16

Accounting for Variability (Reliability)

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16.1 Introduction

The trend in civil engineering today, more than ever before, is toward providing economical designs at specified levels of safety. Often these objectives necessitate a prediction of the performance of a system for which there exists little or no previous experience. Current design procedures, which are generally learned only after many trial-and-error iterations, lacking precedence, often fall short of expectations in new or alien situations. In addition, there is an increasing awareness that the raw data, on which problem solutions are based, themselves exhibit significant variability. It is the aim of this presentation to demonstrate how concepts of probability analysis may be used to supplement the geotechnical engineer's judgment in such matters.

16.2 Probabilistic Preliminaries

Fundamentals

Within the context of engineering usage there are two primary definitions of the concept of probability: *relative frequency* and *subjective* interpretation. Historically, the measure first offered for the probability of an outcome was its relative frequency. If an outcome A can occur T times in N equally likely trials, the probability of the outcome A is

$$P[A] = \frac{T}{N} \tag{16.1a}$$

Implied in Eq. (16.1a) is that the probability of an outcome A equals the number of outcomes favorable to A (within the meaning of the experiment) divided by the total number of possible outcomes, or

$$P[A] = \frac{\text{Favorable outcomes}}{\text{Total possible outcomes}} \tag{16.1b}$$

Example 16.1

Find the probability of drawing a red card from an ordinary well-shuffled deck of 52 cards.

Solution. Of the 52 equally likely outcomes, there are 26 favorable (red card) outcomes. Hence,

$$P[\text{drawing red card}] = \frac{26}{52} = \frac{1}{2}$$

Understood in the example is that if one were to repeat the process a large number of times, a red card would appear in one-half of the trials. This is an example of the relative frequency interpretation. Now, what meaning could be associated with the statement, "The probability of the failure of a proposed structure is 1% ($P[\text{failure}] = 0.01$)"? The concept of repeated trials is meaningless: the structure will be built only once, and it will either fail or be successful during its design lifetime. It cannot do both. Here we have an example of the subjective interpretation of probability. It is a measure of information as to the likelihood of the occurrence of an outcome.

Subjective probability is generally more useful than the relative frequency concept in engineering applications. However, the basic rules governing both are identical. As an example, we note that both concepts specify the probability of an outcome to range from zero to one, inclusive. The lower limit indicates there is no likelihood of occurrence; the upper limit corresponds to a certain outcome.

$$\langle \textbf{Axiom I} \rangle \qquad 0 \le P[A] \le 1 \tag{16.2a}$$

The certainty of an outcome C is a probability of unity:

$$\langle \textbf{Axiom II} \rangle \qquad P[C] = 1 \tag{16.2b}$$

Equations (16.2a) and (16.2b) provide two of the three axioms of the theory of probability. The third axiom requires the concept of *mutually exclusive* outcomes. Outcomes are mutually exclusive if they cannot occur simultaneously. The third axiom states the probability of the occurrence of the sum of a number of mutually exclusive outcomes $A(1), A(2), \ldots, A(N)$ is the sum of their individual probabilities (addition rules), or

$$\langle \textbf{Axiom III} \rangle \qquad P[A(1) + A(2) + \cdots + A(N)] = P[A(1)] + P[A(2)] + \cdots + P[A(N)] \tag{16.2c}$$

As a very important application of these axioms consider the proposed design of a structure. After construction, only one of two outcomes can be obtained in the absolute structural sense: either it is successful or it fails. These are mutually exclusive outcomes. They are also exhaustive in that, within the sense of the example, no other outcomes are possible. Hence, the second axiom, Eq. (16.2b), requires

$$P[\text{success} + \text{failure}] = 1$$

Since they are mutually exclusive, the third axiom specifies that

$$P[\text{success}] + P[\text{failure}] = 1$$

The **probability** of the success of a structure is called its **reliability**, R. Symbolizing the probability of failure as $p(f)$, we have the important expression

$$R + p(f) = 1 \tag{16.3}$$

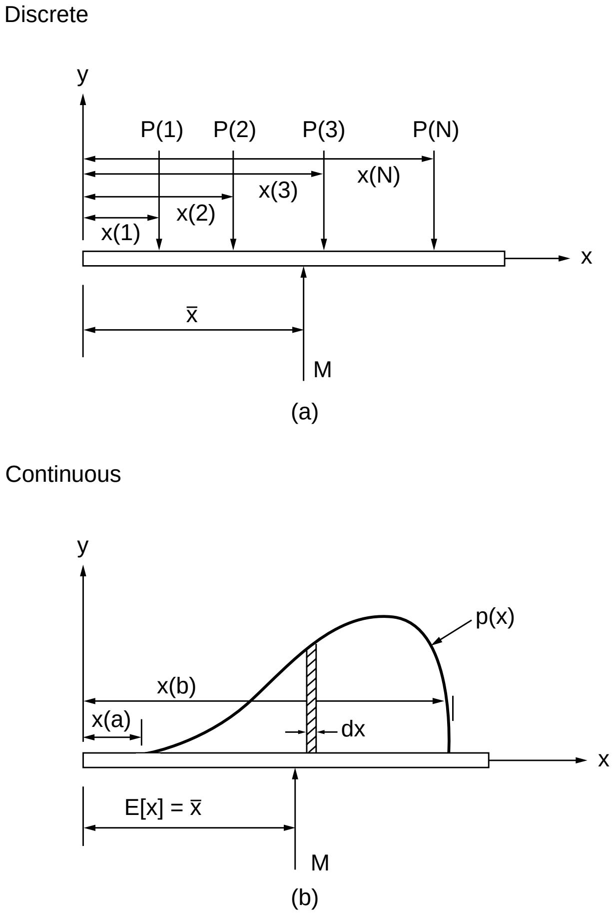


FIGURE 16.1 Equilibrant for discrete and continuous distributions.

Moments

Consider a system of *discrete* parallel (vertical) forces, $P(1)$, $P(2)$, …, $P(N)$, acting on a rigid beam at the respective distances $x(1)$, $x(2)$, …, $x(N)$, as in Fig. 16.1(a). From statics we have that the magnitude of the *equilibrant, M*, is

$$M = \sum_{i=1}^{N} P(i) \tag{16.4a}$$

and its point of application, $\bar{x}$, is

$$\bar{x} = \frac{\sum_{i=1}^{N} x(i) P(i)}{\sum_{i=1}^{N} P(i)} \tag{16.4b}$$

For a continuously distributed parallel force system [Fig. 16.1(b)] over a finite distance, say from $x(a)$ to $x(b)$, the corresponding expressions are

$$M = \int_{x(a)}^{x(b)} p(x)\,dx \tag{16.5a}$$

and

$$\bar{x} = \frac{\int_{x(a)}^{x(b)} xp(x)dx}{M} \tag{16.5b}$$

Suppose now that the discrete forces $P(i)$ in Fig. 16.1(a) represent the frequencies of the occurrence of the N outcomes $x(1), x(2), \ldots, x(N)$. As the distribution is exhaustive, from axiom II, Eq. (16.2b), the magnitude of the equilibrant must be unity, $M = 1$. Hence, Eq. (16.4b) becomes

$$\langle\text{Discrete}\rangle \quad E[x] = \bar{x} = \sum_{i=1}^{N} x(i)P(i) \tag{16.6a}$$

Similarly, for the continuous distribution [Fig. 16.1(b)], as all probabilities $p(x)\,dx$ must lie between $x(a)$ and $x(b)$, in Eq. (16.5a) $M = 1$. Hence, Eq. (16.5b) becomes

$$\langle\text{Continuous}\rangle \quad E[x] = \bar{x} = \int_{x(a)}^{x(b)} xp(x)dx \tag{16.6b}$$

The symbol $E[x]$ in Eqs. (16.6) is called the **expected value** or the **expectation** or simply the *mean* of the variable x. As is true of the equilibrant, it is a measure of the central tendency, the *center of gravity* in statics.

Example 16.2

What is the expected value of the number of dots that will appear if a fair die is tossed?

Solution. Here each of the possible outcomes 1, 2, 3, 4, 5, and 6 has the equal probability of $P(i) = 1/6$ of appearing. Hence, from Eq. (16.6a),

$$E[\text{toss of a fair die}] = \frac{1}{6}[1+2+3+4+5+6] = 3.5$$

We note in the above example that the expected value of 3.5 is an impossible outcome. There is no face on the die that will show 3.5 dots; however, it is still the best measure of the central tendency.

Example 16.3

Find the expected value of a continuous probability distribution wherein all values are equally likely to occur (called a *uniform distribution,* Fig. 16.2) between $y(a) = 0$ and $y(b) = 1/2$.

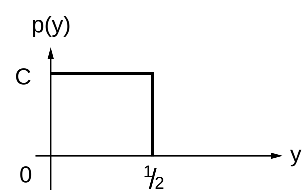


FIGURE 16.2 Uniform distribution.

Solution. From Eq. (16.5a), as $M = 1$ and $p(y) = C$ is a constant, we have

$$1 = C\int_0^{1/2} dy, \qquad C = 2$$

From Eq. (16.6b), $E[y] = 2\int_0^{1/2} y\,dy = 1/4$, as expected.

As the variables x and y in the above examples are determined by the outcomes of random experiments, they are said to be **random variables**. In classical probability theory, random variables are generally represented by capital letters, such as X and Y. The individual values are customarily denoted by their corresponding lowercase letters, x and y; however, no such distinction will be made here.

The expected value (mean) provides the locus of the central tendency of the distribution of a random variable. To characterize other attributes of the distribution, recourse is had to higher moments. Again, returning to statics, a measure of the dispersion of the distribution of the force system about the centroidal axis, at $x = E[x]$ in Fig. 16.1(b), is given by the *moment of inertia* (the *second central moment*),

$$I(y) = \int_{x(a)}^{x(b)} (x-\bar{x})^2 p(x)dx \tag{16.7}$$

The equivalent measure of the scatter (variability) of the distribution of a random variable is called its **variance**, denoted in this text as $v[x]$ and defined as

$$\langle \text{Discrete} \rangle \quad v[x] = \sum_{\text{all } x(i)} [x(i) - \bar{x}]^2 p(i) \tag{16.8a}$$

$$\langle \text{Continuous} \rangle \quad v[x] = \int_{x(a)}^{x(b)} (x-\bar{x})^2 p(x)dx \tag{16.8b}$$

In terms of the expectation these can be written as

$$v[x] = E[(x-\bar{x})^2] \tag{16.9}$$

which, after expansion, leads to a form more amenable to computations:

$$v[x] = E[x^2] - (E[x])^2 \tag{16.10}$$

This expression is the equivalent of the parallel-axis theorem for the moment of inertia.

Example 16.4

Find the expected value and the variance of the *exponential distribution*, $p(x) = a \exp(-ax)$; $x > 0$, a is a constant.

Solution. We first show that $p(x)$ is a valid probability distribution:

$$\int_0^\infty p(x)dx = a\int_0^\infty e^{-ax}dx = 1, \quad \text{Q.E.D.}$$

The expected value is

$$E[x] = a\int_0^\infty xe^{-ax}dx = \frac{1}{a}$$

Continuing,

$$E[x^2] = a\int_0^\infty x^2 e^{-ax}dx = \frac{2}{a^2}$$

whence, using Eq. (16.10),

$$v[x] = \frac{2}{a^2} - \left(\frac{1}{a}\right)^2 = \frac{1}{a^2}$$

It is seen that the variance has the units of the square of those of the random variable. A more meaningful measure of dispersion of a random variable (x) is the positive square root of its variance (compare with radius of gyration of mechanics) called the **standard deviation**, $\sigma[x]$,

$$\sigma[x] = \sqrt{v[x]} \tag{16.11}$$

From the results of the previous example, it is seen that the standard deviation of the exponential distribution is $\sigma[x] = 1/a$.

An extremely useful relative measure of the scatter of a random variable (x) is its *coefficient of variation* $V(x)$, usually expressed as a percentage:

$$V(x) = \frac{\sigma[x]}{E[x]} \times 100(\%) \tag{16.12}$$

For the exponential distribution we found, $\sigma[x] = 1/a$ and $E[x] = 1/a$, hence V (exponential distribution) = 100%. In Table 16.1 representative values of the coefficients of variation of some parameters common to civil engineering design are given. Original sources should be consulted for details.

The coefficient of variation expresses a measure of the reliability of the central tendency. For example, a mean value of a parameter of 10 with a coefficient of variation of 10% would indicate a standard deviation of 1, whereas a similar mean with a coefficient of variation of 20% would demonstrate a standard deviation of 2. The coefficient of variation has been found to be a fairly stable measure of variability for homogeneous conditions. Additional insight into the standard deviation and the coefficient of variation as measures of uncertainty is provided by Chebyshev's inequality [for the derivation see Lipschutz, 1965].

The spread of a random variable is often spoken of as its *range*, the difference between the largest and smallest outcomes of interest. Another useful measure is the range between the mean plus-and-minus h standard deviations, $\bar{x} + h\,\sigma[x]$, called the *h-sigma bounds* (see Fig. 16.3). If x is a random variable with mean value $\bar{x}$ and standard deviation σ, then Chebyshev's inequality states

$$P(\bar{x} - h\sigma) \le x \le (\bar{x} + h\sigma) \ge \frac{1}{h^2} \tag{16.13}$$

In words, it asserts that for any probability distribution (with finite mean and standard deviation) the probability that random values of the variate will lie within h-sigma bounds is at least $[1 - (1/h^2)]$. Some numerical values are given in Table 16.2. It is seen that quantitative probabilistic statements can be made without complete knowledge of the probability distribution function; only its expected value and coefficient of variation (or standard deviation) are required. In this regard, the values for the coefficients of variation given in Table 16.1 may be used in the absence of more definitive information.

Example 16.5

The expected value for the ϕ-strength parameter of a sand is 30°. What is the probability that a random sample of this sand will have a ϕ-value between 20° and 40°?

Solution. From Table 16.1, $V(\phi) = 12\%$; hence, $\sigma[\phi]$ is estimated to be (0.12)(30) = 3.6° and $h = (\phi - \bar{\phi})/\sigma = 10°/3.6° = 2.8$. Hence, $P[20° \le \phi \le 40°] \ge 0.87$. That is, the probability is at least 0.87 that the ϕ-strength parameter will be between 20° and 40°.

If the unknown probability distribution function is symmetrical with respect to its expected value and the expected value is also its maximum value (said to be *unimodal*), it can be shown [Freeman, 1963] that

$$P[(\bar{x} - h\sigma) \le x \le (\bar{x} + h\sigma)] \le 1 - \frac{4}{9h^2} \tag{16.14}$$

This is sometimes called *Gauss's inequality.* Some numerical values are given in Table 16.2.

Example 16.6

Repeat the previous example if it is assumed that the distribution of the ϕ-value is symmetrical with its maximum at the mean value ($\bar{\phi} = 30°$).

Solution. For this case, Gauss's inequality asserts

TABLE 16.1 Representative Coefficients of Variation

Parameter	Coefficient of Variation, %	Source
	Soil	
Porosity	10	Schultze [1972]
Specific gravity	2	Padilla and Vanmarcke [1974]
Water content		
Silty clay	20	Padilla and Vanmarcke [1974]
Clay	13	Fredlund and Dahlman [1972]
Degree of saturation	10	Fredlund and Dahlman [1972]
Unit weight	3	Hammitt [1966]
Coefficient of permeability	(240 at 80% saturation to 90 at 100% saturation)	Nielsen et al. [1973]
Compressibility factor	16	Padilla and Vanmarcke [1974]
Preconsolidation pressure	19	Padilla and Vanmarcke [1974]
Compression index		
Sandy clay	26	Lumb [1966]
Clay	30	Fredlund and Dahlman [1972]
Standard penetration test	26	Schultze [1975]
Standard cone test	37	Schultze [1975]
Friction angle ϕ		
Gravel	7	Schultze [1972]
Sand	12	Schultze [1972]
c, Strength parameter (cohesion)	40	Fredlund and Dahlman [1972]
	Structural Loads, 50-Year Maximum	
Dead load	10	Ellingwood et al. [1980]
Live load	25	Ellingwood et al. [1980]
Snow load	26	Ellingwood et al. [1980]
Wind load	37	Ellingwood et al. [1980]
Earthquake load	>100	Ellingwood et al. [1980]
	Structural Resistance	
Structural steel		
Tension members, limit state, yielding	11	Ellingwood et al. [1980]
Tension members, limit state, tensile strength	11	Ellingwood et al. [1980]
Compact beam, uniform moment	13	Ellingwood et al. [1980]
Beam, column	15	Ellingwood et al. [1980]
Plate, girders, flexure	12	Ellingwood et al. [1980]
Concrete members		
Flexure, reinforced concrete, grade 60	11	Ellingwood et al. [1980]
Flexure, reinforced concrete, grade 40	14	Ellingwood et al. [1980]
Flexure, cast-in-place beams	8–9.5	Ellingwood et al. [1980]
Short columns	12–16	Ellingwood et al. [1980]
	Ice	
Thickness	17	Bercha [1978]
Flexural strength	20	Bercha [1978]
Crushing strength	13	Bercha [1978]
Flow velocity	33	Bercha [1978]
	Wood	
Moisture	3	Borri et al. [1983]
Density	4	Borri et al. [1983]
Compressive strength	19	Borri et al. [1983]
Flexural strength	19	Borri et al. [1983]
Glue-laminated beams		
Live-load	18	Galambos et al. [1982]
Snow load	18	Galambos et al. [1982]

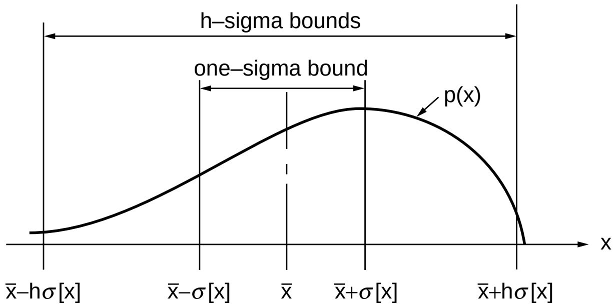


FIGURE 16.3 Range in *h*-sigma bounds.

TABLE 16.2 Probabilities for Range of Expected Values ± *h*-Sigma Units

h	Chebyshev's Inequality	Gauss's Inequality	Exact Exponential Distribution	Exact Normal Distribution	Exact Uniform Distribution
½	0	0	0.78	0.38	0.29
1	0	0.56	0.86	0.68	0.58
2	0.75	0.89	0.95	0.96	1.00
3	0.89	0.95	0.982	0.9973	1.00
4	0.94	0.97	0.993	0.999934	1.00

$$P[20° \le \phi \le 40°] \ge 1 - \frac{4}{9(2.8)^2} \ge 0.94$$

Recognizing symmetry we can also claim $P[\phi \le 20°] = P[\phi \ge 40°] = 0.03$.

Example 16.7

Find the general expression for the probabilities associated with *h*-sigma bounds for the exponential distribution, $h \ge 1$.

Solution. From Example 16.4, we have (with $E[x] = 1/a$, $\sigma[x] = 1/a$)

$$P[(\bar{x} - h\sigma) \le x \le (x + h\sigma)] = \int_0^{(h+1)/a} ae^{-ax}dx = 1 - e^{-(h+1)}$$

Some numerical values are given in Table 16.2. The normal distribution noted in this table will be developed subsequently.

The results in Table 16.2 indicate that lacking information concerning a probability distribution beyond its first two moments, from a practical engineering point of view, it may be taken to range within 3-sigma bounds. That is, in Fig. 16.1(b), $x(a) \approx \bar{x} - 3\sigma[x]$ and $x(b) \approx \bar{x} + 3\,\sigma[x]$.

For a symmetrical distribution all moments of odd order about the mean (central moments) must be zero. Consequently, any odd-ordered moment may be used as a measure of the degree of *skewness*, or *asymmetry*, of a probability distribution. The third central moment $E[(x - \bar{x})^3]$ provides a measure of the peakedness (called *kurtosis*) of a distribution.

As the units of the third central moment are the cube of the units of the variable, to provide an absolute measure of skewness, Pearson [1894, 1895] proposed that its value be divided by the standard deviation cubed to yield the dimensionless *coefficient of skewness*,

$$\beta(1) = \frac{E[(x - \bar{x})^3]}{(\sigma[x])^3} \tag{16.15}$$

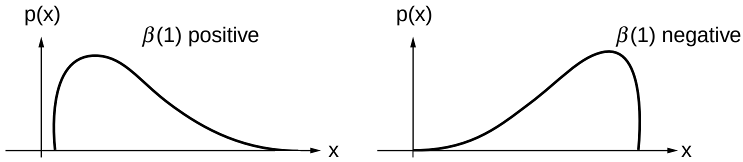


FIGURE 16.4 Coefficient of skewness.

If β(1) is positive, the long tail of the distribution is on the right side of the mean; if it is negative, the long tail is on the left side (see Fig. 16.4). Pearson also proposed the dimensionless *coefficient of kurtosis* as a measure of peakedness:

$$\beta(2) = \frac{E[(x-\bar{x})^4]}{(\sigma[x])^4} \qquad (16.16)$$

In Fig. 16.5 are shown the regions occupied by a number of probability distribution types, as delineated by their coefficients of skewness and kurtosis. Examples of the various types are shown schematically.

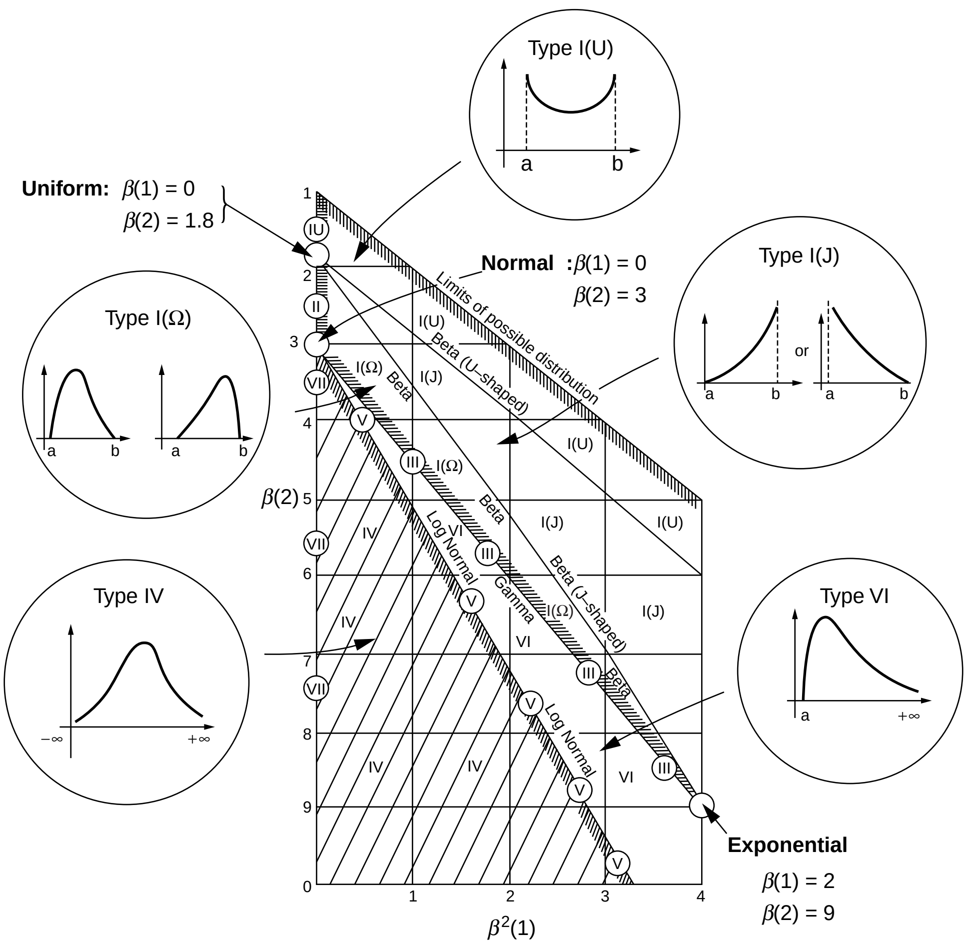


FIGURE 16.5 Space of probability distributions. (After Pearson, E. S. and Hartley, H. O. 1972. *Biometrika Tables for Statisticians*, Vol. II. Cambridge University Press, London.)

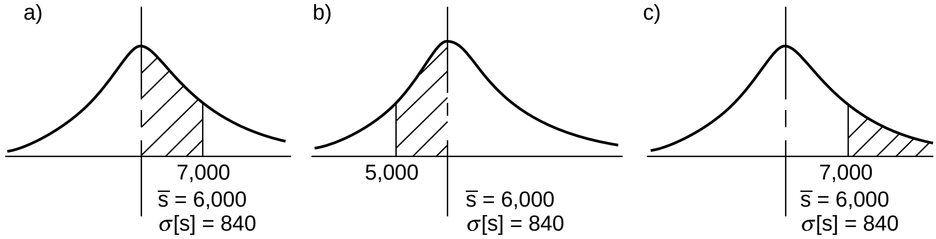


FIGURE 16.6

16.3 Probability Distributions

We note in Fig. 16.5 that the type IV distribution and the symmetrical type VII are unbounded (infinite) below and above. From the point of view of civil engineering applications this represents an extremely unlikely distribution. For example, all parameters or properties (see Table 16.1) are positive numbers (including zero).

The type V (the lognormal distribution), type III (the gamma), and type VI distributions are unbounded above. Hence, their use would be confined to those variables with an extremely large range of possible values. Some examples are the coefficient of permeability, the state of stress at various points in a body, the distribution of annual rainfall, and traffic variations.

The normal (Gaussian) distribution $[\beta(1) = 0, \beta(2) = 3]$, even though it occupies only a single point in the universe of possible distributions, is the most frequently used of probability models. Some associated properties were given in Table 16.2. The normal distribution is the well-known symmetrical bell-shaped curve (see Fig. 16.6). Some tabular values are given in Table 16.3. The table is entered by forming the standardized variable z for the normal variate x as

$$z = \left|\frac{x - \bar{x}}{\sigma[x]}\right| \tag{16.17}$$

Tabular values yield the probabilities associated with the shaded areas shown in the figure: area = $\psi(z)$.

Example 16.8

Assuming the strength s of concrete to be a normal variate with an expected value of $\bar{s} = 6000$ psi and a coefficient of variation of 14%, find (a) $P[6000 \le s \le 7000]$, (b) $P[5000 \le s \le 6000]$, and (c) $P[s \ge 7000]$.

Solution. The standard deviation is $\sigma[s] = (0.14)(6000) = 840$ psi. Hence (see Fig. 16.6),

- $z = |(7000 - 6000)/840| = 1.19$, $\psi(1.19) = 0.383$.
- By symmetry, $P[5000 \le s \le 6000] = 0.383$.
- $P[s \ge 7000] = 0.500 - 0.383 = 0.117$.

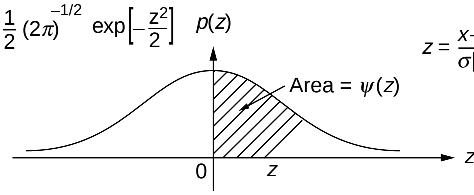
As might be expected from its name, the lognormal distribution (type V) is related to the normal distribution. If x is a normal variate and $x = \ln y$ or $y = \exp(x)$ then y is said to have a lognormal distribution. It is seen that the distribution has a minimum value of zero and is unbounded above. The probabilities associated with lognormal variates can be obtained very easily from those of mathematically corresponding normal variates (see Table 16.3). If $E(y)$ and $V(y)$ are the expected value and coefficient of variation of a lognormal variate, the corresponding normal variate x will have the expected value and standard deviation [Benjamin and Cornell, 1970]:

$$(\sigma[x])^2 = \ln\{1 + [V(y)]^2\} \tag{16.18a}$$

$$E[x] = \ln E(y) - (\sigma[x])^2/2 \tag{16.18b}$$

TABLE 16.3 Standard Normal Probability

for $z > 2.2$, $\psi(z) \approx \frac{1}{2} - \frac{1}{2}(2\pi)^{-1/2} \exp\left[-\frac{z^2}{2}\right]$



$$z = \frac{x - \bar{x}}{\sigma[x]}$$

z	0	1	2	3	4	5	6	7	8	9
0	0	.003969	.007978	.011966	.015953	.019939	.023922	.027903	.031881	.035856
.1	.039828	.043795	.047758	.051717	.055670	.059618	.063559	.067495	.071424	.075345
.2	.079260	.083166	.087064	.090954	.094835	.098706	.102568	.106420	.110251	.114092
.3	.117911	.121720	.125516	.129300	.133072	.136831	.140576	.144309	.148027	.151732
.4	.155422	.159097	.162757	.166402	.170031	.173645	.177242	.180822	.184386	.187933
.5	.191462	.194974	.198466	.201944	.205401	.208840	.212260	.215661	.219043	.222405
.6	.225747	.229069	.232371	.235653	.238914	.242154	.245373	.248571	.251748	.254903
.7	.258036	.261148	.264238	.267305	.270350	.273373	.276373	.279350	.282305	.285236
.8	.288145	.291030	.293892	.296731	.299546	.302337	.305105	.307850	.310570	.313267
.9	.315940	.318589	.321214	.323814	.326391	.328944	.331472	.333977	.336457	.338913
1.0	.341345	.343752	.346136	.348495	.350830	.353141	.355428	.357690	.359929	.362143
1.1	.364334	.366500	.368643	.370762	.372857	.374928	.376976	.379000	.381000	.382977
1.2	.384930	.386861	.388768	.390651	.392512	.394350	.396165	.397958	.399727	.401475
1.3	.403200	.404902	.406582	.408241	.409877	.411492	.413085	.414657	.416207	.417736
1.4	.419243	.420730	.422196	.423641	.425066	.426471	.427855	.429219	.430563	.431888
1.5	.433193	.434476	.435745	.436992	.438220	.439429	.440620	.441792	.442947	.444083
1.6	.445201	.446301	.447384	.448449	.449497	.450529	.451543	.452540	.453521	.454486
1.7	.455435	.456367	.457284	.458185	.459070	.459941	.460796	.461636	.462462	.463273
1.8	.464070	.464852	.465620	.466375	.467116	.467843	.468557	.469258	.469946	.470621
1.9	.471283	.471933	.472571	.473197	.473610	.474412	.475002	.475581	.476148	.476705
2.0	.477250	.477784	.478308	.478822	.479325	.479818	.480301	.480774	.481237	.481691
2.1	.482136	.482571	.482997	.483414	.483823	.484222	.484614	.484997	.485371	.485738
2.2	.486097	.486447	.486791	.487126	.487455	.487776	.488089	.488396	.488696	.488989
2.3	.489276	.489556	.489830	.490097	.490358	.490613	.490863	.491106	.491344	.491576
2.4	.491802	.492024	.492240	.492451	.492656	.492857	.493053	.493244	.493431	.493613
2.5	.493790	.493963	.494132	.494297	.494457	.494614	.494766	.494915	.495060	.495201
2.6	.495339	.495473	.495604	.495731	.495855	.495975	.496093	.496207	.496319	.496427
2.7	.496533	.496636	.496736	.496833	.496928	.497020	.497110	.497197	.497282	.497365
2.8	.497445	.497523	.497599	.497673	.497744	.497814	.497882	.497948	.498012	.498074
2.9	.498134	.498193	.498250	.498305	.498359	.498411	.498462	.498511	.498559	.498605
3.0	.498650	.498694	.498736	.498777	.498817	.498856	.498893	.498930	.498965	.498999
3.1	.499032	.499065	.499096	.499126	.499155	.499184	.499211	.499238	.499264	.499289
3.2	.499313	.499336	.499359	.499381	.499402	.499423	.499443	.499462	.499481	.499499
3.3	.499517	.499534	.499550	.499566	.499581	.499596	.499610	.499624	.499638	.499651
3.4	.499663	.499675	.499687	.499698	.499709	.499720	.499730	.499740	.499749	.499758
3.5	.499767	.499776	.499784	.499792	.499800	.499807	.499815	.499822	.499828	.499835
3.6	.499841	.499847	.499853	.499858	.499864	.499869	.499874	.499879	.499883	.499888
3.7	.499892	.499896	.499900	.499904	.499908	.499912	.499915	.499918	.499922	.499925
3.8	.499928	.499931	.499933	.499936	.499938	.499941	.499943	.499946	.499948	.499950
3.9	.499952	.499954	.499956	.499958	.499959	.499961	.499963	.499964	.499966	.499967

Example 16.9

A live load of 20 kips is assumed to act on a footing. If the loading is assumed to be lognormally distributed, estimate the probability that a loading of 40 kips will be exceeded.

Solution. From Table 16.1 we have that the coefficient of variation for a live load, L, can be estimated as 25%; hence, from Eqs. (16.18) we have for the corresponding normal variate, x,

$$\sigma[x] = \sqrt{\ln[1+(0.25)^2]} = 0.25 \tag{16.19a}$$

and

$$E[x] = \ln 20 - (0.25)^2/2 = 2.96 \tag{16.19b}$$

As $x = \ln L$, the value of the normal variate x equivalent to 40 K is ln 40 = 3.69. We seek the equivalent normal probability $P[3.69 \le x]$. The standardized normal variate is $z = (3.69 - 2.96)/0.25 = 2.92$. Hence, using Table 16.3,

$$P[40 \le L] = 0.50 - \psi(2.92) = 0.500 - 0.498 = 0.002 \tag{16.20}$$

As was noted with respect to Fig. 16.5 many and diverse distributions (as well as the normal, lognormal, uniform, and exponential) can be obtained from the very versatile beta distribution. The beta distribution is treated in great detail by Harr [1977, 1987]. The latter reference also contains FORTRAN programs for beta probability distributions. Additional discussion is given below following Example 16.11.

16.4 Point Estimate Method — One Random Variable

Various probabilistic methods have been developed that yield measures of the distribution of functions of random variables [Harr, 1987]. The simple and very versatile procedure called the *point estimate method* (PEM) is advocated by this writer and will be developed in some detail. This method, first presented by Rosenblueth [1975] and later extended by him in 1981, has since seen considerable use and expansion by this writer and his coauthors (see references). The methodology is presented in considerable detail in Harr [1987].

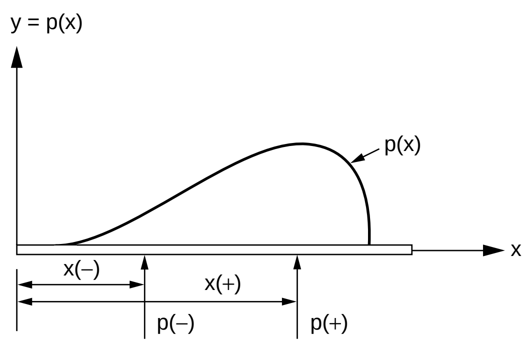


FIGURE 16.7 Point estimate approximations.

Consider $y = p(x)$ to be the probability distribution of the random variable x. With analogy to Fig. 16.1, we replace the load on the beam by two reactions, $p(-)$ and $p(+)$, acting at $x(-)$ and $x(+)$, as shown in Fig. 16.7.

Pleading symmetry, probabilistic arguments produce for a random variable x:

$$p(+) = p(-) = \tfrac{1}{2} \tag{16.21a}$$

$$x(+) = \bar{x} + \sigma[x] \tag{16.21b}$$

$$x(-) = \bar{x} - \sigma[x] \tag{16.21c}$$

With the distribution $p(x)$ approximated by the point estimates $p(-)$ and $p(+)$, the moments of $y = p(x)$ are

$$E[y] = \bar{y} = p(-)y(-) + p(+)y(+) \quad (16.22a)$$

$$E[y^2] = p(-)y^2(-) + p(+)y^2(+) \quad (16.22b)$$

where $y(-)$ and $y(+)$ are the values of the function $p(x)$ at $x(-)$ and $x(+)$, respectively. These reduce to the simpler expressions

$$\bar{y} = \frac{y(+) + y(-)}{2} \quad (16.23a)$$

$$\sigma[y] = \left|\frac{y(+) - y(-)}{2}\right| \quad (16.23b)$$

Example 16.10

Estimate the expected value and the coefficient of variation for the well-known coefficient of active earth pressure $K_A = \tan^2(45 - \phi/2)$, if $\bar{\phi} = 30$.

Solution. With the standard deviation of the ϕ-parameter not given, we again return to Table 16.1, $V(\phi) = 12\%$, and $\sigma[\phi] = 3.6°$. Hence, $\phi(+) = 33.6°$, $\phi(-) = 26.4°$. Hence, $K_A(+) = 0.29$, $K_A(-) = 0.38$, and Eqs. (16.23) produce $\bar{K}_A = 0.34$, $\sigma[K_A] = 0.05$; hence, $V(K_A) = 13\%$.

16.5 Regression and Correlation

Thus far, only one-dimensional (*univariate*) random variables have been considered. More generally, concern is directed toward *multivariate formulations,* wherein there are two or more random variables. As an example, consider the *flexure formula*

$$s = Mc/I$$

where s is the stress at the extreme fiber at a distance c from the neutral axis acted on by a bending moment M for a beam in which I is the moment of inertia of the section. If the parameters M, c, and I are random variables (possess uncertainty), what can be said about the unit stress s? Needless to say, granted the probability distribution function of the stress, statements could be made with respect to the reliability of the beam relative to, say, a maximum allowable stress $\hat{s}$; for example,

$$\text{Reliability} = P[s \le \hat{s}]$$

We first study the functional relationship between random variables called ***regression*** *analysis*. It is regression analysis that provides the grist of being able to predict the value of one variable from that of another or of others. The measure of the degree of correspondence within the developed relationship belongs to *correlation analysis.*

Let us suppose we have N pairs of data $[x(1), y(1)], \ldots, [x(N), y(N)]$ for which we postulate the linear relationship

$$y = Mx + B \quad (16.24)$$

where M and B are constants. Of the procedures available to estimate these constants (including best fit by eye), the most often used is the *method of least squares*. This method is predicated on minimizing the sum of the squares of the distances between the data points and the corresponding points on a straight line. That is, M and B are chosen so that

$$\Sigma(y - Mx - B)^2 = \text{Minimum}$$

This requirement is met by the expressions

$$M = \frac{N\Sigma xy - \Sigma x\Sigma y}{N\Sigma x^2 - (\Sigma x)^2} \tag{16.25a}$$

$$B = \frac{\Sigma x^2\Sigma y - \Sigma x\Sigma xy}{N\Sigma x^2 - (\Sigma x)^2} \tag{16.25b}$$

It should be emphasized that a straight line fit was assumed. The reasonableness of this assumption is provided by the **correlation coefficient** ρ, defined as

$$\rho = \frac{\text{cov}[x, y]}{\sigma[x]\sigma[y]} \tag{16.26}$$

where $\sigma[x]$ and $\sigma[y]$ are the respective standard deviations and cov$[x, y]$ is their *covariance*. The covariance is defined as

$$\text{cov}[x, y] = \frac{1}{N}\sum_{i=1}^{N}[x(i) - \bar{x}][y(i) - \bar{y}] \tag{16.27}$$

With analogy to statics the covariance corresponds to the product of inertia.

In concept, the correlation coefficient is a measure of the tendency for two variables to vary together. This measure may be zero, negative, or positive; wherein the variables are said to be *uncorrelated, negatively correlated,* or *positively correlated*. The variance is a special case of the covariance as

$$\text{cov}[x, x] = v[x] \tag{16.28}$$

Application of their definitions produces [Ditlevsen, 1981] the following identities (a, b, and c are constants):

$$E[a + bx + cy] = a + bE[x] + cE[y] \tag{16.29a}$$

$$v[a + bx + cy] = b^2 v[x] + c^2 v[y] + 2bc\ \text{cov}[x, y] \tag{16.29b}$$

$$\text{cov}[x, y] \le \sigma[x]\sigma[y] \tag{16.29c}$$

$$v[a + bx + cy] = b^2 v[x] + c^2 v[y] + 2bc\sigma[x]\sigma[y]\rho \tag{16.29d}$$

Equation (16.29c) demonstrates that the correlation coefficient, Eq. (16.26), must satisfy the condition

$$-1 \le \rho \le +1 \tag{16.30}$$

If there is perfect correlation between variables in the same direction, $\rho = +1$. If there is perfect correlation in opposite directions (one variable increases as the other decreases), $\rho = -1$. If some scatter exists, $-1 < \rho < +1$, with $\rho = 0$ if there is no correlation. Some examples are shown in Fig. 16.8.

16.6 Point Estimate Method — Several Random Variables

Rosenblueth [1975] generalized the methodology for any number of correlated variables. For example, for a function of three random variables — say, $y = y[x(1), x(2), x(3)]$ — where $\rho(i, j)$ is the correlation coefficient between variables $x(i)$ and $x(j)$,

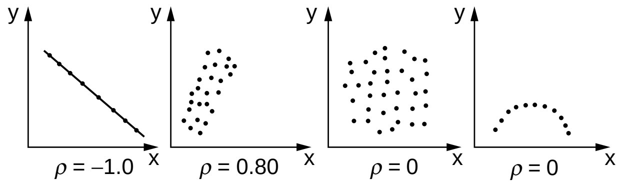


FIGURE 16.8 Example of scatter and correlation coefficients.

$$E[y^N] = p(+++)y^N(+++) + p(++-)y^N(++-) + \cdots + p(---)y^N(---) \qquad (16.31a)$$

where

$$y(\pm\pm\pm) = y[\bar{x}(1) \pm \sigma[x1], \bar{x}(2) \pm \sigma[x2], \bar{x}(3) \pm \sigma[x3]] \qquad (16.31b)$$

$$p(+++) = p(---) = \frac{1}{2^3}[1 + \rho(1,2) + \rho(2,3) + \rho(3,1)]$$

$$p(++-) = p(--+) = \frac{1}{2^3}[1 + \rho(1,2) - \rho(2,3) - \rho(3,1)]$$

$$p(+-+) = p(-+-) = \frac{1}{2^3}[1 - \rho(1,2) - \rho(2,3) + \rho(3,1)]$$

$$p(+--) = p(-++) = \frac{1}{2^3}[1 - \rho(1,2) + \rho(2,3) - \rho(3,1)] \qquad (16.31c)$$

where $\sigma[xi]$ is the standard deviation of $x(i)$. The sign of $\rho(i, j)$ is determined by the multiplication rule of i and j; that is, if the sign of $i = (-)$, and of $j = (+)$, then $(i)(j) = (-)(+) = (-)$.

Equation (16.31a) has $2^3 = 8$ terms, all permutations of the three +s and –s. In general for M variables there are 2^M terms and $M(M - 1)/2$ correlation coefficients, the number of combinations of M objects taken two at a time. The coefficient on the right-hand side of Eqs. (16.31c), in general, is $(1/2)^M$.

Example 16.11

The recommendation of the American Concrete Institute [Galambos et al., 1982] for the design of reinforced concrete structures is (in simplified form)

$$R \geq 1.6D + 1.9L$$

where R is the strength of the element, D is the dead load, and L is the lifetime live load. (a) If $\bar{D} = 10$, $\bar{L} = 8$, $V(D) = 10\%$, $V(L) = 25\%$, and $\rho(D, L) = 0.75$, find the expected value and standard deviation of R for the case $R = 1.6D + 1.9L$. (b) If the results in part (a) generate a normal variate and the maximum strength of the element R is estimated to be 40, estimate the implied probability of failure.

Solution. The solution is developed in Fig. 16.9.

Generalizations of the PEM to more than three random variables are given by Harr [1987]. The PEM procedure yields the first two moments of the dependent random functions under consideration. Functional distributions must then be obtained and statements must be made concerning the probabilities of events. Inherent in the assumption of the form of a particular distribution is the imposition of the limits or range of its applicability. For example, for the normal it is required that the variable range from $-\infty$ to $+\infty$; the range of the lognormal and the exponential is 0 to $+\infty$. Such assignments may not be critical if knowledge of distributions is desired in the vicinity of their expected values and their coefficients

a) R = 1.6D + 1.9L

Variable, x	$\bar{x}$	σ[x]	x(+)	x(–)
D	10	1	11	9
L	8	2	10	6

$\rho(D,L) = +0.75$

	R(ij)	$R(ij)^2$
R(++):	36.6	1340
R(+–):	29.0	841
R(–+):	33.4	1116
R(––):	25.8	666

$p(++) = \frac{1}{4}(1+\rho) = 0.44$

$p(+-) = \frac{1}{4}(1-\rho) = 0.06$

$p(-+) = \frac{1}{4}(1-\rho) = 0.06$

$p(--) = \frac{1}{4}(1+\rho) = 0.44$

$$E[R] = \bar{R} = \Sigma R(ij)p(ij) = 0.44(36.6+25.8) + 0.06(29.0+33.4) = \underline{31.20}$$

$$E[R^2] = \Sigma R(ij)^2 p(ij) = 0.44(1340+666) + 0.06(841+1116) = \underline{1000.06}$$

$$\upsilon[R] = E[R^2] - (E[R])^2 = 1000.06 - (31.20)^2 = \underline{26.62}$$

$\sigma[R] = 5.16$; $V(R) = 16.5\%$

From Eq. (15.29a), the exact solution for $E[R] = 1.6\,\bar{D} + 1.9\bar{L} = \underline{31.20}$

Eq. (15.29d), the exact solution for $\nu[R] = (1.6)^2\nu[D] + (1.9)^2\nu[L] + 2(1.6)(1.9)(1)(2)(0.75) = \underline{26.12}$

Of course, for this example the exact solution is easier to obtain.
This is not generally the case.

b) $P_f = P[R \geq 40] = \frac{1}{2} - \psi\left[\frac{40-31.20}{5.16}\right] = \frac{1}{2} - \psi[1.71] = \underline{0.044}$ (Table 3)

The exact solution is $\underline{0.043}$

FIGURE 16.9 Solution to Example 16.11.

of variation are not excessive (say, less than 25%). On the other hand, estimates of reliability (and of the probability of failure) are vested in the tails of distributions. It is in such characterizations that the beta distribution is of great value. If the limits are known, zero is often an option, and probabilistic statements can readily be obtained. In the event that limits are not defined, the specification of a range of the mean plus or minus three standard deviations would generally place the generated beta distribution well within the accuracy required by most geotechnical engineering applications (see Table 16.2).

16.7 Reliability Analysis

Capacity–Demand

The adequacy of a proposed design in geotechnical engineering is generally determined by comparing the estimated resistance of the system to that of the imposed loading. The resistance is the **capacity** *C* (or strength) and the loading is the induced **demand** *D* imposed on the structure. In the present writing, because of its greater generality, we shall use a *capacity– demand* concept. Some common examples are the bearing capacity of a soil and the column loads, allowable and computed maximum stresses, traffic

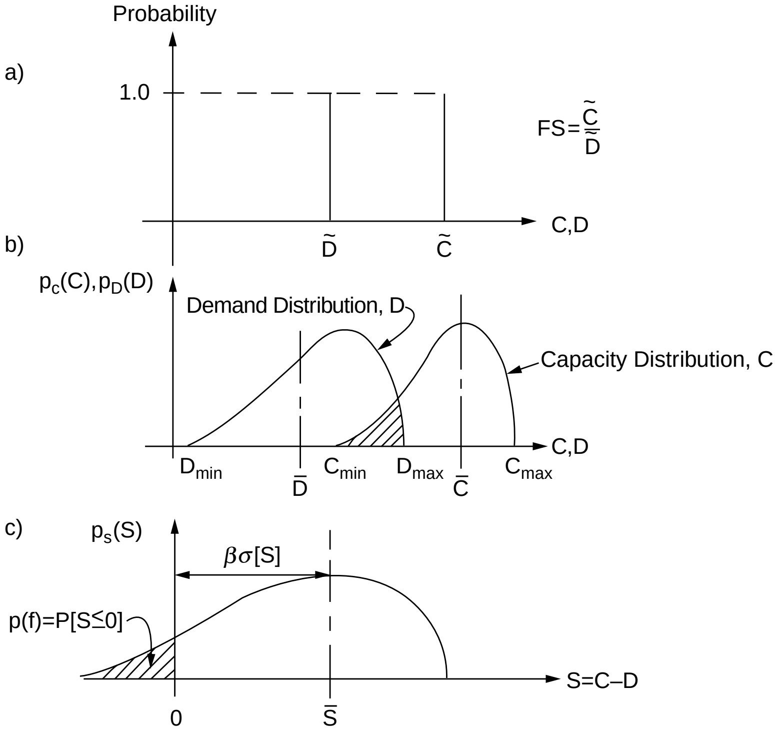


FIGURE 16.10 (a) Conventional factor of safety, (b) capacity–demand model, (c) safety margin.

capacity and anticipated traffic flow on a highway, culvert sizes and the quantity of water to be accommodated, and structural capacity and earthquake loads.

Conventionally, the designer forms the well-known factor of safety as the ratio of the single-valued nominal values of capacity $\bar{C}$ and demand $\bar{D}$ [Ellingwood et al., 1980], depicted in Fig. 16.10(a),

$$FS = \frac{\bar{C}}{\bar{D}} \tag{16.32}$$

For example, if the allowable load is 400 tons per square foot and the maximum calculated load is 250 tons per square foot, the conventional factor of safety would be 1.6. The design is considered satisfactory if the calculated factor of safety is greater than a prescribed minimum value learned from experience with such designs. Thus, in concept, in the above example, if a factor of safety of 1.6 was considered intolerable, the system would be redesigned to decrease the maximum induced load.

In general, the demand function will be the resultant of the many uncertain components of the system under consideration (vehicle loadings, wind loadings, earthquake accelerations, location of the water table, temperatures, quantities of flow, runoff, and stress history, to name only a few). Similarly, the capacity function will depend on the variability of material parameters, testing errors, construction procedures and inspection supervision, ambient conditions, and so on.

A schematic representation of the capacity and demand functions as probability distributions is shown in Fig. 16.10(b). If the maximum demand (D_{max}) exceeds the minimum capacity (C_{min}), the distributions overlap (shown shaded), and there is a nonzero probability of failure.

The difference between the capacity and demand functions is called the safety margin (S); that is,

$$S = C - D \tag{16.33}$$

Obviously, the safety margin is itself a random variable, as shown in Fig. 16.10(c). Failure is associated with that portion of its probability distribution wherein it becomes negative (shown shaded); that is, that portion wherein $S = C - D \leq 0$. As the shaded area is the probability of failure $p(f)$, we have

$$p(f) = P[(C - D) \leq 0] = P[S \leq 0] \tag{16.34}$$

Reliability Index

The number of standard deviations that the mean value of the safety margin is beyond $S = 0$, Fig. 16.10(c), is called the *reliability index*, β; that is,

$$\beta = \frac{\bar{S}}{\sigma[S]} \tag{16.35a}$$

The reliability index is seen (compare also with *h*-sigma bounds in Table 16.2) to be the reciprocal of the coefficient of variation of the safety margin, or

$$\beta = \frac{1}{V(S)} \tag{16.35b}$$

Example 16.12

Obtain a general expression for the reliability index in terms of the first two moments of the capacity and the demand functions.

Solution. From Eq. (16.29a) we have $E[S] = E[C] - E[D] = \bar{C} - \bar{D}$. Equation (16.29c) produces $\sigma^2[S] = \sigma^2[C] + \sigma^2[D] - 2\rho\sigma[C]\sigma[D]$. Hence,

$$\beta = \frac{\bar{C} - \bar{D}}{\sqrt{\sigma^2[C] + \sigma^2[D] - 2\rho\sigma[C]\sigma[D]}} \tag{16.36}$$

It is seen that β is a maximum for a perfect positive correlation and a minimum for a perfect negative correlation.

It can be shown that the sum of difference of two normal variates is also a normal variate [Haugen, 1968]. Hence, if it is assumed that the capacity and demand functions are normal variates, it follows directly from Example 16.12 that

$$p(f) = \frac{1}{2} - \psi[\beta] \tag{16.37}$$

where $\psi[\beta]$ is standard normal probability as given in Table 16.3.

16.8 Recommended Procedure

We list at this point some desirable attributes of a reliability-based design procedure:

1. It should account for the pertinent capacity and demand factors, their components, and their interactions.
2. It should produce outputs that can be related to the expected performance during the design life of the system under consideration.
3. It should employ as input into formulations quantities, parameters, or material characterizations that can be ascertained within the present state of the art.

4. It should not disregard indices currently considered to be pertinent, such as factor of safety or reliability index. It should serve to supplement this knowledge and reduce uncertainty.
5. Ideally, mathematical computations should be reduced to a minimum.

All of the above can be accommodated by an extension of the point estimate method. The recommended procedure is as follows, where applicable:

1. Using PEM, or an equally valid probabilistic formulation, obtain the expected values and standard deviations of the capacity and demand functions: $E[C]$, $E[D]$, $\sigma[C]$, $\sigma[D]$.
2. Calculate the expected value and standard deviation of the safety margin, $E[S]$, $\sigma[S]$.
3. Fit a beta distribution (and normal distribution, as a check) to the safety margin, using appropriate upper and lower bounds. If unknown, take them as $E[S] \pm 3\sigma[S]$; see Table 16.2.
4. Obtain the probability of failure, $p(f) = P[S \leq 0]$, the reliability, central factor of safety, and reliability index, as appropriate.

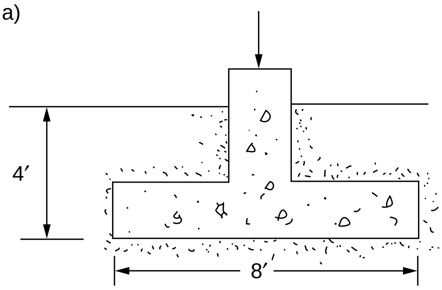
Example 16.13

The ultimate bearing capacity Q per unit length of a long footing of width B founded at a depth D below the ground surface, Fig. 16.11(a), is

$$Q = \frac{\gamma B^2}{2} N_\gamma + \gamma D B N_q + c B N_c$$

where γ is the unit weight of the soil, c is the c-parameter of strength (sometimes called the *cohesion*). The dimensionless factors N_γ, N_q, and N_c, called *bearing capacity factors*, are functions of the friction angle ϕ as given by the tabulated values in Fig. 16.11(b).

a)



b)

Bearing capacity factors

ϕ	N_γ	N_q	N_c
0	0.00	1.00	5.14
5	0.45	1.57	6.49
10	1.22	2.47	8.34
15	2.65	3.94	10.98
20	5.39	6.40	14.83
25	10.88	10.66	20.72
30	22.40	18.40	30.14
35	48.83	33.30	46.12
40	109.41	64.20	75.31
42.5	170.25	91.90	99.20
45	271.75	133.87	134.87
50	762.86	319.06	266.88

FIGURE 16.11 Example 16.13.

(a) A very long footing 8 ft wide is founded at a depth of 4 ft in a soil with the parameters

Parameter, x	Expected Value	Standard Deviation	$x(+)$	$x(-)$
g	110 lb/ft³	0	110	110
f	35°	5°	40	30
c	200 lb/ft²	80 lb/ft²	280	120

The correlation coefficient $\rho(\phi, c) = -0.50$. Estimate the expected value and standard deviation of the bearing capacity.

(b) If a central factor of safety (CFS) of 4 is required and it is assumed that $V(D) = 50\%$, estimate the probability of failure.

Solution. (a) Forming the required values, as the bearing capacity factors are functions of ϕ only. From Fig. 16.11(b),

$$N_\gamma(+) = 109.41 \qquad N_\gamma(-) = 22.40$$
$$N_q(+) = 64.20 \qquad N_q(-) = 18.40$$
$$N_c(+) = 75.31 \qquad N_c(-) = 30.14$$

Forming the respective values, $Q(\phi, c)$ in tons,

$Q(i, j)$	$Q^2(i, j)$
Q(+ +): 389.9	152,020
Q(+ –): 341.7	116,758
Q(– +): 105.6	11,144
Q(– –): 86.3	7,444

and

$$p(++) = p(--) = \frac{1}{4}(1+\rho) = \frac{1}{8}$$

$$p(+-) = p(-+) = \frac{1}{4}(1-\rho) = \frac{3}{8}$$

and

$$E[Q] = \bar{Q} = \Sigma Q(ij)p(ij) = 227.3 \text{ tons/ft}$$
$$E[Q]^2 = \Sigma Q^2(ij)p(ij) = 67{,}896$$
$$v[Q] = E[Q]^2 - (E[Q])^2 = 16{,}231$$

and

$$\sigma[Q] = 127.40 \text{ tons}, \qquad V(Q) = 56\%$$

(b) For a CFS = 4, $\bar{D} = 56.8$. As $V(D) = 50\%$, $\sigma[D] = 28.4$. Forming the characteristics of the safety margin, with $\rho(Q, D) = +3/4$, we have $E[S] = \bar{C} - \bar{D} = 170.50$, $\sigma[S] = 143.72$, $S_{min} = -152.75$, $S_{max} = 493.75$, $\beta = 170.50/107.75 = 1.58$.

If S is taken at a beta variate	$p(f) = 0.059$
If S is taken to be normal	$p(f) = 0.057$

Defining Terms

Capacity — The ability to resist an induced demand; resistance or strength of entity.
Correlation coefficient — Measure of the compliance between two variables.
Demand — Applied loading or energy.
Expected value, expectation — Weighted measure of central tendency of a distribution.
Probability — Quantitative measure of a state of knowledge.
Random variable — An entity whose measure cannot be predicted with certainty.
Regression — Means of obtaining a functional relationship among variables.
Reliability — Probability of an entity (or system) performing its required function adequately for a specified period of time under stated conditions.
Standard deviation — Square root of variance.
Variance — Measure of scatter of variable.

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