APPENDIX B – ANALYSIS FORMULAE

B.1 **Elastic bending formulae**

Bending about a principle axis:

$$\frac{s}{y} = \frac{M}{I} = E\,k \quad ; \qquad \text{curvature-change} \quad k = \frac{1}{R} - \frac{1}{R_0}$$

In general, bending moment is *section modulus* Z times maximum bending stress.
Longitudinal shear force S on material of area A_s , due to transverse shear force F on the beam.

$$S = \frac{F}{I}\int_{A_S} y\,dA = \frac{F\,A_S\,\overline{y}}{I} \qquad \text{per unit length of beam.}$$

B.2 **Elastic torsion formulae**

Round shafts: $\frac{t}{r} = \frac{T}{J} = G\,f$ where ϕ is the angle of twist per unit length

and $J = \int r^2 dA$ is the polar moment of area.

Circular area, radius R: $J = \frac{p\,R^4}{2}$

Thin circular tube, radius R thickness t: $J = 2p\,R^3\,t$

Thin walled tube of arbitrary cross-section:

$$t = \frac{T}{2A_e t} \quad : \quad T = G\frac{4A_e^2}{\oint \frac{ds}{t}}\,f$$

where A_e is the enclosed area to mid thickness, t is the wall thickness. and s is the distance round the permiter.

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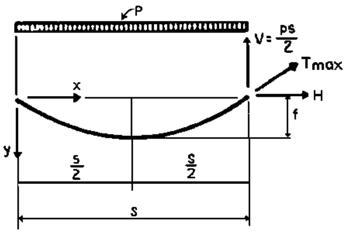
B.3 **Taut wires, cables or chains**

s = span length
f = cable sag
$n = f/s$ = sag ratio
L = length of cable curve
ΔL_s = cable elongation due to axial stress
ΔL_t = cable elongation due to temperature change, t
A = area of cable
E = modulus of elasticity of cable
ε = thermal coefficient of linear expansion
t = temperature change in °F
p = load per unit length

Uniformly loaded cables with horizontal chords



a. $y = \frac{4f}{s^2}\left(sx - x^2\right)$

b. $H = \frac{ps^2}{8f}$

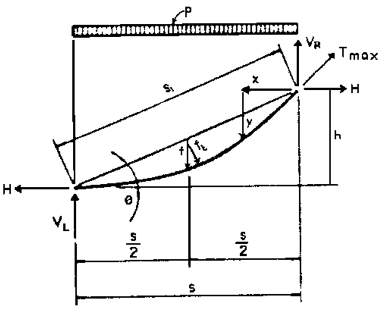
c. $T_{max} = H\sqrt{1 + 16n^2}$

d. $L = s\left(1 + \frac{8}{3}n^2 - \frac{32}{5}n^4 + \ldots\right)$

e. $\Delta L_s \cong \frac{Hs}{AE}\left(1 + \frac{16}{3}n^2\right)$

f. $\Delta L_t = \varepsilon t L \cong \varepsilon t s\left(1 + \frac{8}{3}n^2\right)$

Uniformly loaded cables With inclined chords



a. $y = \frac{4f}{s^2}\left(sx - x^2\right)$

b. $H = ps^2/8f$

c. $T_{max} = H\sqrt{1 + \left(\frac{h}{s} + 4n\right)^2}$

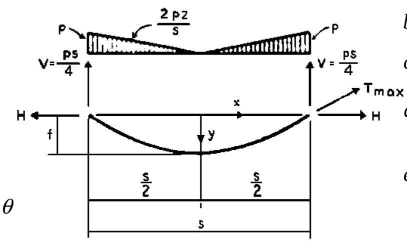
d. $L \cong s\left(1 + \frac{8n^2}{3\sec^4\theta}\right)\sec\theta$

e. $\Delta L_s \cong \frac{Hs}{AE}\left(1 + \frac{16n^2}{3\sec^4\theta}\right)\sec\theta$

f. $\Delta L_t \cong \varepsilon t s\left(1 + \frac{8n^2}{3\sec^4\theta}\right)\sec\theta$

g. $V_s = \frac{Hh}{s} + \frac{ps}{2}$

Triangular loading on cables with horizontal chords



a. $y = f\left(1 - 8\frac{x^3}{s^3}\right)$

b. $H = ps^2/24f$

c. $T_{max} = \sqrt{1 + 36n^2}$

d. $L = s\left(1 + \frac{18}{5}n^2 - 18n^4 + \ldots\right)$

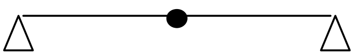
e. $\Delta L_s \cong \frac{Hs}{AE}\left(1 + \frac{36}{5}n^2\right)$

f. $\Delta L_t \cong \varepsilon t s\left(1 + \frac{18}{5}n^2\right)$

B.4 **Vibration**

Typically $f = \frac{18}{\sqrt{y}}$ for most structures

Where: f is in cycles per second
y is the static deflection in mm

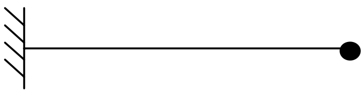


Simply supported
Mass concentrated in centre

$f = \frac{15.8}{\sqrt{y}}$

Simply Supported
Mass and stiffness distributed

$f = \frac{18}{\sqrt{y}}$



Cantilever
Mass concentrated at end

$f = \frac{15.8}{\sqrt{y}}$

Cantilever
Mass and stiffness distributed

$f = \frac{19.7}{\sqrt{y}}$

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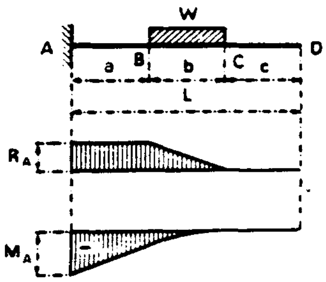
B.5 **Design formulae for beams - cantilever**

CANTILEVER



UNIFORM LOAD ON PART OF SPAN

Span $= L$

Uniform Load $= W$

$R_A = W$

$M_A = -W\left(a+\frac{b}{2}\right)$

$i_C = i_D = \frac{W}{6EI}(3a^2+3ab+b^2)$

$\delta_D = \frac{W}{24EI}\{8a^3+18a^2b+12ab^2+3b^3+4c(3a^2+3ab+b^2)\}$

SPECIAL CASE: UNIFORM LOAD ON FULL SPAN

Span $= L = b$

$a = c = 0$

Uniform Load $= W$

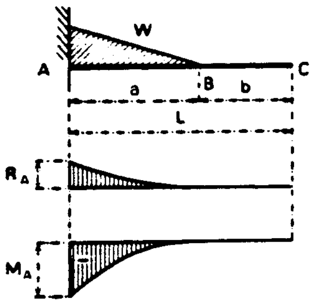
$R_A = W$

$M_A = -\frac{WL}{2}$

$i_D = \frac{WL^2}{6EI}$

$\delta_D = \frac{WL^3}{8EI}$

CANTILEVER



TRIANGULAR LOAD ON PART OF SPAN

Span $= L$

Triangular Load $= W$

$R_A = W$

$M_A = -\frac{Wa}{3}$

$\delta_C = \frac{Wa^2}{15EI}\left(L+\frac{b}{4}\right)$

$i_B = i_C = \frac{Wa^2}{12EI}$

SPECIAL CASE: TRIANGULAR LOAD ON FULL SPAN

Span $= L = a$

$b = 0$

Triangular Load $= W$

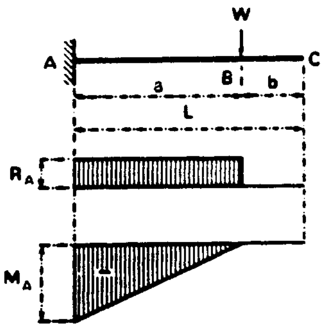
$R_A = W$

$M_A = -\frac{WL}{3}$

$\delta_C = \frac{WL^3}{15EI}$

$i_C = \frac{WL^2}{12EI}$

CANTILEVER



POINT LOAD AT ANY POSITION

Span $= L$

Point Load $= W$

$R_A = W$

$M_A = -Wa$

$\delta_C = \frac{Wa^2}{3EI}\left(L+\frac{b}{2}\right)$

$i_B = i_C = \frac{Wa^2}{2EI}$

SPECIAL CASE: POINT LOAD AT FREE END

Span $= L = a$

$b = 0$

Point Load $= W$

$R_A = W$

$M_A = -WL$

$\delta_C = \frac{WL^3}{3EI}$

$i_C = \frac{WL^2}{2EI}$

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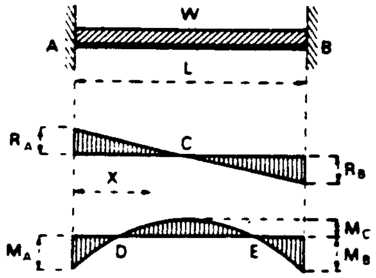
B.6 Design formulae for beams - fixed both ends

BEAM FIXED AT BOTH ENDS



at 0·211L from either end $M_D = M_E = 0$

UNIFORM LOAD ON FULL SPAN

Span = L

Total Uniform Load = W

$R_A = R_B = \frac{W}{2}$

$M_A = M_B = -\frac{WL}{12}$

at mid-span:

$M_C = \frac{WL}{24}$

$\delta_{max} = \frac{WL^3}{384EI}$

at X from A:

$M_X = -\frac{W}{12L}(L^2 - 6LX + 6X^2)$

$\delta_X = \frac{WX^2}{24EIL}(L-X)^2$

$i_X = \frac{WX}{12EIL}(L^2 - 3LX + 2X^2)$

POINT LOAD AT MID-SPAN

Span = L

Point Load = W

$R_A = R_B = \frac{W}{2}$

$M_A = M_B = -\frac{WL}{8}$

at mid-span:

$M_C = \frac{WL}{8}$

$\delta_{max} = \frac{WL^3}{192EI}$

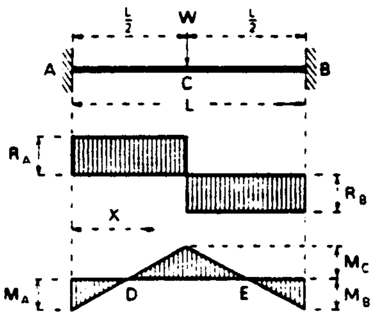
at X from A between A & C:

$M_X = \frac{W}{8}(4X - L)$

$\delta_X = \frac{WX^2}{48EI}(3L - 4X)$

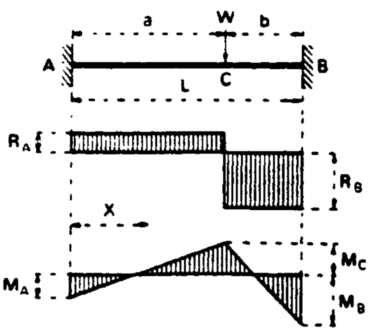
$i_X = \frac{WX}{8EI}(L - 2X)$

BEAM FIXED AT BOTH ENDS



at 0·25L from either end $M_D = M_E = 0$

BEAM FIXED AT BOTH ENDS



When a > b the maximum deflection is at $X = \frac{2La}{L + 2a}$

POINT LOAD AT ANY POSITION

Span = L

Point Load = W

$R_A = \frac{Wb^2(L + 2a)}{L^3}$ $\qquad R_B = \frac{Wa^2(L + 2b)}{L^3}$

$M_A = -\frac{Wab^2}{L^2}$ $\qquad M_B = -\frac{Wa^2b}{L^2}$

at C, under load, $M_C = \frac{2Wa^2b^2}{L^3}$

at X from A between A & C:

$M_X = -\frac{Wab^2}{L^2} + \frac{Wb^2(L + 2a)X}{L^3}$

$\delta_X = \frac{Wb^2X^2[3La - (L + 2a)X]}{6EIL^3}$

$i_X = \frac{Wb^2X[2La - (L + 2a)X]}{2EIL^3}$

$\delta_{max} = \frac{2Wa^3b^2}{3EI(L + 2a)^2}$

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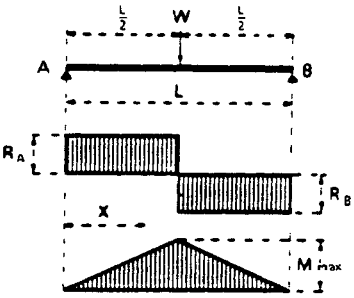
B.7 **Design formulae for beams - simply supported**

SIMPLY SUPPORTED BEAM



POINT LOAD AT MID-SPAN

Span $= L$

Point Load $= W$

$R_A = R_B = \frac{W}{2}$

at mid-span: $M_{max} = \frac{WL}{4}$; $\delta_{max} = \frac{1}{48} \cdot \frac{WL^3}{EI}$

$i_A = i_B = \frac{WL^2}{16EI}$

at X from A between A & centre: $M_X = \frac{WX}{2}$; $\delta_X = \frac{WX}{48EI}(3L^2 - 4X^2)$; $i_X = \frac{W}{16EI}(L^2 - 4X^2)$

POINT LOAD AT ANY POSITION

Span $= L$

Point Load $= W$

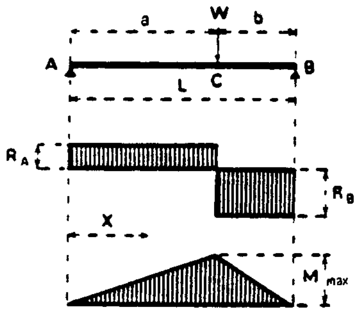
$R_A = \frac{Wb}{L}$ $\quad R_B = \frac{Wa}{L}$

at C under load: $M_{max} = \frac{Wab}{L}$; $\delta_C = \frac{Wa^2b^2}{3EIL}$

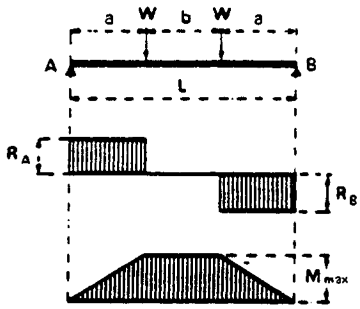
$i_A = \frac{Wab}{6EIL}(L+b)$; $\quad i_B = \frac{Wab}{6EIL}(L+a)$

When $a > b$, δ_{max} is at X from A: $\delta_{max} = \frac{Wab(L+b)}{27EIL}\sqrt{3a(L+b)}$; $X = \sqrt{\frac{a(L+b)}{3}}$

SIMPLY SUPPORTED BEAM



SIMPLY SUPPORTED BEAM



TWO EQUAL SYMMETRICAL POINT LOADS

Span $= L$

Two Point Loads, each $= W$

$R_A = R_B = W$

M_{max} over length b $= Wa$

δ_{max} at mid-span $= \frac{Wa}{24EI}(3L^2 - 4a^2)$

δ under either load $= \frac{Wa^2}{6EI}(3L - 4a)$

$i_A = i_B = \frac{Wa}{2EI}(L - a)$

If $a = b = \frac{L}{3}$, $\delta_{max} = \frac{23}{648} \cdot \frac{WL^3}{EI}$

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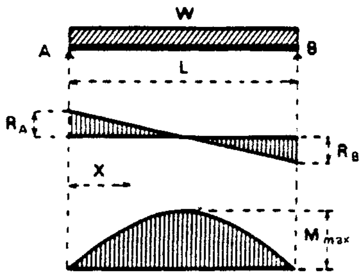
[B.7 **Design formulae for beams - simply supported** (cont..)]

SIMPLY SUPPORTED BEAM



UNIFORM LOAD ON FULL SPAN

Span $= L$

Total Uniform Load $= W$

$R_A = R_B = \frac{W}{2}$

at mid-span:
$M_{max} = \frac{WL}{8}$
$\delta_{max} = \frac{5}{384} \cdot \frac{WL^3}{EI}$

$i_A = i_B = \frac{WL^2}{24EI}$

at X from A:
$M_X = \frac{WX}{2L}(L-X)$
$\delta_X = \frac{WX}{24EIL}(X^3-2X^2L+L^3)$
$i_X = \frac{W}{24EIL}(4X^3-6X^2L+L^3)$

UNIFORM LOAD ON PART OF SPAN

Span $= L$

Total Uniform Load $= W$

Let $r = \frac{0{\cdot}5b+c}{L}$

$R_A = Wr$; $R_B = W(1-r)$

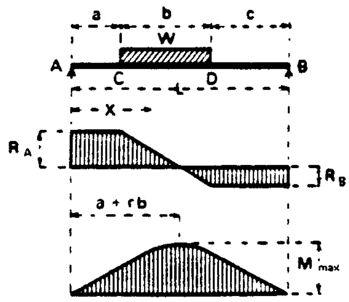
at $X = a+rb$ $M_{max} = Wr(a+0{\cdot}5rb)$

$i_A = \frac{Wr}{6EI}(L^2-c^2-Lbr)$; $i_B = \frac{W(1-r)}{6EI}\{L^2-a^2-Lb(1-r)\}$

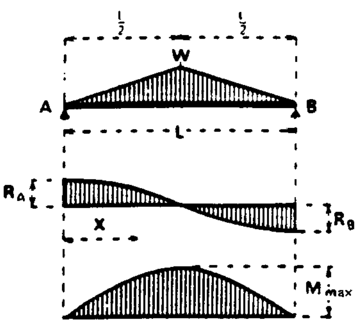
Equation to elastic line between C and D, i.e. $a \le X \le a+b$

$$\delta_x = \frac{W}{24EIb}\left[X^4-4(a+rb)X^3+6a^2X^2+4\left\{rb\left(L^2-c^2-cb-\frac{b^2}{2}\right)-a^3\right\}X+a^4\right]$$

SIMPLY SUPPORTED BEAM



SIMPLY SUPPORTED BEAM



TRIANGULAR LOAD ON FULL SPAN

Span $= L$

Total Load $= W$

$R_A = R_B = \frac{W}{2}$

at mid-span:
$M_{max} = \frac{WL}{6}$
$\delta_{max} = \frac{WL^3}{60EI}$

$i_A = i_B = \frac{5WL^2}{96EI}$

at X from A between A & centre:
$M_X = \frac{WX}{6L^2}(3L^2-4X^2)$
$\delta_X = \frac{WX}{480EIL^2}(16X^4-40X^2L^2+25L^4)$
$i_X = \frac{W}{96EIL^2}(16X^4-24X^2L^2+5L^4)$

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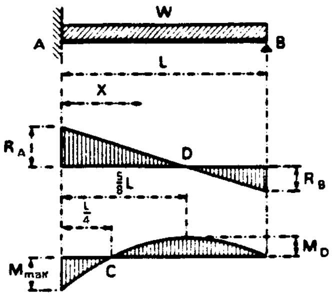
B.8 **Design formulae for beams - propped cantilever**

PROPPED CANTILEVER



at $\frac{1}{4}L$ from A, $M_C = 0$

UNIFORM LOAD ON FULL SPAN

Span $= L$

Total Uniform Load $= W$

$R_A = \frac{5}{8}W \qquad R_B = \frac{3}{8}W$

at A $\quad M_{max} = -\frac{WL}{8}$

at $\frac{5}{8}L$ from A $\quad M_D = \frac{9}{128}WL$

at 0·5785L from A $\quad \delta_{max} = \frac{WL^3}{185EI}$

at B $\quad \theta_B = \frac{WL^2}{48EI}$

at X from A:

$M_X = -\frac{W}{8L}(L^2 - 5LX + 4X^2)$

$\delta_X = \frac{WX^2}{48EIL}(3L^2 - 5LX + 2X^2)$

$\theta_X = \frac{WX}{48EIL}(6L^2 - 15LX + 8X^2)$

POINT LOAD AT MID-SPAN

Span $= L$

Point Load $= W$

$R_A = \frac{11}{16}W \qquad R_B = \frac{5}{16}W$

at A $\quad M_{max} = -\frac{3}{16}WL$

at mid-span under load:

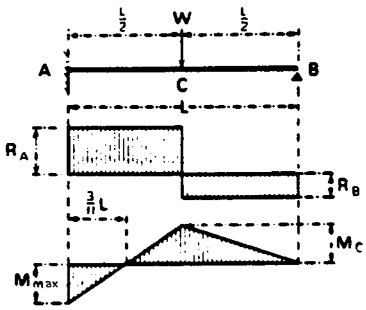
$M_C = \frac{5}{32}WL$

$\delta_C = \frac{7WL^3}{768EI}$

at 0·5528L from A, $\delta_{max} = \frac{WL^3}{107EI}$

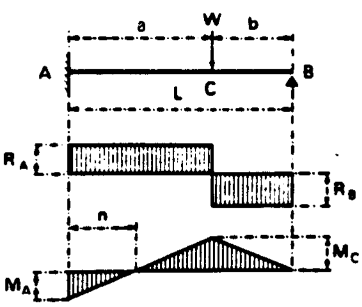
at B $\quad \theta_B = \frac{WL^2}{32EI}$

PROPPED CANTILEVER



at $\frac{3}{11}L$ from A, $M = 0$

PROPPED CANTILEVER



at $n = aL\frac{L+b}{3L^2-b^2}$ from A, $M = 0$

POINT LOAD AT ANY POSITION

Span $= L$

Point Load $= W$

$R_A = \frac{Wb(3L^2-b^2)}{2L^3} \qquad R_B = \frac{Wa^2(2L+b)}{2L^3}$

$M_A = -\frac{Wab(L+b)}{2L^2} \qquad M_C = \frac{Wa^2b(2L+b)}{2L^3}$

$\theta_B = \frac{Wa^2b}{4EIL}$

Absolute max deflection is under the load when $a = b\sqrt{2} = 0{\cdot}5858L$ $\quad \delta_{max\,max} = \frac{WL^3}{102EI}$

When $a > b\sqrt{2}$ max deflection is between A and C $\quad \delta_{max} = \frac{Wa^3b}{3EI}\cdot\frac{(L+b)^3}{(3L^2-b^2)^2}$

When $a < b\sqrt{2}$ max deflection is between C and B $\quad \delta_{max} = \frac{Wa^2b}{6EI}\sqrt{\frac{b}{2L+b}}$

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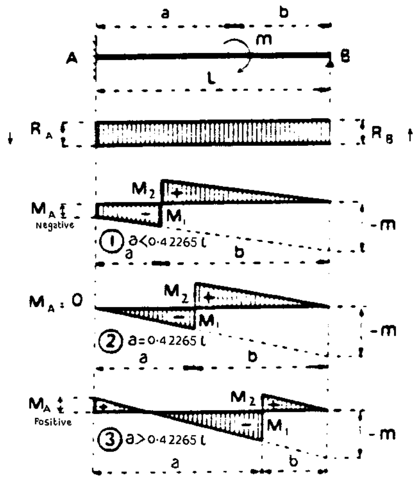
[B.8 **Design formulae for beams - propped cantilever** (cont..)]

PROPPED CANTILEVER



MOMENT APPLIED AT ANY POINT

Span = L   Applied Moment = m

$M_A = \frac{L^2-3b^2}{2L^2}m$ $\theta_B = \frac{ma}{4EIL}(2b-a)$

$R_A = -R_B = -\frac{3(L^2-b^2)}{2L^3}m = -\frac{m+M_A}{L}$

$$\begin{cases} M_1 = -\frac{m}{L^3}\left(a^3+\frac{3}{2}a^2b+b^3\right) \\ M_2 = \frac{3\,mab}{L^3}\left(b+\frac{a}{2}\right) = m+M_1 \end{cases}$$

$$\begin{cases} M_1 = -0{\cdot}42265\ m \\ M_2 = 0{\cdot}57735\ m \end{cases}$$

$$\left.\begin{matrix} M_1 = \\ M_2 = \end{matrix}\right\}\text{as for ①}$$

UNIFORM LOAD ON LENGTH BEYOND PROP

Span = L   Full Length = S

Uniform Load = W

$R_A = -\frac{3Wa}{4L}$ $R_B = \frac{W}{L}\left(S-\frac{a}{4}\right)$

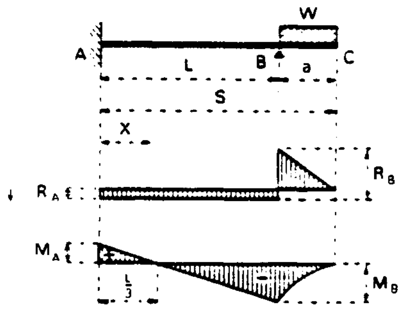
$M_A = \frac{Wa}{4}$ $M_B = -\frac{Wa}{2}$

Deflection at C = $\delta_{max} = \frac{Wa^2S}{8EI}$

Max. Negative Deflection at $X = \frac{2}{3}L$ } $\delta_{neg} = -\frac{WL^2a}{54EI}$

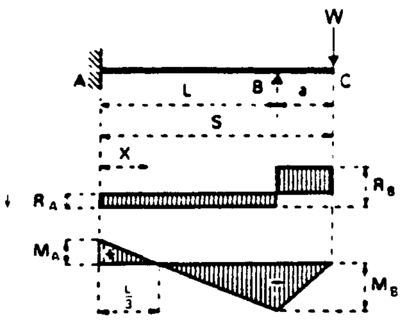
Slope at C = $\theta_C = \frac{Wa}{8EI}\left(S+\frac{a}{3}\right)$

PROPPED CANTILEVER



at $X = \frac{L}{3}$ from A, M = 0

PROPPED CANTILEVER



at $X = \frac{L}{3}$ from A, M = 0

POINT LOAD AT FREE END

Span = L   Full Length = S

Point Load = W

$R_A = -\frac{3Wa}{2L}$ $R_B = \frac{W}{L}\left(S+\frac{a}{2}\right)$

$M_A = \frac{Wa}{2}$ $M_B = -Wa$

Deflection at C = $\delta_{max} = \frac{Wa^2}{4EI}\left(S+\frac{a}{3}\right)$

Max. Negative Deflection at $X = \frac{2}{3}L$ } $\delta_{neg} = -\frac{WL^2a}{27EI}$

Slope at C = $\theta_C = \frac{Wa}{4EI}(S+a)$

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