20

Stress Distribution

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From a microscopic point of view, all soil bodies are composed of discrete particles that are connected to each other by forces of mutual attraction and repulsion. Given an initial state of equilibrium, if an additional force system is applied, deformations may occur; particle arrangements may be altered; and changes in the distribution of the resultant forces may take place. Although the intensity of the generated forces may be high at points of contact, their range of influence is very short. Generally, effects extend only over a distance of molecular size or *in the very near vicinity* of the particles. The internal forces generated at these points by the induced loadings are called **stresses.**

Because stresses reflect distributional changes induced by boundary loadings, they can be thought of as providing a measure of the transmission of induced energy throughout the body. The transformation from real soil bodies, composed of discrete particles, into a form such that useful deductions can be made through the exact processes of mathematics is accomplished by introducing the abstraction of a *continuum,* or *continuous medium,* and pleading *statistical macroscopic equivalents* through the introduction of *material properties*. In concept, the soil is assumed to be continuously distributed in the region of space under consideration. This supposition then brings the continuous space required by mathematical formulations and the material points of real bodies into conformity.

This chapter will present some basic solutions for increases in vertical stresses due to some commonly encountered boundary loadings. Solutions will be presented both from the linear theory of elasticity and from particulate mechanics. Special efforts have been made to present the results in easy-to-use form. Many solutions can also be found in computer software packages.

20.1 Elastic Theory (Continuum)

Three-Dimensional Systems

Soil in the neighborhood of a point is called **isotropic** if its defining parameters are the same in all directions emanating from that point. Isotropy reduces the number of elastic constants at a point to two: *E*, the *modulus of elasticity;* and μ, *Poisson's ratio* [Harr, 1966]. If the elastic constants are the same at all points within a region of a soil body, that region is said to be **homogeneous.** Invoking *linear constitutive* equations produces

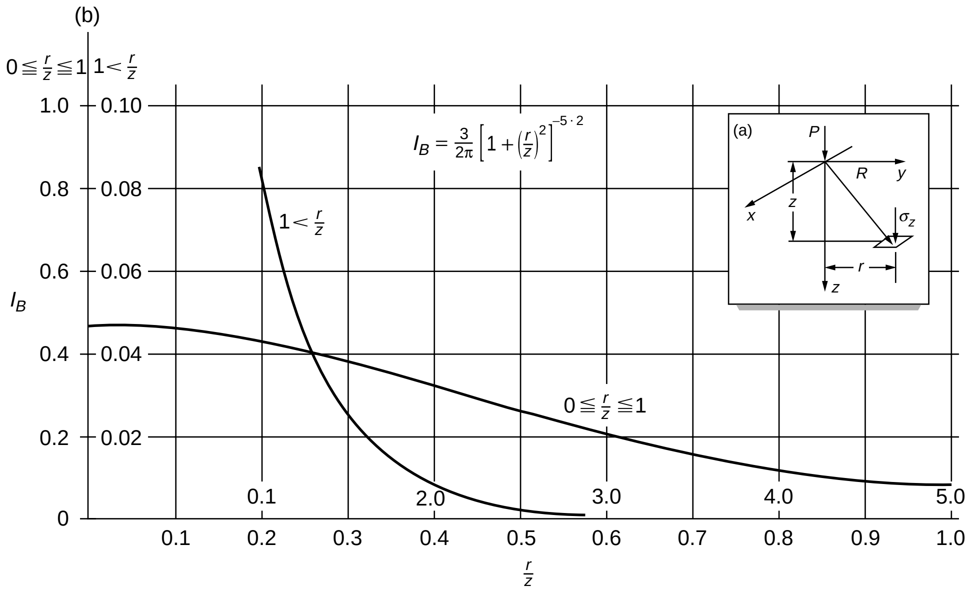

FIGURE 20.1 Concentrated force acting at and normal to surface; $\sigma_z = I_B P/z^2$.

$$
\begin{aligned}
\varepsilon_x &= \frac{1}{E}[\sigma_x - \mu(\sigma_y - \sigma_z)] \\
\varepsilon_y &= \frac{1}{E}[\sigma_y - \mu(\sigma_x - \sigma_z)] \\
\varepsilon_z &= \frac{1}{E}[\sigma_z - \mu(\sigma_x - \sigma_y)] \\
\gamma_{yz} &= \frac{2(1+\mu)}{E}\tau_{yz} \\
\gamma_{xz} &= \frac{2(1+\mu)}{E}\tau_{xz} \\
\gamma_{xy} &= \frac{2(1+\mu)}{E}\tau_{xy}
\end{aligned}
\tag{20.1}
$$

where ε_i and σ_i are the normal **strains** and stresses, respectively, and γ_i and τ_i are the shearing strains and stresses in the $i = x, y, z$ directions, respectively.

Force Normal to Surface (Boussinesq Problem)

Assuming that the z direction coincides with the direction of gravity, the vertical stress under a concentrated load P as shown in Fig. 20.1(a), where $R^2 = x^2 + y^2 + z^2$, is [Boussinesq, 1885]

$$\sigma_z = \frac{3Pz^3}{2\pi R^5} \tag{20.2}$$

It should be noted that the vertical normal stress (σ_z) is independent of the so-called elastic parameters. Equation (20.2) can also be written as

$$\sigma_z = I_B \frac{P}{z^2} \tag{20.3}$$

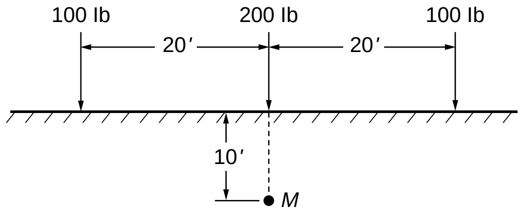

FIGURE 20.2 Example 20.1.

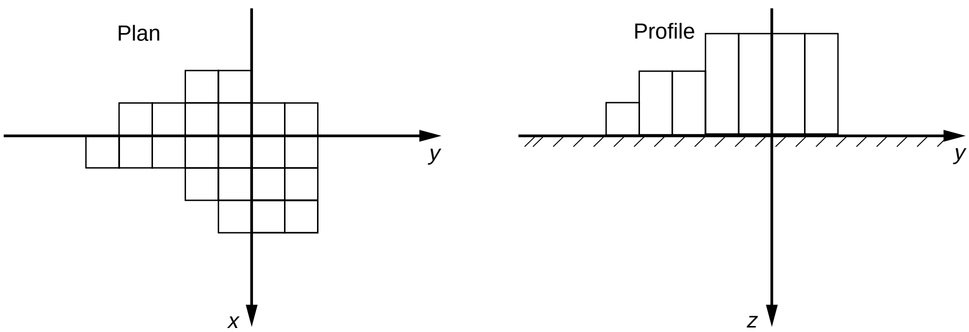

FIGURE 20.3 Distributed loads.

where

$$I_B = \frac{3}{2\pi}\left[1 + \left(\frac{r}{z}\right)^2\right]^{-5/2}$$

A plot of the **influence factor** I_B is given in Fig. 20.1(b).

Because superposition is valid, the effects of a number of normal forces acting on the surface of a body can be accounted for by adding their relative influence values.

Example 20.1

Find the vertical stress at point M in Fig. 20.2 due to the three loads shown acting in a line at the surface.

Solution. For the 200 lb force $r/z = 0$ and, from Fig. 20.1(b), $I_B = 0.478$. For the 100 lb forces, $r/z = 20/10 = 2$, $I_B = 0.009$. Hence, the corresponding vertical stress at point M is

$$\sigma_z = \frac{200(0.478)}{100} + \frac{2(100)(0.009)}{100} = 0.974 \text{ lb/ft}^2$$

By applying the principle of superposition, the increased stress due to distributed loadings over flexible areas at the surface can be obtained by dividing the loading into increments (see Fig. 20.3) and treating each increment as a concentrated force.

For a concentrated force parallel to the boundary surface (Fig. 20.4), the vertical stress [Cerruti, 1882] is

$$\sigma_z = \frac{3Qxz^2}{2\pi R^5} \tag{20.4}$$

By combining Eqs. (20.2) and (20.4), the increase in vertical stress, consistent with the assumptions of linear elasticity, can be determined for a concentrated force at the surface with any inclination.

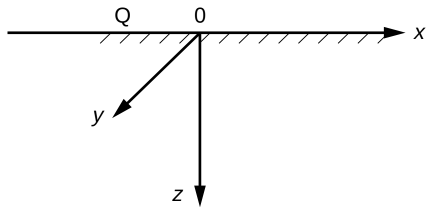

FIGURE 20.4 Concentrated force parallel to surface.

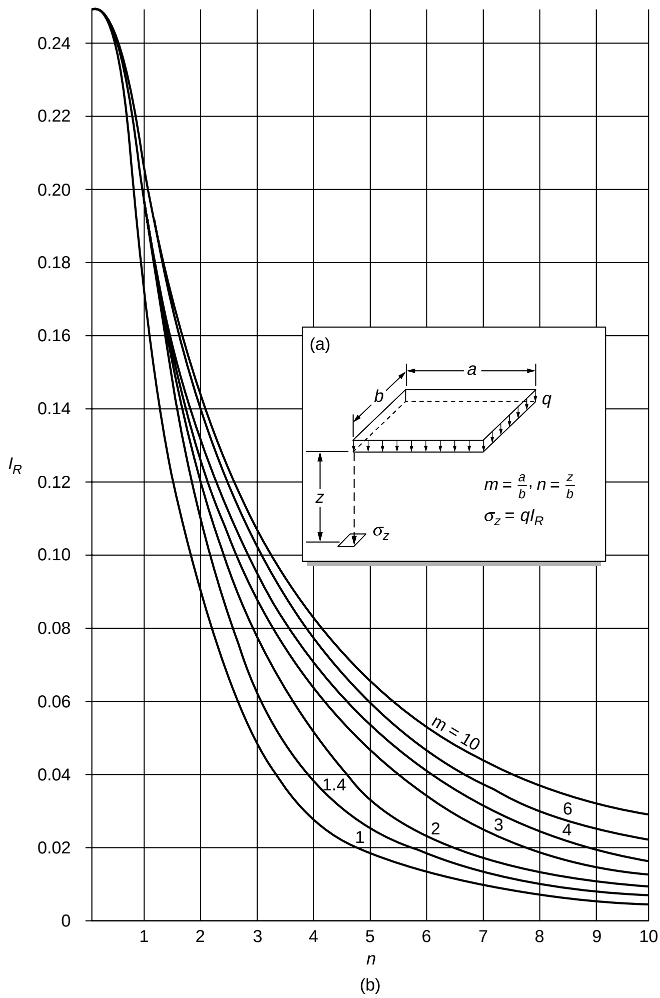

FIGURE 20.5 Normal uniform load over rectangular area. *Source:* After Steinbrenner, W. 1936 A rational method for the determination of the vertical normal stresses under foundations. *Proc. 1st Int. Conf. Soil Mech. Found. Eng.*, Vol. 2.

Uniform Flexible Load over Rectangular Area

The vertical stress under the corner of a flexible, uniformly loaded, rectangular area with sides *a* by *b*, as in Fig. 20.5(a), is

$$\sigma_z = \frac{q}{2\pi}\left[\frac{mn}{\sqrt{1+m^2+n^2}}\,\frac{1+m^2+2n^2}{(1+n^2)(m^2+n^2)} + \sin^{-1}\frac{m}{\sqrt{m^2+mn^2}\sqrt{1+n^2}}\right] \tag{20.5}$$

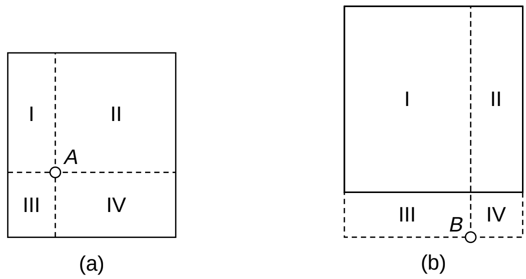

FIGURE 20.6 Vertical stress. (a) Interior. (b) Exterior.

where $m = a/b$ and $n = z/b$. This can also be written as

$$\sigma_z = qI_R$$

where I_R is a dimensionless influence factor. Figure 20.5(b) gives a plot of I_R as a function of the dimensionless ratios m and n. This form of the solution was given by Steinbrenner [1936].

By superposition, the distribution of vertical stress can be obtained anywhere under uniformly loaded, flexible, rectangular loadings. Two cases will be examined.

Vertical Stress for a Point Interior to a Loaded Area

For this case, Fig. 20.6(a), the influence factor I_R is determined from Fig. 20.5(b) for each of the rectangular areas (Roman numerals) with their corresponding m and n values and add them to obtain

$$\sigma_{zA} = q(I_{R\mathrm{I}} - I_{R\mathrm{II}} + I_{R\mathrm{III}} + I_{R\mathrm{IV}})$$

Vertical Stress for a Point Exterior to a Loaded Area

For a point such as B in Fig. 20.6(b), the stress is computed as

$$\sigma_{zB} = q(I_{R\mathrm{I+III}} + I_{R\mathrm{II+IV}} - I_{R\mathrm{III}} - I_{R\mathrm{IV}})$$

Influence Chart Normal Load

Although the above procedure can also be used to approximate irregularly shaped loadings, an influence chart, developed by Newmark [1942] greatly reduces the work required. Such a chart is shown in Fig. 20.7. To use the chart, the shape of the loading is drawn (generally on tracing paper) to scale so that the length *AB* on the chart represents the depth *z* at which the vertical stress is desired. The scaled drawing is then oriented so that the point under which the stress is sought is directly over the center of the circles on the chart. The number of blocks covered by the area of loading multiplied by the influence value (each square is $0.001q$ for this chart) yields, for the vertical stress, that part of the uniformly distributed load. By repeating this procedure and varying the size of the drawings, the complete distribution of vertical stresses can be found with depth. A separate drawing of the area is required for each depth. Although the chart was developed for uniform loadings over the whole area, the effects of varying loads can be treated as a series of uniform loadings.

Two-Dimensional Systems

In soil mechanics and foundation engineering, problems such as the analysis of retaining walls or of continuous footings and slopes generally offer one dimension very large in comparison with the other two. Hence, if boundary forces are perpendicular to and independent of this dimension, all cross sections will be the same. In Fig. 20.8 the *y* dimension is taken to be large, and it is assumed that the state of affairs existing in the *xz* plane holds for all planes parallel to it. These conditions are said to define the state of **plane strain.**

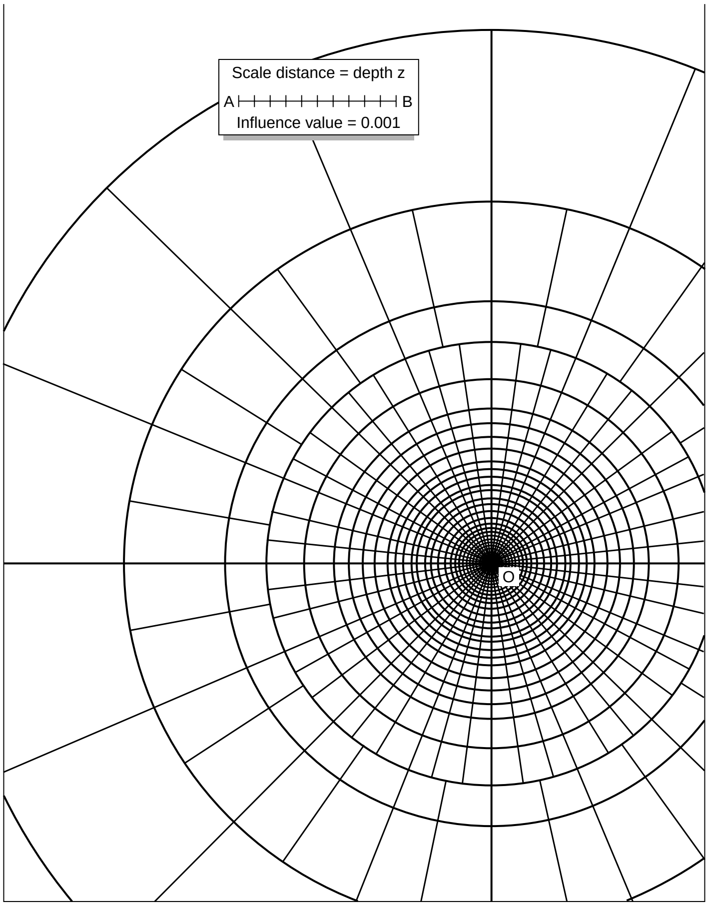

FIGURE 20.7 Influence chart for uniform vertical normal stress. *Source:* After Newmark, N. M. 1942. Influence charts for computation of stresses in elastic foundations. *Univ. of Illinois Bull. 338.*

Infinite Line Load Normal to Surface (Flamant Problem)

Figure 20.9(a) shows a semi-infinite plane with a concentrated load (line load) of intensity P (per unit run) normal to the surface. The solution, given by Flamant [1892], is

$$\sigma_z = \frac{2Pz^3}{\pi(x^2 + z^2)^2} \tag{20.6}$$

In Fig. 20.9(b) is given a plot of this equation taken as $\sigma_z = I_{\sigma z}(P/z)$. Also plotted are the influence factors for $I_{\sigma x}$ and the tangential stress, I_τ.

Example 20.2

Find the vertical normal stress at a depth corresponding to point A in Fig. 20.10, due to the three normal line forces N_1, N_2 and N_3.

Solution. The influence curve is oriented so that the origin (point O) is directly above point A (as shown). The magnitude of the force multiplied by the ordinate of the σ_z curve immediately under it for $P = 1$ (such as $\overline{B_1C_1}$ under N_1) gives that part of the stress at point A due to the particular force. By superposition the total vertical stress at point A is obtained as the algebraic sum of the contributions of each of the

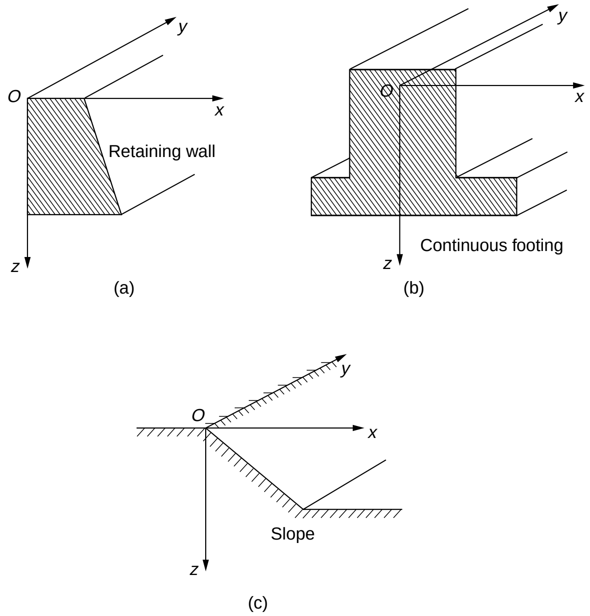

FIGURE 20.8 Examples of plane strain models.

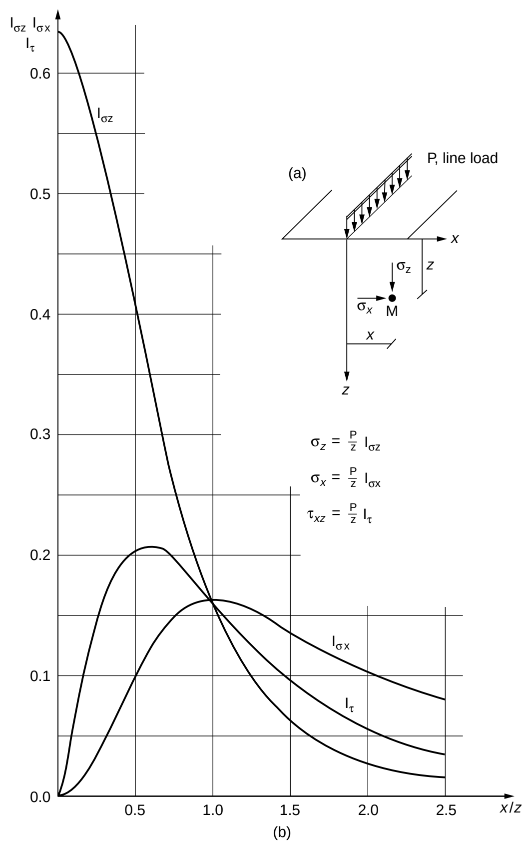

FIGURE 20.9 Infinite line load, normal to surface.

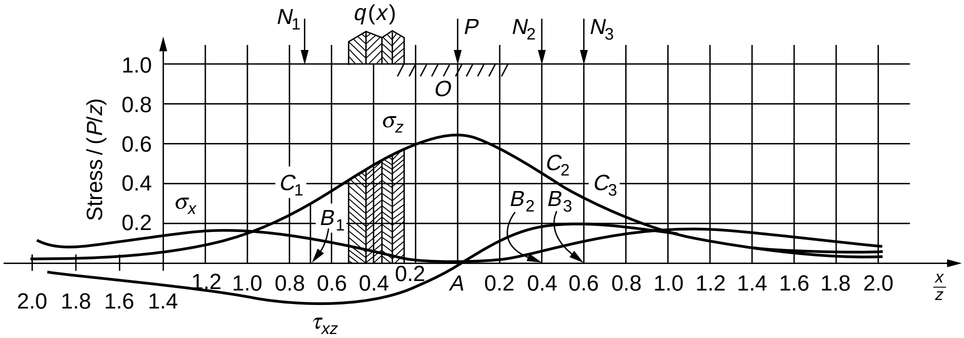

FIGURE 20.10 Example 20.2.

forces. Thus, for the three forces N_1, N_2, and N_3, the increase in vertical stress at point A is $\sigma_{zA} = (N_1(\overline{B_1C_1}) + N_2(\overline{B_2C_2}) + N_3(\overline{B_3C_3}))$.

Superposition also permits the determination of the stress under any distribution of flexible surface loading. For example, to obtain the vertical stress at point A in Fig. 20.10 under the distributed line load $q(x)$, the load is first divided into a number of increments, and each increment is then treated, as just described, as a concentrated force.

Infinite Strip of Width *b*

An influence chart [Giroud, 1973] provides the stress σ_z at point $P(x, z)$ in Fig. 20.11, due to the distributed vertical load q over a flexible strip of width b in the form

$$\sigma_z = I_{\sigma z} \cdot q \qquad (20.7)$$

Plots of this equation as well as the influence factor $I_{\sigma x}$ are also shown.

20.2 Particulate Medium

Two-Dimensional Systems

Infinite Line Load Normal to Surface

On the basis of a probabilistic model, Harr [1977] gave the expression for the vertical stress due to a concentrated load (line load) of intensity P (per unit run) normal to the surface (see Fig. 20.9) as

$$\bar{s}_z = \frac{P}{z\sqrt{2\nu\pi}} \exp\left[-\frac{x^2}{2\nu z^2}\right] \qquad (20.8)$$

The symbol $\bar{s}_z$ will be used to designate the vertical normal stress for the probabilistic model. The overlaid bar implies that this is the expected value of the stress. The customary symbol σ_z will be reserved for the linear theory of elasticity. Harr called the parameter ν (Greek nu) the *coefficient of lateral stress* and showed that it can be approximated by the conventional *coefficient of lateral earth pressure, K*. For comparisons with the linear theory of elasticity, $\nu \approx 1/3$.

Example 20.3

Compare the distribution of the vertical normal stress of the particulate theory with that given by the theory of elasticity, Eq.(20.6), using $\nu = K_{\text{active}} = 1/3$.

Solution. Some numerical values of these two functions are given in Table 20.1. The correspondence is seen to be very good.

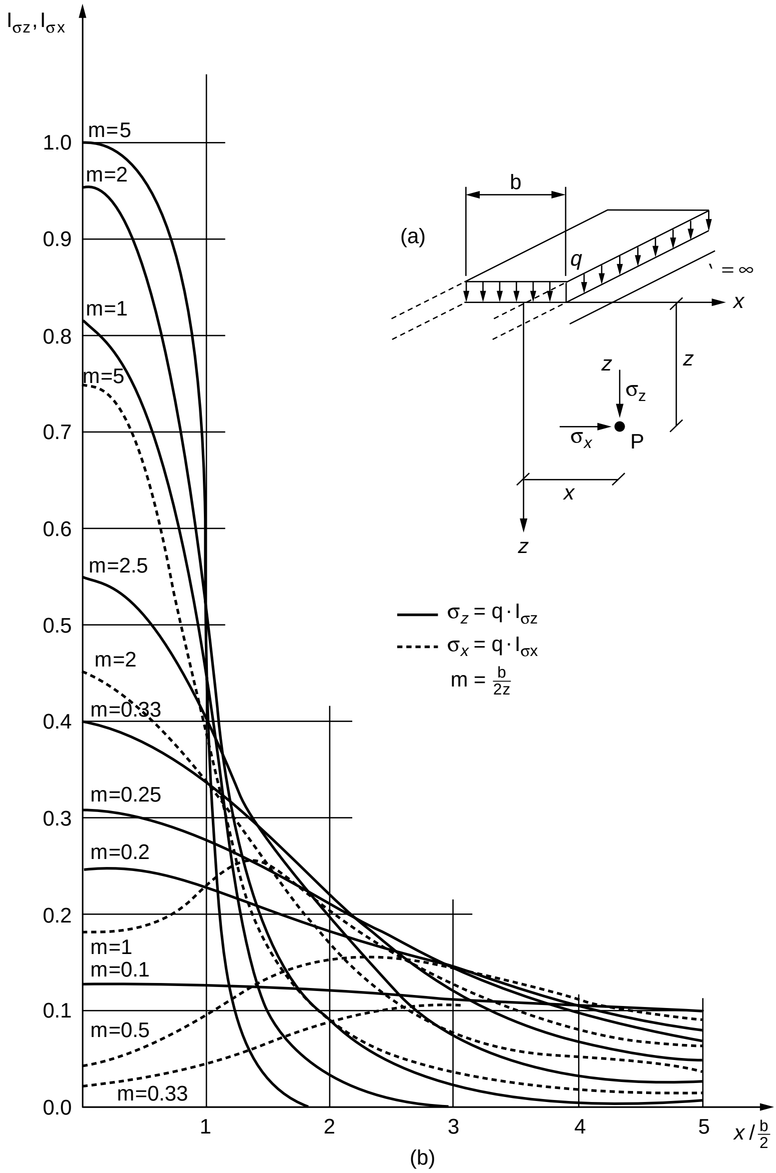

FIGURE 20.11 Stresses under infinite strip loading. *Source:* Giroud, J. P. 1973. *Tables pour le Calcul des Fondations.* 2 Vols. Dunod, Paris.

Infinite Strip of Width *b*

The counterpart of Eq. (20.7), Fig. 20.11, for the particulate model is

$$\bar{s}_z = q\left\{\psi\left[\frac{x+b/2}{z\sqrt{\nu}}\right] - \psi\left[\frac{x-b/2}{z\sqrt{\nu}}\right]\right\} \tag{20.9a}$$

and under the center line ($x = 0$),

$$\bar{s}_z(0) = 2q\psi\left[\frac{b}{2z\sqrt{\nu}}\right] \tag{20.9b}$$

TABLE 20.1 Comparison of Particulate and Elastic Solutions (Infinite Normal Line Load)

$\frac{x}{z}$	$z\bar{s}_z/P = \left(\frac{1}{2\nu\pi}\right)^{1/2}\exp\left[-\frac{x^2}{2\nu z^2}\right]$	$z\sigma_z/P = \left(\frac{2}{\pi}\right)\left[1+\frac{x^2}{z^2}\right]^{-2}$
0.0	0.69	0.64
0.1	0.68	0.62
0.2	0.65	0.59
0.3	0.60	0.54
0.4	0.54	0.47
0.5	0.47	0.41
0.6	0.40	0.34
0.8	0.26	0.24
1.0	0.15	0.16
1.2	0.08	0.11
1.5	0.02	0.06
1.8	0.01	0.04
2.0	0.004	0.03

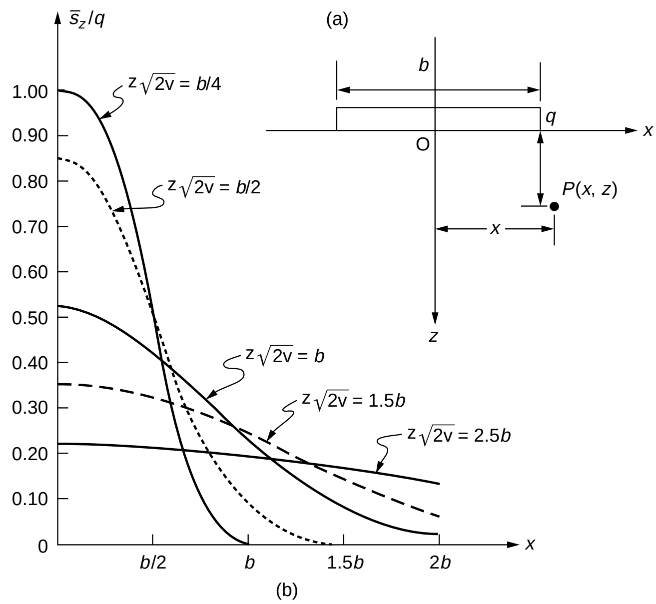

FIGURE 20.12 Uniform normal load over strip.

Values of the function ψ [ ] are given in Table 16.3 of Chapter 16. For example, ψ[0.92] = 0.321. In Fig. 20.12 a plot of $\bar{s}_z/q$ is given for a range of values (compare with Fig. 20.11 for ν = 1/3).

Example 20.4

Find the expected value for the vertical normal stress at point $x = 2$ ft, $z = 4$ ft for a uniformly distributed load $q = 100$ lb/ft² acting over a strip 8 ft wide. Take $\nu = \pi/8$.

Solution. Equation (20.9a) becomes for this case

$$\bar{s}_z = 100\left\{\psi\left[\frac{2+4}{4(0.63)}\right] - \psi\left[\frac{(2-4)}{4(0.63)}\right]\right\}$$

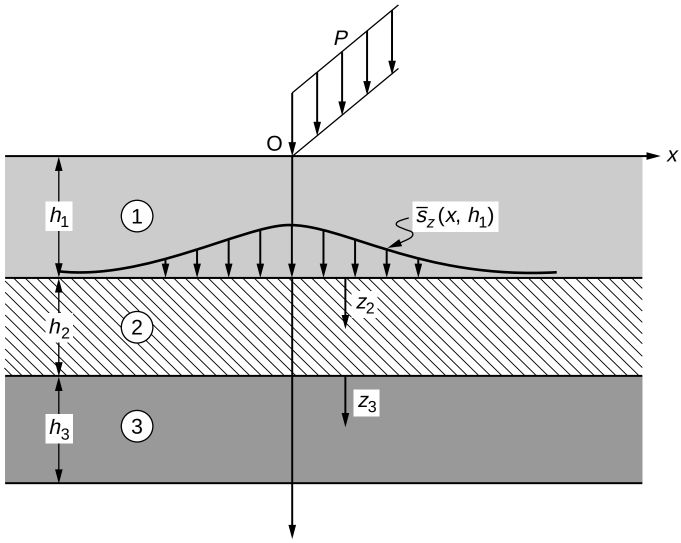

FIGURE 20.13 Multilayer system.

From Table 16.3 of Chapter 16,

$$\bar{s}_z = 100\{0.4916 + 0.2881\} = 78.0 \text{ lb/ft}^2$$

The theory of elasticity gives for the vertical stress in this case, from Fig. 20.11, $\sigma_z = 73$ lb/ft².

Multilayer System

Given a system with N layers in which the ith layer has thickness h_i and coefficient of lateral stress ν_i, Fig. 20.13, Kandaurov [1966] showed that the equivalent thickness for the upper $N - 1$ layers can be found as

$$\bar{h}_{N-1} = h_1\sqrt{\frac{\nu_1}{\nu_N}} + h_2\sqrt{\frac{\nu_2}{\nu_N}} + \cdots + h_{N-1}\sqrt{\frac{\nu_{N-1}}{\nu_N}} \tag{20.10}$$

Hence, the expected vertical normal stress in the Nth layer (z_N is the vertical distance into the Nth layer as measured from its upper boundary), for a line load P normal to the layers, will be

$$\bar{s}_z(x, z) = \frac{P}{\bar{h}_{N-1} + z_N}\sqrt{\frac{1}{2\pi\nu_N}}\exp\left[-\frac{x^2}{2\nu_N(\bar{h}_{N-1} + z_N)^2}\right] \tag{20.11}$$

Example 20.5

A three-layer system is subjected to a line load of 9000 lb/ft with the following information: $h_1 = 1$ ft, $\nu_1 = 0.4$; $h_2 = 2$ ft, $\nu_2 = 0.3$; $\nu_3 = 0.2$, h_3 is unbounded. Find the expected value for the vertical normal stress 3 ft into the third layer immediately under the line load.

Solution. From Eq. (20.10), the equivalent thickness is

$$\bar{h} = 1\sqrt{\frac{0.4}{0.2}} + 2\sqrt{\frac{0.3}{0.2}} = 3.86 \text{ ft}$$

Thus, from Eq. (20.11) the expected vertical normal stress at a depth of 3 ft in the third layer, immediately under the line load ($x = 0$), is

$$\bar{s}_z = \frac{9000}{3.86 + 3}\sqrt{\frac{1}{2\pi(0.2)}} = 1169.7 \text{ lb/ft}^2$$

The theory of elasticity would give for this case, assuming a single homogeneous layer, Eq. (20.6), σ_z = 954.9 lb/ft^2.

Three-Dimensional System

Force Normal to Surface

The probabilistic counterpart of the Boussinesq solution, Eq. (20.2), for the expected vertical normal stress is

$$\bar{s}_z = \frac{P}{2\pi\nu z^2}\exp\left[-\frac{r^2}{2\nu z^2}\right] \qquad (20.12)$$

where $r^2 = x^2 + y^2$.

Example 20.6

Find the coefficient ν in Eq. (20.12) that would yield the same value for the maximum expected value of normal vertical stress as that given by the theory of elasticity.

Solution. For the theory of elasticity Eq. (20.2) gives as the maximum vertical stress

$$\sigma_{z_{\max}} = \frac{3P}{2\pi z^2}$$

Equation (20.12) gives

$$\bar{s}_{z_{\max}} = \frac{P}{2\pi\nu z^2}$$

Equating the two produces $\nu = 1/3$.

Example 20.7

Compare the lateral attenuation of vertical normal stress for the probabilistic theory with that given by the theory of elasticity (for three dimensions) using $\nu = 1/3$.

Solution. Substituting $\nu = 1/3$ into Eq. (20.12) obtains

$$\bar{s}_z = 3P/2\pi z^2 \exp[-3r^2/z^2]$$

The theory of elasticity, Eq. (20.2), yields $\sigma_z = 3P/2\pi z^2(1 + r^2/z^2)^{-5/2}$. Some values of the two functions are given in Table 20.2. In Fig. 20.14 a nomograph of the expected vertical normal stress is given for a range of ν values.

Example 20.8

Find the expected value of the vertical normal stress at the point r = 6 ft, z = 10 ft if $\nu = 1/5$ under a concentrated vertical force of 100 lb.

Solution. The arrow in Fig. 20.14 indicates that for the given conditions $\nu z^2 \bar{s}_z/P = 0.06$. Hence, the expected value of the vertical normal stress is $\bar{s}_z$ = 0.065(100)(5)(1/100) = 0.33 lb/ft^2. The theory of elasticity gives for this case, Fig. 20.1, σ_z = 0.22 lb/ft^2.

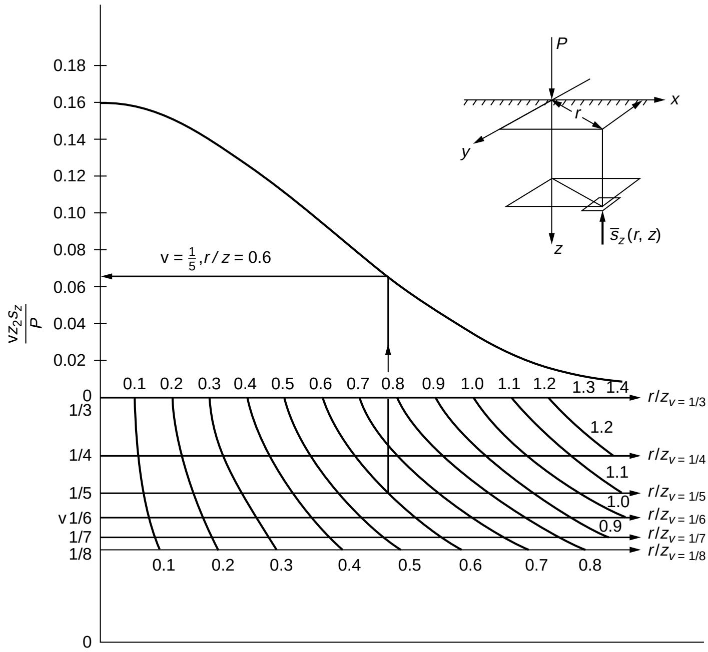

FIGURE 20.14 Expected value of vertical stress, particulate model.

TABLE 20.2 Comparison of Particulate and Elastic Solutions (Normal Point Load)

r/z	$z^2\bar{s}_z/P$	$z^2\sigma_z/P$
0.0	0.48	0.48
0.1	0.47	0.47
0.2	0.45	0.43
0.4	0.38	0.33
0.6	0.28	0.22
0.8	0.18	0.14
1.0	0.11	0.08
1.2	0.06	0.05
1.5	0.02	0.03
2.0	0.001	0.01

Distributed Vertical Loads at Surface

Normal Uniform Load over a Rectangular Area

The probabilistic counterpart of Eq. (20.5) for the expected vertical normal stress under the corner of a rectangular area with sides a by b, as in Fig. 20.5(a), is

$$\bar{s}_z/q = \psi\left[\frac{a}{z\sqrt{v}}\right]\psi\left[\frac{b}{z\sqrt{v}}\right] \tag{20.13}$$

Values of the function ψ[ ] are given in Table 16.3 of Chapter 16.

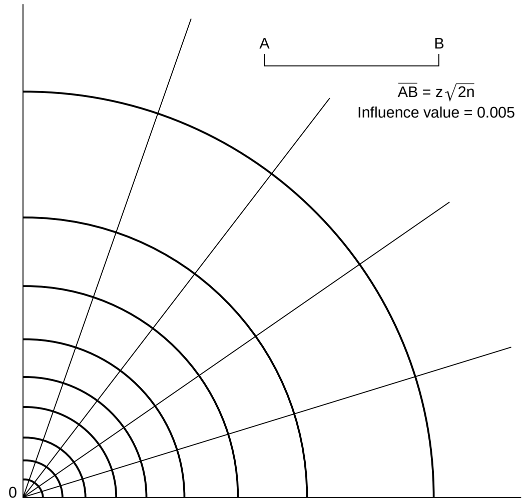

FIGURE 20.15 Influence chart.

Example 20.9

A uniformly distributed vertical load of magnitude 25 lb/ft^2 acts over an area 4 ft by 8 ft at the surface of a particulate medium. Find the expected vertical normal stress 6 ft below the center of the area. Take $\nu = 1/3$.

Solution. From Eq. (20.13), taking the 4 × 8 ft area as four areas, each with sides 2 × 4 ft, we have

$$\bar{s}_z = 4(25)\psi\left[\frac{2}{6\sqrt{1/3}}\right]\psi\left[\frac{4}{6\sqrt{1/3}}\right] = 100\psi[0.58]\psi[1.15] = 100(0.219)(0.375)$$

and $\bar{s}_z = 8.2$ lb/ft^2. The theory of elasticity, as shown in Fig. 5(b), yields for this case $\sigma_z = 7.3$ lb/ft^2.

Uniform Normal Load over Circular Area

The expected vertical normal stress under the center of a circular area, of radius a, subject to a uniform normal load q was obtained by Kandaurov [1959] as

$$\bar{s}_z(z) = q\left[1 - \exp\left(-\frac{a^2}{2\nu z^2}\right)\right] \tag{20.14}$$

The theory of elasticity gives for this case [Harr, 1966]

$$\sigma_z(z) = q\left[1 - \frac{1}{[(a/z)^2 + 1]^{3/2}}\right] \tag{20.15}$$

The influence chart shown in Fig. 20.15 was prepared using Eq. (20.14). The chart is used in the same way as was previously explained in reference to Fig. 20.7. For layered systems, the equivalent depth $\overline{h}_{N-1} + z_N$, Eq. (20.10) can be substituted for z.

Example 20.10

Compare Eqs. (20.14) and (20.15) for $\nu = 1/3$.

Solution. Some values are given in Table 20.3. Again we see that for $\nu = 1/3$ the correspondence between the two is very good for the special value of the coefficient of lateral stress.

TABLE 20.3 Comparison of Particulate and Elastic Solutions (Normal Vertical Distributed Load)

a/z	$\bar{s}_z/q$	σ_z/q
0.2	0.06	0.06
0.4	0.21	0.20
0.6	0.42	0.37
0.8	0.62	0.52
1.0	0.78	0.65
1.2	0.88	0.74
1.4	0.95	0.80
2.0	1.00	0.91

Parabolic Loading over Circular Area

The parabolic form of the loading on the surface is defined as

$$\bar{s}_z(r, 0) = \begin{cases} q\left(1 - \dfrac{r^2}{a^2}\right) & 0 \le r \le a \\ 0 & r > a \end{cases} \tag{20.16a}$$

For the special, but important, case under the center of the loading ($r = 0$), the expected vertical stress reduces to the expression

$$\bar{s}_z(0, z) = q\left\{1 - \frac{2\nu z^2}{a^2}\left[1 - \exp\left(-\frac{a^2}{2\nu z^2}\right)\right]\right\} \tag{20.16b}$$

Tangential (Horizontal Force at Surface)

For a concentrated force of intensity Q parallel to the boundary surface (the equivalent of Cerruti's solution, Fig. 20.4), Muller [1962] obtained the expression

$$\bar{s}_x = \frac{Q}{2\pi\nu x^2}\exp\left[-\frac{y^2 + z^2}{2\nu x^2}\right] \tag{20.17}$$

Example 20.11

Compare the expression for the vertical stress in Eq. (20.17) with that given by Cerruti (Eq. 20.4) for the line $y = 0$, $z = 1$.

Solution. Cerruti obtained $\sigma_z = 3Qxz^2/2\pi R^5$ where $R^2 = x^2 + y^2 + z^2$. From Eq. (20.17) for $y = 0$, $z = 1$,

$$\frac{2\pi\bar{s}_z}{Q} = \frac{1}{\nu x^2}\exp\left[-\frac{1}{2\nu x^2}\right]$$

It can be shown that $\lim_{x\to 0}\bar{s}_z = 0$. Some numerical values for two values of ν are given in Table 20.4. It is seen that the vertical stresses for the developed theory are smaller than those given by the theory of elasticity in the near vicinity of the applied load. However, the general patterns of the distributions are quite similar away from the point of loading.

TABLE 20.4 Comparison of Particulate and Elastic Solutions (Concentrated Force Parallel to Boundary Surface)

x	$2\pi\sigma_z/Q$	$2\pi\bar{s}_z/Q$	
		$\nu = 1/3$	$\nu = 1/2$
0	0	0.00	0.00
0.25	0.64	0+	0+
0.5	0.86	0.03	0.15
0.75	0.74	0.37	0.60
1.0	0.53	0.67	0.74
1.5	0.24	0.68	0.57
2.0	0.11	0.52	0.39
2.5	0.05	0.38	0.27

Defining Terms

Homogeneous — Having elastic parameters the same at all points in a region of a body.
Influence factor — A dimensionless cluster of factors generally obtained from a graphical presentation.
Isotropic — Having defining parameters the same in all directions emanating from a point.
Plane strain — A two-dimensional simplification used when there is little or no variation in strain in the direction normal to the plane.
Strain — Changes in displacements due to changes in stress.
Stress — Intensity of internal forces within a soil body induced by loadings. In the classical theory of elasticity stress is single-valued. The particulate theory deals with a distribution, the best measure of which is the *expected* (mean) value.

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Further Information

Many compilations of elastic solutions are available in the literature; primary among these is that of Poulos and Davis [1974], which has recently been reprinted (see references). Historical background can be found in Timoshenko [1953], Todhunter and Pearson [1960], and Love [1944]. Many particulate solutions can be found in Kandaurov [1988] and Harr [1977].