

Rosowsky, D. V. "Structural Reliability"  
*Structural Engineering Handbook*  
Ed. Chen Wai-Fah  
Boca Raton: CRC Press LLC, 1999

# Structural Reliability<sup>1</sup>

---

## 26.1 Introduction

Definition of Reliability • Introduction to Reliability-Based Design Concepts

## 26.2 Basic Probability Concepts

Random Variables and Distributions • Moments • Concept of Independence • Examples • Approximate Analysis of Moments • Statistical Estimation and Distribution Fitting

## 26.3 Basic Reliability Problem

Basic  $R - S$  Problem • More Complicated Limit State Functions Reducible to  $R - S$  Form • Examples

## 26.4 Generalized Reliability Problem

Introduction • FORM/SORM Techniques • Monte Carlo Simulation

## 26.5 System Reliability

Introduction • Basic Systems • Introduction to Classical System Reliability Theory • Redundant Systems • Examples

## 26.6 Reliability-Based Design (Codes)

Introduction • Calibration and Selection of Target Reliabilities • Material Properties and Design Values • Design Loads and Load Combinations • Evaluation of Load and Resistance Factors

## 26.7 Defining Terms

## Acknowledgments

## References

## Further Reading

## Appendix

D. V. Rosowsky  
*Department of Civil Engineering,  
Clemson University,  
Clemson, SC*

## 26.1 Introduction

---

### 26.1.1 Definition of Reliability

Reliability and **reliability-based design** (RBD) are terms that are being associated increasingly with the design of civil engineering structures. While the subject of reliability may not be treated explicitly in the civil engineering curriculum, either at the graduate or undergraduate levels, some basic knowledge of the concepts of structural reliability can be useful in understanding the development and bases for many modern design codes (including those of the American Institute of Steel Construction [AISC],

---

<sup>1</sup>Parts of this chapter were previously published by CRC Press in *The Civil Engineering Handbook*, W.F. Chen, Ed., 1995.

the American Concrete Institute [ACI], the American Association of State Highway Transportation Officials [AASHTO], and others).

Reliability simply refers to some probabilistic measure of satisfactory (or safe) performance, and as such, may be viewed as a complementary function of the probability of failure.

$$\text{Reliability} = fcn (1 - P_{\text{failure}}) \quad (26.1)$$

When we talk about the reliability of a structure (or member or system), we are referring to the probability of safe performance for a particular limit state. A limit state can refer to ultimate failure (such as collapse) or a condition of unserviceability (such as excessive vibration, deflection, or cracking). The treatment of structural loads and resistances using probability (or reliability) theory, and of course the theories of structural analysis and mechanics, has led to the development of the latest generation of probability-based, reliability-based, or limit states design codes.

If the subject of structural reliability is generally not treated in the undergraduate civil engineering curriculum, and only a relatively small number of universities offer graduate courses in structural reliability, why include a basic (introductory) treatment in this handbook? Besides providing some insight into the bases for modern codes, it is likely that future generations of structural codes and specifications will rely more and more on probabilistic methods and reliability analyses. The treatment of (1) structural analysis, (2) structural design, and (3) probability and statistics in most civil engineering curricula permits this introduction to structural reliability without the need for more advanced study. This section by no means contains a complete treatment of the subject, nor does it contain a complete review of probability theory. At this point in time, structural reliability is usually only treated at the graduate level. However, it is likely that as RBD becomes more accepted and more prevalent, additional material will appear in both the graduate and undergraduate curricula.

### 26.1.2 Introduction to Reliability-Based Design Concepts

The concept of RBD is most easily illustrated in Figure 26.1. As shown in that figure, we consider the

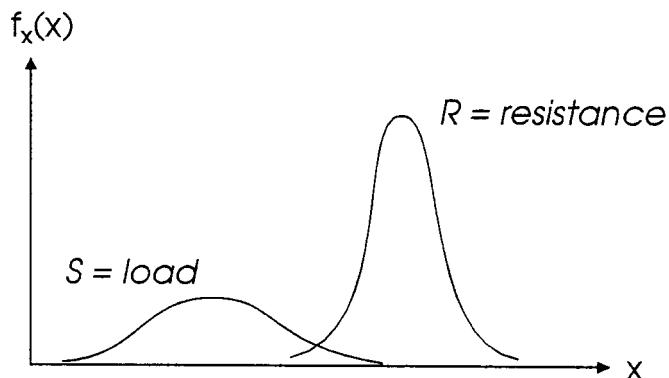


FIGURE 26.1: Basic concept of structural reliability.

acting load and the structural resistance to be random variables. Also as the figure illustrates, there is the possibility of a resistance (or strength) that is inadequate for the acting load (or conversely, that the load exceeds the available strength). This possibility is indicated by the region of overlap on Figure 26.1 in which realizations of the load and resistance variables lead to failure. The objective

of RBD is to ensure the probability of this condition is acceptably small. Of course, the load can refer to any appropriate structural, service, or environmental loading (actually, its effect), and the resistance can refer to any limit state capacity (i.e., flexural strength, bending stiffness, maximum tolerable deflection, etc.). If we formulate the simplest expression for the probability of failure ( $P_f$ ) as

$$P_f = P [(R - S) < 0] \quad (26.2)$$

we need only ensure that the units of the resistance ( $R$ ) and the load ( $S$ ) are consistent. We can then use probability theory to estimate these limit state probabilities.

Since RBD is intended to provide (or ensure) uniform and acceptably small failure probabilities for similar designs (limit states, materials, occupancy, etc.), these acceptable levels must be predetermined. This is the responsibility of code development groups and is based largely on previous experience (i.e., calibration to previous design philosophies such as [allowable stress design](#) [ASD] for steel) and engineering judgment. Finally, with information describing the statistical variability of the loads and resistances, and the target probability of failure (or target reliability) established, factors for codified design can be evaluated for the relevant load and resistance quantities (again, for the particular limit state being considered). This results, for instance, in the familiar form of design checking equations:

$$\phi R_n \geq \sum_i \gamma_i Q_{n,i} \quad (26.3)$$

referred to as [load and resistance factor design \(LRFD\)](#) in the U.S., and in which  $R_n$  is the nominal (or design) resistance and  $Q_n$  are the [nominal load effects](#). The factors  $\gamma_i$  and  $\phi$  in Equation 26.3 are the load and resistance factors, respectively. This will be described in more detail in later sections. Additional information on this subject may be found in a number of available texts [3, 21].

## 26.2 Basic Probability Concepts

---

This section presents an introduction to basic probability and statistics concepts. Only a sufficient presentation of topics to permit the discussion of reliability theory and applications that follows is included herein. For additional information and a more detailed presentation, the reader is referred to a number of widely used textbooks (i.e., [2, 5]).

### 26.2.1 Random Variables and Distributions

Random variables can be classified as being either discrete or continuous. Discrete random variables can assume only discrete values, whereas continuous random variables can assume any value within a range (which may or may not be bounded from above or below). In general, the random variables considered in structural reliability analyses are continuous, though some important cases exist where one or more variables are discrete (i.e., the number of earthquakes in a region). A brief discussion of both discrete and continuous random variables is presented here; however, the reliability analysis (theory and applications) sections that follow will focus mainly on continuous random variables.

The relative frequency of a variable is described by its probability mass function (PMF), denoted  $p_X(x)$ , if it is discrete, or its probability density function (PDF), denoted  $f_X(x)$ , if it is continuous. (A histogram is an example of a PMF, whereas its continuous analog, a smooth function, would represent a PDF.) The cumulative frequency (for either a discrete or continuous random variable) is described by its cumulative distribution function (CDF), denoted  $F_X(x)$ . (See Figure 26.2.)

There are three basic axioms of probability that serve to define valid probability assignments and provide the basis for probability theory.

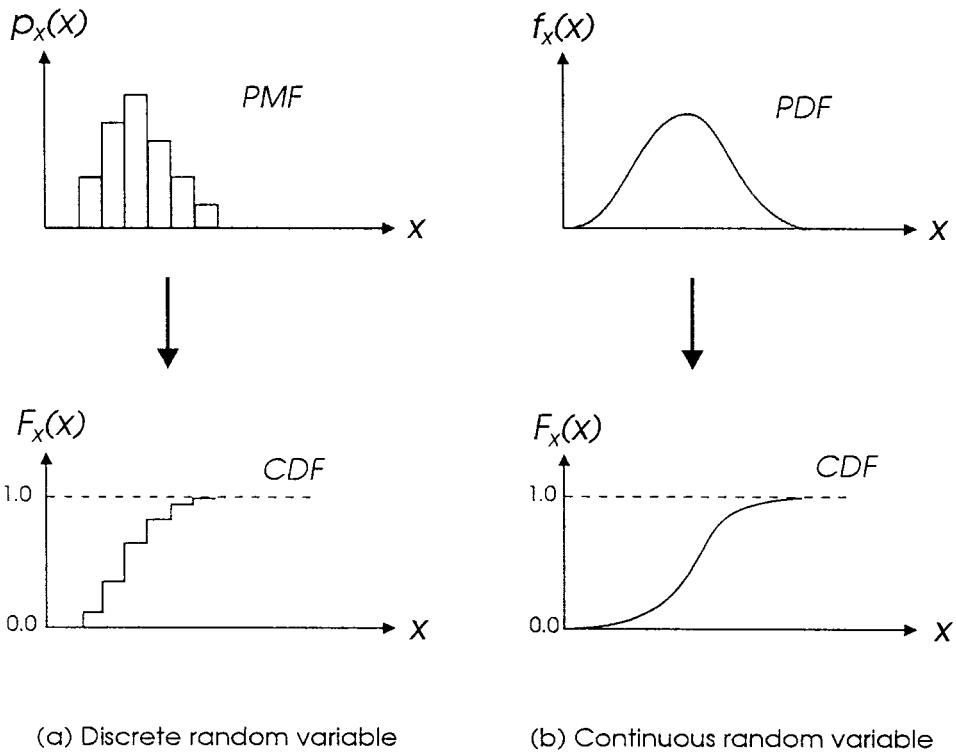


FIGURE 26.2: Sample probability functions.

1. The probability of an event is bounded by zero and one (corresponding to the cases of zero probability and certainty, respectively).
2. The sum of all possible outcomes in a sample space must equal one (a statement of collectively exhaustive events).
3. The probability of the union of two mutually exclusive events is the sum of the two individual event probabilities,  $P[A \cup B] = P[A] + P[B]$ .

The PMF or PDF, describing the relative frequency of the random variable, can be used to evaluate the probability that a variable takes on a value within some range.

$$P[a < X_{\text{discr}} \leq b] = \sum_a^b p_X(x) \quad (26.4)$$

$$P[a < X_{\text{cts}} \leq b] = \int_a^b f_X(x) dx \quad (26.5)$$

The CDF is used to describe the probability that a random variable is less than or equal to some value. Thus, there exists a simple integral relationship between the PDF and the CDF. For example, for a continuous random variable,

$$F_X(a) = P[X \leq a] = \int_{-\infty}^a f_X(x) dx \quad (26.6)$$

There are a number of common distribution forms. The probability functions for these distribution forms are given in Table 26.1.

**TABLE 26.1** Common Distribution Forms and Their Parameters

Distribution	PMF or PDF	Parameters	Mean and variance
Binomial	$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$ $x = 0, 1, 2, \dots, n$	$p$	$E[X] = np$ $\text{Var}[X] = np(1-p)$
Geometric	$p_X(x) = p(1-p)^{x-1}$ $x = 0, 1, 2, \dots$	$p$	$E[X] = 1/p$ $\text{Var}[X] = (1-p)/p^2$
Poisson	$p_X(x) = \frac{(vt)^x}{x!} e^{-vt}$ $x = 0, 1, 2, \dots$	$v$	$E[X] = vt$ $\text{Var}[X] = vt$
Exponential	$f_X(x) = \lambda e^{-\lambda x}$ $x \geq 0$	$\lambda$	$E[X] = 1/\lambda$ $\text{Var}[X] = 1/\lambda^2$
Gamma	$f_X(x) = \frac{v(vx)^{k-1} e^{-vx}}{\Gamma(k)}$ $x \geq 0$	$v, k$	$E[X] = k/v$ $\text{Var}[X] = k/v^2$
Normal	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$ $-\infty < x < \infty$	$\mu, \sigma$	$E[X] = \mu$ $\text{Var}[X] = \sigma^2$
Lognormal	$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta x} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right]$ $x \geq 0$	$\lambda, \zeta$	$E[X] = \exp\left(\lambda + \frac{1}{2}\zeta^2\right)$ $\text{Var}[X] = E^2[X]\left(\exp\left(\zeta^2\right) - 1\right)$
Uniform	$f_X(x) = \frac{1}{b-a}$ $a < x < b$	$a, b$	$E[X] = \frac{(a+b)}{2}$ $\text{Var}[X] = \frac{1}{12}(b-a)^2$
Extreme	$f_X(x) = \alpha \exp\left[-\alpha(x-u) - e^{-\alpha(x-u)}\right]$	$\alpha, u$	$E[X] = u + \frac{\gamma}{\alpha}$
Type I (largest)	$-\infty < x < \infty$		$(\gamma \cong 0.5772)$ $\text{Var}[X] = \frac{\pi^2}{6\alpha^2}$
Extreme	$f_X(x) = \frac{k}{x} \left(\frac{u}{x}\right)^k e^{-(\frac{u}{x})^k}$	$k, u$	$E[X] = u\Gamma\left(1 - \frac{1}{k}\right)$
Type II (largest)	$x \geq 0$		$(k > 1)$ $\text{Var}[X] = u^2 \left[ \Gamma\left(1 - \frac{2}{k}\right) - \Gamma^2\left(1 - \frac{1}{k}\right) \right]$
Extreme	$f_X(x) = \frac{k}{w-\varepsilon} \left(\frac{x-\varepsilon}{w-\varepsilon}\right)^{k-1} \exp\left[-\left(\frac{x-\varepsilon}{w-\varepsilon}\right)^k\right]$	$k, w, \varepsilon$	$E[X] = \varepsilon + (u-\varepsilon)\Gamma\left(1 + \frac{1}{k}\right)$
Type III (smallest)	$x \geq \varepsilon$		$\text{Var}[X] = (u-\varepsilon)^2 \left[ \Gamma\left(1 + \frac{2}{k}\right) - \Gamma^2\left(1 + \frac{1}{k}\right) \right]$

An important class of distributions for reliability analysis is based on the statistical theory of extreme values. Extreme value distributions are used to describe the distribution of the largest or smallest of a set of independent and identically distributed random variables. This has obvious implications for reliability problems in which we may be concerned with the largest of a set of 50 annual-extreme snow loads or the smallest (lowest) concrete strength from a set of 100 cylinder tests, for example. There are three important extreme value distributions (referred to as Type I, II, and III, respectively), which are also included in Table 26.1. Additional information on the derivation and application of extreme value distributions may be found in various texts (e.g., [3, 21]).

In most cases, the solution to the integral of the probability function (see Equations 26.5 and 26.6) is available in closed form. The exceptions are two of the more common distributions, the normal and lognormal distributions. For these cases, tables are available (i.e., [2, 5, 21]) to evaluate the integrals. To simplify the matter, and eliminate the need for multiple tables, the standard normal distribution is most often tabulated. In the case of the normal distribution, the probability is evaluated:

$$P[a < X \leq b] = F_X(b) - F_X(a) = \Phi\left(\frac{b - \mu_x}{\sigma_x}\right) - \Phi\left(\frac{a - \mu_x}{\sigma_x}\right) \quad (26.7)$$

where  $F_X(\cdot)$  = the particular normal distribution,  $\Phi(\cdot)$  = the standard normal CDF,  $\mu_x$  = mean of random variable  $X$ , and  $\sigma_x$  = standard deviation of random variable  $X$ . Since the standard normal variate is therefore the variate minus its mean, divided by its standard deviation, it too is a normal random variable with mean equal to zero and standard deviation equal to one. Table 26.2 presents the standard normal CDF in tabulated form.

In the case of the lognormal distribution, the probability is evaluated (also using the standard normal probability tables):

$$P[a < Y \leq b] = F_y(b) - F_y(a) = \Phi\left(\frac{\ln b - \lambda_y}{\xi_y}\right) - \Phi\left(\frac{\ln a - \lambda_y}{\xi_y}\right) \quad (26.8)$$

where  $F_Y(\cdot)$  = the particular lognormal distribution,  $\Phi(\cdot)$  = the standard normal CDF, and  $\lambda_y$  and  $\xi_y$  are the lognormal distribution parameters related to  $\mu_y$  = mean of random variable  $Y$  and  $V_y$  = coefficient of variation (COV) of random variable  $Y$ , by the following:

$$\lambda_y = \ln \mu_y - \frac{1}{2} \xi_y^2 \quad (26.9)$$

$$\xi_y^2 = \ln(V_y^2 + 1) \quad (26.10)$$

Note that for relatively low coefficients of variation ( $V_y \approx 0.3$  or less), Equation 26.10 suggests the approximation,  $\xi \approx V_y$ .

### 26.2.2 Moments

Random variables are characterized by their distribution form (i.e., probability function) and their moments. These values may be thought of as shifts and scales for the distribution and serve to uniquely define the probability function. In the case of the familiar normal distribution, there are two moments: the mean and the standard deviation. The mean describes the central tendency of the distribution (the normal distribution is a symmetric distribution), while the standard deviation is a measure of the dispersion about the mean value. Given a set of  $n$  data points, the sample mean and the sample variance (which is the square of the sample standard deviation) are computed as

$$m_x = \frac{1}{n} \sum_i X_i \quad (26.11)$$

$$\hat{\sigma}_x^2 = \frac{1}{n-1} \sum_i (X_i - m_x)^2 \quad (26.12)$$

Many common distributions are two-parameter distributions and, while not necessarily symmetric, are completely characterized by their first two moments (see Table 26.1). The population mean, or first moment of a continuous random variable, is computed as

$$\mu_x = E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx \quad (26.13)$$

where  $E[X]$  is referred to as the expected value of  $X$ . The population variance (the square of the population standard deviation) of a continuous random variable is computed as

$$\sigma_x^2 = \text{Var}[X] = E[(X - \mu_x)^2] = \int_{-\infty}^{+\infty} (x - \mu_x)^2 f_X(x) dx \quad (26.14)$$

**TABLE 26.2** Complementary Standard Normal Table,  
 $\Phi(-\beta) = 1 - \Phi(\beta)$

$\beta$	$\Phi(-\beta)$	$\beta$	$\Phi(-\beta)$	$\beta$	$\Phi(-\beta)$
.00	.50000 + 00	.47	.3192E + 00	.94	.1736E + 00
.01	.4960E + 00	.48	.3156E + 00	.95	.1711E + 00
.02	.4920E + 00	.49	.3121E + 00	.96	.1685E + 00
.03	.4880E + 00	.50	.3085E + 00	.97	.1660E + 00
.04	.4840E + 00	.51	.3050E + 00	.98	.1635E + 00
.05	.4801E + 00	.52	.3015E + 00	.99	.1611E + 00
.06	.4761E + 00	.53	.2981E + 00	1.00	.1587E + 00
.07	.4721E + 00	.54	.2946E + 00	1.01	.1562E + 00
.08	.4681E + 00	.55	.2912E + 00	1.02	.1539E + 00
.09	.4641E + 00	.56	.2877E + 00	1.03	.1515E + 00
.10	.4602E + 00	.57	.2843E + 00	1.04	.1492E + 00
.11	.4562E + 00	.58	.2810E + 00	1.05	.1469E + 00
.12	.4522E + 00	.59	.2776E + 00	1.06	.1446E + 00
.13	.4483E + 00	.60	.2743E + 00	1.07	.1423E + 00
.14	.4443E + 00	.61	.2709E + 00	1.08	.1401E + 00
.15	.4404E + 00	.62	.2676E + 00	1.09	.1379E + 00
.16	.4364E + 00	.63	.2643E + 00	1.10	.1357E + 00
.17	.4325E + 00	.64	.2611E + 00	1.11	.1335E + 00
.18	.4286E + 00	.65	.2578E + 00	1.12	.1314E + 00
.19	.4247E + 00	.66	.2546E + 00	1.13	.1292E + 00
.20	.4207E + 00	.67	.2514E + 00	1.14	.1271E + 00
.21	.4168E + 00	.68	.2483E + 00	1.15	.1251E + 00
.22	.4129E + 00	.69	.2451E + 00	1.16	.1230E + 00
.23	.4090E + 00	.70	.2420E + 00	1.17	.1210E + 00
.24	.4052E + 00	.71	.2389E + 00	1.18	.1190E + 00
.25	.4013E + 00	.72	.2358E + 00	1.19	.1170E + 00
.26	.3974E + 00	.73	.2327E + 00	1.20	.1151E + 00
.27	.3936E + 00	.74	.2297E + 00	1.21	.1131E + 00
.28	.3897E + 00	.75	.2266E + 00	1.22	.1112E + 00
.29	.3859E + 00	.76	.2236E + 00	1.23	.1093E + 00
.30	.3821E + 00	.77	.2207E + 00	1.24	.1075E + 00
.31	.3783E + 00	.78	.2177E + 00	1.25	.1056E + 00
.32	.3745E + 00	.79	.2148E + 00	1.26	.1038E + 00
.33	.3707E + 00	.80	.2119E + 00	1.27	.1020E + 00
.34	.3669E + 00	.81	.2090E + 00	1.28	.1003E + 00
.35	.3632E + 00	.82	.2061E + 00	1.29	.9853E - 01
.36	.3594E + 00	.83	.2033E + 00	1.30	.9680E - 01
.37	.3557E + 00	.84	.2005E + 00	1.31	.9510E - 01
.38	.3520E + 00	.85	.1977E + 00	1.32	.9342E - 01
.39	.3483E + 00	.86	.1949E + 00	1.33	.9176E - 01
.40	.3446E + 00	.87	.1922E + 00	1.34	.9012E - 01
.41	.3409E + 00	.88	.1894E + 00	1.35	.8851E - 01
.42	.3372E + 00	.89	.1867E + 00	1.36	.8691E - 01
.43	.3336E + 00	.90	.1841E + 00	1.37	.8534E - 01
.44	.3300E + 00	.91	.1814E + 00	1.38	.8379E - 01
.45	.3264E + 00	.92	.1788E + 00	1.39	.8226E - 01
.46	.3228E + 00	.93	.1762E + 00	1.40	.8076E - 01
.47	.7927E - 01	1.88	.3005E - 01	2.35	.9387E - 02
.48	.7780E - 01	1.89	.2938E - 01	2.36	.9138E - 02
.49	.7636E - 01	1.90	.2872E - 01	2.37	.8894E - 02
.50	.7493E - 01	1.91	.2807E - 01	2.38	.8656E - 02
.51	.7353E - 01	1.92	.2743E - 01	2.39	.8424E - 02
.52	.7215E - 01	1.93	.2680E - 01	2.40	.8198E - 02
.53	.7078E - 01	1.94	.2619E - 01	2.41	.7976E - 02
.54	.6944E - 01	1.95	.2559E - 01	2.42	.7760E - 02
.55	.6811E - 01	1.96	.2500E - 01	2.43	.7549E - 02
.56	.6681E - 01	1.97	.2442E - 01	2.44	.7344E - 02
.57	.6552E - 01	1.98	.2385E - 01	2.45	.7143E - 02
.58	.6426E - 01	1.99	.2330E - 01	2.46	.6947E - 02
.59	.6301E - 01	2.00	.2275E - 01	2.47	.6756E - 02
.60	.6178E - 01	2.01	.2222E - 01	2.48	.6569E - 02
.61	.6057E - 01	2.02	.2169E - 01	2.49	.6387E - 02
.62	.5938E - 01	2.03	.2118E - 01	2.50	.6210E - 02
.63	.5821E - 01	2.04	.2068E - 01	2.51	.6037E - 02
.64	.5705E - 01	2.05	.2018E - 01	2.52	.5868E - 02
.65	.5592E - 01	2.06	.1970E - 01	2.53	.5703E - 02
.66	.5480E - 01	2.07	.1923E - 01	2.54	.5543E - 02
.67	.5370E - 01	2.08	.1876E - 01	2.55	.5386E - 02
.68	.5262E - 01	2.09	.1831E - 01	2.56	.5234E - 02
.69	.5155E - 01	2.10	.1786E - 01	2.57	.5085E - 02

**TABLE 26.2** Complementary Standard Normal Table,  
 $\Phi(-\beta) = 1 - \Phi(\beta)$  (continued)

$\beta$	$\Phi(-\beta)$	$\beta$	$\Phi(-\beta)$	$\beta$	$\Phi(-\beta)$
1.64	.5050E - 01	2.11	.1743E - 01	2.58	.4940E - 02
1.65	.4947E - 01	2.12	.1700E - 01	2.59	.4799E - 02
1.66	.4846E - 01	2.13	.1659E - 01	2.60	.4661E - 02
1.67	.4746E - 01	2.14	.1618E - 01	2.61	.4527E - 02
1.68	.4648E - 01	2.15	.1578E - 01	2.62	.4396E - 02
1.69	.4551E - 01	2.16	.1539E - 01	2.63	.4269E - 02
1.70	.4457E - 01	2.17	.1500E - 01	2.64	.4145E - 02
1.71	.4363E - 01	2.18	.1463E - 01	2.65	.4024E - 02
1.72	.4272E - 01	2.19	.1426E - 01	2.66	.3907E - 02
1.73	.4182E - 01	2.20	.1390E - 01	2.67	.3792E - 02
1.74	.4093E - 01	2.21	.1355E - 01	2.68	.3681E - 02
1.75	.4006E - 01	2.22	.1321E - 01	2.69	.3572E - 02
1.76	.3920E - 01	2.23	.1287E - 01	2.70	.3467E - 02
1.77	.3836E - 01	2.24	.1255E - 01	2.71	.3364E - 02
1.78	.3754E - 01	2.25	.1222E - 01	2.72	.3264E - 02
1.79	.3673E - 01	2.26	.1191E - 01	2.73	.3167E - 02
1.80	.3593E - 01	2.27	.1160E - 01	2.74	.3072E - 02
1.81	.3515E - 01	2.28	.1130E - 01	2.75	.2980E - 02
1.82	.3438E - 01	2.29	.1101E - 01	2.76	.2890E - 02
1.83	.3363E - 01	2.30	.1072E - 01	2.77	.2803E - 02
1.84	.3288E - 01	2.31	.1044E - 01	2.78	.2718E - 02
1.85	.3216E - 01	2.32	.1017E - 01	2.79	.2635E - 02
1.86	.3144E - 01	2.33	.9903E - 02	2.80	.2555E - 02
1.87	.3074E - 01	2.34	.9642E - 02	2.81	.2477E - 02
2.82	.2401E - 02	3.29	.5009E - 03	3.76	.8491E - 04
2.83	.2327E - 02	3.30	.4834E - 03	3.77	.8157E - 04
2.84	.2256E - 02	3.31	.4664E - 03	3.78	.7836E - 04
2.85	.2186E - 02	3.32	.4500E - 03	3.79	.7527E - 04
2.86	.2118E - 02	3.33	.4342E - 03	3.80	.7230E - 04
2.87	.2052E - 02	3.34	.4189E - 03	3.81	.6943E - 04
2.88	.1988E - 02	3.35	.4040E - 03	3.82	.6667E - 04
2.89	.1926E - 02	3.36	.3897E - 03	3.83	.6402E - 04
2.90	.1866E - 02	3.37	.3758E - 03	3.84	.6147E - 04
2.91	.1807E - 02	3.38	.3624E - 03	3.85	.5901E - 04
2.92	.1750E - 02	3.39	.3494E - 03	3.86	.5664E - 04
2.93	.1695E - 02	3.40	.3369E - 03	3.87	.5437E - 04
2.94	.1641E - 02	3.41	.3248E - 03	3.88	.5218E - 04
2.95	.1589E - 02	3.42	.3131E - 03	3.89	.5007E - 04
2.96	.1538E - 02	3.43	.3017E - 03	3.90	.4804E - 04
2.97	.1489E - 02	3.44	.2908E - 03	3.91	.4610E - 04
2.98	.1441E - 02	3.45	.2802E - 03	3.92	.4422E - 04
2.99	.1395E - 02	3.46	.2700E - 03	3.93	.4242E - 04
3.00	.1350E - 02	3.47	.2602E - 03	3.94	.4069E - 04
3.01	.1306E - 02	3.48	.2507E - 03	3.95	.3902E - 04
3.02	.1264E - 02	3.49	.2415E - 03	3.96	.3742E - 04
3.03	.1223E - 02	3.50	.2326E - 03	3.97	.3588E - 04
3.04	.1183E - 02	3.51	.2240E - 03	3.98	.3441E - 04
3.05	.1144E - 02	3.52	.2157E - 03	3.99	.3298E - 04
3.06	.1107E - 02	3.53	.2077E - 03	4.00	.3162E - 04
3.07	.1070E - 02	3.54	.2000E - 03	4.10	.2062E - 04
3.08	.1035E - 02	3.55	.1926E - 03	4.20	.1332E - 04
3.09	.1001E - 02	3.56	.1854E - 03	4.30	.8524E - 05
3.10	.9676E - 03	3.57	.1784E - 03	4.40	.5402E - 05
3.11	.9354E - 03	3.58	.1717E - 03	4.50	.3391E - 05
3.12	.9042E - 03	3.59	.1653E - 03	4.60	.2108E - 05
3.13	.8740E - 03	3.60	.1591E - 03	4.70	.1298E - 05
3.14	.8447E - 03	3.61	.1531E - 03	4.80	.7914E - 06
3.15	.8163E - 03	3.62	.1473E - 03	4.90	.4780E - 06
3.16	.7888E - 03	3.63	.1417E - 03	5.00	.2859E - 06
3.17	.7622E - 03	3.64	.1363E - 03	5.10	.1694E - 06
3.18	.7363E - 03	3.65	.1311E - 03	5.20	.9935E - 07
3.19	.7113E - 03	3.66	.1261E - 03	5.30	.5772E - 07
3.20	.6871E - 03	3.67	.1212E - 03	5.40	.3321E - 07
3.21	.6636E - 03	3.68	.1166E - 03	5.50	.1892E - 07
3.22	.6409E - 03	3.69	.1121E - 03	6.00	.9716E - 09
3.23	.6189E - 03	3.70	.1077E - 03	6.50	.3945E - 10
3.24	.5976E - 03	3.71	.1036E - 03	7.00	.1254E - 11
3.25	.5770E - 03	3.72	.9956E - 04	7.50	.3116E - 13
3.26	.5570E - 03	3.73	.9569E - 04	8.00	.6056E - 15
3.27	.5377E - 03	3.74	.9196E - 04	8.50	.9197E - 17
3.28	.5190E - 03	3.75	.8837E - 04	9.00	.1091E - 18

The population variance can also be expressed in terms of expectations as

$$\sigma_x^2 = E[X^2] - E^2[X] = \int_{-\infty}^{+\infty} x^2 f_X(x) dx - \left( \int_{-\infty}^{+\infty} x f_X(x) dx \right)^2 \quad (26.15)$$

The COV is defined as the ratio of the standard deviation to the mean, and therefore serves as a nondimensional measure of variability.

$$\text{COV} = V_X = \frac{\sigma_x}{\mu_x} \quad (26.16)$$

In some cases, higher order ( $> 2$ ) moments exist, and these may be computed similarly as

$$\mu_x^{(n)} = E[(X - \mu_x)^n] = \int_{-\infty}^{+\infty} (x - \mu_x)^n f_X(x) dx \quad (26.17)$$

where  $\mu_x^{(n)}$  = the  $n$ th central moment of random variable  $X$ . Often, it is more convenient to define the probability distribution in terms of its parameters. These parameters can be expressed as functions of the moments (see Table 26.1).

### 26.2.3 Concept of Independence

The concept of statistical independence is very important in structural reliability as it often permits great simplification of the problem. While not all random quantities in a reliability analysis may be assumed independent, it is certainly reasonable to assume (in most cases) that loads and resistances are statistically independent. Often, the assumption of independent loads (actions) can be made as well.

Two events,  $A$  and  $B$ , are statistically independent if the outcome of one in no way affects the outcome of the other. Therefore, two random variables,  $X$  and  $Y$ , are statistically independent if information on one variable's probability of taking on some value in no way affects the probability of the other random variable taking on some value. One of the most significant consequences of this statement of independence is that the joint probability of occurrence of two (or more) random variables can be written as the product of the individual marginal probabilities. Therefore, if we consider two events ( $A$  = probability that an earthquake occurs and  $B$  = probability that a hurricane occurs), and we assume these occurrences are statistically independent in a particular region, the probability of both an earthquake and a hurricane occurring is simply the product of the two probabilities:

$$P[A \text{ "and" } B] = P[A \cap B] = P[A]P[B] \quad (26.18)$$

Similarly, if we consider resistance ( $R$ ) and load ( $S$ ) to be continuous random variables, and assume independence, we can write the probability of  $R$  being less than or equal to some value  $r$  and the probability that  $S$  exceeds some value  $s$  (i.e., failure) as

$$\begin{aligned} P[R \leq r \cap S > s] &= P[R \leq r]P[S > s] \\ &= P[R \leq r](1 - P[S \leq s]) = F_R(r)(1 - F_S(s)) \end{aligned} \quad (26.19)$$

Additional implications of statistical independence will be discussed in later sections. The treatments of dependent random variables, including issues of correlation, joint probability, and conditional probability are beyond the scope of this introduction, but may be found in any elementary text (e.g., [2, 5]).

## 26.2.4 Examples

Three relatively simple examples are presented here. These examples serve to illustrate some important elements of probability theory and introduce the reader to some basic reliability concepts in structural engineering and design.

### EXAMPLE 26.1:

The Richter magnitude of an earthquake, given that it has occurred, is assumed to be exponentially distributed. For a particular region in Southern California, the exponential distribution parameter ( $\lambda$ ) has been estimated to be 2.23. What is the probability that a given earthquake will have a magnitude greater than 5.5?

$$\begin{aligned} P[M > 5.5] &= 1 - P[M \leq 5.5] = 1 - F_X(5.5) \\ &= 1 - \left[1 - e^{-5.5\lambda}\right] \\ &= e^{-2.23 \times 5.5} = e^{-12.265} \\ &\approx 4.71 \times 10^{-6} \end{aligned}$$

Given that two earthquakes have occurred in this region, what is the probability that both of their magnitudes were greater than 5.5?

$$\begin{aligned} P[M_1 > 5.5 \cap M_2 > 5.5] &= P[M_1 > 5.5]P[M_2 > 5.5] \text{ (assumed independence)} \\ &= (P[M > 5.5])^2 \text{ (identically distributed)} \\ &= (4.71 \times 10^{-6})^2 \\ &\approx 2.22 \times 10^{-11} \text{ (very small!)} \end{aligned}$$

### EXAMPLE 26.2:

Consider the cross-section of a reinforced concrete column with 12 reinforcing bars. Assume the load-carrying capacity of each of the 12 reinforcing bars ( $R_i$ ) is normally distributed with mean of 100 kN and standard deviation of 20 kN. Further assume that the load-carrying capacity of the concrete itself is  $r_c = 500$  kN (deterministic) and that the column is subjected to a known load of 1500 kN. What is the probability that this column will fail?

First, we can compute the mean and standard deviation of the column's total load-carrying capacity.

$$\begin{aligned} E[R] &= m_R = r_c + \sum_{i=1}^{12} E[R_i] = 500 + 12(100) = 1700 \text{ kN} \\ \text{Var}[R] &= \sigma_R^2 = \sum_{i=1}^{12} \sigma_{R_i}^2 = 12(20)^2 = 4800 \text{ kN}^2 \quad \therefore \sigma_R = 69.28 \text{ kN} \end{aligned}$$

Since the total capacity is the sum of a number of normal variables, it too is a normal variable (central limit theorem). Therefore, we can compute the probability of failure as the probability that the load-carrying capacity,  $R$ , is less than the load of 1500 kN.

$$P[R < 1500] = F_R(1500) = \Phi\left(\frac{1500 - 1700}{69.28}\right) = \Phi(-2.89) \approx 0.00193$$

**EXAMPLE 26.3:**

The moment capacity ( $M$ ) of the simply supported beam ( $l = 10$  ft) shown in Figure 26.3 is assumed to be normally distributed with mean of 25 ft-kips and COV of 0.20. Failure occurs if the

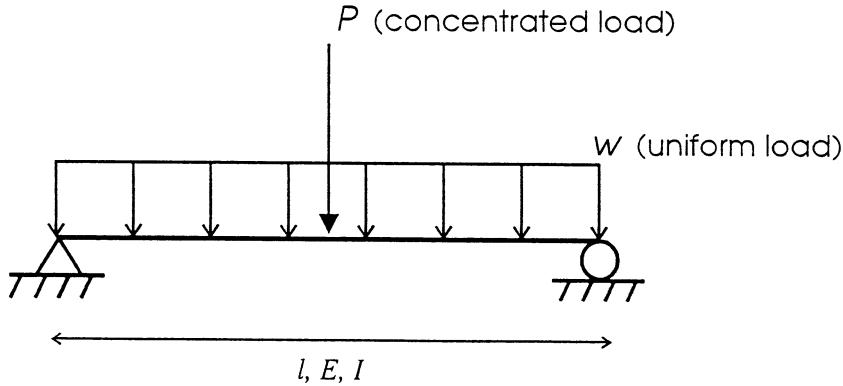


FIGURE 26.3: Simply supported beam (for Example 26.3).

maximum moment exceeds the moment capacity. If only a concentrated load  $P = 4$  kips is applied at midspan, what is the failure probability?

$$\begin{aligned} M_{\max} &= \frac{Pl}{4} = \frac{4(10')}{4} = 10 \text{ ft-kips} \\ P_f &= P[M < M_{\max}] = F_M(10) = \Phi\left(\frac{10 - 25}{5}\right) = \Phi(-3.0) \approx 0.00135 \end{aligned}$$

If only a uniform load  $w = 1$  kip/ft is applied along the entire length of the beam, what is the failure probability?

$$\begin{aligned} M_{\max} &= \frac{wl^2}{8} = \frac{1(10')^2}{8} = 12.5 \text{ ft-kips} \\ P_f &= P[M < M_{\max}] = F_M(12.5) = \Phi\left(\frac{12.5 - 25}{5}\right) = \Phi(-2.5) \approx 0.00621 \end{aligned}$$

If the beam is subjected to both  $P$  and  $w$  simultaneously, what is the probability the beam performs safely?

$$\begin{aligned} M_{\max} &= \frac{Pl}{4} + \frac{wl^2}{8} = 10 + 12.5 = 22.5 \text{ ft-kips} \\ P_f &= P[M < M_{\max}] = F_M(22.5) = \Phi\left(\frac{22.5 - 25}{5}\right) = \Phi(-0.5) \approx 0.3085 \\ \therefore P(\text{"safety"}) &= P_S = (1 - P_f) = 0.692 \end{aligned}$$

Note that this failure probability is not simply the sum of the two individual failure probabilities computed previously. Finally, for design purposes, suppose we want a probability of safe performance  $P_s = 99.9\%$ , for the case of the beam subjected to the uniform load ( $w$ ) only. What value of  $w_{\max}$

(i.e., maximum allowable uniform load for design) should we specify?

$$\begin{aligned}
 M_{\text{allow.}} &= \frac{w_{\max} (l^2)}{8} = w_{\max} \left( \frac{10^2}{8} \right) = 12.5 (w_{\max}) \\
 \text{goal : } P[M > 12.5w_{\max}] &= 0.999 \\
 1 - F_M(12.5w_{\max}) &= 0.999 \\
 1 - \Phi \left( \frac{12.5w_{\max} - 25}{5} \right) &= 0.999 \\
 \therefore \Phi^{-1}(1.0 - 0.999) &= \frac{12.5w_{\max} - 25}{5} \\
 \therefore w_{\max} &= \frac{(-3.09)(5) + 25}{12.5} \approx 0.76 \text{ kips/ft}
 \end{aligned}$$

#### EXAMPLE 26.4:

The total annual snowfall for a particular location is modeled as a normal random variable with mean of 60 in. and standard deviation of 15 in. What is the probability that in any given year the total snowfall in that location is between 45 and 65 in.?

$$\begin{aligned}
 P[45 < S \leq 65] &= F_S(65) - F_S(45) = \Phi \left( \frac{65 - 60}{15} \right) - \Phi \left( \frac{45 - 60}{15} \right) \\
 &= \Phi(0.33) - \Phi(-1.00) = \Phi(0.33) - [1 - \Phi(1.00)] \\
 &= 0.629 - (1 - 0.841) \approx 0.47 \text{ (about 47%)}
 \end{aligned}$$

What is the probability the total annual snowfall is at least 30 in. in this location?

$$\begin{aligned}
 1 - F_S(30) &= 1 - \Phi \left( \frac{30 - 60}{15} \right) \\
 &= 1 - \Phi(-2.0) = 1 - [1 - \Phi(2.0)] \\
 &= \Phi(2.0) \approx 0.977 \text{ (about 98%)}
 \end{aligned}$$

Suppose for design we want to specify the 95th percentile snowfall value (i.e., a value that has a 5% exceedence probability). Estimate the value of  $S_{.95}$ .

$$\begin{aligned}
 P[S > S_{.95}] &\equiv 0.05 \quad P[S < S_{.95}] = .95 \\
 \Phi \left( \frac{S_{.95} - 60}{15} \right) &= 0.95 \\
 \therefore S_{.95} &= \left[ 15 \times \Phi^{-1}(0.95) \right] + 60 \\
 &= (15)(1.64) + 60 = 84.6 \text{ in.} \\
 &\text{(so, specify 85 in.)}
 \end{aligned}$$

Now, assume the total annual snowfall is lognormally distributed (rather than normally) with the same mean and standard deviation as before. Recompute  $P[45 \text{ in.} \leq S \leq 65 \text{ in.}]$ . First, we obtain the lognormal distribution parameters:

$$\begin{aligned}
 \xi^2 &= \ln(V_S^2 + 1) = \ln \left( \left( \frac{15}{60} \right)^2 + 1 \right) = 0.061 \\
 \xi &= 0.246 \ (\approx 0.25 = V_S; \quad \text{o.k. for } V \approx 0.3 \text{ or less}) \\
 \lambda &= \ln(m_S) - 0.5\xi^2 = \ln(60) - 0.5(0.61) = 4.064
 \end{aligned}$$

Now, using these parameters, recompute the probability:

$$\begin{aligned}
 P[45 < S_{LN} \leq 65] &= F_S(65) - F_S(45) = \Phi\left(\frac{\ln(65) - 4.06}{0.25}\right) - \Phi\left(\frac{\ln(45) - 4.06}{0.25}\right) \\
 &= \Phi(0.46) - \Phi(-1.01) = \Phi(0.46) - [1 - \Phi(1.01)] \\
 &= 0.677 - (1 - 0.844) \approx 0.52 \text{ (about 52%)}
 \end{aligned}$$

Note that this is slightly higher than the value obtained assuming the snowfall was normally distributed (47%). Finally, again assuming the total annual snowfall to be lognormally distributed, recompute the 5% exceedence limit (i.e., the 95th percentile value):

$$\begin{aligned}
 P[S < S_{.95}] &= .95 \\
 \Phi\left(\frac{\ln(S_{.95}) - 4.06}{0.25}\right) &= 0.95 \\
 \therefore \ln(S_{.95}) &= [.25 \times \Phi^{-1}(.95)] + 4.06 \\
 &= (.25)(1.64) + 4.06 = 4.47 \therefore S_{.95} = \exp(4.47) \approx 87.4 \text{ in.} \\
 &\text{(specify 88 in.)}
 \end{aligned}$$

Again, this value is slightly higher than the value obtained assuming the total snowfall was normally distributed (about 85 in.).

### 26.2.5 Approximate Analysis of Moments

In some cases, it may be desired to estimate approximately the statistical moments of a function of random variables. For a function given by

$$Y = g(X_1, X_2, \dots, X_n) \quad (26.20)$$

approximate estimates for the moments can be obtained using a first-order Taylor series expansion of the function about the vector of mean values. Keeping only the 0th- and 1st-order terms results in an approximate mean

$$E[Y] \approx g(\mu_1, \mu_2, \dots, \mu_n) \quad (26.21)$$

in which  $\mu_i$  = mean of random variable  $X_i$ , and an approximate variance

$$\text{Var}[Y] \approx \sum_{i=1}^n c_i^2 \text{Var}[X_i] + \sum_{i \neq j}^n \sum_{j=1}^n c_i c_j \text{Cov}[X_i, X_j] \quad (26.22)$$

in which  $c_i$  and  $c_j$  are the values of the partial derivatives  $\partial g / \partial X_i$  and  $\partial g / \partial X_j$ , respectively, evaluated at the vector of mean values  $(\mu_1, \mu_2, \dots, \mu_n)$ , and  $\text{Cov}[X_i, X_j]$  = covariance function of  $X_i$  and  $X_j$ . If all random variables  $X_i$  and  $X_j$  are mutually uncorrelated (statistically independent), the approximate variance reduces to

$$\text{Var}[Y] \approx \sum_{i=1}^n c_i^2 \text{Var}[X_i] \quad (26.23)$$

These approximations can be shown to be valid for reasonably linear functions  $g(X)$ . For nonlinear functions, the approximations are still reasonable if the variances of the individual random variables,  $X_i$ , are relatively small.

The estimates of the moments can be improved if the second-order terms from the Taylor series expansions are included in the approximation. The resulting second-order approximation for the mean assuming all  $X_i$ ,  $X_j$  uncorrelated is

$$E[Y] \approx g(\mu_1, \mu_2, \dots, \mu_n) + \frac{1}{2} \sum_{i=1}^n \left( \frac{\partial^2 g}{\partial X_i^2} \right) \text{Var}[X_i] \quad (26.24)$$

For uncorrelated  $X_i$ ,  $X_j$ , however, there is no improvement over Equation 26.23 for the approximate variance. Therefore, while the second-order analysis provides additional information for estimating the mean, the variance estimate may still be inadequate for nonlinear functions.

### 26.2.6 Statistical Estimation and Distribution Fitting

There are two general classes of techniques for estimating statistical moments: point-estimate methods and interval-estimate methods. The method of moments is an example of a point-estimate method, while confidence intervals and hypothesis testing are examples of interval-estimate techniques. These topics are treated generally in an introductory statistics course and therefore are not covered in this chapter. However, the topics are treated in detail in Ang and Tang [2] and Benjamin and Cornell [5], as well as many other texts.

The most commonly used tests for goodness-of-fit of distributions are the Chi-Squared ( $\chi^2$ ) test and the Kolmogorov-Smirnov (K-S) test. Again, while not presented in detail herein, these tests are described in most introductory statistics texts. The  $\chi^2$  test compares the observed relative frequency histogram with an assumed, or theoretical, PDF. The K-S test compares the observed cumulative frequency plot with the assumed, or theoretical, CDF. While these tests are widely used, they are both limited by (1) often having only limited data in the tail regions of the distribution (the region most often of interest in reliability analyses), and (2) not allowing evaluation of goodness-of-fit in specific regions of the distribution. These methods do provide established and effective (as well as statistically robust) means of evaluating the relative goodness-of-fit of various distributions over the entire range of values. However, when it becomes necessary to assure a fit in a particular region of the distribution of values, such as an upper or lower tail, other methods must be employed. One such method, sometimes called the inverse CDF method, is described here. The inverse CDF method is a simple, graphical technique similar to that of using probability paper to evaluate goodness-of-fit.

It can be shown using the theory of order statistics [5] that

$$E[F_X(y_i)] = \frac{i}{n+1} \quad (26.25)$$

where  $F_X(\cdot)$  = cumulative distribution function,  $y_i$  = mean of the  $i$ th order statistic, and  $n$  = number of independent samples. Hence, the term  $i/(n+1)$  is referred to as the  $i$ th rank mean plotting position. This well-known plotting position has the properties of being nonparametric (i.e., distribution independent), unbiased, and easy to compute. With a sufficiently large number of observations,  $n$ , a cumulative frequency plot is obtained by plotting the rank-ordered observation  $x_i$  versus the quantity  $i/(n+1)$ . As  $n$  becomes large, this observed cumulative frequency plot approaches the true CDF of the underlying phenomenon. Therefore, the plotting position is taken to approximate the CDF evaluated at  $x_i$ :

$$F_X(x_i) \approx \left( \frac{i}{n+1} \right) \quad i = 1, \dots, n \quad (26.26)$$

Simply examining the resulting estimate for the CDF is limited as discussed previously. That is, assessing goodness-of-fit in the tail regions can be difficult. Furthermore, relative goodness-of-fit

over all regions of the CDF is essentially impossible. To address this shortcoming, the inverse CDF is considered. For example, taking the inverse CDF of both sides of Equation (26.26) yields

$$F_X^{-1}[F_X(x_i)] \approx F_X^{-1}\left[\left(\frac{i}{n+1}\right)\right] \quad (26.27)$$

where the left-hand side simply reduces to  $x_i$ . Therefore, an estimate for the  $i$ th observation can be obtained provided the inverse of the assumed underlying CDF exists (see Table 26.5). Finally, if the  $i$ th (rank-ordered) observation is plotted against the inverse CDF of the rank mean plotting position, which serves as an estimate of the  $i$ th observation, the relative goodness-of-fit can be evaluated over the entire range of observations. Essentially, therefore, one is seeking a close fit to the 1:1 line. The better this fit, the better the assumed underlying distribution  $F_X(\cdot)$ . Figure 26.4 presents an example of a relatively good fit of an Extreme Type I largest (Gumbel) distribution to annual maximum wind speed data from Boston, Massachusetts.

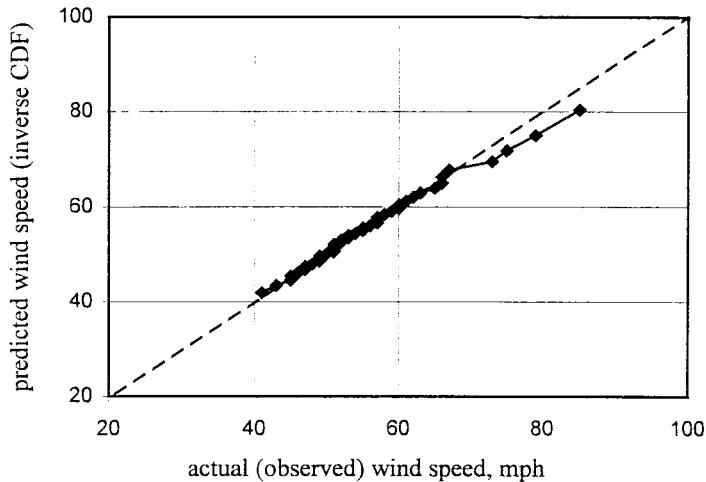


FIGURE 26.4: Inverse CDF (Extreme Type I largest) of annual maximum wind speeds, Boston, MA (1936–1977).

Caution must be exercised in interpreting goodness-of-fit using this method. Clearly, a perfect fit will not be possible, unless the phenomenon itself corresponds directly to a single underlying distribution. Furthermore, care must be taken in evaluating goodness-of-fit in the tail regions, as often limited data exists in these regions. A poor fit in the upper tail, for instance, may not necessarily mean that the distribution should be rejected. This method does have the advantage, however, of permitting an evaluation over specific ranges of values corresponding to specific regions of the distribution. While this evaluation is essentially qualitative, as described herein, it is a relatively simple extension to quantify the relative goodness-of-fit using some measure of correlation, for example.

Finally, the inverse CDF method has advantages over the use of probability paper in that (1) the method can be generalized for any distribution form without the need for specific types of plotting paper, and (2) the method can be easily programmed.

## 26.3 Basic Reliability Problem

---

A complete treatment of structural reliability theory is not included in this section. However, a number of texts are available (in varying degrees of difficulty) on this subject [3, 10, 21, 23]. For the purpose of an introduction, an elementary treatment of the basic (two-variable) reliability problem is provided in the following sections.

### 26.3.1 Basic $R - S$ Problem

As described previously, the simplest formulation of the failure probability problem may be written:

$$P_f = P[R < S] = P[R - S < 0] \quad (26.28)$$

in which  $R$  = resistance and  $S$  = load. The simple function,  $g(X) = R - S$ , where  $X$  = vector of basic random variables, is termed the limit state function. It is customary to formulate this limit state function such that the condition  $g(X) < 0$  corresponds to failure, while  $g(X) > 0$  corresponds to a condition of safety. The limit state surface corresponds to points where  $g(X) = 0$  (where the term "surface" implies it is possible to have problems involving more than two random variables). For the simple two-variable case, if the assumption can be made that the load and resistance quantities are statistically independent, and that the population statistics can be estimated by the sample statistics, the failure probabilities for the cases of normal or lognormal variates ( $R$ ,  $S$ ) are given by

$$P_{f(N)} = \Phi\left(\frac{0 - m_M}{\hat{\sigma}_M}\right) = \Phi\left(\frac{m_S - m_R}{\sqrt{\hat{\sigma}_S^2 + \hat{\sigma}_R^2}}\right) \quad (26.29)$$

$$P_{f(LN)} = \Phi\left(\frac{0 - m_M}{\hat{\sigma}_M}\right) = \Phi\left(\frac{\lambda_S - \lambda_R}{\sqrt{\xi_S^2 + \xi_R^2}}\right) \quad (26.30)$$

where  $M = R - S$  is the safety margin (or limit state function). The concept of a safety margin and the reliability index,  $\beta$ , is illustrated in Figure 26.5. Here, it can be seen that the reliability index,  $\beta$ , corresponds to the distance (specifically, the number of standard deviations) the mean of the

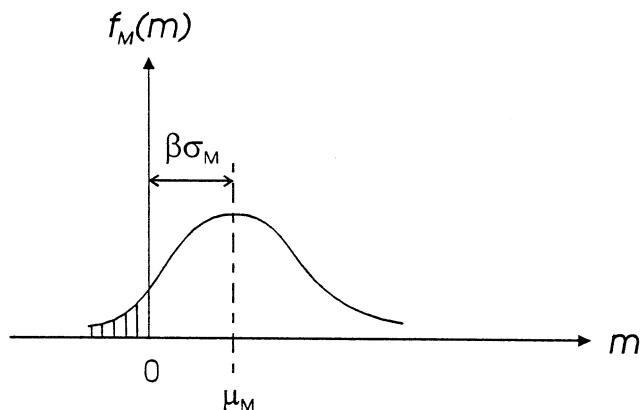


FIGURE 26.5: Safety margin concept,  $M = R - S$ .

safety margin is away from the origin (recall,  $M = 0$  corresponds to failure). The most common, generalized definition of reliability is the second-moment reliability index,  $\beta$ , which derives from this simple two-dimensional case, and is related (approximately) to the failure probability by

$$\beta \approx \Phi^{-1}(1 - P_f) \quad (26.31)$$

where  $\Phi^{-1}(\cdot)$  = inverse standard normal CDF. Table 26.2 can also be used to evaluate this function. (In the case of normal variates, Equation 26.31 is exact. Additional discussion of the reliability index,  $\beta$ , may be found in any of the texts cited previously.) To gain a feel for relative values of the reliability index,  $\beta$ , the corresponding failure probabilities are shown in Table 26.3. Based on the above discussion (Equations 26.29 through 26.31), for the case of  $R$  and  $S$  both distributed normal or lognormal, expressions for the reliability index are given by

$$\beta_{(N)} = \frac{m_M}{\hat{\sigma}_M} = \frac{m_R - m_S}{\sqrt{\hat{\sigma}_R^2 + \hat{\sigma}_S^2}} \quad (26.32)$$

$$\beta_{(LN)} = \frac{m_M}{\hat{\sigma}_M} = \frac{\lambda_R - \lambda_S}{\sqrt{\xi_R^2 + \xi_S^2}} \quad (26.33)$$

For the less generalized case where  $R$  and  $S$  are not necessarily both distributed normal or lognormal

**TABLE 26.3** Failure Probabilities and Corresponding Reliability Values

Probability of failure, $P_f$	Reliability index, $\beta$
.5	0.00
.1	1.28
.01	2.32
.001	3.09
$10^{-4}$	3.71
$10^{-5}$	4.75
$10^{-6}$	5.60

(but are still independent), the failure probability may be evaluated by solving the convolution integral shown in Equation 26.34a or 26.34b either numerically or by simulation:

$$P_f = P[R < S] = \int_{-\infty}^{+\infty} F_R(x) f_S(x) dx \quad (26.34a)$$

$$P_f = P[R < S] = \int_{-\infty}^{+\infty} [1 - F_S(x)] f_R(x) dx \quad (26.34b)$$

Again, the second-moment reliability is approximated as  $\beta = \Phi^{-1}(1 - P_f)$ . Additional methods for evaluating  $\beta$  (for the case of multiple random variables and more complicated limit state functions) are presented in subsequent sections.

### 26.3.2 More Complicated Limit State Functions Reducible to $R - S$ Form

It may be possible that what appears to be a more complicated limit state function (i.e., more than two random variables) can be reduced, or simplified, to the basic  $R - S$  form. Three points may be useful in this regard:

1. If the COV of one random variable is very small relative to the other random variables, it may be able to be treated as a deterministic quantity.
2. If multiple, statistically independent random variables ( $X_i$ ) are taken in a summation function ( $Z = aX_1 + bX_2 + \dots$ ), and the random variables are assumed to be normal, the summation can be replaced with a single normal random variable ( $Z$ ) with moments:

$$E[Z] = aE[X_1] + bE[X_2] + \dots \quad (26.35)$$

$$\text{Var}[Z] = \sigma_z^2 = a^2\sigma_{x_1}^2 + b^2\sigma_{x_2}^2 + \dots \quad (26.36)$$

3. If multiple, statistically independent random variables ( $Y_i$ ) are taken in a product function ( $Z' = Y_1 Y_2 \dots$ ), and the random variables are assumed to be lognormal, the product can be replaced with a single lognormal random variable ( $Z'$ ) with moments (shown here for the case of the product of two variables):

$$E[Z'] = E[Y_1]E[Y_2] \quad (26.37)$$

$$\text{Var}[Z'] = \mu_{Y_1}^2 \sigma_{Y_2}^2 + \mu_{Y_2}^2 \sigma_{Y_1}^2 + \sigma_{Y_1}^2 \sigma_{Y_2}^2 \quad (26.38)$$

Note that the last term in Equation 26.38 is very small if the coefficients of variation are small. In this case, and more generally, for the product of  $n$  random variables, the COV of the product may be expressed:

$$V_Z \approx \sqrt{V_{Y_1}^2 + V_{Y_2}^2 + \dots + V_{Y_n}^2} \quad (26.39)$$

When it is not possible to reduce the limit state function to the simple  $R - S$  form, and/or when the random variables are not both normal or lognormal, more advanced methods for the evaluation of the failure probability (and hence the reliability) must be employed. Some of these methods will be described in the next section after some illustrative examples.

### 26.3.3 Examples

The following examples all contain limit state functions that are in, or can be reduced to, the form of the basic  $R - S$  problem. Note that in all cases the random variables are all either normal or lognormal. Additional information suggesting when such distribution assumptions may be reasonable (or acceptable) is also provided in these examples.

#### EXAMPLE 26.5:

Consider the statically indeterminate beam shown in Figure 26.6, subjected to a concentrated load,  $P$ . The moment capacity,  $M_{\text{cap}}$ , is a random variable with mean of 20 ft-kips and standard deviation of 4 ft-kips. The load,  $P$ , is a random variable with mean of 4 kips and standard deviation of 1 kip. Compute the second-moment reliability index assuming  $P$  and  $M_{\text{cap}}$  are normally distributed and statistically independent.

$$\begin{aligned} M_{\text{max}} &= \frac{Pl}{2} \\ P_f &= P \left[ M_{\text{cap}} < \frac{Pl}{2} \right] = P \left[ M_{\text{cap}} - \frac{Pl}{2} < 0 \right] = P \left[ M_{\text{cap}} - 2P < 0 \right] \end{aligned}$$

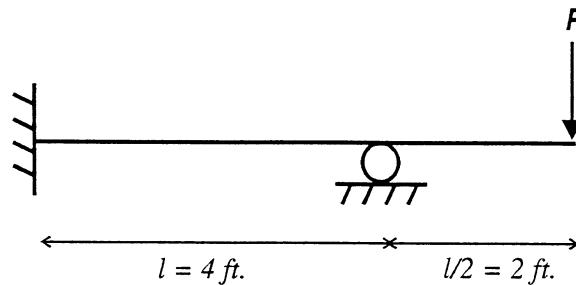


FIGURE 26.6: Cantilever beam subject to point load (Example 26.5).

Here, the failure probability is expressed in terms of  $R - S$ , where  $R = M_{\text{cap}}$  and  $S = 2P$ . Now, we compute the moments of the safety margin given by  $M = R - S$ :

$$\begin{aligned} m_M &= E[M] = E[R - S] = E[R] - E[S] = m_{M_{\text{cap}}} - 2m_p = 20 - 2(4) = 12 \text{ ft-kips} \\ \hat{\sigma}_M^2 &= \text{Var}[M] = \text{Var}[R] + \text{Var}[S] = \hat{\sigma}_{M_{\text{cap}}}^2 + (2)^2 \hat{\sigma}_p^2 = (4)^2 + 4(1)^2 = 20 \text{ (ft-kips)}^2 \end{aligned}$$

Finally, we can compute the second-moment reliability index,  $\beta$ , as

$$\beta = \frac{m_M}{\hat{\sigma}_M} = \frac{m_R - m_S}{\sqrt{\hat{\sigma}_R^2 + \hat{\sigma}_S^2}} = \frac{12}{\sqrt{20}} \approx 2.68$$

(The corresponding failure probability is therefore  $P_f \approx \Phi(-\beta) = \Phi(-2.68) \approx 0.00368$ .)

**EXAMPLE 26.6:**

When designing a building, the total force acting on the columns must be considered. For a particular design situation, the total column force may consist of components of dead load (self-weight), live load (occupancy), and wind load, denoted  $D$ ,  $L$ , and  $W$ , respectively. It is reasonable to assume these variables are statistically independent, and here we will further assume them to be normally distributed with the following moments:

Variable	Mean( $m$ )	$\text{SD}(\sigma)$
$D$	4.0 kips	0.4 kips
$L$	8.0 kips	2.0 kips
$W$	3.4 kips	0.7 kips

If the column has a strength that is assumed to be deterministic,  $R = 20$  kips, what is the probability of failure and the corresponding second-moment reliability index,  $\beta$ ?

First, we compute the moments of the combined load,  $S = D + L + W$ :

$$\begin{aligned} m_S &= m_D + m_L + m_W = 4.0 + 8.0 + 3.4 = 15.4 \text{ kips} \\ \hat{\sigma}_S &= \sqrt{\hat{\sigma}_D^2 + \hat{\sigma}_L^2 + \hat{\sigma}_W^2} = \sqrt{(0.4)^2 + (2.0)^2 + (0.7)^2} = 2.16 \text{ kips} \end{aligned}$$

Since  $S$  is the sum of a number of normal random variables, it is itself a normal variable. Now, since the resistance is assumed to be deterministic, we can simply compute the failure probability directly in terms of the standard normal CDF (rather than formulating the limit state function).

$$P_f = P[S > R] = 1 - P[S < R] = 1 - F_S(20)$$

$$= 1 - \Phi\left(\frac{20 - 15.4}{2.16}\right) = 1 - \Phi(2.13) \approx 1 - (.9834) = .0166$$

$$(\therefore \beta = 2.13)$$

If we were to formulate this in terms of a limit state function (of course, the same result would be obtained), we would have  $g(X) = R - S$ , where the moments of  $S$  are given above and the moments of  $R$  would be  $m_R = 20$  kips and  $\sigma_R = 0$ . Now, if we assume the resistance,  $R$ , is a random variable (rather than being deterministic), with mean and standard deviation given by  $m_R = 20$  kips and  $\sigma_R = 2$  kips (i.e.,  $\text{COV} = 0.10$ ), how would this additional uncertainty affect the probability of failure (and the reliability)? To answer this, we analyze this as a basic  $R - S$  problem, assuming normal variables, and making the reasonable assumption that the loads and resistance are independent quantities. Therefore, from Equation 26.29:

$$P_f = P[R - S < 0] = \Phi\left(\frac{m_S - m_R}{\sqrt{\hat{\sigma}_S^2 + \hat{\sigma}_R^2}}\right) = \Phi\left(\frac{15.4 - 20}{\sqrt{(2.16)^2 + (2)^2}}\right)$$

$$= \Phi\left(\frac{-4.6}{\sqrt{8.67}}\right) = \Phi(-1.56) \approx 0.0594$$

$$(\therefore \beta = 1.56)$$

As one would expect, the uncertainty in the resistance serves to increase the failure probability (in this case, fairly significantly), thereby decreasing the reliability.

### EXAMPLE 26.7:

The fully plastic flexural capacity of a steel beam section is given by the product  $YZ$ , where  $Y$  = steel yield strength and  $Z$  = section modulus. Therefore, for an applied moment,  $M$ , we can express the limit state function as  $g(X) = YZ - M$ , where failure corresponds to the condition  $g(X) < 0$ . Given the statistics shown below and assuming all random variables are lognormally distributed (this ensures non-negativity of the load and resistance variables), reduce this to the simple  $R - S$  form and estimate the second-moment reliability index.

Variable	Distribution	Mean	COV
$Y$	Lognormal	40 ksi	0.10
$Z$	Lognormal	50 in. <sup>3</sup>	0.05
$M$	Lognormal	1000 in.-kip	0.20

First, we obtain the moments of  $R$  and  $S$  as follows:

" $R$ " =  $YZ$ :

$$E[R] = m_R = m_Y m_Z = (40)(50) = 2000 \text{ in.-kips}$$

$$V_R = \text{COV} \approx \sqrt{V_Y^2 + V_Z^2} = 0.112 \text{ (since COVs are "small")}$$

" $S$ " =  $M$ :

$$E[S] = m_M = 1000 \text{ in.-kips}$$

$$V_S = \text{COV} = V_M = 0.20$$

Now, we can compute the lognormal parameters ( $\lambda$  and  $\xi$ ) for  $R$  and  $S$ :

$$\xi_R \approx V_R = 0.112 \text{ (since small COV)}$$

$$\begin{aligned}
\lambda_R &= \ln m_R - \frac{1}{2}\xi_R^2 = \ln(2000) - \frac{1}{2}(.112)^2 = 7.595 \\
\xi_S &\approx V_S = 0.20 \text{ (since small COV)} \\
\lambda_S &= \ln m_S - \frac{1}{2}\xi_S^2 = \ln(1000) - \frac{1}{2}(.2)^2 = 6.888
\end{aligned}$$

Finally, the second-moment reliability index,  $\beta$ , is computed:

$$\beta_{LN} = \frac{\lambda_R - \lambda_S}{\sqrt{\xi_R^2 + \xi_S^2}} = \frac{7.595 - 6.888}{\sqrt{(.112)^2 + (.2)^2}} \approx 3.08$$

Since the variability in the section modulus,  $Z$ , is very small ( $V_Z = 0.05$ ), we could choose to neglect it in the reliability analysis (i.e., assume  $Z$  deterministic). Still assuming variables  $Y$  and  $M$  to be lognormally distributed, and using Equation 26.33 to evaluate the reliability index, we obtain  $\beta = 3.17$ . If we further assumed  $Y$  and  $M$  to be normal (instead of lognormal) random variables, the reliability index computed using Equation 26.32 would be  $\beta = 3.54$ . This illustrates the relative error one might expect from (a) assuming certain variables with low COVs to be essentially deterministic (i.e., 3.17 vs. 3.08), and (b) assuming the incorrect distributions, or simply using the normal distribution when more statistical information is available suggesting another distribution form (i.e., 3.54 vs. 3.08).

#### EXAMPLE 26.8:

Consider again the simply supported beam shown in Figure 26.3, subjected to a uniform load,  $w$  (only), along its entire length. Assume that, in addition to  $w$  being a random variable, the member properties  $E$  and  $I$  are also random variables. (The length, however, may be assumed to be deterministic.) Formulate the limit state function for excessive deflection (assume a maximum allowable deflection of  $l/360$ , where  $l$  = length of the beam) and then reduce it to the simple  $R - S$  form. (Set-up only.)

$$\begin{aligned}
\delta_{\max} &= \frac{5wl^4}{384EI} \\
P_f &= P[\delta_{\text{allow.}} - \delta_{\max} < 0]
\end{aligned}$$

The failure probability is in the  $R - S$  form ( $R = \delta_{\text{allow.}}$  and  $S = \delta_{\max}$ ); however, we still must express the limit state function in terms of the basic variables.

$$\begin{aligned}
g(X) &= \frac{l}{360} - \frac{5wl^4}{384EI} < 0 \text{ (for failure)} \\
&= \frac{EI}{360} - \frac{5wl^3}{384} < 0 \\
&= \frac{384}{360}(EI) - 5wl^3 < 0 \\
&= 1.067(EI) - 5l^3(w) < 0
\end{aligned}$$

Note that the limit state function is now expressed in the simple  $R - S$  form, with  $R = EI$  and  $S = w$ . If  $E$  and  $I$  are assumed to be lognormally distributed, their product,  $EI$ , is also a lognormal random variable, and the moments can be computed as was done in the previous example. Finally, if the uniform load,  $w$ , can be assumed lognormal as well, the second-moment reliability index could be computed (also as done in the previous example).

## 26.4 Generalized Reliability Problem

---

### 26.4.1 Introduction

As discussed previously for the simple two-variable ( $R - S$ ) case in which  $R$  and  $S$  are assumed to be independent, identically distributed (normal or lognormal) random variables permits a closed-form solution to the failure probability. However, such a two-variable simplification of the limit state is often not possible for many structural reliability problems. Furthermore, the joint probability function for the random variables in the limit state equation is seldom known precisely, due to limited data. Even if the basic variables are mutually statistically independent and all marginal density functions are known, it is often impractical (or impossible) to perform the numerical integration of the multidimensional convolution integral over the failure domain. In this section, a number of widely used techniques for evaluating structural reliability under general conditions are presented.

### 26.4.2 FORM/SORM Techniques

First-order second-moment (FOSM) methods were the first techniques used to evaluate structural reliability. The name refers to the way in which the limit state is linearized (first-order) and the way in which the method characterizes the basic variables (second moment). Later, more advanced methods were developed to include information about the complete distributions characterizing the random variables. These advanced FOSM techniques became known as first-order reliability methods (FORM). Finally, among the most recent developments has been the refined curve-fitting of the limit state surface in the analysis, giving rise to the so-called second-order reliability methods (SORM). Details of these reliability analysis techniques may be found in the literature [3, 21, 23].

When the simple limit state (safety margin) is defined by  $M = R - S$ , we have already seen that the reliability index,  $\beta$ , can be expressed (see Figure 26.5):

$$\beta = \frac{\mu_M}{\sigma_M} = \frac{E[g(X)]}{SD[g(X)]} \quad (26.40)$$

where  $E[g(X)]$  and  $SD[g(X)]$  are the mean and standard deviation of the limit state function, respectively. Therefore, for the simple  $R - S$  case,  $\beta$  is the distance from the mean of the safety margin ( $\mu_M = \mu_R - \mu_S$ ) to the origin in units of standard deviations of  $M$ . This is illustrated in Figure 26.1. In this simple second-moment formulation, no mention is made of the underlying probability distributions. The reliability index,  $\beta$ , depends only on measures of central tendency and dispersion of the margin of safety,  $M$ , for the limit state function.

For the more general case where the number of random variables may be greater than two, the limit state surface may be nonlinear, and the random variables may not be normal, a number of iterative solution techniques have been developed. These techniques are all very similar, and differ mainly in the approach taken to a minimization problem. One general procedure is presented at the end of this section. Other approaches may be found in the literature [3, 12, 21, 23]. What follows is a summary of the mathematics behind the formulation of FORM techniques. It is not necessary to fully understand the development of these methods, and those wishing to skip this material can go directly to the algorithm provided later in this section.

To simplify the presentation herein, the basic variables,  $X_i$ , are assumed to be statistically independent and therefore uncorrelated. This assumption, as discussed earlier, is often reasonable for many structural reliability problems. Further, it can be shown that weak correlation (i.e.,  $\rho < 0.2$ , where  $\rho$  is the correlation coefficient) can generally be neglected and that strong correlation (i.e.,  $\rho > 0.8$ ) can be considered to imply fully dependent variables. Additional discussion of correlated variables in FORM/SORM may be found in the references [21, 23]. The limit state function, expressed in terms of the basic variables,  $X_i$ , is first transformed to reduced variables,  $u_i$ , having zero mean and unit

standard deviation:

$$u_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \quad (26.41)$$

A transformed limit state function can then be expressed in terms of the reduced variables:

$$g_1(u_1, \dots, u_n) = 0 \quad (26.42)$$

with failure now being defined as  $g_1(u) < 0$ . The space corresponding to the reduced variables can be shown to have rotational symmetry, as indicated by the concentric circles of equiprobability shown on Figure 26.7. The reliability index,  $\beta$ , is now defined as the shortest distance between the

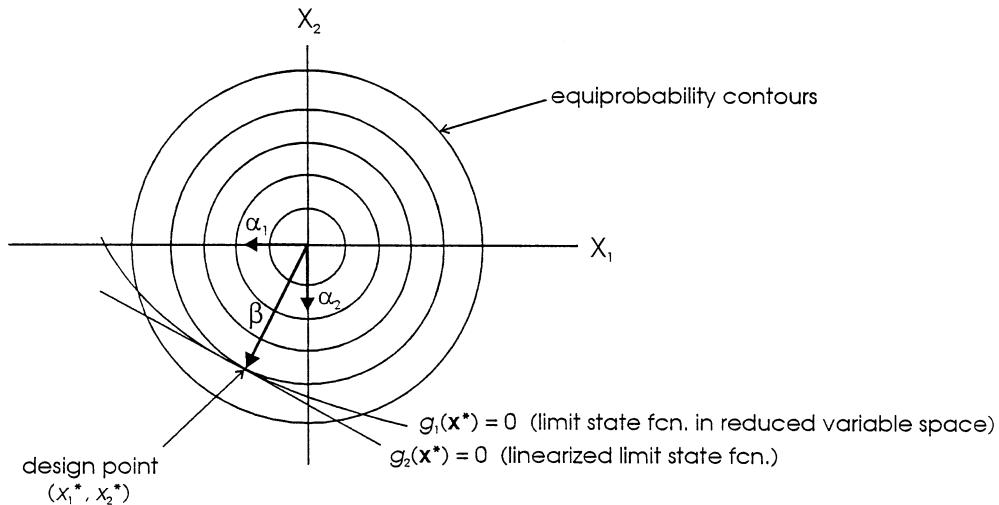


FIGURE 26.7: Formulation of reliability analysis in reduced variable space. (Adapted from Ellingwood, B., Galambos, T.V., MacGregor, J. G. and Cornell, C. A. 1980. Development of a Probability Based Load Criterion for American National Standard A58, NBS Special Publication SP577, National Bureau of Standards, Washington, D.C.)

limit state surface,  $g_1(u) = 0$ , and the origin in reduced variable space (see Figure 26.7). The point  $(u_1^*, \dots, u_n^*)$  on the limit state surface that corresponds to this minimum distance is referred to as the checking (or design) point and can be determined by simultaneously solving the set of equations:

$$\alpha_i = \frac{\frac{\partial g_1}{\partial u_i}}{\sqrt{\sum_i \left( \frac{\partial g_1}{\partial u_i} \right)^2}} \quad (26.43)$$

$$u_i^* = -\alpha_i \beta \quad (26.44)$$

$$g_1(u_1^*, \dots, u_n^*) = 0 \quad (26.45)$$

and searching for the direction cosines,  $\alpha_i$ , that minimize  $\beta$ . The partial derivatives in Equation 26.43 are evaluated at the reduced space design point  $(u_1^*, \dots, u_n^*)$ . This procedure, and Equations 26.43 through 26.45, result from linearizing the limit state surface (in reduced space) and computing the reliability as the shortest distance from the origin in reduced space to the limit state hyperplane. It

may be useful at this point to compare Figures 26.5 and 26.7 to gain some additional insight into this technique.

Once the convergent solution is obtained, it can be shown that the checking point in the original random variable space corresponds to the points:

$$X_i^* = \mu_{X_i} (1 - \alpha_i \beta V_{X_i}) \quad (26.46)$$

such that  $g(X_1^*, \dots, X_n^*) = 0$ . These variables will correspond to values in the upper tails of the probability distributions for load variables and the lower tails for resistance (or geometric) variables.

The formulation described above provides an exact estimate of the reliability index,  $\beta$ , for cases in which the basic variables are normal and in which the limit state function is linear. In other cases, the results are only approximate. As many structural load and resistance quantities are known to be non-normal, it seems reasonable that information on distribution type be incorporated into the reliability analysis. This is especially true since the limit state probabilities can be affected significantly by different distributions' tail behaviors. Methods that include distribution information are known as full-distribution methods or advanced FOSM methods. One commonly used technique is described below.

Because of the ease of working with normal variables, the objective here is to transform the non-normal random variables into equivalent normal variables, and then to perform the analysis for a solution of the reliability index, as described previously. This transformation is accomplished by approximating the true distribution by a normal distribution at the value corresponding to the design point on the failure surface. By fitting an equivalent normal distribution at this point, we are forcing the best approximation to be in the tail of interest of the particular random variable. The fitting is accomplished by determining the mean and standard deviation of the equivalent normal variable such that, at the value corresponding to the design point, the cumulative probability and the probability density of the actual (non-normal) and the equivalent normal variable are equal. (This is the basis for the so-called Rackwitz-Fiessler algorithm.) These moments of the equivalent normal variable are given by

$$\sigma_i^N = \frac{\phi(\Phi^{-1}(F_i(X_i^*)))}{f_i(X_i^*)} \quad (26.47)$$

$$\mu_i^N = X_i^* - \Phi^{-1}(F_i(X_i^*)) \sigma_i^N \quad (26.48)$$

in which  $F_i(\cdot)$  and  $f_i(\cdot)$  are the non-normal CDF and PDF, respectively,  $\phi(\cdot)$  = standard normal PDF, and  $\Phi^{-1}(\cdot)$  = inverse standard normal CDF. Once the equivalent normal mean and standard deviation given by Equations 26.47 and 26.48 are determined, the solution proceeds exactly as described previously. Since the checking point,  $X_i^*$ , is updated at each iteration, the equivalent normal mean and standard deviation must be updated at each iteration cycle as well. While this can be rather laborious by hand, the computer handles this quite efficiently. Only in the case of highly nonlinear limit state functions does this procedure yield results that may be in error.

One possible procedure for computing the reliability index,  $\beta$ , for a limit state with non-normal basic variables is shown below:

1. Define the appropriate limit state function.
2. Make an initial guess at the reliability index,  $\beta$ .
3. Set the initial checking point values,  $X_i^* = \mu_i$ , for all  $i$  variables.
4. Compute the equivalent normal mean and standard deviation for non-normal variables.
5. Compute the partial derivatives  $(\partial g / \partial X_i)$  evaluated at the design point  $X_i^*$ .
6. Compute the direction cosines,  $\alpha_i$ , as

$$\alpha_i = \frac{\frac{\partial g}{\partial X_i} \sigma_i^N}{\sqrt{\sum_i \left( \frac{\partial g}{\partial X_i} \sigma_i^N \right)^2}} \quad (26.49)$$

7. Compute the new values of design point  $X_i^*$  as

$$X_i^* = \mu_i^N - \alpha_i \beta \sigma_i^N \quad (26.50)$$

8. Repeat steps 4 through 7 until estimates of  $\alpha_i$  stabilize (usually fast).
9. Compute the value of  $\beta$  such that  $g(X_1^*, \dots, X_n^*) = 0$ .
10. Repeat steps 4 through 9 until the value for  $\beta$  converges. (This normally occurs within five cycles or less, depending on the nonlinearity of the limit state function.)

As with the previous procedure, this method is easily programmed on the computer. Many spreadsheet programs and other numerical analysis software packages also have considerable statistical capabilities, and therefore can be used to perform these types of analyses. This procedure can also be modified to estimate a design parameter (i.e., a section modulus) such that a specific target reliability is achieved. Other procedures are presented elsewhere in the literature [3, 12, 21, 23] including a somewhat different technique in which the equivalent normal mean and standard deviation are used directly in the reduction of the variables to standard normal form (i.e.,  $u_i$  space). Additional information on SORM techniques may be found in the literature [8, 9].

### 26.4.3 Monte Carlo Simulation

An alternative to integration of the relevant joint probability equation over the domain of random variables corresponding to failure is to use Monte Carlo simulation (MCS). While FORM/SORM techniques are approximate in the case of nonlinear limit state functions, or with non-normal random variables (even when advanced FORM/SORM techniques are used), MCS offers the advantage of providing an exact solution to the failure probability. The potential disadvantage of MCS is the amount of computing time needed, especially when very small probabilities of failure are being estimated. Still, as computing power continues to increase and with the development and refinement of variance reduction techniques (VRTs) MCS is becoming more accepted and more utilized, especially for the analysis of increasingly complicated structural systems. VRTs such as importance sampling, stratified sampling, and Latin hypercube sampling can often be used to significantly reduce the number of simulations required to obtain reliable estimates of the failure probability.

A brief description of MCS is presented here. Additional information may be found elsewhere [21, 22]. The concept behind MCS is to generate sets of realizations of the random variables in the limit state function (with the assumed known probability distributions) and to record the number of times the resulting limit state function is less than zero (i.e., failure). The estimate of the probability of failure ( $P_f$ ) then is simply the number of failures divided by the total number of simulations ( $N$ ). Clearly, the accuracy of this estimate increases as  $N$  increases, and a larger number of simulations are required to reliably estimate smaller failure probabilities. Table 26.4 presents the number of simulations required to obtain three different confidence intervals on the estimate of  $P_f$  for some typical values in structural reliability analyses.

The generation of random variates is a relatively simple task (provided the random variables may be assumed independent) and requires only (1) that the relevant CDF is invertable (or in the case of normal and lognormal variates, numerical approximations exist for the inverse CDF), and (2) that a uniform random number generator is available. (See the Appendix for two examples of uniform

**TABLE 26.4** Approximate Number of Simulations Required for Given Confidence Intervals ( $\alpha \times 100\%$ ) on Reliability Index

$\beta \pm \varepsilon$	$\alpha = 0.90$ ( $k = 1.64$ )	$\alpha = 0.95$ ( $k = 1.96$ )	$\alpha = 0.99$ ( $k = 2.58$ )
$1.5 \pm .10$	1,000	1,400	2,500
$1.5 \pm .05$	4,000	5,700	9,800
$1.5 \pm .01$	100,000	142,000	246,000
$2.0 \pm .10$	2,000	3,000	5,100
$2.0 \pm .05$	8,200	12,000	20,500
$2.0 \pm .01$	240,000	342,000	592,000
$3.0 \pm .10$	18,000	25,600	44,300
$3.0 \pm .05$	75,000	107,000	186,000
$3.0 \pm .01$	2,270,000	3,240,000	5,610,000

random number generators. Random number generators for other distributions may be available to you, and would further simplify the simulation analysis.) The generation of correlated variates is not described here, but information may be found in the literature [9, 21, 23]. As shown in Figure 26.8, the value of the CDF for random variable  $X$  is (by definition) uniformly distributed on  $\{0, 1\}$ . Therefore,

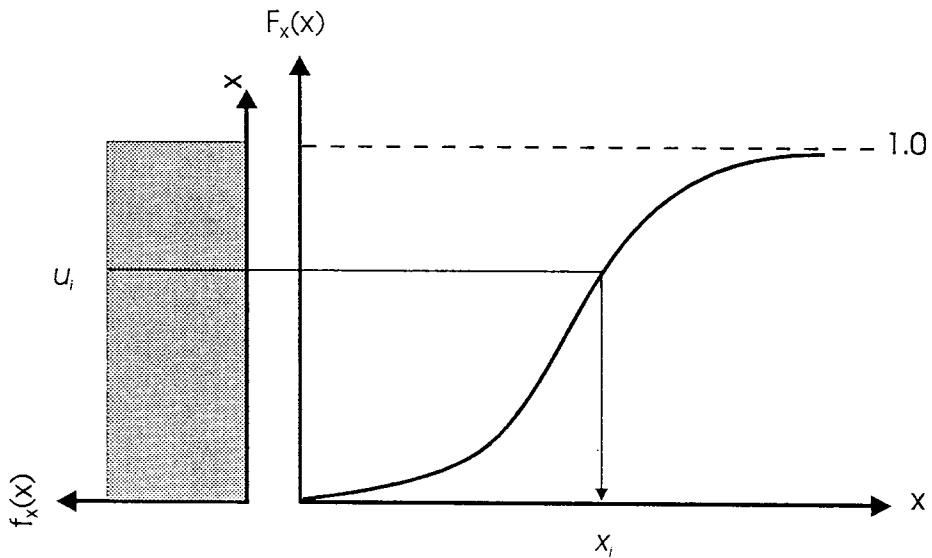


FIGURE 26.8: Random variable simulation.

if we generate a uniform  $\{0, 1\}$  deviate and substitute this into the inverse of the CDF of interest (with the relevant parameters or moments), we obtain a realization of a variate with this CDF. For example, consider the generation of an exponential variate with parameter  $\lambda$ . The CDF is expressed:

$$F_X(x) = 1 - \exp(-\lambda x) \quad (26.51)$$

If we substitute  $u_i$  (a uniform  $\{0, 1\}$  deviate; see the Appendix) for  $F_X(x)$  and invert the CDF to solve for  $x_i$ , we obtain

$$x_i = -\frac{1}{\lambda} \ln(1 - u_i) \quad (26.52)$$

Here,  $x_i$  is an exponential variate with parameter  $\lambda$ . As another example, consider the normal

distribution, for which no closed-form expression exists for the CDF or its inverse. The generalized normal CDF can be written as a function of the standard normal CDF as

$$F_X(x) = \Phi\left(\frac{x - \mu_x}{\sigma_x}\right) \quad (26.53)$$

Therefore, an expression for a generalized normal variate would be:

$$x_i = \mu_x + \sigma_x \Phi^{-1}(u_i) \quad (26.54)$$

where  $\mu_x$  and  $\sigma_x$  are the mean and standard deviation, respectively,  $u_i$  = uniform {0, 1} deviate, and  $\Phi^{-1}(\cdot)$  = inverse standard normal CDF. While not available in closed form, numerical approximations for  $\Phi^{-1}(\cdot)$  (i.e., in the form of algorithms or subroutines) are available (e.g., [12]). The Appendix presents approximate functions for both  $\Phi(\cdot)$  and  $\Phi^{-1}(\cdot)$ . Table 26.5 presents the inverse CDFs for a number of common distribution types.

**TABLE 26.5** Common Distributions, CDFs, and Inverse CDFs

Distribution	CDF ( $= u_i$ )	Inverse CDF
Normal	$F_X(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$	$x_i = (\Phi^{-1}(u_i) \times \sigma) + \mu$
Lognormal	$F_X(x) = \Phi\left(\frac{\ln x - \lambda}{\xi}\right)$	$x_i = \exp\left[\left(\Phi^{-1}(u_i) \times \xi\right) + \lambda\right]$
Uniform	$F_X(x) = \frac{x - a}{b - a}$	$x_i = a + (b - a)u_i$
Exponential	$F_X(x) = 1 - \exp(-\lambda x)$	$x_i = -\frac{1}{\lambda} \ln(1 - u_i)$
Extreme Type I (largest), "Gumbel"	$F_X(x) = \exp(-\exp(-\alpha(x - u)))$	$x_i = -\frac{1}{\alpha} \ln(-\ln u_i) + u$
Extreme Type II (largest)	$F_X(x) = \exp(-(u/x)^k)$	$x_i = u(-\ln u_i)^{-1/k}$
Extreme Type III (smallest), "Weibull"	$F_X(x) = 1 - \exp\left[-\left(\frac{x - \varepsilon}{w - \varepsilon}\right)^k\right]$	$x_i = (-\ln(1 - u_i))^{1/k} (w - \varepsilon) + \varepsilon$

MCS can provide a very powerful tool for the solution of a wide variety of problems. Improvements in efficiency over crude or direct MCS can be realized by improved algorithmic design (programming) and by the utilization of VRTs. Monte Carlo techniques can also be used for the simulation of discrete and continuous random processes.

## 26.5 System Reliability

### 26.5.1 Introduction

While most structural codes in the U.S. treat design on a member-by-member basis, most elements within a structure are actually performing as part of an often complicated structural system. Interest in characterizing the performance and safety of structural systems has led to an increased interest in the area of system reliability. The classical theories of series and parallel system reliability are well developed and have been applied to the analysis of such complicated structural systems as nuclear power plants and offshore structures. In the following sections, a brief introduction to system reliability is presented along with some examples. This subject within the broad field of structural reliability is relatively new, and advances both in the theory and application of system reliability concepts to civil engineering design can be expected in the coming years.

### 26.5.2 Basic Systems

The two types of systems in classical theory are the series (or weakest link) system and the parallel system. The literature is replete with formulations for the reliability of these systems, including the possibility of correlated element strengths (e.g., [23, 24]). The relevant limit state is defined by the system type. For a series system, the system limit state is taken by definition to correspond to the first member failure, hence the name “weakest link.” In the case of the strictly parallel system, the system limit state is taken by definition to correspond to the failure of all members. Formulations for the system reliability of a parallel system in which the load-deformation behavior of the members is assumed to be ductile or brittle are both well developed and presented in the literature (see [24], for example). In all cases, the system reliabilities are expressed in terms of the component (or member) reliabilities.

Classical system reliability theory has been able to be extended somewhat to model more complicated systems using combinations of series and parallel systems. These formulations, however, are still subject to limitations with regard to possible load sharing (distribution of load among components of the system) and time-dependent effects, such as degrading member resistances.

### 26.5.3 Introduction to Classical System Reliability Theory

For a system limit state defined by  $g(x_1, \dots, x_m) = 0$ , where  $x_i$  are the basic variables, the failure probability is computed as the integral over the failure domain ( $g(X) < 0$ ) of the joint probability density function of  $X$ . In general, the failure of any system can be expressed as a union and/or intersection of events. For example, the failure of an ideal series (or weakest link) system may be expressed,

$$F_{\text{sys}} = F_1 \cup F_2 \cup \dots \cup F_m \quad (26.55)$$

in which  $\cup$  denotes the Boolean OR operator and  $F_i = i$ th component (element) failure event. A statically determinate truss is modeled as a series system since the failure of the truss corresponds to the failure of any single member. Both first-order and second-order (which includes information on the joint probability behavior) bounds have been developed to express the system failure probability as a function of the individual element failure probabilities. These formulations are well developed and presented in the literature [3, 10, 21, 24].

The failure of a strictly parallel system may be expressed,

$$F_{\text{sys}} = F_1 \cap F_2 \cap \dots \cap F_m \quad (26.56)$$

in which  $\cap$  denotes the Boolean AND operator. Such is the case for the classical “Daniels” system of parallel, ductile rods or cables subject to equal deformation. In this case, system failure corresponds to the failure of all members or elements. First- and second-order bounds are also available for this system idealization (e.g., [17]). Furthermore, bounds that account for possible dependence of failure modes (modal correlation) have been developed [3]. If the parallel system is composed of brittle elements, the analysis may be further complicated by having to account for load redistribution following member failure. This total failure may therefore be the result of progressive element failures.

Returning again to the two fundamental system types, series and parallel, we can examine the probability distributions for the strength of these systems as functions of the distributions of the strengths of the individual members (elements). In the simple structural idealization of a series system of  $n$  elements (for which the characterization of the member failures as brittle or ductile is irrelevant since system failure corresponds to first-member failure), the distribution function for the system strength,  $R_{\text{sys}}$ , can be expressed:

$$F_{R_{\text{sys}}}(r) = 1 - \prod_{i=1}^n (1 - F_{R_i}(r)) \quad (26.57)$$

where the individual member strengths are assumed independent. In Equation 26.57,  $F_{R_i}(r) =$  distribution function (CDF) for the individual member resistance. If the  $n$  individual member strengths are also identically distributed (i.e., have the same parent distribution,  $F_R(r)$ , with the same moments), Equation 26.57 can be simplified to

$$F_{R_{\text{sys}}}(r) = 1 - (1 - F_R(r))^n \quad (26.58)$$

In the case of the idealized parallel system of  $n$  elements, the system failure is dependent on whether the member behavior is perfectly brittle or perfectly ductile. In the simple case of the parallel system with  $n$  perfectly ductile elements, the system strength is given by

$$R_{\text{sys}} = \sum_{i=1}^n R_i \quad (26.59)$$

where  $R_i$  = strength of element  $i$ . The central limit theorem (see [2, 5]) suggests that as the number of members in this system gets large, the system strength approaches a normal random variable, regardless of the distributions of the individual member strengths. When the member behavior is perfectly brittle, the system behavior is dependent on the degree of indeterminacy (redundancy) of the system and the ability of the system to redistribute loads to other members. For some applications, it may be reasonable to model structures idealized as parallel systems with brittle members as series systems, if the brittle failure of one member is likely to overload the remaining members. The issue of correlated member strengths (and correlated failure modes) is beyond the scope of this introduction, but information may be found in [3, 23, 24].

It is appropriate at this point to present the simple first-order bounds for the two fundamental systems. Additional information on the development and application of these as well as the second-order bounds may be found in the literature cited previously. The first-order bounds for a series system are given by

$$\max_{i=1}^n \{P_{f_i}\} \leq P_{f_{\text{sys}}} \leq 1 - \left( \prod_{i=1}^n (1 - P_{f_i}) \right) \quad (26.60)$$

where  $P_{f_i}$  = failure probability for member (element)  $i$ . The first-order bounds for a parallel system are given by

$$\prod_{i=1}^n P_{f_i} \leq P_{f_{\text{sys}}} \leq \min_{i=1}^n \{P_{f_i}\} \quad (26.61)$$

Improved (second-order) bounds (the first-order bounds are often too broad to be of practical use) that include information on the joint probability behavior (i.e., member or modal correlation) have been developed and are described in the literature (e.g., [3, 10, 24]).

Classical system reliability theory, as briefly introduced above, is limited in that it cannot account for more complicated load-deformation behavior and the time dependencies associated with load redistribution following (brittle) member failure. Generalized formulations for the reliability of systems that are neither strictly series nor strictly parallel type systems are not available. Analyses of these systems are often based on combined series and parallel system models in which the complete system is modeled as some arrangement of these classical subsystems. These solutions tend to be problem specific and still do not address any possible time-dependent or load-sharing issues.

#### 26.5.4 Redundant Systems

A redundant (indeterminate) system may be defined as having some overload capacity following the failure of an element. The level of redundancy (or degree of indeterminacy) refers to the number of

element failures that can be tolerated without the system failing. The reliability of such a structure is dependent on the nature (type) of redundancy. The level of redundancy dictates how many members can fail prior to collapse, and therefore answers the question, “Would the failure of member  $j$  lead to impending collapse?” Furthermore, load-deformation behavior of the individual members specifies whether or not the limit states are load-path dependent. For ductile element behavior (i.e., the Daniels system), the limit state is effectively load-path independent, implying the order of member failures is not significant. For a system of brittle elements, however, the limit state may be load-path dependent. In this case, the performance of the system is related to the load redistribution behavior following member failure, and hence the order (or relative position) of member failures becomes important. The parallel-member system model with brittle elements (i.e., perfectly elastic load-deformation behavior) is appropriate for (and has been used to model) a wide range of redundant structural systems including floors, roofs, and wall systems.

### 26.5.5 Examples

Three examples are described in this section. The first example considers a series system in which the elements are considered to represent different modes of failure. Modal failure analysis is often treated using the concepts of system reliability (i.e [3]). Here, the structure being considered (actually, the simply supported beam element, i.e., Figure 26.3) may fail in any one of three different modes: flexure, shear, and excessive deflection. (The last mode corresponds to a serviceability-type limit state rather than an ultimate strength type.) The “failure” of the structural element is assumed to occur when any of these limit states is violated. For simplicity, the modal failure probabilities are assumed to be uncorrelated. (For information on handling correlated failure modes, see [3]). In other words, the element (system) fails when it fails in flexure, or it fails in shear, or it experiences excessive deflection:

$$F_{\text{sys}} = F_M \cup F_V \cup F_\delta \quad (26.62)$$

If, for example, the probabilities of moment, shear, and deflection failure, respectively, are given by  $F_M = 0.0015$ ,  $F_V = 0.002$ , and  $F_\delta = 0.005$ , the first-order bounds shown in Equation 26.60 result in

$$\begin{aligned} 0.005 &\leq P_{f_{\text{sys}}} \leq 1 - (1 - 0.0015)(1 - 0.002)(1 - 0.005) \\ 0.005 &\leq P_{f_{\text{sys}}} \leq 0.0085 \end{aligned} \quad (26.63)$$

This corresponds to a range for  $\beta$  of  $2.39 \leq \beta_{\text{sys}} \leq 2.58$ .

The second example considers a strictly parallel system of five cables supporting a load (see Figure 26.9). In this case, the system failure corresponds to the condition where the cable system can no longer carry any load. Therefore, all of the cables must have failed for the system to have failed. In this simple example, the issue of load redistribution following the failure of one of the cables is not addressed; however, this problem has been studied extensively (e.g., [19]). Here, the five cable strengths are assumed to be statistically independent, and the system failure probability is the probability that  $P$  is large enough to fail all of the cables simultaneously:

$$F_{\text{sys}} = F_1 \cap F_2 \cap \dots \cap F_5 \quad (26.64)$$

If, for example, the probability of failure of an individual cable is 0.001, and the cable strengths are assumed to be independent, identically distributed random variables, the first-order bounds on the system failure probability given by Equation 26.61 become

$$(0.001)^5 \leq P_{f_{\text{sys}}} \leq 0.001 \quad (26.65)$$

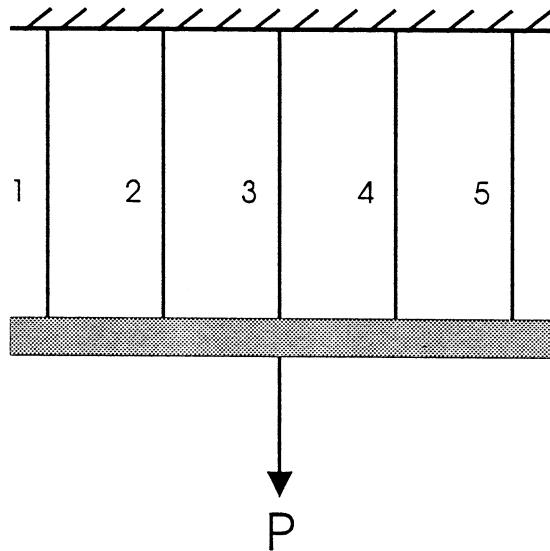


FIGURE 26.9: Five-element parallel system.

Here, the lower bound corresponds to the case of perfectly uncorrelated member strengths (i.e., independent cable failures), while the upper bound corresponds to the case of perfect correlation. These first-order bounds, as indicated by Equation 26.65, become very wide with increasing  $n$ . Here, information on correlation can be important in computing narrower and more useful bounds.

Finally, as a third example, a combined (series and parallel) system is considered. In this case, the event probabilities correspond to the failure probabilities of different components required for a safe shutdown of a nuclear power plant. While these events are assumed to be independent, their arrangement describing safe system performance (see Figure 26.10) forms a combined series-parallel system. In this case, the three subsystems are arranged in series: subsystem A is a series system and

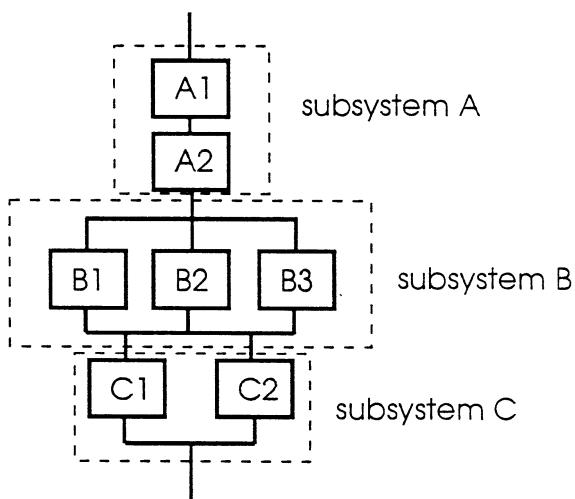


FIGURE 26.10: Safe shutdown of a nuclear power plant.

subsystems  $B$  and  $C$  are parallel systems. In this case, the system failure probability is given by

$$F_{\text{sys}} = F_A \cup F_B \cup F_C \quad (26.66)$$

or, expressed in terms of the individual component failure probabilities:

$$F_{\text{sys}} = [F_{A_1} \cup F_{A_2}] \cup [F_{B_1} \cap F_{B_2} \cap F_{B_3}] \cup [F_{C_1} \cap F_{C_2}] \quad (26.67)$$

## 26.6 Reliability-Based Design (Codes)

---

### 26.6.1 Introduction

This section will provide a brief introduction to reliability-based design concepts in civil engineering, with specific emphasis on structural engineering design. Since the 1970s, the theories of probability and statistics and reliability have provided the bases for modern structural design codes and specifications. Thus, probabilistic codes have been replacing previous deterministic-format codes in recent years. RBD procedures are intended to provide more predictable levels of safety and more risk-consistent (i.e., design-to-design) structures, while utilizing the most up-to-date statistical information on material strengths, as well as structural and environmental loads. An excellent discussion of RBD in the U.S. as well as other countries is presented in [21]. Other references are also available that deal specifically with probabilistic code development in the U.S. [12, 13, 15]. The following sections provide some basic information on the application of reliability theory to aspects of RBD.

### 26.6.2 Calibration and Selection of Target Reliabilities

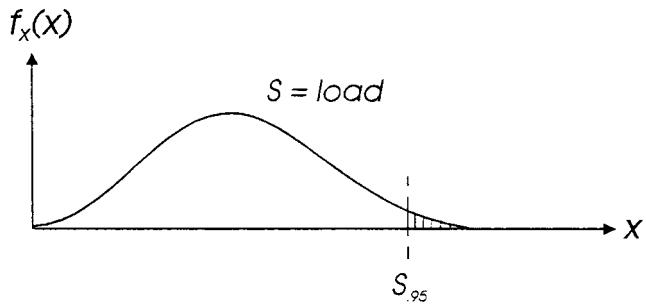
Calibration refers to the linking of new design procedures to previous existing design philosophies. Much of the need for calibration arises from making any new code changes acceptable to the engineering and design communities. For purely practical reasons, it is undesirable to make drastic changes in the procedures for estimating design values, for example, or in the overall formats of design checking equations. If such changes are to be made, it is impractical and uneconomical to make them often. Hence, code development is an often slow process, involving many years and many revisions. The other justification for code calibration has been the notion that previous design philosophies (i.e., ASD) have resulted in safe designs (or designs with acceptable levels of performance), and that therefore these previous levels of safety should serve as benchmarks in the development of new specifications or procedures (i.e., LRFD).

The actual process of calibration is relatively simple. For a given design procedure (i.e., ASD for steel beams in flexure), estimate the reliability based on the available statistical information on the loads and resistances and the governing checking equation. This becomes the target reliability and is used to develop the appropriate load and resistance factors, for example, for the new procedure (i.e., LRFD). In the development of LRFD for both steel and wood, for example, the calibration process revealed an inconsistency in the reliability levels for different load combinations. As this was undesirable, a single target reliability was selected and the new LRFD procedures were able to correct this problem. For more information on code calibration, the reader is referred to the literature [12, 15, 21].

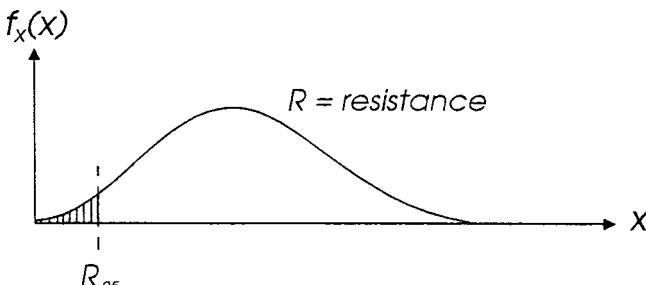
### 26.6.3 Material Properties and Design Values

The basis for many design values encountered in structural engineering design is now probabilistic. Earlier design values were often based on mean values of member strength, for example, with the

**factor of safety** intended to account for all forms of uncertainty, including material property variability. Later, as more statistical information became available, as people became more aware of the concept of relative uncertainty, and with the use of probabilistic methods in code development, characteristic values were selected for use in design. The characteristic values (referred to as nominal or design values in most specifications) are generally selected from the lower tail of the distribution describing the material property (see Figure 26.11). Typically, the 5th percentile value (that value



(a) design load (e.g., 5% exceedence probability)



(b) nominal resistance (e.g., 5th-percentile)

FIGURE 26.11: Typical specification of design (nominal) load and resistance values.

below which 5% of the probability density lies) is selected as the **nominal resistance** (i.e., nominal strength), though in some cases, a different percentile value may be selected. While this value may serve as the starting point for establishing the design value, modifications are often needed to account for such things as size effects, system effects, or (in the case of wood) moisture content effects, etc. The bases for the design resistance values for specifications in the U.S. are described in the literature (e.g., [12, 16, 20]). An excellent review of resistance modeling and a summary of statistical properties for structural elements is presented in [21]. Table 26.6 presents some typical resistance statistics for concrete and steel members. Additional statistics are available, along with statistics for masonry, aluminum, and wood members in [12] as well. The mean values are presented in ratio to their nominal (or design) values,  $m_R/R_n$ . In addition, the coefficient of variation,  $V_R$ , and the PDF are

listed in Table 26.6.

**TABLE 26.6** Typical Resistance Statistics for Concrete and Steel Members

	Type of member	$m_R/R_n$	$V_R$
Concrete elements			
Flexure, reinforced concrete	Continuous one-way slabs	1.22	0.16
	Two-way slabs	1.12-1.16	0.15
	One-way pan joists	1.13	0.14
	Beams, grade 40, $f'_c = 5$ ksi	1.14-118	0.14
	Beams, grade 60, $f'_c = 5$ ksi	1.01-1.09	0.08-0.12
	Overall values	1.05	0.11
Flexure, prestressed concrete	Plant precast pretensioned	1.06	0.08
	Cast-in-place post-tensioned	1.04	0.10
Axial load and flexure	Short columns, compression	0.95-1.05	0.14-0.16
	Short columns, tension	1.05	0.12
	Slender columns, compression	1.10	0.17
	Slender columns, tension	0.95	0.12
Shear	Beams with $a/d < 2.5$ , $\rho_w = 0.008$ :		
	No stirrups	0.93	0.21
	Minimum stirrups	1.00	0.19
	Moderate stirrups	1.09	0.17
Hot-rolled steel elements			
	Tension member, yield	1.05	0.11
	Tension member, ultimate	1.10	0.11
	Compact beam, uniform moment	1.07	0.13
	Compact beam, continuous	1.11	0.13
	Elastic beam, LTB	1.03	0.12
	Inelastic beam, LTB	1.11	0.14
	Beam columns	1.07	0.15
	Plate-girders, flexure	1.08	0.12
	Plate girders, shear	1.14	0.16
	Compact composite beams	1.04	0.14
	Fillet welds	0.88	0.18
	ASS bolts in tension, A325	1.20	0.09
	ASS bolts in tension, A490	1.07	0.05
	HSS bolts in shear, A325	0.60	0.10
	HSS bolts in shear, A490	0.52	0.07

Adapted from Ellingwood, B., Galambos, T.V., MacGregor, J.G., and Cornell, C.A. 1980. "Development of a Probability Based Load Criterion for American National Standard A58," NBS Special Publication SP577, National Bureau of Standards, Washington, D.C.

#### 26.6.4 Design Loads and Load Combinations

The selection of design load values, such as those found in the ASCE 7-95 standard [4] (formerly the ANSI A58.1 standard), *Minimum Design Loads for Buildings and Other Structures*, is also largely probability based. Though somewhat more complicated than the selection of design resistance values as described above, the concept is quite similar. Of course, greater complexity is introduced since we may be concerned with both spatial and temporal variations in the load effects. In addition, because of the difficulties in conducting load surveys, and the large amount of variability associated with naturally occurring phenomena giving rise to many structural and environmental loadings, there is a high degree of uncertainty associated with these quantities. A number of load surveys have been conducted, and the valuable data collected have formed the basis for many of our design values (e.g., [7, 11, 14, 18]). When needed, such as in the case where data simply are not available or able to be collected with any reasonable amount of effort, this information is supplemented by engineering judgment and expert opinion. Therefore, design load values are based on (1) statistical information, such as load survey data, and (2) engineering judgment, including past experience, and scenario analysis. As shown in Figure 26.11, the design load value can be visualized as some characteristic value in the upper tail of the distribution describing the load. For example, the 95th percentile wind

speed is that value of wind speed that has a 5% ( $1 - .95$ ) exceedence probability. Probabilistic load modeling represents an extensive area of research, and a significant amount of work is reported on in the literature [21, 28]. A summary of load statistics is presented in Table 26.7.

**TABLE 26.7** Typical Load Statistics

Load type	Mean-to-nominal	COV	Distribution
Dead load	1.05	0.10	Normal
Live load			
Sustained component	0.30	0.60	Gamma
Extraordinary component	0.50	0.87	Gamma
Total (max., 50 years)	1.00	0.25	Type I
Snow load (annual max.)			
General site (northeast U.S.)	0.20	0.87	Lognormal
Wind load			
50-year maximum	0.78	0.37	Type I
Annual maximum	0.33	0.59	Type I
Earthquake load	0.5-1.0	0.5-1.4	Type II

In most codes, a number of different load combinations are suggested for use in the appropriate checking equation format. For example, the ASCE 7-95 standard recommends the following load combinations [4]:

$$\begin{aligned}
 U &= 1.4D_n \\
 U &= 1.2D_n + 1.6L_n \\
 U &= 1.2D_n + 1.6S_n + (0.5L_n \text{ or } 0.8W_n) \\
 U &= 1.2D_n + 1.3W_n + 0.5L_n \\
 U &= 1.2D_n + 1.0E_n + 0.5L_n + 0.2S_n \\
 U &= 0.9D_n + (-1.3W_n \text{ or } 1.0E_n)
 \end{aligned} \tag{26.68}$$

where  $D_n$ ,  $L_n$ ,  $S_n$ ,  $W_n$ , and  $E_n$  are the nominal (design) values for dead load, live load, snow load, wind load, and earthquake load, respectively. A similar set of load combinations may be found in both the ACI and AISC specifications, though in the case of the ACI code the load factors (developed earlier) are slightly different. These load combinations were developed in order to ensure essentially equal exceedence probabilities for all combinations,  $U$ . A discussion of the bases for these load combinations may be found in [12]. A comparison of LRFD with other countries' codes may be found in [21].

One important tool used in the development of the load combinations is known as Turkstra's Rule [25, 26], developed as an alternative to more complicated load combination analysis. This rule states that, in effect, the maximum of a combination of two or more load effects will occur when one of the loads is at its maximum value while the other loads take on their instantaneous or arbitrary point-in-time values. Therefore, if  $n$  time-varying loads are being considered, there are at least  $n$  corresponding load combinations that would need to be considered. This rule may be written generally as

$$\max \{Z\} = \max_i \left[ \max_T X_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^n X_j(t) \right] \tag{26.69}$$

where  $\max \{Z\}$  = maximum combined load,  $X_i(t)$ ,  $i = 1, \dots, n$  are the time-varying loads being considered in combination, and  $t$  = time. In the equation above, the first term in the brackets represents the maximum in the lifetime ( $T$ ) of load  $X_i$ , while the second term is the sum of all other

loads at their point-in-time values. This approximation may be unconservative in some cases where the maximum load effect occurs as a result of the combination of multiple loads at near maximum values. However, in most cases, the probability of this occurring is small, and thus Turkstra's Rule has been shown to be a good approximation for most structural load combinations [27].

### 26.6.5 Evaluation of Load and Resistance Factors

Recall that for the generalized case of non-normal random variables, the following expression was developed (see Equation 26.50):

$$X_i^* = \mu_i^N - \alpha_i \beta \sigma_i^N \quad (26.70)$$

If we further define the design point value  $X_i^*$  in terms of a nominal (design) value  $X_n$ :

$$X_i^* = \gamma_i X_n \quad (26.71)$$

where  $\gamma_i$  = partial factor on load  $X_i$  (or the inverse of the resistance factor). Therefore, for the popular LRFD format in the U.S. in which the design equation has the form

$$\phi R_n \geq \sum_i \gamma_i X_{n,i} \quad (26.72)$$

the **load factors** may be computed as

$$\gamma_i = \frac{\mu_i^N - \alpha_i \hat{\beta} \sigma_i^N}{X_{n,i}} \quad (26.73)$$

and the **resistance factor** is given by

$$\phi = \frac{R_n}{\mu_i^N - \alpha_i \hat{\beta} \sigma_i^N} \quad (26.74)$$

In Equations 26.73 and 26.74,  $\alpha_i$  = direction cosine from the convergent iterative solution for random variable  $i$ ,  $\beta$  =convergent reliability index (i.e., the target reliability), and  $X_{n,i}$  and  $R_n$  are the nominal load and resistance values, respectively.

Additional information on the evaluation of load and resistance factors based on FORM/SORM techniques, as well as comparisons between different code formats, may be found in the literature [3, 12, 21].

## 26.7 Defining Terms<sup>2</sup>

---

**Allowable stress design** (or **working stress design**): A method of proportioning structures such that the computed elastic stress does not exceed a specified limiting stress.

**Calibration**: A process of adjusting the parameters in a new standard to achieve approximately the same reliability as exists in a current standard or specification.

**Factor of safety**: A factor by which a designated limit state force or stress is divided to obtain a specified limiting value.

**Failure**: A condition where a limit state is reached.

---

<sup>2</sup>Selected terms taken from [12].

**FORM/SORM (FOSM):** First- and Second-order reliability methods (first-order second-moment reliability methods). Methods that involve (1) a first- or second-order Taylor series expansion of the limit state surface, and (2) computing a notional reliability measure that is a function only of the means and variances (first two moments) of the random variables. (Advanced FOSM includes full distribution information as well as any possible correlations of random variables.)

**Limit state:** A criterion beyond which a structure or structural element is judged to be no longer useful for its intended function (serviceability limit state) or beyond which it is judged to be unsafe (ultimate limit state).

**Limit states design:** A design method that aims at providing safety against a structure or structural element being rendered unfit for use.

**Load factor:** A factor by which a nominal load effect is multiplied to account for the uncertainties inherent in the determination of the load effect.

**LRFD:** Load and resistance factor design. A design method that uses load factors and resistance factors in the design format.

**Nominal load effect:** Calculated using a nominal load; the nominal load frequently is determined with reference to a probability level; e.g., 50-year mean recurrence interval wind speed used in calculating the wind load for design.

**Nominal resistance:** Calculated using nominal material and cross-sectional properties and a rationally developed formula based on an analytical and/or experimental model of limit state behavior.

**Reliability:** A measure of relative safety of a structure or structural element.

**Reliability-based design (RBD):** A design method that uses reliability (probability) theory in the safety checking process.

**Resistance factor:** A factor by which the nominal resistance is multiplied to account for the uncertainties inherent in its determination.

## Acknowledgments

---

The author is grateful for the comments and suggestions provided by Professor James T. P. Yao at Texas A&M University and Professor Theodore V. Galambos at the University of Minnesota. In addition, discussions with Professor Bruce Ellingwood at Johns Hopkins University were very helpful in preparing this chapter.

## References

---

- [1] Abramowitz, M. and Stegun, I.A., Eds. 1966. *Handbook of Mathematical Functions*, Applied Mathematics Series No. 55, National Bureau of Standards, Washington, D.C.
- [2] Ang, A.H.-S. and Tang, W.H. 1975. *Probability Concepts in Engineering Planning and Design, Volume I: Basic Principles*, John Wiley & Sons, New York.
- [3] Ang, A.H.-S. and Tang, W.H. 1975. *Probability Concepts in Engineering Planning and Design, Volume II: Decision, Risk, and Reliability*, John Wiley & Sons, New York.
- [4] American Society of Civil Engineers. 1996. *Minimum Design Loads for Buildings and Other Structures*, ASCE 7-95, New York.
- [5] Benjamin, J.R. and Cornell, C.A. 1970. *Probability, Statistics, and Decision for Civil Engineers*, McGraw-Hill, New York.

- [6] Bratley, P., Fox, B.L., and Schrage, L.E. 1987. *A Guide to Simulation*, Second Edition, Springer-Verlag, New York.
- [7] Chalk, P. and Corotis, R.B. 1980. A Probability Model for Design Live Loads, *J. Struct. Div.*, ASCE, 106(10):2017-2033.
- [8] Chen, X. and Lind, N.C. 1983. Fast Probability Integration by Three-Parameter Normal Tail Approximation, *Structural Safety*, 1(4):269-276.
- [9] Der Kiureghian, A. and Liu, P.L. 1986. Structural Reliability Under Incomplete Probability Information, *J. Eng. Mech.*, ASCE, 112(1):85-104.
- [10] Ditlevsen, O. 1981. *Uncertainty Modelling*, McGraw-Hill, New York.
- [11] Ellingwood, B. and Culver, C.G. 1977. Analysis of Live Loads in Office Buildings, *J. Struct. Div.*, ASCE, 103(8):1551-1560.
- [12] Ellingwood, B., Galambos, T.V., MacGregor, J.G., and Cornell, C.A. 1980. Development of a Probability Based Load Criterion for American National Standard A58, NBS Special Publication SP577, National Bureau of Standards, Washington, D.C.
- [13] Ellingwood, B., MacGregor, J.G., Galambos, T.V. and Cornell, C.A. 1982. Probability Based Load Criteria: Load Factors and Load Combinations, *J. Struct. Div.*, ASCE, 108(5):978-997.
- [14] Ellingwood, B. and Redfield, R. 1982. Ground Snow Loads for Structural Design, *J. Struct. Eng.*, ASCE, 109(4):950-964.
- [15] Galambos, T.V., Ellingwood, B., MacGregor, J.G., and Cornell, C.A. 1982. Probability Based Load Criteria: Assessment of Current Design Practice, *J. Struct. Div.*, ASCE, 108(5):959-977.
- [16] Galambos, T.V. and Ravindra, M.K. 1978. Properties of Steel for Use in LRFD, *J. Struct. Div.*, ASCE, 104(9):1459-1468.
- [17] Grigoriu, M. 1989. Reliability of Daniels Systems Subject to Gaussian Load Processes, *Structural Safety*, 6(2-4):303-309.
- [18] Harris, M.E., Corotis, R.B., and Bova, C.J. 1981. Area-Dependent Processes for Structural Live Loads, *J. Struct. Div.*, ASCE, 107(5):857-872.
- [19] Hohenbichler, M. and Rackwitz, R. 1983. Reliability of Parallel Systems Under Imposed Uniform Strain, *J. Eng. Mech. Div.*, ASCE, 109(3):896-907.
- [20] MacGregor, J.G., Mirza, S.A., and Ellingwood, B. 1983. Statistical Analysis of Resistance of Reinforced and Prestressed Concrete Members, *ACIJ*, 80(3):167-176.
- [21] Melchers, R.E. 1987. *Structural Reliability: Analysis and Prediction*, Ellis Horwood Limited, distributed by John Wiley & Sons, New York.
- [22] Rubinstein, R.Y. 1981. *Simulation and the Monte Carlo Method*, John Wiley & Sons, New York.
- [23] Thoft-Christensen, P. and Baker, M.J. 1982. *Structural Reliability Theory and Its Applications*, Springer-Verlag, Berlin.
- [24] Thoft-Christensen, P. and Murotsu, Y. 1986. *Application of Structural Systems Reliability Theory*, Springer-Verlag, Berlin.
- [25] Turkstra, C.J. 1972. Theory of Structural Design Decisions, *Solid Mech. Study No. 2*, University of Waterloo, Ontario, Canada.
- [26] Turkstra, C.J. and Madsen, H.O. 1980. Load Combinations in Codified Structural Design, *J. Struct. Div.*, ASCE, 106(12):2527-2543.
- [27] Wen, Y.-K. 1977. Statistical Combinations of Extreme Loads, *J. Struct. Div.*, ASCE, 103(6):1079-1095.
- [28] Wen, Y.-K. 1990. *Structural Load Modeling and Combination for Performance and Safety Evaluation*, Elsevier, Amsterdam.

## Further Reading

---

Melchers [21] provides one of the best overall presentations of structural reliability, both its theory and applications. Ang and Tang [3] also provides a good summary. For a more advanced treatment, refer to Ditlevsen [10], Thoft-Christensen and Baker [23], or Thoft-Christensen and Murotsu [24].

The International Conference on Structural Safety and Reliability (ICOSSAR) and the International Conference on the Application of Statistics and Probability in Civil Engineering (ICASP) are each held every 4 years. The proceedings from these conferences include short papers on a variety of state-of-the-art topics in structural reliability. The conference proceedings may be found in the engineering libraries at most universities. A number of other conferences, including periodic specialty conferences cosponsored by ASCE, also include sessions pertaining to reliability.

## Appendix

---

### Some Useful Functions for Simulation

1.  $\Phi(\cdot)$  = standard normal cumulative distribution function

Approximate algorithm [1]:

$$\Phi(x) = 1 - \frac{1}{2} \left( 1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 \right)^{-4} + \varepsilon(x) \quad ; \quad x \geq 0$$
$$|\varepsilon(x)| < 2.5 \times 10^{-4}$$
$$c_1 = 0.196854 \quad c_2 = 0.115194 \quad c_3 = 0.000344 \quad c_4 = 0.019527$$

*Note:*  $\Phi(-x) = 1 - \Phi(x)$

2.  $\Phi^{-1}(\cdot)$  = inverse standard normal cumulative distribution function.

Simple approximate routine for inverse standard normal CDF:

Input: CDF

Output: Z (also: CDF is returned unchanged)

Format:  $Z = \Phi^{-1}(\text{CDF})$

Usage: call `invcdf(arg1, arg2)`

```
subroutine invcdf(cdf,z)
data a0,a1,a2 / 2.515517, 0.802853, 0.010328/
data b1,b2,b3 / 1.432788, 0.189269, 0.001308/
y = cdf
if(cdf .gt. 0.5) y = 1.0 - cdf
if(y .le. 0.0) y = 3.0E-39      (note: machine-dependent)
v = sqrt(-2.0*alog(y))
z = v-(a0+v*(a1+v*a2))/(1.0+v*(b1+v*(b2+v*b3)))
if(cdf .lt. 0.5) z = -z
return
end
```

3. RNUM = uniform {0, 1} random number generator.

Simple routine to generate an array "arrcdf" of "inum" uniform {0, 1} random deviates.  
(The parameter "dseed" is just a starting random seed.)

```

subroutine RNUM(dseed,inum,arrcdf)
dimension arrcdf(50000)
double precision dseed
k = 5701
j = 3612
m = 566927
rm = 566927.0
do 100 i = 1,inum
    ix = int(dseed* rm)
    irand = mod (j* ix + k, m)
    arrcdf(i) = (real(irand) + 0.5)/rm
    dseed = arrcdf(i)
100  continue
      return
      end

```

4. UNIF = uniform random deviate between 0 and 1.

Recursive uniform {0, 1} random number generator [6]:

INPUT:  $I_X$  = any integer between 0 and 2147483647

OUTPUT: new pseudorandom value, and uniform variate between 0 and 1

$$\begin{aligned}
K_1 &= \frac{I_X}{127773} \\
I_X &= 16807 \left( I_X - \frac{K_1}{127773} \right) - 2836(K_1) \\
\text{if } (I_X < 0) \text{ then } (I_X &= I_X + 2147483647) \\
\text{UNIF} &= I_X \times 4.656612875E^{-10}
\end{aligned}$$