$. For $\theta < 5°$ the error in approximating the pressure head by y is less than 1%.
- The width of a rectangular channel is its *breadth*, b.

Special importance attaches to *prismatic channels*: those that have a constant cross sectional shape, longitudinal slope, and alignment. The generators of prismatic channels are parallel straight lines. The most common prismatic channel cross sections are trapezoids, rectangles, and partially full circles. Constructed channels often consist of long prismatic reaches connected by short transition sections. Natural channels are never prismatic, although the assumption that they are is sometimes tolerable.

The direction of flow is indicated by the spatial variable x; the two coordinates orthogonal to each other and to x are called y' and z'. For a parallel flow, the total volume of water flowing per unit time across an orthogonal flow area, is the *flowrate* or *discharge*, Q, given by

$$Q(x,t) = \iint_A v(x, y', z', t)dy'dz' = V(x,t)A(x,t) \tag{30.1}$$

where $v(x, y', z', t)$ is the local x-velocity at coordinates x, y', z' and time t.

The integral extends across the whole flow area, and V is the *mean velocity*.

Classification of Flows

Steady flows are time invariant and *unsteady flows* are time dependent. Because open channel flows are typically turbulent, and thus inherently unsteady in detail, these terms are understood to apply to the time-averaged components of the flow variables. *Uniform* or *normal flow* is the important special case of constant thickness flow in a prismatic channel. More common is *gradually varied flow* in which streamwise changes in the flow area are sufficiently gradual that the time-averaged streamlines can be assumed parallel. When the deviation of the time-averaged streamlines from being parallel cannot be neglected, the flow is termed *rapidly varied*. If the flowrate changes along the direction of flow (due to addition or withdrawal of liquid) it is a *spatially varied flow*.

Flow Regimes

Since free surface flows are affected by gravitational, viscous, and surface tension forces, the relevant dimensionless parameters are the Froude number, the Reynolds number, and the Weber number. The most important of these is the Froude number, $Fr = V/c$ where c, the celerity, is the velocity of propagation of a small amplitude, shallow water gravity wave. For an arbitrary cross section $c = (g\ D)^{1/2}$. For a rectangular cross section this reduces to $c = (g\ y)^{1/2}$. The Froude number compares the speed of the liquid to the speed at which small disturbances of the free surface propagate relative to the liquid. When $Fr < 1$, small disturbances can propagate upstream as well as downstream, and the flow regime is called *subcritical, tranquil,* or *streaming*. When $Fr > 1$, small disturbances are too slow to propagate upstream. This regime is called *supercritical, rapid*, or *shooting*. This distinction is of great practical importance because if the flow at a given cross section is supercritical, downstream events cannot influence the flow unless they are large enough to force the flow to change to subcritical. The rare case of $Fr = 1$ is called *critical flow*. The Froude number can also be interpreted as being proportional to the square root of the ratio of the inertial forces to the gravitational forces. Some authors define the Froude number as the square of the present definition.

The Reynolds number may be defined for open channel flow as $Re = 4\rho eRV/\mu$, where ρ is the *mass density* and μ is the *dynamic viscosity* of the liquid. (Many authors omit the factor of 4.) The Reynolds number is proportional to the ratio of inertial forces to viscous forces. For $Re < 2000$, open channel flow is laminar. When Re exceeds about 8000, it is turbulent. At intermediate values the flow is transitional. In hydraulic engineering practice, laminar and transitional flows are rare, occurring mostly in shallow sheet storm runoff from roofs and pavements.

The Weber number for open channel flow is defined as $We = \rho eDV^2/\sigma$, where σ is the surface tension coefficient. The Weber number is a measure of the ratio of inertial forces to surface tension forces. Although threshold values have not been determined, the high values typical of hydraulic engineering applications indicate that surface tension effects may be neglected.

30.2 Balance and Conservation Principles

As shown in Chapter 28, the fundamental principles of nature may be written in a balance form for an arbitrarily specified region called a *control volume*. In this chapter we consider a control volume which contains all of the liquid between an upstream flow area (A_1) and a downstream flow area (A_2). The lateral boundaries coincide with the wetted channel lining and the free surface.

Conservation of Mass

The principle of conservation of mass states that the time rate of change of mass inside a control volume is equal to the balance between the inflowing and outflowing mass through the control surfaces. For liquids of constant density, conservation of mass implies conservation of volume. In the case of steady flow in a control volume which contains all of the liquid between an upstream flow area and a downstream flow area, we have $Q_{in} = Q_{out} = Q$ or

$$Q = A_1V_1 = A_2V_2 \tag{30.2}$$

This is known as the *continuity equation*. For a rectangular channel of constant breadth

$$q = V_1\, d_1 = V_2\, d_2 \tag{30.3}$$

where $q = Q/b$ is the specific flowrate or flowrate per unit breadth.

Conservation of Momentum

The principle of conservation of momentum states that the time rate of change of the momentum inside a control volume is equal to the sum of all the forces acting on the control volume plus the difference between the incoming and outgoing momentum flowrates. Note that it is a vector equation. For steady flow with constant density along a straight channel, the streamwise component of the momentum equation for a control volume that contains all of the liquid between an upstream flow area and a downstream flow area becomes

$$\sum F' = \rho Q\left(\beta_2 V_2 - \beta_1 V_1\right) \tag{30.4}$$

where $\Sigma F'$ is the sum of the streamwise forces on the control volume.

These forces typically include the hydrostatic pressure forces on the flow areas, the streamwise component of the weight of liquid within the control volume, and the streamwise force exerted by the wetted surface of the channel. β is called the *momentum correction factor* and accounts for the fact that the velocity is not constant across the flow areas:

$$\beta = \frac{1}{V^2 A}\int_A v^2 dA \tag{30.5}$$

The value of β is 1.0 for a flat velocity profile, but its value increases as the irregularity of the velocity distribution increases.

Piezometric Head

As a first application of the momentum equation, consider the control volume shown in Fig. 30.2 having sides AB and DC of length Δx parallel to the streamlines, sides AD and BC of length $\Delta y'$, and breadth $\Delta z'$ normal to the page. Since the momentum fluxes and shears along the faces AB and DC have no normal component, the net pressure force in the y'-direction $(\partial p/\partial y')\ \Delta y'\Delta x\Delta z'$, must be balanced by the normal component of the weight of the liquid in the control volume $\rho\ g\Delta x\Delta y'\Delta z'\cos\theta = \rho\ g\Delta x\Delta y'\Delta z'(\partial z/\partial y')$, thus $\partial/\partial y'[z + p/(\rho\ g)] = 0$.

Integration gives

$$z + \frac{p}{\gamma} = h \tag{30.6}$$

where $\gamma = \rho\ g$ is the *specific weight* of the liquid
$h =$ the elevation of the free surface

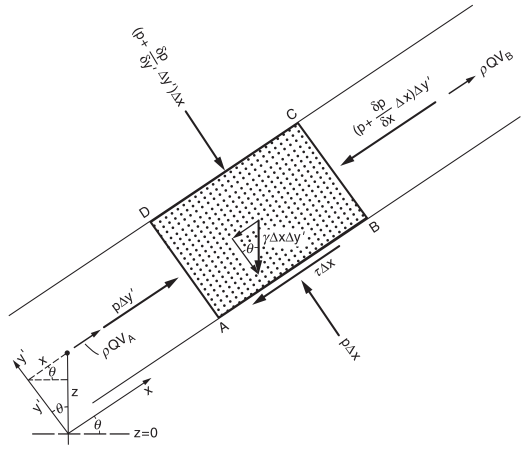

FIGURE 30.2 Forces and momentum flowrate on a fluid element.

The sum of the elevation head and the pressure head, called the *piezometric head*, has the same value at all the points of the same cross section of a parallel flow. This value is the elevation of the free surface for that cross section. This result is of fundamental importance for open channel hydraulics and suggests two corollaries: the pressure distribution within a given cross section of a parallel flow is hydrostatic, and the free surface profile of a parallel flow may be defined as its *piezometric head line* or *hydraulic grade line*.

Boundary Shear

Apply the streamwise momentum Eq. (30.4) to a control volume that contains all of the liquid contained between an upstream flow area and a downstream flow area in a prismatic channel. For uniform flow the momentum flowrates and the pressure forces on the entry and exit faces of the control volume cancel. Therefore the streamwise component of the weight of the fluid in the control volume must be balanced by the shear force acting on the wetted perimeter. Thus, if τ is the average shear stress on the channel lining, $\gamma A \, \Delta x \sin\theta = \tau \, P \Delta x$, or

$$\tau = \gamma R \sin\theta = \gamma R S \qquad (30.7)$$

where $S = \sin\theta$ is the bottom slope of the channel.

Because the flow is uniform, S is also the slope of the piezometric head line and the total head line. When Eq. (30.7) is applied to gradually varied flow, S is interpreted as the slope of the total head line and is usually called the friction slope.

Total Thrust and Specific Force

Consider a control volume containing all of the liquid between upstream flow area A_1 and downstream flow area A_2 in a prismatic channel. In this case the flow depth may vary in the streamwise direction. The forces in the streamwise momentum Eq. (30.4) are the hydrostatic forces on the end surfaces and the component of the weight of the liquid in the flow direction. Friction forces are neglected. The momentum equation in the flow direction is thus

$$\gamma\delta_1 A_1 - \gamma\delta_2 A_2 + W \sin\theta = \beta_2 \rho Q^2 / A_2 - \beta_1 \rho Q^2 / A_1 \qquad (30.8)$$

where δ_1 represents the vertical depth of the centroid of flow area A_1 below the free surface of A_1, and W is the weight of the liquid in the control volume.

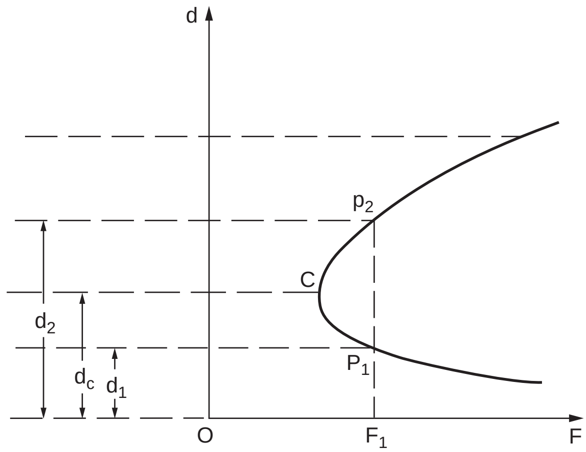

FIGURE 30.3 Total thrust curve.

This can be rearranged as

$$\left(\gamma\delta_1 A_1 + \beta_1 \rho Q^2 / A_1\right) = \left(\gamma\delta_2 A_2 + \beta_2 \rho Q^2 / A_2\right) - W\sin\theta \tag{30.9}$$

where the expression in parentheses is called the *total thrust, F.*

$$F = \gamma\delta A + \beta\rho Q^2 / A \tag{30.10}$$

The total thrust is the sum of the *hydrostatic thrust* $\gamma\delta A$, which increases with flow thickness, and the *momentum flowrate* $\beta\rho Q^2/A$, which decreases with flow thickness if Q and β are assumed constant. Therefore the total thrust reaches a minimum at the *critical stream thickness*, d_c, as illustrated in Fig. 30.3. This value can be found by setting the derivative of F with respect to d equal to zero with the result that

$$\left(A^3/T\right)_{d_c} = \beta Q^2 / \left(g\cos\theta\right) \tag{30.11}$$

For most cross sections, Eq. (30.11) must be solved numerically. The function F/γ is known variously as the *specific force*, *momentum function*, or *thrust function*.

Balance of Mechanical Energy

The mechanical energy of a body of mass m is the sum of its gravitational potential energy mgz and of its kinetic energy $mv^2/2$, where z is the elevation of the mass m above a reference datum. The principle of conservation of mechanical energy states that the time rate of change of the mechanical energy in a control volume is equal to the net flowrate of mechanical energy at the inlet and outlet sections, plus the work done by the pressure forces at the inlet and outlet sections, plus the loss of mechanical energy in the control region. For the steady flow of an incompressible liquid through a control volume where the flow is parallel and normal to a single plane inflow area and a single plane outflow area, the mechanical energy equation in terms of head becomes

$$z_1 + \frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} = z_2 + \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + h_L \tag{30.12}$$

where h_L is the *head loss* and α is the *kinetic energy correction factor* that accounts for the nonuniformity of the velocity across the flow area.

$$\alpha = \frac{1}{V^3 A}\int_A v^3 dA \tag{30.13}$$

The value of α is 1.0 for a flat velocity profile, but increases as the velocity profile becomes more irregular.

The total head H is defined as the sum of the elevation head, pressure head, and velocity head, or as the sum of the piezometric head and the velocity head.

$$H = z + \frac{p}{\gamma} + \alpha\frac{V^2}{2g} = h + \alpha\frac{V^2}{2g} \tag{30.14}$$

Therefore Eq. (30.12) states that the total head at cross section 1 exceeds that at cross section 2 by the head loss between the sections. In terms of the variables shown in Fig. 30.1, Eq. (30.12) can be written as

$$z_1 + d_1\cos\theta_1 + \alpha_1 V_1^2/(2g) = z_2 + d_2\cos\theta_2 + \alpha_2 V_2^2/(2g) + h_L \tag{30.15}$$

Specific Energy

Bakhmeteff (1932) first emphasized the importance of the quantity E, where

$$E = d\cos\theta + \alpha V^2/(2g) = d\cos\theta + \alpha Q^2/(2gA^2) \tag{30.16}$$

which he called the *specific energy*. (Because of its dimensions, E is more properly called *specific head*.) In terms of the specific energy, Eq. (30.12) becomes

$$z_1 + E_1 = z_2 + E_2 + h_L \tag{30.17}$$

which shows that the specific energy is conserved when $h_L = z_1 - z_2$, i.e., in uniform flow. It is this property that gives the specific energy its importance.

Equations (30.16) and (30.17) are the foundations for the calculation of gradually varied water surface profiles. Given Q, θ, α_1, d_1, and the geometry of cross-section 1, E_1 can be evaluated using Eq. (30.16). Then Eq. (30.17) can be solved for E_2, if an estimate of the head losses h_L is possible. Once E_2 is known, Eq. (30.16) can be solved for d_2.

Equation (30.16) shows that E is the sum of two terms. Assuming that α and Q are constant, the first term increases and the second decreases as d increases. Hence there is a *critical thickness*, d_c, for which E is a minimum. By differentiating Eq. (30.16) with respect to d and equating the derivative to zero, the following condition for critical flow is obtained.

$$\left(A^3/T\right)_{d_c} = \alpha Q^2/(g\cos\theta) \tag{30.18}$$

The expression A^3/T is called the *section factor for critical flow*. For most cross sections, Eq. (30.18) must be solved numerically. Note that the critical depth is independent of the channel slope and roughness. It is interesting to recognize that the critical thickness which satisfies Eq. (30.18) differs from that which satisfies Eq. (30.11) unless α = β, which is true only for a flat velocity profile. In practice, the difference is usually negligible; and Eq. (30.18) is used in the calculation. The velocity corresponding to d_c, called the critical velocity, V_c, is given by

$$V_c = \left(g\cos\theta\, D/\alpha\right)^{1/2} \tag{30.19}$$

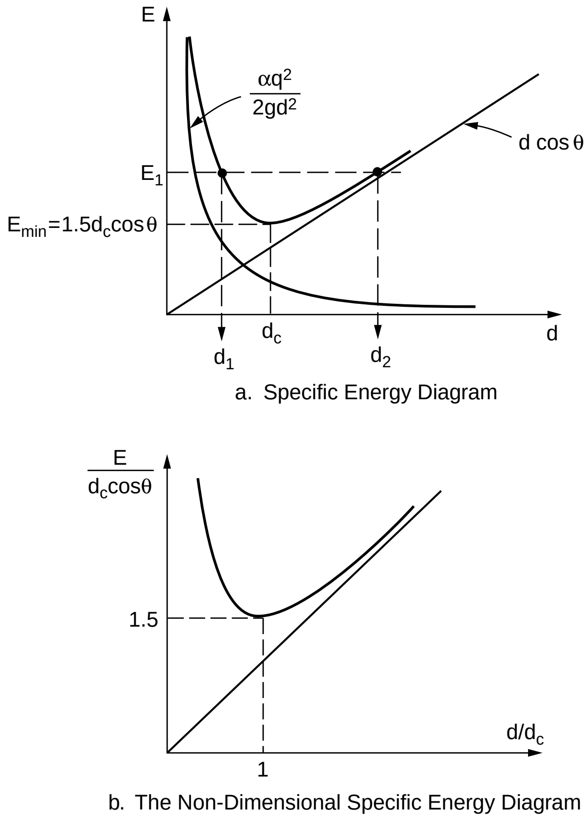

FIGURE 30.4 (a) Specific energy diagram for rectangular channel; (b) Dimensionless specific energy diagram for rectangular channel.

If the slope is small and the velocity profile is flat, this reduces to the previous expression for the wave celerity.

The behavior of the specific energy function can be clarified by considering the rectangular channel, for which Eq. (30.16) becomes

$$E = d\cos\theta + \alpha q^2 / \left(2gd^2\right) \tag{30.20}$$

The variation of E with d for a rectangular channel assuming constant α and q is shown in Fig. 30.4(a). For any value of $E > E_{min}$ there are two possible flow thicknesses, $d_R < d_c$ corresponding to supercritical flow and $d_T > d_c$ for subcritical flow. These are called *alternate thicknesses* (or *alternate depths* in the small slope case). For $E = E_{min}$ only one thickness is possible. It is impossible to transmit the specified flowrate for $E < E_{min}$. Eq. (30.20) can also be presented in dimensionless form as shown in Fig. 30.4(b) and in Table 30.1. In a rectangular channel, Eq. (30.18) can be solved explicitly for the critical thickness.

$$d_c = \left[\alpha q^2 / \left(g\cos\theta\right)\right]^{1/3} \tag{30.21}$$

and the corresponding minimum of specific energy for a rectangular channel is

$$E_{min} = 1.5 d_c \cos\theta \tag{30.22}$$

In reality, α always varies with d. When the variation is appreciable, as in the case of a river and its flood plain, there may be multiple local extrema in the specific energy vs. thickness curve and hence

TABLE 30.1 Dimensionless Specific Energy for a Rectangular Channel (Giorgini, 1987)

$E/d_c \cos\theta$ E/y_c	d_R/d_c y_R/y_c	d_T/d_c y_T/y_c	$E/d_c \cos\theta$ E/y_c	d_R/d_c y_R/y_c	d_T/d_c y_T/y_c
1.500	1.000	1.000	2.500	0.500	2.414
1.505	.944	1.060	2.600	0.486	2.521
1.510	.923	1.086	2.800	4.62	2.733
1.515	.906	1.107	3.000	0.442	2.942
1.520	.893	1.125	3.500	0.402	3.458
1.525	.881	1.141	4.000	0.371	3.963
1.530	.871	1.156	4.500	0.347	4.475
1.540	.853	1.182	5.000	0.327	4.980
1.550	.838	1.207	5.500	0.310	5.483
1.560	.825	1.229	6.000	0.296	5.986
1.570	.812	1.250	7.000	0.273	6.990
1.580	.801	1.270	8.000	0.254	7.992
1.590	.791	1.289	9.000	0.239	8.994
1.600	.782	1.308	10.000	0.226	9.995
1.625	.761	1.351	11.000	0.215	10.996
1.650	.742	1.392	12.000	0.206	11.997
1.675	.726	1.431	14.000	0.190	13.997
1.700	.711	1.468	16.000	0.178	15.998
1.750	.685	1.539	18.000	0.167	17.998
1.800	.663	1.606	20.000	0.159	19.999
1.850	.644	1.671	25.000	0.142	24.999
1.900	.627	1.734	30.000	0.129	29.999
2.000	.597	1.855	35.000	0.120	35.000
2.100	.572	1.971	40.000	0.112	40.000
2.200	.551	2.085	45.000	0.106	45.000
2.300	.532	2.196	50.000	0.100	50.000
2.400	.515	2.306			

multiple critical depths. Discussion of this recently recognized phenomenon is found in Chaudhry (1993), Sturm (2000), and Jain (2001). Channels having a single critical depth are called *regular*.

Hydraulic Jump

A *hydraulic jump* is a sudden increase in depth that occurs when the flow changes from supercritical to subcritical as a result of a rapid deceleration. For a stationary hydraulic jump in a horizontal rectangular channel, the depth before the jump, y_1, and the depth after the jump, y_2, are related by the expression

$$y_2 = y_1\left[\left(1+8Fr_1^2\right)^{1/2} - 1\right] \Big/ 2 \tag{30.23}$$

where $Fr_1 = V_1/(gy_1)^{1/2}$ is the Froude number of the flow before the jump.

The depths y_1 and y_2 are called *conjugate depths* (they are *not* alternate depths). The formula obtained by interchanging the subscripts is also valid. The head loss in this hydraulic jump, h_J, is given by

$$h_J = E_1 - E_2 = \left(y_2 - y_1\right)^3 \Big/ \left(4y_1y_2\right) \tag{30.24}$$

The length of the jump is approximately $6y_2$ for $Fr_1 > 4.5$. The hydraulic jump is an effective means of dissipating excess kinetic energy in supercritical flows.

Example 30.1 Subcritical Flow on a Step

Consider the subcritical flow in the rectangular horizontal channel shown in Fig. 30.5, which presents an upward step of height *s* with respect to the direction of the current. The specific energy upstream of

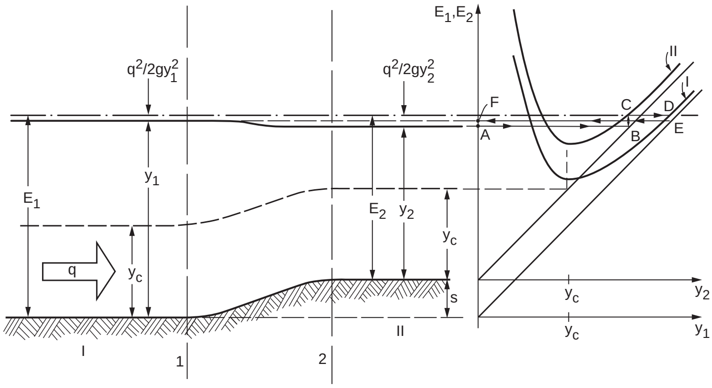

FIGURE 30.5 Subcritical stream on upward step.

the step is larger than the minimum ($3y_c/2$) and the flow is subcritical. The water surface elevation at section 1 is higher than at section 2. Given $q = 10$ m^2/s, $y_2 = 3.92$ m, and $s = 0.5$ m, find y_1, assuming no losses.

1. Find $y_c = (q^2/g)^{1/3} = 2.17$ m
2. Find $E_2 = y_2 + q^2/(2gy_2{}^2) = 4.25$ m
3. Find $E_1 = E_2 + s = 4.75$ m
4. Enter Table 30.1 with $E_1/y_c = 2.19$
5. Find $y_1/y_c = 2.07$. Thus $y_1 = 4.49$ m.

This result can also be obtained by solving Eq. (30.20). If the bottom of the upward step increases very gradually, one can find intermediate points along the step as shown in Fig. 30.5. It is useful to draw the critical depth line in order to visualize the relative distance of the free surface profile from it. Fig. 30.5 also illustrates a graphical technique based on the curve $E(y)$. Follow the path A, B, C, D, E, F. The limit case is where the specific energy on the upper part of the step is a minimum, i.e., where the depth is critical.

Example 30.2 Supercritical Flow on a Step

Consider the supercritical flow in the rectangular horizontal channel shown in Fig. 30.6, which presents an upward step of height s with respect to the direction of the current. The specific energy upstream of the step is larger than the minimum specific energy ($3y_c/2$) and the flow is supercritical. The water depth at cross section 2 is larger than at section 1 due to the decreased kinetic energy of the stream. Given $q = 10$ m^2/s, $y_1 = 1.2$ m, $s = 0.5$ m, find y_2, assuming no losses.

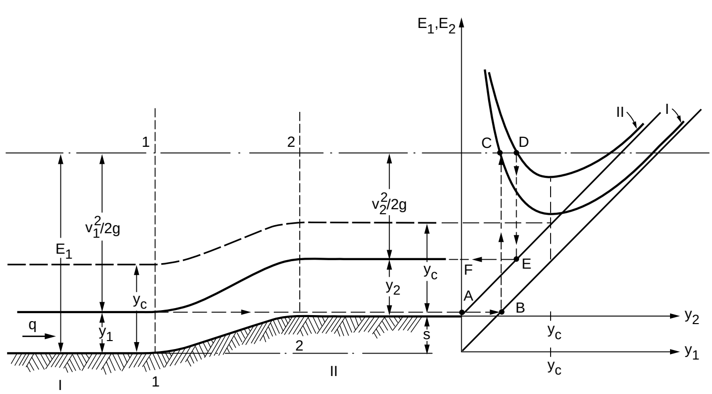

FIGURE 30.6 Supercritical stream on upward step.

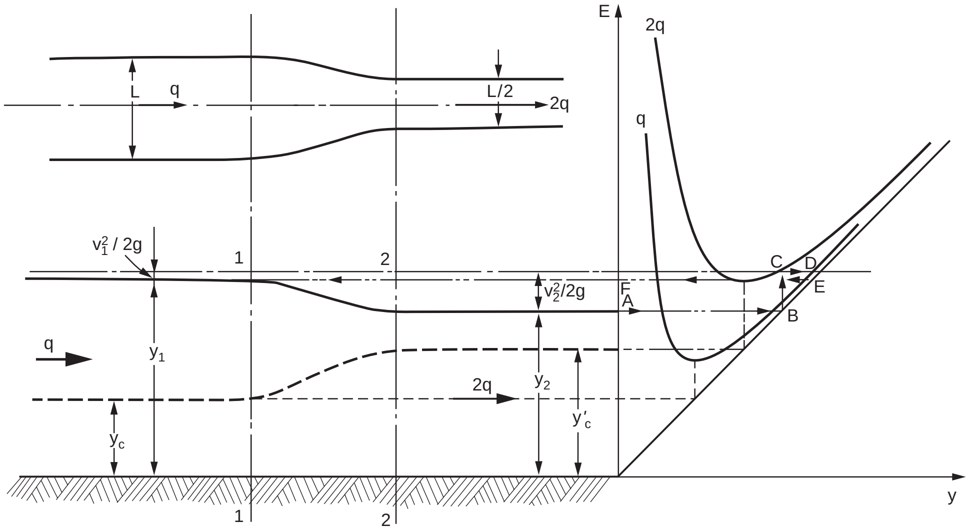

FIGURE 30.7 Subcritical stream in a channel contraction.

1. Find $y_c = (q^2/g)^{1/3} = 2.17$ m
2. Find $E_1 = y_1 + q^2/(2gy_1{}^2) = 4.74$ m
3. Find $E_2 = E_1 - s = 4.24$ m
4. Enter Table 30.1 with $E_2/y_c = 1.95$
5. Find $y_2/y_c = 0.612$, thus $y_2 = 1.33$ m.

This result can also be obtained from Eq. (30.20). Fig. 30.6 shows a graphical technique based on the curve $E(y)$. Follow the path A, B, C, D, E, F. The limit case occurs where the specific energy on the upper part of the step is a minimum, i.e., where the depth is critical.

Example 30.3 Contraction in a Subcritical Stream

Consider the horizontal rectangular channel of Fig. 30.7 with a lateral contraction in the direction of the current. The width of the contracted channel is half that of the upstream channel. The specific energy upstream of the contraction is larger than $E_{\text{min}} = 3y_c/2$. The water depth at section 2 is shallower than at section 1 as the kinetic energy of the stream increases. Given $q = 10$ m²/s, $y_2 = 5.46$ m, find y_1, assuming no losses.

1. Find $y_{c1} = (q_1{}^2/g)^{1/3} = 2.17$ m and $y_{c2} = [(2q_1)^2/g]^{1/3} = 3.44$ m
2. Find $E_1 = E_2 = y_2 + (2q_1)^2/2gy_2{}^2 = 6.14$ m
3. Enter Table 30.1 with $E_1/y_{c1} = 2.83$.
4. Find $y_1/y_{c1} = 2.76$, thus $y_1 = 5.99$ m.

It is seen that $E_2 > 1.5y_{c2} = 5.16$ m, thus the flow is subcritical in the contraction. Fig. 30.7 shows a graphical technique based on the curve $E(y)$. Follow the path A, B, C, D, E, F. The limiting case occurs when $y_2 = y_{c2}$. Further information on sills, contractions, and expansions for subcritical and supercritical flows can be found in Ippen (1950).

30.3 Uniform Flow

When a steady flowrate is maintained in a prismatic channel, a constant depth flow will be reached somewhere in the channel if it is long enough. Such a flow is called *uniform flow* or *normal flow*. In uniform flow there is a perfect balance between the component of fluid weight in the direction of flow

TABLE 30.2 Manning Roughness Coefficient (Giorgini, 1987)

Nature of Surface	n_{min}	n_{max}
Neat cement surface	0.010	0.013
Concrete, precast	0.011	0.013
Cement mortar surfaces	0.011	0.015
Concrete, monolithic	0.012	0.016
Cement rubble surfaces	0.017	0.030
Canals and ditches, smooth earth	0.017	0.025
Canals:		
Dredged in earth, smooth	0.025	0.033
In rock cuts, smooth	0.025	0.035
Rough beds and weeds on sides	0.025	0.040
Rock cuts, jagged and irregular	0.035	0.045
Natural streams:		
Smoothest	0.025	0.033
Roughest	0.045	0.060
Very weedy	0.075	0.150

and the resistance to flow exerted by the channel lining. Assuming that the average shear stress τ given by Eq. (30.7) is proportional to the square of the average velocity gives

$$V = C\sqrt{RS} \tag{30.25}$$

Equation (30.25) is called the *Chezy equation* and C is the *Chezy C*, a dimensional factor which characterizes the resistance to flow. The Chezy C may depend on the channel shape, size, and roughness and on the Reynolds number. Equation (30.25) and the equations derived from it are often applied to gradually varied flows. In these cases, S is interpreted as the slope of the total head line, called the *friction slope*. Numerous equations have been suggested for the evaluation of C. The most popular was proposed independently by several engineers including Manning for high Reynolds numbers where the flow is independent of Reynolds number (fully rough flow). This is

$$C = BR^{1/6}/n \tag{30.26}$$

where B is a dimensional constant equal to 1 $m^{1/3}$/sec in the International System (SI) or 1.486 $ft^{1/3}$/sec in the U. S. Customary System, and n is a dimensionless number characterizing the roughness of a surface.

Table 30. 2 lists typical values of n; a much more comprehensive table is given by Chow (1959). Many investigators have proposed equations relating n to a typical particle size in particle lined streams. As an example, Subramanya (1982) proposed

$$n = 0.047 d_{50}^{1/6} \tag{30.27}$$

where d_{50} is the diameter in meters chosen so that 50% of the particles by weight are smaller.

Substituting Eq. (30.26) in Eq. (30.25) gives

$$V = \frac{B}{n} R^{2/3} S^{1/2} \tag{30.28}$$

Multiplying this by the flow area A yields

$$Q = \frac{B}{n} A R^{2/3} S^{1/2} = K S^{1/2} \tag{30.29}$$

Equations (30.28) and (30.29) are both called the *Manning equation*. The multiplier of $S^{1/2}$ in Eq. (30.29) is called the *conveyance of the cross-section, K*.

Because Eq. (30.26) is valid only for fully rough flow, the Manning equation is likewise limited. Henderson (1966) proposed that for water at ordinary temperatures, the Manning equation is valid if

$$n^{12}RS \geq 1.100 \times 10^{-26}\,\text{m} = 3.61 \times 10^{-26}\,\text{ft} \tag{30.30}$$

In the event that the *Henderson criterion* is violated, but the flow is nonetheless turbulent ($Re > 8000$), C can be calculated iteratively from a transformation of the Colebrook equation of pipe flow.

$$\frac{C}{\sqrt{8g}} = -2.0\log_{10}\left[\frac{\varepsilon}{14.8R} + \frac{2.51\,C}{Re\sqrt{8g}}\right] \tag{30.31}$$

Here, ε is the equivalent sand grain roughness (Henderson, 1966, Table 4.1, p. 95).

The thickness of a uniform flow is called the *normal thickness*, d_o. Assuming that the Manning equation is valid, the normal thickness for a general cross section is found by solving Eq. (30.29) for d_o. This is facilitated by writing the Manning equation so that the factors which depend on d are on the left side.

$$\left(AR^{2/3}\right)_{d_O} = nQ/\left(BS^{1/2}\right) \tag{30.32}$$

The expression $AR^{2/3}$ is called the *section factor for uniform flow*. For most cross sections, Eq. (30.32) must be solved numerically, but for a wide rectangular channel d_o can be calculated directly from

$$d_o = \left[nq/\left(BS^{1/2}\right)\right]^{3/5} \tag{30.33}$$

For open top cross sections, d_o is unique; but when the channel has a gradually closing top, such as a circular pipe, there will be two normal depths for some flowrates. For the case of an open top section, if $d_o > d_c$ the normal flow is subcritical and the channel slope is *mild*; if $d_o < d_c$ the normal flow is supercritical and the channel slope is *steep*. The *critical slope*, S_c, for which uniform flow is also critical flow, can be found for a wide rectangular channel by equating the thicknesses given by Eqs (30.21) and (30.33) and solving for S_c.

$$S_c = (n/B)^2(g\cos\theta/\alpha)^{10/9}q^{-2/9} \tag{30.34}$$

A channel whose bottom slope is less then S_c is called *mild* and a channel whose bottom slope is larger than S_c is called *steep*. For a wide rectangular channel, Eq. (30.34) can be solved for the *critical discharge*, q_c, when S, n, and α are fixed, or the *critical Manning coefficient*, n_c, when S, α, and q are fixed.

30.4 Composite Cross-Sections

Consider the determination of the discharge, global Manning n, normal depth, and critical depth of a composite section as shown in Fig. 30.8(a) having a small slope. The three parts of the global channel are assumed to behave as three different channels in parallel with the same slope and the same water surface elevation. The wetted perimeters of the subsections include only the portions of the solid boundary belonging to that subsection. According to the Manning equation, the total discharge is

$$Q = B\left(\frac{1}{n_l}\frac{A_l^{5/3}}{P_l^{2/3}} + \frac{1}{n_m}\frac{A_m^{5/3}}{P_m^{2/3}} + \frac{1}{n_r}\frac{A_r^{5/3}}{P_r^{2/3}}\right)S^{1/2} \tag{30.35}$$

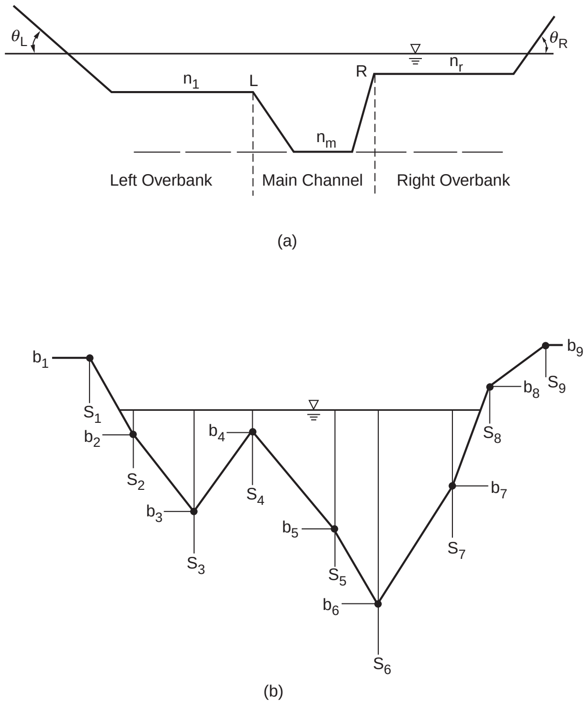

FIGURE 30.8 Composite cross-sections.

where the subscripts *l, m, r* refer to the left overbank, the main channel, and the right overbank.

From Eq. (30.35) with A representing the total cross sectional area and P the total wetted perimeter, the global value of n is

$$\frac{1}{n} = \sum_{l}^{r} \frac{1}{n_i} \frac{\left(A_i/A\right)^{5/3}}{\left(P_i/P\right)^{2/3}} \tag{30.36}$$

Given the quantities Q, S, n_1, n_m, and n_r, and given the functions $A_1(y)$, $A_m(y)$, $A_r(y)$, $P_1(y)$, $P_m(y)$, and $P_r(y)$, the *normal depth* is the root y_o of Eq. (30.35). Then y_o can be substituted to find the actual values of the areas and wetted perimeters of the subsections to obtain Q_1, Q_m, and Q_r.

To find the critical depth it is necessary to obtain an estimate of the kinetic energy correction factor. For an irregular cross section, α defined in Eq. (30.13) becomes

$$\alpha = \frac{1}{A} \sum_{l}^{r} \left(\frac{V_i}{V}\right)^3 A_i = \sum_{l}^{r} \left(\frac{Q_i}{Q}\right)^3 \left(\frac{A}{A_i}\right)^2 = \frac{B^3 A^2 S^{3/2}}{Q^3} \sum_{l}^{r} \frac{1}{n_i^3} \frac{A_i^3}{P_i^2} \tag{30.37}$$

Substituting Eq. (30.37) into Eq. (30.18) (for small slopes) yields

$$\frac{gQ}{B^3 S^{3/2}} = \left(\frac{T}{A} \sum_{l}^{r} \frac{1}{n_i^3} \frac{A_i^3}{P_i^2}\right)_{y_c} \tag{30.38}$$

The *critical depth* is obtained by solving Eq. (30.38) for $y = y_c$. Numerical techniques are generally needed to obtain the root(s).

Usually river cross sections are given as sequences of station abscissas, s_i, and bottom elevations, b_i, and a general cross section is approximated as shown in Fig. 30.8(b). With z_i as the water surface elevation, the flowrate Q_i in trapezoidal element i is

$$Q_i = \frac{B}{n_i}\frac{A_i^{5/3}}{P_i^{2/3}}S^{1/2} = \frac{B}{n_i}\frac{\left(s_{i+1}-s_i\right)^{5/3}\left\{z_i - \left[\left(b_{i+1}+b_i\right)/2\right]\right\}^{5/3}}{\left[\left(s_{i+1}-s_i\right)^2+\left(b_{i+1}-b_i\right)^2\right]^{1/3}}S^{1/2} \tag{30.39}$$

30.5 Gradually Varied Flow

In gradually varied flow, the depth changes in the flow direction slowly enough that the piezometric head can be assumed constant on every cross section. The channel need not be prismatic, but it cannot change abruptly in the streamwise direction. In contrast to uniform flow, the slope of the channel bottom, S_o, the slope of the water surface, S_w, and the slope of the total head line (the friction slope), S_f, must be distinguished. Consider the development of an expression for the rate of change of the depth along the channel, called the *gradually varied flow equation*. In the interest of simplicity, consider only the special case of a regular open top prismatic channel of small bottom slope in which the velocity profile is flat ($\alpha = 1$) and Q does not change with x. For such a channel, there is one normal depth and one critical depth.

From Fig. 30.1, the total head H for an open channel flow of small slope with $\alpha = 1$ is

$$H = z + y + V^2/(2g) \tag{30.40}$$

Differentiate Eq. (30.40) with respect to x; recognize that $dH/dx = -S_f$, $dz/dx = -S_o$, and $dA/dy = T$; and solve for dy/dx to obtain

$$\frac{dy}{dx} = \frac{S_o - S_f}{1+\dfrac{d}{dy}\left(\dfrac{V^2}{2g}\right)} = S_o\frac{1-\left(S_f/S_o\right)}{1-\dfrac{Q^2T}{gA^3}} = S_o\frac{1-\left(S_f/S_o\right)}{1-\dfrac{V^2}{gD}} \tag{30.41}$$

Recalling the definition of the Froude number, the gradually varied flow equation for a prismatic channel may also be written as

$$\frac{dy}{dx} = \frac{S_o - S_f}{1-Fr^2} = \frac{\mathcal{N}(y)}{\mathcal{D}(y)} \tag{30.42}$$

where $\mathcal{N}$ is the numerator and $\mathcal{D}$ is the denominator of the right side of Eq. (30.42). Eqs. (30.41) and (30.42) are forms of the differential equation for the water surface profile $y(x)$ in a prismatic channel. They are some of the ways of writing the *gradually varied flow equation* for regular open top prismatic channels. Note that dy/dx is the slope of the water surface with respect to the bottom of the channel, not necessarily with respect to the horizontal. Eq. (30.42) is a separable first order ordinary differential equation, whose formal solution is

$$x = x_1 + \int_{y_1}^{y}\frac{\mathcal{D}(\xi)}{\mathcal{N}(\xi)}d\xi \tag{30.43}$$

where y_1 is the boundary condition at x_1 and ξ is a dummy variable.

Because of the physical restrictions on the direction of surface wave propagation discussed in Section 30.1, the boundary condition for a given reach is usually given at the downstream boundary if the type of flow in the reach is subcritical, and at the upstream boundary if the flow in the reach is supercritical.

As Eq. (30.43) is integrable in closed form only under very particular conditions of channel geometry and of resistance law, some general observations, valid for any regular open top prismatic channel and for the Manning resistance law, will be made here. In Eq. (30.42), $\mathcal{N} \to S_o$ as $y \to \infty$ because $S_f \to 0$. As $y \to y_o$, $\mathcal{N} \to 0$. Thus the derivative of y with respect to x is zero when either the curve coincides with the y_o line (normal flow) or as the curve approaches the y_o line asymptotically. As $y \to 0$, $S_f \to \infty$, so $\mathcal{N} \to -\infty$.

As $y \to \infty$, $Fr \to 0$ and $\mathcal{D} \to 1$. As $y \to y_c$, $Fr \to 1$, so $\mathcal{D} \to 0$ which implies that if a free surface profile approaches the y_c line, it must do so with infinite slope. But in such an event, the assumption of quasi-parallel flow becomes invalid; and Eq. (30.42) no longer represents the physics of the flow. This means that where the mathematical water surface profiles cut the y_c line, they do not represent accurately what happens in nature. Fortunately this phenomenon is of limited extent. In reality, the water surface approaches the y_c line at an angle which is large, but less than 90°. A similar discrepancy occurs as $y \to 0$: namely $Fr \to \infty$, so $\mathcal{D} \to -\infty$. This makes dy/dx indeterminate, but it can be shown that $dy/dx \to \infty$ as $y \to 0$. Observe that, for any reach of given constant slope S_o, the lines $y = y_c$, $y = y_o$ (if it exists), and the bottom line $y = 0$, divide the x,y plane into three regions if $y_o \neq y_c$, or two regions if $y_o = y_c$. With these observations in mind, a brief presentation of all possible types of water surface profiles is made in the next section.

30.6 Water Surface Profile Analysis

Once again consider only the special case of a regular open top prismatic channel of small bottom slope in which the velocity profile is flat ($\alpha = 1$) and Q does not change with x. Figure 30.9 illustrates the different classes of profiles which can be distinguished according to the relative magnitude of the critical depth, y_c, calculated from Eq. (30.18) and the normal depth, y_o, calculated from Eq. (30.32). The sign of dy/dx is determined by considering the gradually varied flow equation Eq. (30.42).

The Mild Slope Profiles ($y_c < y_o$)

The M_1 Profile ($y_o < y$)

With the actual depth y exceeding the normal depth y_o, the friction slope S_f is less than the bottom slope S_o so that $\mathcal{N}$ is positive. The actual depth y also exceeds the critical depth y_c so that the flow is subcritical and $\mathcal{D}$ is also positive. Thus dy/dx is always positive, and y grows as the stream proceeds downstream. As the depth increases, $S_f \to 0$ and $Fr \to 0$, so $\mathcal{D} \to 1$ and $\mathcal{N} \to S_o$. Thus dy/dx asymptotically approaches S_o. Since the slope is with respect to the channel bottom, the water surface becomes horizontal. The water surface asymptotically approaches the normal depth line in the upstream direction for reasons given previously. Since the M_1 profile is subcritical, it is drawn from downstream to upstream, starting from a known depth such as P_6. The M_1 profile is called a *backwater profile*.

The M_2 Profile ($y_c < y < y_o$)

Because the actual depth is less than the normal depth, the friction slope must exceed the bottom slope so that $\mathcal{N} < 0$. With the actual depth greater than the critical depth, the flow is subcritical so that $\mathcal{D} > 0$. Consequently dy/dx is always negative and the depth decreases from upstream, where the surface profile is asymptotic to the y_o line, to downstream, where it approaches the y_c line vertically. Since the stream is subcritical it is drawn from downstream to upstream, starting from a known depth such as P_7. The M_2 profile is a *drawdown profile.*

The M_3 Profile ($0 < y < y_c$)

In this case $\mathcal{N} < 0$ and $\mathcal{D} < 0$, therefore $dy/dx > 0$. Since dy/dx tends to infinity as y approaches either zero or y_c, the profile has an inverted S shape. It can be shown that the tangent to the profile at the

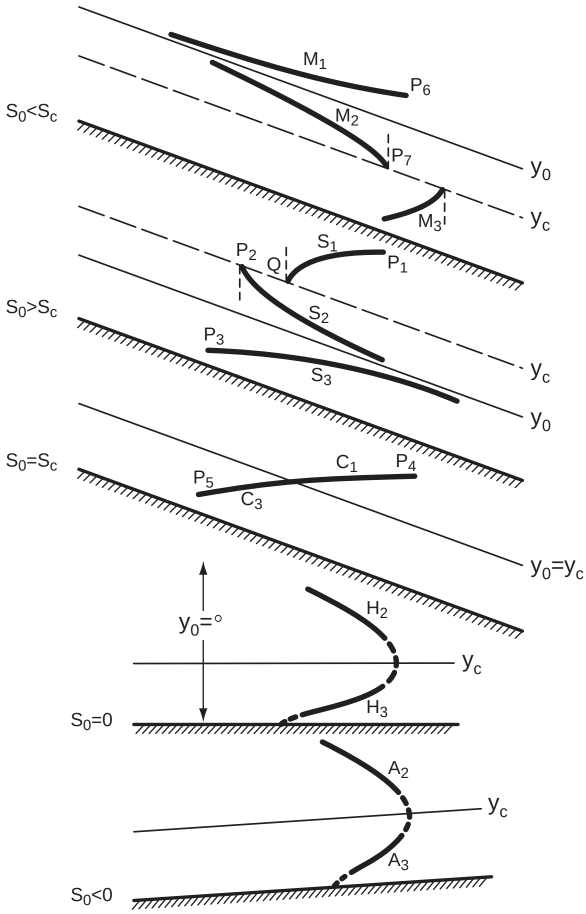

FIGURE 30.9 Gradually varied flow water surface curves.

inflection point is slightly larger than S_o, which means that the profile has always a slope above the horizontal line.

The Steep Slope Profiles ($y_o < y_c$)

The S_1 Profile ($y_c < y$)

In this case $\mathcal{N} > 0$, and $\mathcal{D} > 0$, so dy/dx is always positive; and y grows as x increases from a starting datum. If this datum coincides with the critical depth y_c, then the profile grows from it with infinite slope. The slope decreases very rapidly to become S_o for large values of x (where the values of y are large). This means that the S_1 profile has a horizontal asymptote for $x \rightarrow \infty$. Since S_1 is a subcritical profile, it is drawn from downstream to upstream, always below a horizontal line through the known control point P_1. It cuts the y_c line vertically at Q.

The S_2 Profile ($y_o < y < y_c$)

Since in this case $\mathcal{N} > 0$ and $\mathcal{D} < 0$, dy/dx is always negative; and y decreases as x increases from the starting datum. If this datum coincides with the critical depth y_c, then the profile decreases from an initial infinite slope to approach y_o asymptotically. Since S_2 is a supercritical profile, it is drawn from upstream to downstream through a control point such as P_2.

The S_3 Profile ($0 < y < y_o$)

In this case $\mathcal{N} < 0$ and $\mathcal{D} < 0$, therefore dy/dx is always positive; and the water surface profile tends to the y_o line asymptotically downstream. Moving upstream, the depth approaches zero vertically. The S_3 profile is drawn from upstream to downstream because it is supercritical.

The Critical Slope Profiles ($y_o = y_c$)

Critical slope profiles rarely occur in practice because equality of the critical and normal depths is highly unlikely.

The C_1 Profile ($y_c = y_o < y$)

In this case $\mathcal{N} > 0$ and $\mathcal{D} > 0$, so $dy/dx > 0$. Eq. (30.42) is indeterminate at y_c, but it can be shown that the slope has a finite value there that decreases asymptotically to S_o as x increases. Since the stream is subcritical, it is drawn from downstream to upstream, starting from a known datum point such as P_4.

The C_2 Profile ($y = y_c = y_o$)

Region 2 degenerates to the $y_o = y_c$ line. In reality this profile is unstable and a wavy surface occurs.

The C_3 Profile ($0 < y < y_c = y_o$)

In this case $\mathcal{N} < 0$ and $\mathcal{D} < 0$, so $dy/dx > 0$. According to the mathematical model, the profile rises vertically from the channel bottom and decreases in slope until it intersects the $y_c = y_o$ with a finite slope. Because the stream is supercritical, it is drawn from upstream to downstream, starting from a known depth such as P_5.

The Horizontal Slope Profiles ($y_c < y_o \to \infty$)

As the slope of a channel is reduced, the depth of a uniform flow of magnitude Q tends to infinity. Therefore region 1 does not exist for a horizontal channel. Because $S_o = 0$, $\mathcal{N} = -S_f < 0$.

The H_2 Profile ($y_c < y$)

The flow is subcritical, so $\mathcal{D} > 0$. This implies that $dy/dx < 0$. As x increases, y decreases toward the y_c line, which it approaches vertically.

The H_3 Profile ($y < y_c$)

The flow is supercritical so $\mathcal{D} < 0$. This implies that $dy/dx > 0$. As x increases, y increases from the channel bottom toward the y_c line, which it approaches vertically.

The Adverse Slope Profiles ($S_o < 0$)

It is impossible to have uniform flow in a channel of negative bottom slope because both gravity and friction oppose the motion. As a consequence, the normal depth does not exist. In all adverse slope cases, $\mathcal{N} < 0$.

The A_2 Profile ($y_c < y$)

The flow is subcritical so $\mathcal{D} > 0$. This implies that $dy/dx < 0$. As x increases, y decreases toward the y_c line, which it approaches vertically.

The A_3 Profile ($y < y_c$)

The flow is supercritical so $\mathcal{D} < 0$. This implies that $dy/dx > 0$. As x increases, y increases from the channel bottom toward the y_c line, which it approaches vertically. Persons who believe that "water always flows downhill" are often astounded by the A_3 profile.

30.7 Qualitative Solution of Flow Profiles

The steady water surface profile in a series of regular open top prismatic channels connected by short transition sections will consist of the elementary solutions discussed above, connected by short reaches of gradually or rapidly varied flow in the transition sections. In addition, hydraulic jumps may occur in the prismatic reaches. Among the possible transition sections are upward steps, downward steps, channel contractions, channel expansions, weirs, gates, spillways, and culverts. If the flowrate is given, the first step in deducing the form of the profile is to draw the channel bottom and the normal and critical depth lines in each reach. This determines the slope family (M, S, C, H, or A) to which the profile in each prismatic reach belongs.

As the gradually varied flow equation is a first order differential equation, it needs only one boundary condition to define the elementary water surface profile in a given reach. In some cases, this condition is fixed by the requirement of continuity with the water surface elevation in the upstream or downstream reach. But there will always be one or more features which provide a definite relationship between flowrate and depth. Such features are usually called *controls* in open channel hydraulics. The second step in sketching the profile is the determination of the control points. By fixing the zone (1, 2, or 3) through which the profile must pass, the shape of the profile in that reach can be determined. Fig. 30.10 shows a long mild channel flowing into a long steep channel. In any long channel, the flow must tend toward y_o far away from a disturbance. Therefore, the only possible profiles in the mild channel are M_1 and M_2. If the M_1 profile occurred, the profile in the steep reach would be an S_1, which could not approach y_o. The same problem happens if the mild channel contains an M_2 profile unless it crosses the y_c line at the break in grade. In that case, the flow in the steep section follows an S_2 profile that does approach y_o. In obtaining the numerical solution, it is important to start the process of integration for both reaches at the known depth y_c because the normal depth is approached, but not equaled, in both reaches. Thus the control in this example is at the break in grade. In general, the correct solution is deduced by the process of elimination; however there may be several possible alternatives in more complicated situations.

Figure 30.11 illustrates a complex channel system in which each prismatic reach is a wide rectangular cross section of small slope. As is usual, the vertical scale is drawn much larger than the horizontal scale for clarity. The prismatic reaches *AB, CD, DE, FG, HI,* and *JK* are intermingled with shorter transition sections: *BC*, a downward step; *EF*, a sluice gate; *GH*, an upward step; and *IJ*, a contraction. Discharges, slopes and roughness coefficients are shown on the figure. The channel terminates at *K* in a free overfall. The solution process can be divided in three phases as follows:

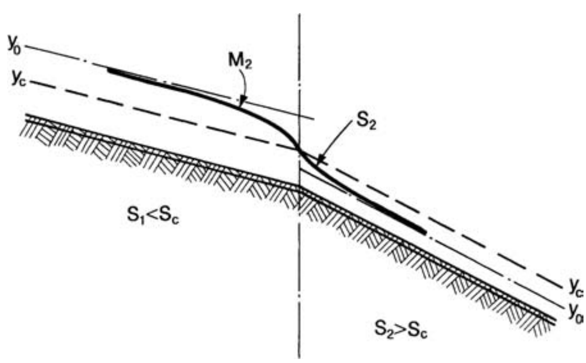

FIGURE 30.10 Channel with abrupt change of slope from mild to steep.

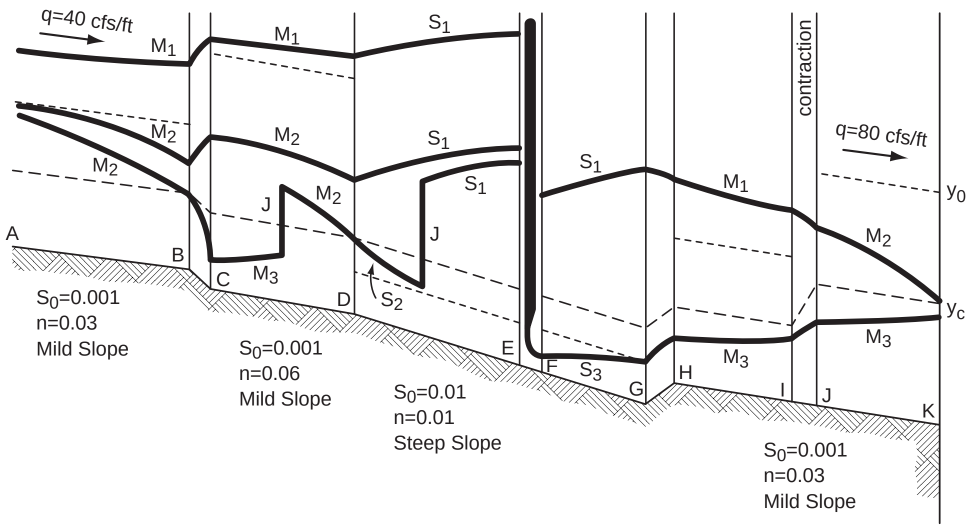

FIGURE 30.11 Complex open channel system with some possible water surface profiles. Profiles U1, U7, U12, D1, and D3 are shown. Refer to Tables 30.3 and 30.4 for details.

Phase I — Determination of Critical Depths and Normal Depths

Calculate the values of y_c and of y_o throughout the channel length:

From Eq. (30.21) $y_c = (q^2/g)^{1/3} = 3.68$ ft from *A* to *I*. Similarly, $y_c = 5.84$ ft from *J* to *K*.
From Eq. (30.33) $y_o = [nq/(1.486\ S_o^{1/2})]^{3/5} = 6.99$ ft on *AB*. Similarly, $y_o = 10.59$ ft on *CD*, 1.81 ft on *DE* and *FG*, 6.99 ft on *HI*, and 10.59 ft on *JK*.

The y_c and y_o lines are then sketched as shown in Fig. 30.11. Reaches *AB*, *CD*, *HI*, and *JK* are mild; reaches *DE* and *FG* are steep.

Phase II — Virtual Control Section (VCS) Determination

The adjective "virtual" indicates the uncertainty that these cross sections are actually "controlling" the water surface profile. Only at the end of the solution process will it be determined whether a virtual control section actually does control the flow. Recall that subcritical streams are controlled from downstream and supercritical streams are controlled from upstream.

Start the search process by inquiring whether the cross section *A* is a VCS. No indication of this is given by the sketch of Fig. 30.11. It is permissible to assume that *AB* is long enough to allow the depth at *A* to approach y_o.

Next consider the downward step *BC* as a possible site for a control section. It is obvious that if any cross section between *B* on *C* is a control section, it should be *B*, because if no loss of energy occurs between *B* and *C*, *B* is the point of lowest specific energy. This would make cross section *B* a *natural control*, that is a section that controls both upstream and downstream.

As seen in Fig. 30.11, there is the possibility of another natural control at *D*. Moving downstream, consider the segment *EF* where a sluice gate is situated. The flow from *E* to *F* is a strongly curved rapidly varied flow. If the efflux from the sluice gate is not submerged, then the water surface elevation at *F*, the vena contracta of the efflux, is known to be $C_c a$ where C_c is a contraction coefficient and *a* is the gate opening. Notice that the water depth at the cross section through *E* can then be found as the alternate of the depth at *F*, assuming negligible loss of energy occurs between *E* and *F*.

Go now to the upward step *GH*. If any cross section between *G* and *H* is a control section, it is *H* because, with no energy loss, *H* has the lowest specific energy. It could possibly be a natural control if the water surface profile would cut the cross section through *H* at the critical depth, but this is not possible because *HI* is a mild reach and no mild elementary solution starts at critical depth and moves away from it.

The same argument is valid for the contraction *IJ*: neither *I* nor *J* nor any cross section in between can be a control. The only section left is *K*, which is a virtual control. If *K* is a control the water depth is critical there.

Phase III — Profile Sketching

The action of the sluice gate allows the profile analysis of the channel to be divided into two parts. Fig. 30.11 shows the opening of the sluice gate to be in zone 3, implying that the flow at the vena contracta *F* is supercritical, unless a drowned hydraulic jump occurs. Assuming no loss of energy between *E* and *F* allows the depth at *E* to be determined, providing the relationship between the profiles upstream and downstream of the sluice gate. The depth upstream of the gate at *E* is the subcritical alternate depth corresponding to $E_E = E_F = y_F + q^2/(2gC_c^2a^2)$. Here y_F is the actual water depth, whether or not a submerged hydraulic jump occurs.

Table 30.3 lists 15 possible profiles upstream of the sluice gate. Fig. 30.11 illustrates U1, U7, and U12. Solutions U1 through U7 are entirely subcritical and are controlled from *E*. Solutions U8 through U12, U14, and U15 contain one or two hydraulic jumps. For a hydraulic jump to occur, the depths before and after the jump must be conjugate, i.e., they must have the same value of specific force. Solution U13 passes from subcritical to supercritical at *B* and from supercritical back to subcritical at *D* without a hydraulic jump. In the transcritical cases, the subcritical segments are controlled from downstream, and the supercritical parts are controlled from upstream.

Table 30.4 lists the eight possible water surface profiles downstream of the sluice gate. Fig. 30.11 shows solutions D1 and D3. Solutions D1 and D2, being entirely subcritical, are controlled from *K*. Solution D3 is completely supercritical and has its control at *F*. Solutions D4 through D8 have a supercritical portion controlled from *F* followed by a subcritical portion controlled at *K*.

Each of the 15 upstream solutions could be followed by any of the eight downstream solutions so that there are 120 possible qualitatively distinct water surface profiles for this problem. Only a detailed numerical calculation, based on the lengths of each prismatic reach and the dimensions of each transition, can determine which would actually occur.

TABLE 30.3 Water Surface Profiles Upstream of Sluice Gate

Profile	AB	BC	CD	DE	Comments
U1	M_1	Rise	M_1	S_1	Control at *E*
U2	M_1	Rise	M_2	S_1	Control at *E*
U3	Uniform	Rise	M_1	S_1	Control at *E*
U4	Uniform	Rise	Uniform	S_1	Control at *E*
U5	Uniform	Rise	M_2	S_1	Control at *E*
U6	M_2	Rise	M_1	S_1	Control at *E*
U7	M_2	Rise	M_2	S_1	Control at *E*
U8	M_2	Rise	M_2	S_2-J-S_1	y_c at *D*, Control at *D* and *E*
U9	M_2	Drop	M_3-J-M_1	S_1	y_c at *B*, Control at *B* and *E*
U10	M_2	Drop	M_3-J-uniform	S_1	y_c at *B*, Control at *B* and *E*
U11	M_2	Drop	M_3-J-M_2	S_1	y_c at *B*, Control at *B* and *E*
U12	M_2	Drop	M_3-J-M_2	S_2-J-S_1	y_c at *B* and *D*, Control at *B*, *D*, and *E*
U13	M_2	Drop	M_3	S_1	y_c at *B* and *D*, Control at *B*, *D*, and *E*
U14	M_2	Drop	M_3	S_2-J-S_1	y_c at *B*, Control at *B* and *E*
U15	M_2	Drop	M_3	S_3-J-S_1	y_c at *B*, Control at *B* and *E*

J = hydraulic jump

TABLE 30.4 Water Surface Profiles Downstream of Sluice Gate

Profile	FG	GH	HI	IJ	JK	Comments
D1	S_1	Drop	M_1	Drop	M_2	y_c at *K*, control at *K*
D2	S_1	Drop	M_2	Drop	M_2	y_c at *K*, control at *K*
D3	S_3	Rise	M_3	Rise	M_3	Control at *F*
D4	S_3	Rise	M_3	Rise	M_3-J-M_2	y_c at *K*, control at *F* and *K*
D5	S_3	Rise	M_3-J-M_1	Drop	M_2	y_c at *K*, control at *F* and *K*
D6	S_3	Rise	M_3-J-M_2	Drop	M_2	y_c at *K*, control at *F* and *K*
D7	S_3-J-S_1	drop	M_1	drop	M_2	y_c at *K*, control at *F* and *K*
D8	S_3-J-S_1	drop	M_2	drop	M_2	y_c at *K*, control at *F* and *K*

J = hydraulic jump

30.8 Methods of Calculation of Flow Profiles

The integration of the gradually varied flow Eq. (30.42) can be performed by either direct or numerical integration. The *direct integration method* is based on tables for the evaluation of the integral in Eq. (30.43). This integral cannot be evaluated in terms of elementary functions except for some simplified cases such as an infinitely wide rectangular channel, but a much wider class of channels can be treated by the introduction of special tabulated functions. This requires the definitions of the *hydraulic exponent for critical flow* (Chow, 1959, p. 66) and of the *hydraulic exponent for uniform flow* (Chow, 1959, p. 131). The *varied flow function* for regular open top prismatic channels was developed and tabulated by Bakhmeteff (1932) and extended by Chow (1959, p. 254–261, tables p. 641–655). Keifer and Chu applied the method of direct integration to circular pipes (Chow, 1959, p. 261–262, Tables p. 657–661). With the widespread availability of computers, the various methods of *numerical integration* are the most widely used methods of solving the gradually varied flow equation. We shall consider only two: the *direct step method*, a non-iterative method suitable only for prismatic channels, and the *standard step method*, an iterative method which can be used for both prismatic and nonprismatic channels. Either can be readily adapted for computer solution using a programming language such as Fortran or a spreadsheet program. To establish the equations for the direct step method, a new form of the gradually varied flow equation is derived by differentiating the specific energy *E* with respect to streamwise distance *x*.

$$\frac{dE}{dx} = \frac{d(H - z)}{dx} = -S_f + S_o \tag{30.44}$$

Multiplying by *dx* and integrating from station 1 to station 2 gives

$$E_2 - E_1 = S_o(x_2 - x_1) - \int_{x_1}^{x_2} S_f dx = \left(S_o - \langle S_f \rangle\right)(x_2 - x_1) \tag{30.45}$$

Here $\langle S_f \rangle$, is the average friction slope over the reach from 1 to 2. It can be approximated by the arithmetic average of the friction slopes calculated at 1 and 2 using the Manning equation to find S_f.

$$\langle S_f \rangle = \left(S_{f_1} + S_{f_2}\right)/2 = 0.5\left[\left(\frac{nQ}{BAR^{2/3}}\right)_1^2 + \left(\frac{nQ}{BAR^{2/3}}\right)_2^2\right] \tag{30.46}$$

Solving Eq. (30.45) for x_2 gives the equation for the direct step method.

$$x_2 = x_1 + \left(E_2 - E_1\right) / \left(S_o - \langle S_f \rangle\right) \tag{30.47}$$

In a prismatic channel, knowing y permits the calculation of A, $V = Q/A$, $E = y + \alpha V^2/2g$, and S_f. The calculation starts at x_1 where y_1 is known. By specifying y_2, Eq. (30.47) can be solved for the corresponding x_2. (The restriction to prismatic channels occurs because of the need to calculate $A(y_2)$ at an unknown x_2.) After y_2 is found, station 2 becomes the new station 1 and the process is repeated for the next reach. The computation should proceed upstream if the flow is subcritical and downstream otherwise. The accuracy of the solution is improved by using smaller y increments. Care must be taken to specify depths that lie within the zone of the solution, i.e., which do not cross the normal or critical depths.

Example 30.4 Direct Step Method

A trapezoidal channel having a bottom width b = 10 ft, side slopes 2 horizontal on 1 vertical, n = 0.02 and S_o = 0.0016 carries a discharge of 160 cfs. Assume α = 1.0. A dam creates a depth of 4.5 ft. Calculate the backwater profile.

1. Find y_o = 2.47 ft by iteration from Eq. (30.32):

$$y_o(10+2y_o)\left[\frac{y_o(10+2y_o)}{\left(10+2\sqrt{5}y_o\right)}\right]^{2/3} = \frac{0.02(160)}{1.486\sqrt{0.0016}}$$

2. Find y_c = 1.76 ft by iteration from Eq. (30.18):

$$\left[y_c(10+2y_c)\right]^3 \big/ \left[10+2(2)y_c\right] = 1(160)^2 \big/ 32.2$$

3. Find profile type: since $y_1 = 4.5$ ft $> y_o > y_c$ the solution is an M_1 curve.
4. The direct step method is illustrated in Table 30.5 using only two steps for brevity. For an accurate solution, much shorter steps should be used.

In contrast to the direct step method, the standard step method solves for the depth at specified values of x. This method is especially suited for natural rivers for which cross-sections have been surveyed only at specific stations. The calculation begins at x_1 where the depth y_1 and the total head H_1 are known. The standard step method uses two equations to calculate the unknown total head at x_2. The first is simply the definition of total head in a channel of small slope.

$$H_2' = z_2 + y_2 + \alpha_2 V_2^2 \big/ (2g) \tag{30.48}$$

The second equation is the mechanical energy Eq. (30.12) written in terms of the total head

$$H_2'' = H_1 - h_L = H_1 - \langle S_f \rangle (x_2 - x_1) - h_E \tag{30.49}$$

where h_E is a minor loss term to account for local expansions, contractions, or other localized irregularities.

The standard step method proceeds by choosing x_2, guessing a trial value for y_2, estimating α_2, finding A_2 and V_2, and calculating H_2' from Eq. (30.48). Then h_L is calculated based on the assumed y_2 and Eq. (30.49) is used to get H_2''. If the correct y_2 has been used, $H_2' = H_2''$. If the difference between the estimates is too

TABLE 30.5 Example 30.4: Direct Step Method

y ft	A ft²	V ft/s	R ft	E ft	E_2-E_1 ft	S_f	$\langle S_f \rangle$	So-$\langle S_f \rangle$	x_2-x_1 ft	x ft
4.5	85.5	1.871	2.838	4.554378		0.0001570				0
3.5	59.5	2.689	2.319	3.612285	–0.942093	0.0004243	0.0002907	0.0013093	–719.5	–719.5
2.5	37.5	4.267	1.771	2.782678	–0.829607	0.0015313	0.0009778	0.0006222	–1333.4	–2052.9

TABLE 30.6 Example 30.5: Standard Step Method

x ft	z ft	y ft	A ft^2	V ft/s	$\alpha V^2/2g$ ft	H' ft	H'' ft	S_f	$\langle S_f \rangle$	$\langle S_f \rangle (x_2\text{-}x_1)$ ft
0.0	500	4.500	85.500	1.871	0.054	504.554	504.554	0.0001570		
–719.5	501.15	3.500	59.500	2.689	0.112	504.763	504.764	0.0004243	0.0002907	–0.2091403
–2052.9	503.28	2.500	37.500	4.267	0.283	506.067	506.067	0.0015313	0.0009778	–1.3038386

large, a new y_2 is chosen and the process is repeated. After the calculation has converged, station 2 becomes the new station 1 and the process is repeated for the next reach. The computation should proceed upstream if the flow is subcritical and downstream otherwise. The accuracy of the solution is improved by using smaller x increments. Care must be taken to specify depths that lie within the zone of the solution.

Example 30.5 Standard Step Method

Although this method is generally used for nonprismatic channels, this example repeats the flow of Example 30.4, specifying in Table 30.6 the same x values calculated in Table 30.5. In Table 30.6 the y values are guessed and adjusted until H' and H'' are nearly equal. It can be seen that the agreement between the two methods is excellent. Table 6 contains only two steps for brevity. For an accurate solution, much smaller steps should be used.

30.9 Unsteady Flows

Applying the conservation of mass and conservation of streamwise momentum principles to unsteady flow in a prismatic open channel control volume of infinitesimal length which contains all of the liquid between an upstream flow area and a downstream flow area yields the *St. Venant equations*:

$$\frac{\partial y}{\partial t} + V\frac{\partial y}{\partial x} + D\frac{\partial V}{\partial x} = 0 \tag{30.50}$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} - gS_o + gS_f = 0 \tag{30.51}$$

This is one non-conservative form of the St. Venant equations; several other forms are also common.

The St. Venant equations are nonlinear hyperbolic partial differential equations that can be solved by the *method of characteristics*. This means that there are two families of special lines in x,t space called *characteristics* along which Eqs. (30.50) and (30.51) can be replaced by ordinary differential equations. The characteristics, along which information propagates, are themselves defined by the solution of the following ordinary differential equations:

$$\frac{dx}{dt} = V \pm (gD)^{1/2} = V \pm c \tag{30.52, 30.53}$$

The characteristics defined by these equations are known as the C^+ and C^- families. Excluding singularities, one characteristic of each family passes through each point in the computational domain, a subregion of the x,t plane. If a flow is subcritical, $V < c$ and the C^+ characteristics will have a positive slope in the computational domain while the C^- characteristics have a negative slope. In supercritical regions, $V > c$ so both families have positive slopes.

Because the St. Venant equations are partial differential equations with x and t as independent variables, both initial and boundary conditions must be specified. A typical initial condition would be the specification of y and V at every point of the channel at $t = 0$. To understand how the boundary conditions are specified, it is useful to consider the characteristic equations. One boundary condition must be specified for each characteristic where it enters the computational domain. For a subcritical flow, one condition must be specified at the upstream boundary and the second must be at the downstream boundary. For a supercritical flow, both conditions are specified at the upstream boundary; none can be given on the downstream boundary.

The method of characteristics can be solved by hand in some simplified cases, yielding important insights; but for more realistic applications two variants have evolved for computer solution: the *characteristic grid method* and the *method of specified intervals*. The programming of these methods is rather complex, and most recent work has used the *finite difference* or *finite element methods* in which the St. Venant differential equations themselves are discretized and solved numerically.

30.10 Software

Numerous software packages have been developed to solve problems of open channel hydraulics under the approximation of one-dimensional flow. Perhaps the most powerful program presently available is the U. S. Army Corps of Engineers Hydrologic Engineering Center River Analysis System (HEC-RAS). This Windows-based program can solve both steady and unsteady flows in single channels, dendritic systems, or complex networks. It can handle mixed subcritical and supercritical flows with hydraulic jumps, and can model the effects of obstructions such as bridge piers, culverts, and weirs. HEC-RAS has superceded the U. S. Army Corps of Engineers' HEC-2 (formerly the industry standard) and the Natural Resources Conservation Service's WSP-2, both of which are limited to steady state simulations. Another commonly used steady state program is WSPRO, developed by the U. S. Geological Survey (USGS) for the Federal Highway Administration. The USGS has developed several programs for unsteady open channel hydraulics including FEQ, BRANCH, and FOURPT. All of the programs mentioned in this section are in the public domain. Those that have not been superceded can be downloaded from the Websites of the appropriate agencies.

Defining Terms

Alternate depths — The subcritical or tranquil depth, y_T, and the supercritical or rapid depth y_R for a given specific energy and flowrate.

Conjugate depths — The depths before and after a hydraulic jump corresponding to a given total thrust (or specific force) and flowrate.

Control section — A section in a nonuniform flow at which the depth is known *a priori* and which serves as a boundary condition for the calculation of a water surface profile. Often the flow goes through critical depth at the control section. In general, a control is any feature which determines a relationship between depth and flowrate.

Critical depth — The depth of flow corresponding to an extremum (usually a minimum) of specific energy. The Froude number is unity at critical depth.

Critical slope — The slope of the bottom of a channel in which the normal depth and the critical depth coincide.

Critical velocity — The velocity occurring at critical depth. It is equal to the celerity or speed of propagation of an infinitesimal gravity wave in still shallow water.

Hydraulic jump — The sudden increase in depth that occurs when the flow passes from supercritical to subcritical, usually accompanied by turbulence and large energy loss.

Mild slope — A slope less than the critical slope.

Specific energy — The sum of the vertical component of the thickness of flow plus the velocity head.

Specific force — The total thrust per unit weight of liquid.

Steep slope — A slope larger than the critical slope.

Subcritical flow — The flow that occurs when the velocity is smaller than the critical velocity and the depth is larger than the critical depth.

Supercritical flow — The flow that occurs when the velocity is larger than the critical velocity and the depth is smaller than the critical depth.

Total thrust — The sum of the hydrostatic force on a flow area plus the momentum flowrate through that area.

References and Bibliography

Bakhmeteff, B.A. 1932. *Hydraulics of Open Channels*, Engineering Societies Monographs, McGraw-Hill, New York.

Chanson, H. 1999. *The Hydraulics of Open Channel Flow*, Arnold, London, UK.

Chaudhry, M.H. 1993. *Open-Channel Flow*, Prentice Hall, Englewood Cliffs, NJ.

Chow, Ven Te 1959. *Open-Channel Hydraulics*, McGraw-Hill, New York.

Fread, D.L., 1993. *Flood Routing* in Maidment, D.R., ed., *Handbook of Hydrology,* McGraw-Hill, New York.

French, R.H., 1985. *Open-Channel Hydraulics*, McGraw-Hill, New York.

Giorgini, A., 1987. *Open Channel Hydraulics*, unpublished class notes, School of Civil Engineering, Purdue University, West Lafayette, IN.

Gray, D.D., 2000. *A First Course in Fluid Mechanics for Civil Engineers,* Chapter 13, Water Resources Publications, Highlands Ranch, CO.

Henderson, F.M., 1966, *Open Channel Flow,* Macmillan, New York.

Ippen, A.T., 1950, *Channel Transitions and Controls,* in Rouse, H., ed. *Engineering Hydraulics*, John Wiley & Sons, New York.

Jain, S.C. 2001. *Open-Channel Flow*, John Wiley & Sons, New York.

Lai, C., 1986. *Numerical Modeling of Unsteady Open-channel Flow*, *Advances in Hydroscience,* Vol. 14, pp. 162–323, Academic Press, New York.

Nezu, I. and Nakagawa, H., 1993. *Turbulence in Open-Channel Flows,* (IAHR Monograph Series) A.A. Balkema, Rotterdam, Netherlands.

Subramanya, K. 1982. *Flow in Open Channels*, Vol. 1,Tata McGraw-Hill, New Delhi, India.

Sturm, T.W. 2001. *Open Channel Hydraulics*, McGraw-Hill, New York.

Yen, B.C., ed. 1992. *Channel Flow Resistance: Centennial of Manning's Formula*, Water Resources Publications, Littleton, CO.

Further Information

Gray (2000) provides an extended introductory treatment of open channel hydraulics.

Chow (1959) wrote an extensive text that remains the classic reference on the subject. Although Chow's numerical examples predate the computer era, his book contains much useful material which can be found nowhere else.

Henderson (1966) is a pedagogically superb textbook covering open channel flow including unsteady flows, flood routing, sediment transport, and physical hydraulic models.

French (1985), in addition to classical open channel hydraulics, provides chapters on turbulent diffusion and buoyant surface jets.

Chaudhry (1993) emphasizes unsteady flow problems and their solution by numerical methods. Also included are two-dimensional flows and finite elements applied to both one- and two-dimensional flows. Several simple Fortran computer programs are listed and provided on a diskette.

Chanson (1999) covers sediment transport, hydraulic modeling, and hydraulic structures in addition to open channel hydraulics.

Jain (2001) and **Sturm** (2001) are modern, rigorous treatments of steady and unsteady open channel hydraulics.

Yen (1992) is the proceedings of a symposium on the history of the Manning equation and on new developments in the calculation of resistance to flow in open channels.

Nezu and Nakagawa (1993) deal with open channel turbulence, boundary layers, and turbulent transport processes in rivers and estuaries.