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Effective Length Factors of Compression Members

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17.1 Introduction

The concept of the effective length factors of columns has been well established and widely used by practicing engineers and plays an important role in compression member design. The most structural design codes and specifications have provisions concerning the effective length factor. The aim of this chapter is to present a state-of-the-art engineering practice of the effective length factor for the design of columns in structures. In the first part of this chapter, the basic concept of the effective length factor is discussed. And then, the design implementation for isolated columns, framed columns, crossing bracing systems, latticed members, tapered columns, crane columns, as well as columns in gable frames is presented. The determination of whether a frame is braced or unbraced is also addressed. Several detailed examples are given to illustrate the determination of effective length factors for different cases of engineering applications.

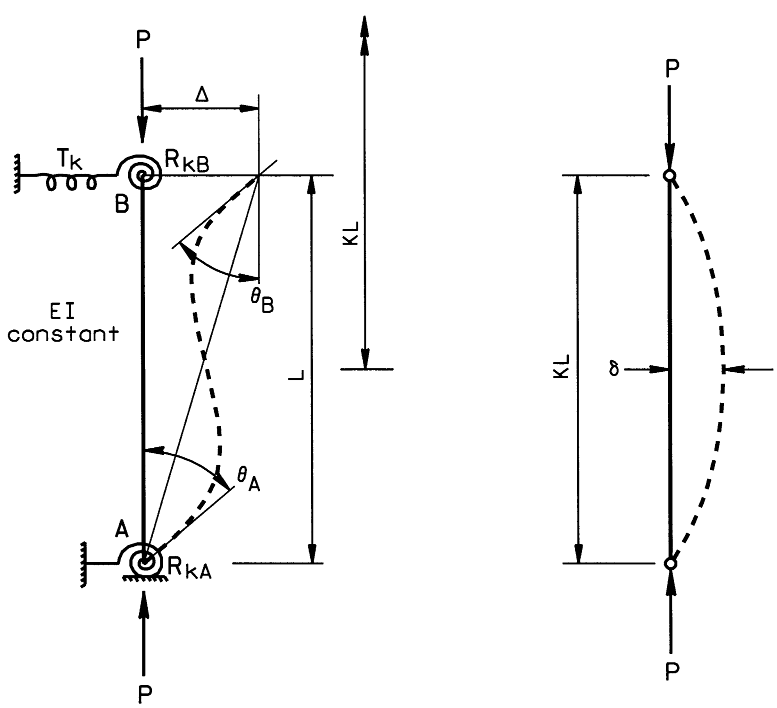
17.2 Basic Concept

Mathematically, the effective length factor or the *elastic* K-factor is defined as:

$$K = \sqrt{\frac{P_e}{P_{cr}}} = \sqrt{\frac{\pi^2 EI}{L^2 P_{cr}}} \tag{17.1}$$

where P_e is the Euler load, the elastic buckling load of a pin-ended column; P_{cr} is the elastic buckling load of an end-restrained framed column; E is the modulus of elasticity; I is the moment of inertia in the flexural buckling plane; and L is the unsupported length of column.

Physically, the K-factor is a factor that when multiplied by actual length of the end-restrained column (Figure 17.1a) gives the length of an equivalent pin-ended column (Figure 17.1b) whose buckling load is the same as that of the end-restrained column. It follows that effective length, KL,



(a) End-restrained Columns (b) Pin-ended Column

FIGURE 17.1: Isolated columns.

of an end-restrained column is the length between adjacent inflection points of its pure flexural buckling shape.

Specifications provide the resistance equations for pin-ended columns, while the resistance of framed columns can be estimated through the K-factor to the pin-ended columns strength equation. Theoretical K-factor is determined from an elastic eigenvalue analysis of the entire structural system, while practical methods for the K-factor are based on an elastic eigenvalue analysis of selected subassemblages. The effective length concept is the only tool currently available for the design of compression members in engineering structures, and it is an essential part of analysis procedures.

17.3 Isolated Columns

From an eigenvalue analysis, the general K-factor equation of an end-restrained column as shown in Figure 17.1 is obtained as:

$$\det \begin{vmatrix} C + \frac{R_{kA}L}{EI} & S & -(C+S) \\ S & C + \frac{R_{kB}L}{EI} & -(C+S) \\ -(C+S) & -(C+S) & 2(C+S) - \left(\frac{\pi}{K}\right)^2 + \frac{T_k L^3}{EI} \end{vmatrix} = 0 \tag{17.2}$$

where the stability functions C and S are defined as:

$$C = \frac{(\pi/K)\sin(\pi/K) - (\pi/K)^2\cos(\pi/K)}{2 - 2\cos(\pi/K) - (\pi/K)\sin(\pi/K)} \tag{17.3}$$

$$S = \frac{(\pi/K)^2 - (\pi/K)\sin(\pi/K)}{2 - 2\cos(\pi/K) - (\pi/K)\sin(\pi/K)} \tag{17.4}$$

The largest value of K that satisfies Equation 17.2 gives the elastic buckling load of an end-restrained column.

Figure 17.2 [1, 3, 4] summarizes the theoretical K-factors for columns with some idealized end conditions. The recommended K-factors are also shown in Figure 17.2 for practical design applications. Since actual column conditions seldom comply fully with idealized conditions used in buckling analysis, the recommended K-factors are always equal to or greater than their theoretical counterparts.

17.4 Framed Columns—Alignment Chart Method

In theory, the effective length factor K for any column in a framed structure can be determined from a stability analysis of the entire structural analysis—eigenvalue analysis. Methods available for stability analysis include the slope-deflection method [17, 35, 71], three-moment equation method [13], and energy methods [42]. In practice, however, such analysis is not practical, and simple models are often used to determine the effective length factors for framed columns [38, 47, 55, 72]. One such practical procedure that provides an approximate value of the elastic K-factor is the alignment chart method [46]. This procedure has been adopted by the AISC [3, 4], ACI 318-95 [2], and AASHTO [1] specifications, among others. At present, most engineers use the alignment chart method in lieu of an actual stability analysis.

17.4.1 Alignment Chart Method

The structural models employed for determination of K-factor for framed columns in the alignment chart method are shown in Figure 17.3. The assumptions used in these models are [4, 17]:

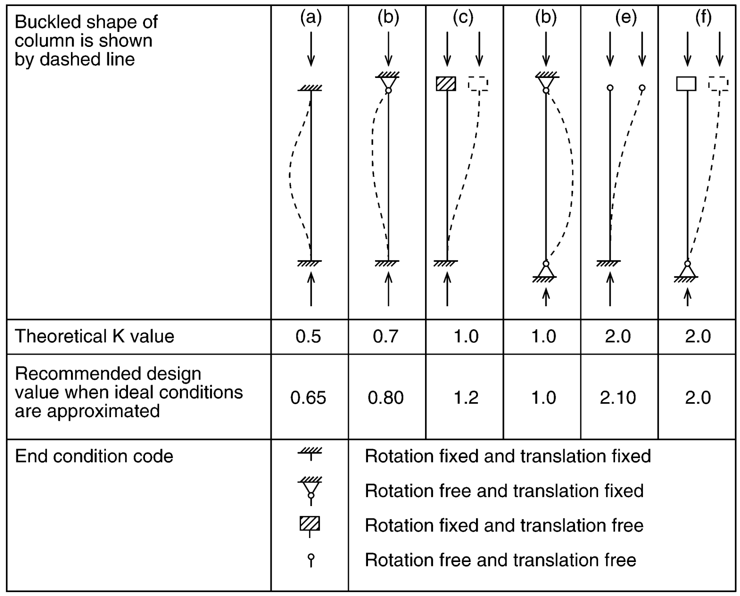


FIGURE 17.2: Theoretical and recommended K-factors for isolated columns with idealized end conditions.

1. All members have constant cross-section and behave elastically.
2. Axial forces in the girders are negligible.
3. All joints are rigid.
4. For braced frames, the rotations at the near and far ends of the girders are equal in magnitude and opposite in direction (i.e., girders are bent in single curvature).
5. For unbraced frames, the rotations at the near and far ends of the girders are equal in magnitude and direction (i.e., girders are bent in double curvature).
6. The stiffness parameters, $L\sqrt{P/EI}$, of all columns are equal.
7. All columns buckle simultaneously.

Using the slope-deflection equation method and stability functions, the effective length factor equations of framed columns are obtained as follows.

For columns in braced frames:

$$\frac{G_A G_B}{4}(\pi/K)^2 + \left(\frac{G_A + G_B}{2}\right)\left(1 - \frac{\pi/K}{\tan(\pi/K)}\right) + \frac{2\tan(\pi/2K)}{\pi/K} - 1 = 0 \tag{17.5}$$

For columns in unbraced frames:

$$\frac{G_A G_B (\pi/K)^2 - 36^2}{6\,(G_A + G_B)} - \frac{\pi/K}{\tan\,(\pi/K)} = 0 \tag{17.6}$$

where G_A and G_B are stiffness ratios of columns and girders at two end joints, A and B, of the column section being considered, respectively. They are defined by:

$$G_A = \frac{\sum_A (E_c I_c / L_c)}{\sum_A \left(E_g I_g / L_g\right)} \tag{17.7}$$

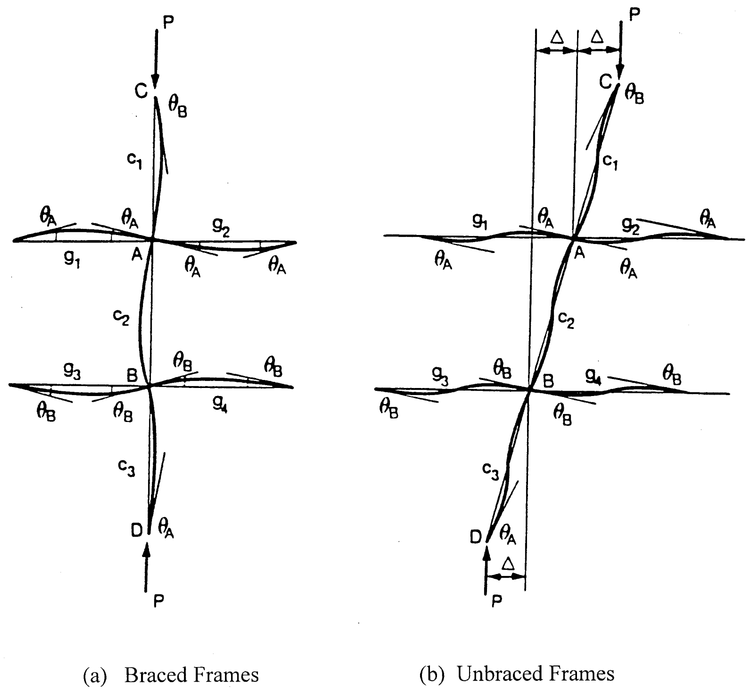


FIGURE 17.3: Subassemblage models for K-factors of framed columns.

$$G_B = \frac{\sum_B (E_c I_c / L_c)}{\sum_B (E_g I_g / L_g)} \tag{17.8}$$

where $\sum$ indicates a summation of all members rigidly connected to the joint and lying in the plane in which buckling of column is being considered; subscripts c and g represent columns and girders, respectively.

Equations 17.5 and 17.6 can be expressed in the form of alignment charts, as shown in Figure 17.4. It is noted that for columns in braced frames, the range of K is $0.5 \leq K \leq 1.0$; for columns in unbraced frames, the range is $1.0 \leq K \leq \infty$. For column ends supported by but not rigidly connected to a footing or foundations, G is theoretically infinity, but, unless actually designed as a true friction free pin, may be taken as 10 for practical design. If the column end is rigidly attached to a properly designed footing, G may be taken as 1.0 [4].

EXAMPLE 17.1:

Given: A two-story steel frame is shown in Figure 17.5. Using the alignment chart, determine the K-factor for the elastic column DE. $E = 29{,}000$ ksi (200 GPa) and $Fy = 36$ ksi (248 MPa).

Solution

1. For the given frame, section properties are

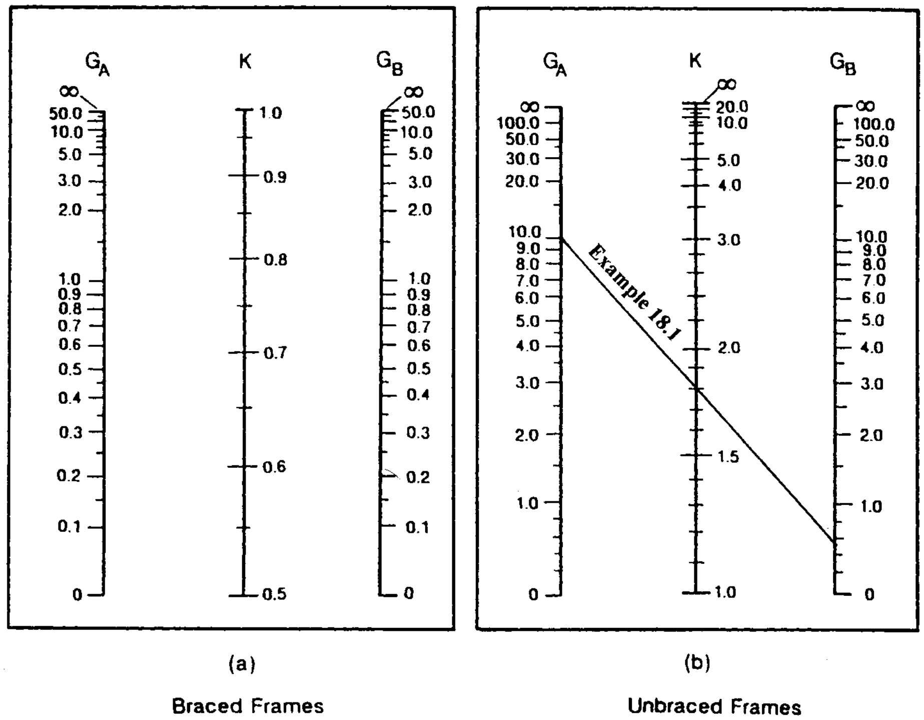


FIGURE 17.4: Alignment charts for effective length factors of framed columns.

Members	Section	I_x in.^{4} (mm^4 × 10^8)	L in. (mm)	I_x/L in.^{3} (mm^3)
AB and GH	W 10x22	118 (0.49)	180 (4,572)	0.656(10,750)
BC and HI	W10x22	118 (0.49)	144 (3,658)	0.819(13,412)
DE	W10x45	248 (1.03)	180 (4,572)	1.378(22,581)
EF	W10x45	248 (1.03)	144 (3,658)	1.722(28,219)
BE	W18x50	800 (3.33)	300 (7,620)	2.667(43,704)
EH	W18x86	1530 (6.37)	360 (9,144)	4.250(69,645)
CF	W16x40	518 (2.16)	300 (7,620)	1.727(28,300)
FI	W16x67	954 (3.97)	360 (9,144)	2.650(43,426)

2. Calculate G-factor for column DE:

$$G_E = \frac{\sum_E (E_c I_c/L_c)}{\sum_E \left(E_g I_g/L_g\right)} = \frac{1.378 + 1.722}{2.667 + 4.250} = 0.448$$

$$G_D = 10 \quad \text{(AISC-LRFD, 1993)}$$

3. From the alignment chart in Figure 17.4b, $K = 1.8$ is obtained.

17.4.2 Requirements for Braced Frames

In stability design, one of the major decisions engineers have to make is the determination of whether a frame is braced or unbraced. The AISC-LRFD [4] states that a frame is braced when "lateral stability is provided by diagonal bracing, shear walls or equivalent means". However, there is no specific provision for the "amount of stiffness required to prevent sidesway buckling" in the AISC,

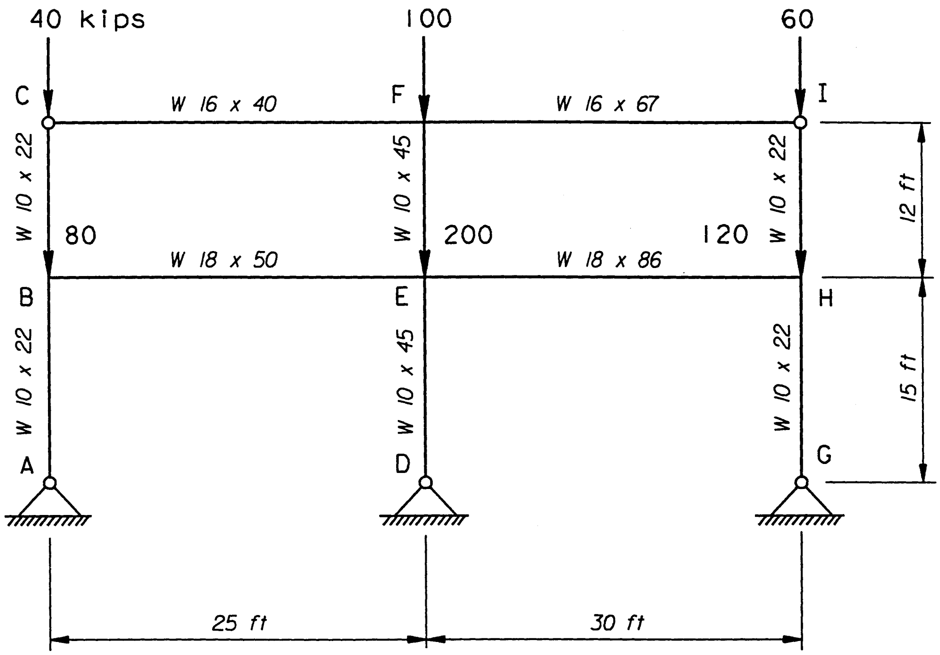


FIGURE 17.5: An unbraced two-story frame.

AASHTO, and other specifications. In actual structures, a completely braced frame seldom exists. But in practice, some structures can be analyzed as braced frames as long as the lateral stiffness provided by the bracing system is large enough. The following brief discussion may provide engineers with the tools to make engineering decisions regarding the basic requirements for a braced frame.

1. Lateral Stiffness Requirement
   Galambos [34] presented a simple conservative procedure to evaluate minimum lateral stiffness provided by a bracing system so that the frame is considered braced.

$$\text{Required lateral stiffness,} \quad T_k = \frac{\sum P_n}{L_c} \tag{17.9}$$

   where $\sum$ represents the summation of all columns in one story, P_n is the nominal axial compression strength of a column using the effective length factor $K = 1$, and L_c is the unsupported length of a column.
2. Bracing Size Requirement
   Galambos [34] applied Equation 17.9 to a diagonal bracing (Figure 17.6) and obtained minimum requirements of diagonal bracing for a braced frame as

$$A_b = \frac{\left[1 + (L_b/L_c)^2\right]^{3/2} \sum P_n}{(L_b/L_c)^2\, E} \tag{17.10}$$

   where A_b is the cross-sectional area of diagonal bracing and L_b is the span length of the beam.
   A recent study by Aristizabal-Ochoa [8] indicates that the size of the diagonal bracing required for a totally braced frame is about 4.9 and 5.1% of the column cross-section for

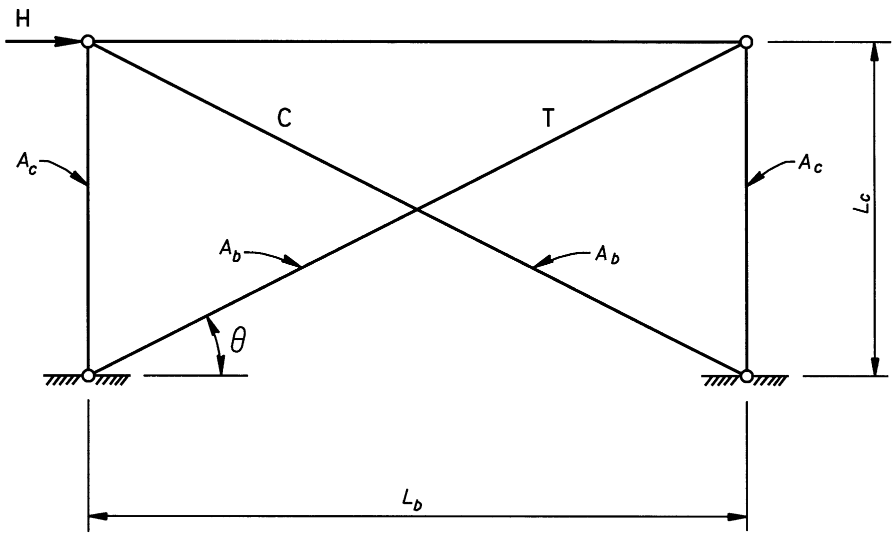


FIGURE 17.6: Diagonal cross bracing system.

a "rigid frame" and "simple framing", respectively, and increases with the moment inertia of the column, the beam span, and the beam-to-column span ratio, L_b/L_c.

17.4.3 Simplified Equations to Alignment Charts

1. ACI 318-95 Equations
The ACI Building Code [2] recommends the use of alignment charts as the primary design aid for estimating K-factors, following two sets of simplified K-factor equations as an alternative:
For braced frames [19]:

$$K = 0.7 + 0.05\,(G_A + G_B) \leq 1.0 \tag{17.11}$$

$$K = 0.85 + 0.05 G_{\min} \leq 1.0 \tag{17.12}$$

The smaller of the above two expressions provides an upper bound to the effective length factor for braced compression members.

For unbraced frames [32]:

For $G_m < 2$

$$K = \frac{20 - G_m}{20}\sqrt{1 + G_m} \tag{17.13}$$

For $G_m \geq 2$

$$K = 0.9\sqrt{1 + G_m} \tag{17.14}$$

For columns hinged at one end

$$K = 2.0 + 0.3G \quad (17.15)$$

where G_m is the average of G values at the two ends of the columns.

2. Duan-King-Chen Equations
   A graphical alignment chart determination of the K-factor is easy to perform, while solving the chart Equations 17.5 and 17.6 always involves iteration. Although the ACI code provides simplified K-factor equations, generally, they may not lead to an economical design [40]. To achieve both accuracy and simplicity for design purposes, the following alternative K-factor equations were proposed by Duan, King and Chen [48].
   For braced frames:

$$K = 1 - \frac{1}{5 + 9G_A} - \frac{1}{5 + 9G_B} - \frac{1}{10 + G_A G_B} \quad (17.16)$$

   For unbraced frames:

   For $K < 2$

$$K = 4 - \frac{1}{1 + 0.2G_A} - \frac{1}{1 + 0.2G_B} - \frac{1}{1 + 0.01G_A G_B} \quad (17.17)$$

   For $K \geq 2$

$$K = \frac{2\pi a}{0.9 + \sqrt{0.81 + 4ab}} \quad (17.18)$$

   where

$$a = \frac{G_A G_B}{G_A + G_B} + 3 \quad (17.19)$$

$$b = \frac{36}{G_A + G_B} + 6 \quad (17.20)$$

3. French Equations
   For braced frames:

$$K = \frac{3G_A G_B + 1.4\,(G_A + G_B) + 0.64}{3G_A G_B + 2.0\,(G_A + G_B) + 1.28} \quad (17.21)$$

   For unbraced frames:

$$K = \sqrt{\frac{1.6G_A G_B + 4.0\,(G_A + G_B) + 7.5}{G_A + G_B + 7.5}} \quad (17.22)$$

Equations 17.21 and 17.22 first appeared in the *French Design Rules for Steel Structure* [31] since 1966, and were later incorporated into the *European Recommendation for Steel Construction* [28]. They provide a good approximation to the alignment charts [26].

17.5 Modifications to Alignment Charts

In using the alignment charts in Figure 17.4 and Equations 17.5 and 17.6, engineers must always be aware of the assumptions used in the development of these charts. When actual structural conditions differ from these assumptions, unrealistic design may result [4, 43, 53]. The SSRC (Structural Stability Research Council) guide [43] provides methods that enable engineers to make simple modifications of the charts for some special conditions, such as unsymmetrical frames, column base conditions, girder far end conditions, and flexible conditions. A procedure that can be used to account for far ends of restraining columns being hinged or fixed was proposed by Duan and Chen [21, 22]. Consideration of effects of material inelasticity on the K-factor was developed originally by Yura [73] and expanded by Disque [20]. LeMessurier [52] presented an overview of unbraced frames with or without leaning columns. An approximate procedure is also suggested by AISC-LRFD [4]. Special attention should also be paid to calculation of the proper G values [10, 49] when partially restrained (PR) connections are used in frames. Several commonly used modifications are summarized in this section.

17.5.1 Different Restraining Girder End Conditions

When the end conditions of restraining girders are not rigidly jointed to columns, the girder stiffness (I_g/L_g) used in the calculation of G_A and G_B in Equations 17.7 and 17.8 should be multiplied by a modification factor, α_k, as:

$$G = \frac{\sum (E_c I_c / L_c)}{\sum \alpha_k \left(E_g I_g / L_g\right)} \tag{17.23}$$

where the modification factor, α_k, for braced frames developed by Duan and Lu [25] and for unbraced frames proposed by Kishi, Chen, and Goto [49] are given in Table 17.1 and 17.2. In these tables, R_{kN} and R_{kF} are elastic spring constants at the near and far ends of a restraining girder, respectively. R_{kN} and R_{kF} are the tangent stiffness of a semi-rigid connection at buckling.

TABLE 17.1 Modification Factor α_k for Braced Frames with Semi-Rigid Connections

End conditions of restraining girder		
Near end	Far end	α_k
Rigid	Rigid	1.0
Rigid	Hinged	1.5
Rigid	Semi-rigid	$\left(1+\frac{6E_g I_g}{L_g R_{kF}}\right) / \left(1+\frac{4E_g I_g}{L_g R_{kF}}\right)$
Rigid	Fixed	2.0
Semi-rigid	Rigid	$1/\left(1+\frac{4E_g I_g}{L_g R_{kN}}\right)$
Semi-rigid	Hinged	$1.5/\left(1+\frac{3E_g I_g}{L_g R_{kN}}\right)$
Semi-rigid	Semi-rigid	$\left(1+\frac{6E_g I_g}{L_g R_{kF}}\right)/R^*$
Semi-rigid	Fixed	$2/\left(1+\frac{4E_g I_g}{L_g R_{kN}}\right)$

Note: $R^* = \left(1+\frac{4E_g I_g}{L_g R_{kN}}\right)\left(1+\frac{4E_g I_g}{L_g R_{kF}}\right) - \left(\frac{E_g I_g}{L_g}\right)^2 \frac{4}{R_{kN} R_{kF}}$

TABLE 17.2 Modification Factor, α_k, for Unbraced Frames with Semi-Rigid Connections

End conditions of restraining girder		
Near end	Far end	α_k
Rigid	Rigid	1
Rigid	Hinged	0.5
Rigid	Semi-rigid	$\left(1+\frac{2E_gI_g}{L_gR_{kF}}\right)/\left(1+\frac{4E_gI_g}{L_gR_{kF}}\right)$
Rigid	Fixed	2/3
Semi-rigid	Rigid	$1/\left(1+\frac{4E_gI_g}{L_gR_{kN}}\right)$
Semi-rigid	Hinged	$0.5/\left(1+\frac{3E_gI_g}{L_gR_{kN}}\right)$
Semi-rigid	Semi-rigid	$\left(1+\frac{2E_gI_g}{L_gR_{kF}}\right)/R^*$
Semi-rigid	Fixed	$(2/3)/\left(1+\frac{4E_gI_g}{L_gR_{kN}}\right)$

Note: $R^* = \left(1+\frac{4E_gI_g}{L_gR_{kN}}\right)\left(1+\frac{4E_gI_g}{L_gR_{kF}}\right)-\left(\frac{E_gI_g}{L_g}\right)^2\frac{4}{R_{kN}R_{kF}}$

EXAMPLE 17.2:

Given: A steel frame is shown in Figure 17.5. Using the alignment chart with the necessary modifications, determine the K-factor for elastic column EF. $E = 29{,}000$ ksi (200 GPa) and $F_y =$ 36 ksi (248 MPa).

Solution

1. Calculate G-factor with modification for column EF.

   Since the far end of restraining girders are hinged, girder stiffness should be multiplied by 0.5 (see Table 17.2). Using section properties in Example 17.1, we obtain:

$$G_F = \frac{\sum(E_cI_c/L_c)}{\sum\alpha_k\left(E_gI_g/L_g\right)} = \frac{1.722}{0.5(1.727)+0.5(2.650)} = 0.787$$
$$GE = 0.448$$

2. From the alignment chart in Figure 17.4b, $K = 1.22$ is obtained.

17.5.2 Different Restraining Column End Conditions

To consider different far end conditions of restraining columns, the general effective length factor equations for column $C2$ (Figure 17.3) were derived by Duan and Chen [21, 22, 23]. By assuming that the far ends of columns $C1$ and $C3$ are hinged and using the slope-deflection equation approach for the subassemblies shown in Figure 17.3, we obtain the following.

1. For a Braced Frame [21]:

$$C^2 - S^2\left[G_{AC1}+G_{BC3}+G_{AC2}G_{BC2}+\frac{\frac{2G_{BC3}}{G_A}+\frac{2G_{AC1}}{G_B}}{C}-G_{AC1}G_{BC3}\left(\frac{S}{C}\right)^2\right] + 2C\left(\frac{1}{G_A}+\frac{1}{G_B}\right)+\frac{4}{G_AG_B} = 0 \quad (17.24)$$

where C and S are stability functions as defined by Equations 17.3 and 17.4; G_A and G_B are defined in Equations 17.7 and 17.8; G_{AC1}, G_{AC2}, G_{BC2}, and G_{BC3} are stiffness ratios of columns at A-th and B-th ends of the columns being considered, respectively. They are defined as:

$$G_{Ci} = \frac{E_{ci} I_{ci} / L_{ci}}{\sum (E_{ci} I_{ci} / L_{ci})} \tag{17.25}$$

where $\sum$ indicates a summation of all columns rigidly connected to the joint and lying in the plane in which buckling of the column is being considered.

Although Equation 17.24 was derived for the special case in which the far ends of both columns $C1$ and $C3$ are hinged, this equation is also applicable if adjustment to G_{Ci} is made as follows: (1) if the far end of column $Ci(C1$ or $C3)$ is fixed, then take $G_{Ci} = 0$ (except for G_{C2}); (2) if the far end of the column $Ci(C1$ or $C3)$ is rigidly connected, then take $G_{Ci} = 0$ and $G_{C2} = 1.0$. Therefore, Equation 17.24 can be specialized for the following conditions:

(a) If the far ends of both columns $C1$ and $C3$ are fixed, we have $G_{AC1} = G_{BC3} = 0$ and Equation 17.24 reduces to

$$C^2 - S^2 (G_{AC2} G_{BC2}) + 2C \left(\frac{1}{G_A} + \frac{1}{G_B} \right) + \frac{4}{G_A G_B} = 0 \tag{17.26}$$

(b) If the far end of column $C1$ is rigidly connected and the far end of column $C3$ is fixed, we have $G_{AC2} = 1.0$ and $G_{AC1} = G_{BC3} = 0$, and Equation 17.24 reduces to

$$C^2 - S^2 + G_{BC2} + 2C \left(\frac{1}{G_A} + \frac{1}{G_B} \right) + \frac{4}{G_A G_B} = 0 \tag{17.27}$$

(c) If the far end of column $C1$ is rigidly connected and the far end of column $C3$ is hinged, we have $G_{AC1} = 0$ and $G_{AC2} = 1.0$, and Equation 17.24 reduces to

$$\begin{aligned} C^2 \quad - \quad & S^2 \left(G_{BC3} + G_{BC2} + \frac{2G_{BC3}}{G_A C} \right) \\ & + 2C \left(\frac{1}{G_A} + \frac{1}{G_B} \right) + \frac{4}{G_A G_B} = 0 \end{aligned} \tag{17.28}$$

(d) If the far end of column $C1$ is hinged and the far end of column $C3$ is fixed, we have $G_{BC3} = 0$ and Equation 17.24 reduces to

$$\begin{aligned} C^2 \quad - \quad & S^2 \left(G_{AC1} + G_{AC2} G_{BC2} + \frac{2G_{AC1}}{G_B C} \right) \\ & + 2C \left(\frac{1}{G_A} + \frac{1}{G_B} \right) + \frac{4}{G_A G_B} = 0 \end{aligned} \tag{17.29}$$

(e) If the far ends of both columns $C1$ and $C3$ are rigidly connected (i.e., assumptions used in developing the alignment chart), we have $G_{C2} = 1.0$ and $G_{Ci} = 0$, and Equation 17.24 reduces to

$$C^2 - S^2 + 2C\left(\frac{1}{G_A} + \frac{1}{G_B}\right) + \frac{4}{G_A G_B} = 0 \tag{17.30}$$

which can be rewritten in the form of Equation 17.5.

2. For an Unbraced Frame [22, 23]:

$$\det\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \tag{17.31}$$

or

$$\begin{aligned} &a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{23}a_{12} - a_{31}a_{22}a_{13} \\ &- a_{21}a_{12}a_{33} + a_{11}a_{23}a_{32} = 0 \end{aligned} \tag{17.32}$$

where

$$a_{11} = C + \frac{6}{G_A} - G_{AC1}\frac{S^2}{C} \tag{17.33}$$

$$a_{22} = C + \frac{6}{G_B} - G_{BC3}\frac{S^2}{C} \tag{17.34}$$

$$a_{33} = -2\left[C + S - \frac{1}{2}\left(\frac{\pi}{K}\right)^2\right] \tag{17.35}$$

$$a_{12} = G_{AC2}S \tag{17.36}$$

$$a_{21} = G_{BC2}S \tag{17.37}$$

$$a_{31} = a_{32} = C + S \tag{17.38}$$

$$a_{13} = -(C + S) + G_{AC1}\left(S + \frac{S^2}{C}\right) \tag{17.39}$$

$$a_{23} = -(C + S) + G_{BC3}\left(S + \frac{S^2}{C}\right) \tag{17.40}$$

Although Equation 17.31 was derived for the special case in which the far ends of both columns $C1$ and $C3$ are hinged, it can be adjusted to account for the following cases: (1) if the far end of column $Ci(C1$ or $C3)$ is fixed, then take $G_{Ci} = 0$ (except for G_{C2}); (2) if the far end of column $Ci(C1$ or $C3)$ is rigidly connected, then take $G_{Ci} = 0$ and $G_{C2} = 1.0$. Therefore, Equation 17.31 can be used for the following conditions:

(a) If the far ends of both columns $C1$ and $C3$ are fixed, we take $G_{C1} = G_{C3} = 0$, and obtain from Equations 17.33, 17.34, 17.39, and 17.40,

$$a_{11} = C + \frac{6}{G_A} \tag{17.41}$$

$$a_{22} = C + \frac{6}{G_B} \tag{17.42}$$

$$a_{13} = a_{23} = -(C + S) \tag{17.43}$$

(b) If the far end of column $C1$ is rigidly connected and the far end of column $C3$ is fixed, we take $G_{AC2} = 1.0$ and $G_{AC1} = G_{BC3} = 0$, and obtain from Equations 17.33, 17.34, 17.36, 17.39, and 17.40,

$$a_{11} = C + \frac{6}{G_A} \tag{17.44}$$

$$a_{22} = C + \frac{6}{G_B} \tag{17.45}$$

$$a_{12} = S \tag{17.46}$$

$$a_{13} = a_{23} = -(C + S) \tag{17.47}$$

(c) If the far end of column $C1$ is rigidly connected and the far end of column $C3$ is hinged, we take $G_{AC1} = 0$ and $G_{AC2} = 1.0$, and obtain from Equations 17.33, 17.36, and 17.39,

$$a_{11} = C + \frac{6}{G_A} \tag{17.48}$$

$$a_{12} = S \tag{17.49}$$

$$a_{13} = -(C + S) \tag{17.50}$$

(d) If the far end of column $C1$ is hinged and the far end of column $C3$ is fixed, we have $G_{BC3} = 0.0$, and obtain from Equations 17.34 and 17.40,

$$a_{22} = C + \frac{6}{G_B} \tag{17.51}$$

$$a_{23} = -(C + S) \tag{17.52}$$

(e) If the far ends of both columns $C1$ and $C3$ are rigidly connected (i.e., assumptions used in developing the alignment chart, that is $\theta_C = \theta_B$ and $\theta_D = \theta_A$), we take $G_{C2} = 1.0$ and $G_{Ci} = 0$, and obtain from Equations 17.33 to 17.40,

$$a_{11} = C + \frac{6}{G_A} \tag{17.53}$$

$$a_{22} = C + \frac{6}{G_B} \tag{17.54}$$

$$a_{12} = a_{21} = S \tag{17.55}$$

$$a_{13} = a_{23} = -(C + S) \tag{17.56}$$

Equation 17.31 is reduced to the form of Equation 17.6.

The procedures to obtain the K-factor directly from the alignment charts without resorting to solve Equations 17.24 and 17.31 were also proposed by Duan and Chen [21, 22].

17.5.3 Column Restrained by Tapered Rectangular Girders

A modification factor, α_T, was developed by King et al. [48] for those framed columns restrained by tapered rectangular girders with different far end conditions. The following modified G-factor is introduced in connection with the use of alignment charts:

$$G = \frac{\sum (E_c I_c / L_c)}{\sum \alpha_T \left(E_g I_g / L_g\right)} \tag{17.57}$$

where I_g is the moment of inertia of the girder at near end. Both closed-form and approximate solutions for modification factor α_T were derived. It is found that the following two-parameter power function can describe the closed-form solutions very well:

$$\alpha_T = D\,(1 - r)^{\beta} \tag{17.58}$$

in which the parameter D is a constant depending on the far end conditions, and β is a function of the far end conditions and tapering factor, α and r, as defined in Figure 17.7.

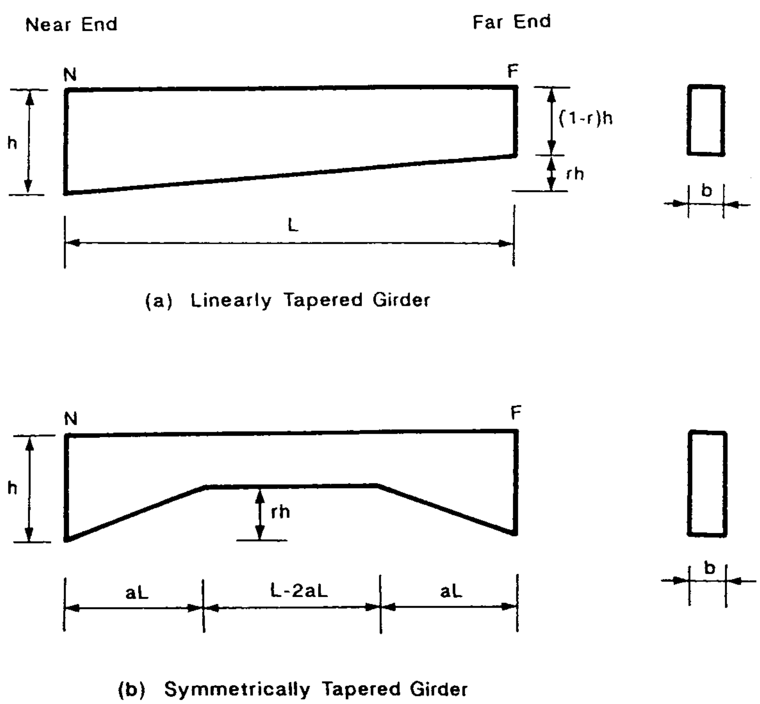


FIGURE 17.7: Tapered rectangular girders. (From King, W.S., Duan, L., et al., *Eng. Struct.*, 15(5), 369, 1993. With kind permission from Elsevier Science, Ltd, The Boulevard, Langford Lane, Kidlington OX5 IGB, UK.)

For a braced frame:

$$D = \left\{ \begin{array}{ll} 1.0 & \text{rigid far end} \\ 2.0 & \text{fixed far end} \\ 1.5 & \text{hinged far end} \end{array} \right\} \tag{17.59}$$

For an unbraced frame:

$$D = \left\{ \begin{array}{ll} 1.0 & \text{rigid far end} \\ 2/3 & \text{fixed far end} \\ 0.5 & \text{hinged far end} \end{array} \right\} \tag{17.60}$$

1. For a linearly tapered rectangular girder (Figure 17.7a)
   For a braced frame:

$$\beta = \left\{ \begin{array}{ll} 0.02 + 0.4r & \text{rigid far end} \\ 0.75 - 0.1r & \text{fixed far end} \\ 0.75 - 0.1r & \text{hinged far end} \end{array} \right\} \tag{17.61}$$

   For an unbraced frame:

$$\beta = \left\{ \begin{array}{ll} 0.95 & \text{rigid far end} \\ 0.70 & \text{fixed far end} \\ 0.70 & \text{hinged far end} \end{array} \right\} \tag{17.62}$$

2. For a symmetrically tapered rectangular girder (Figure 17.7b)
   For a braced frame:

$$\beta = \left\{ \begin{array}{ll} 3 - 1.7a^2 - 2a & \text{rigid far end} \\ 3 + 2.5a^2 - 5.55a & \text{fixed far end} \\ 3 - a^2 - 2.7a & \text{hinged far end} \end{array} \right\} \tag{17.63}$$

   For an unbraced frame:

$$\beta = \left\{ \begin{array}{ll} 3 + 3.8a^2 - 6.5a & \text{rigid far end} \\ 3 + 2.3a^2 - 5.45a & \text{fixed far end} \\ 3 - 0.3a & \text{hinged far end} \end{array} \right\} \tag{17.64}$$

EXAMPLE 17.3:

Given: A one-story frame with a symmetrically tapered rectangular girder is shown in Figure 17.8. Assuming $r = 0.5$, $a = 0.2$, and $I_g = 2I_c = 2I$, determine K-factor for column AB.

Solution

1. Using the alignment chart with modification

   For joint A, since the far end of the girder is rigid, use Equations 17.64 and 17.58,

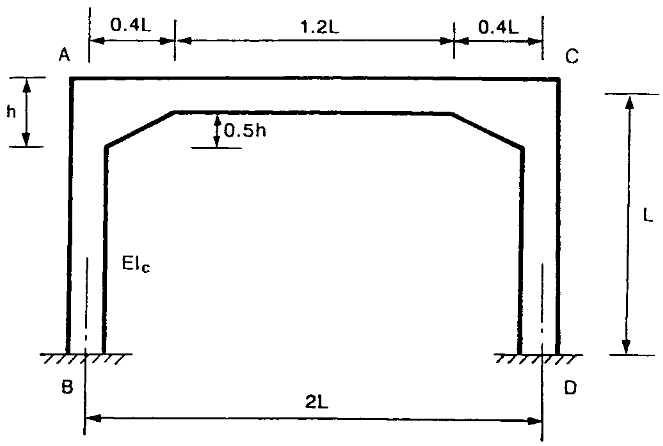


FIGURE 17.8: A simple frame with rectangular sections. (From King, W.S., Duan, L., et al., *Eng. Struct.*, 15(5), 369, 1993. With kind permission from Elsevier Science, Ltd, The Boulevard, Langford Lane, Kidlington OX5 IGB, UK.)

$$\begin{aligned}
\beta &= 3+3.8\,(0.2)^2-6.5\,(0.2)=1.852\\
\alpha_T &= (1-0.5)^{1.852}=0.277\\
G_A &= \frac{\sum E_c I_c/L_c}{\sum \alpha_T E_g I_g/L_g}=\frac{EI/L}{0.277E(2I)/2L}=3.61\\
G_B &= 1.0
\end{aligned}$$

(AISC – LRFD 1993)

From the alignment chart in Figure 17.4b, $K=1.59$ is obtained.

2. Using the alignment chart without modification

   A direct use of Equations 17.7 and 17.8 with an average section $(0.75\,h)$ results in:

$$\begin{aligned}
I_g &= 0.75^3(2I)=0.844I\\
G_A &= \frac{EI/L}{0.844EI/2L}=2.37 \quad G_B=1.0
\end{aligned}$$

   From the alignment chart in Figure 17.4b $K=1.49$, or $(1.49-1.59)/1.59=-6\,\%$ in error on the less conservative side.

17.5.4 Unsymmetrical Frames

When the column sizes or column loads are not identical, adjustments to the alignment charts are necessary to obtain a correct K-factor. SSRC Guide [43] presents a set of curves as shown in Figure 17.9 for a modification factor, β, originally developed by Chu and Chow [18].

$$K_{\text{adjusted}}=\beta K_{\text{alignment chart}} \tag{17.65}$$

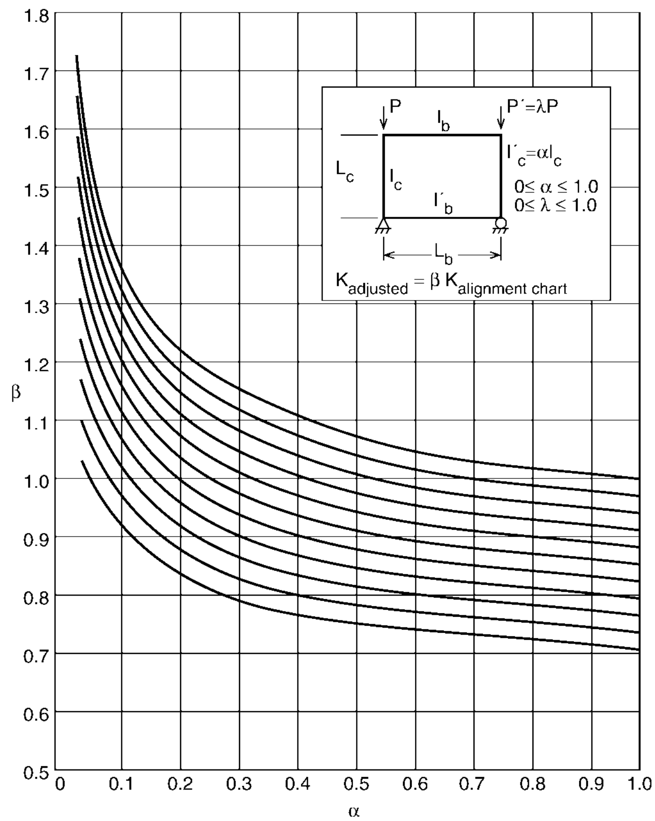


FIGURE 17.9: Chart for the modification factor β in an unsymmetrical frame.

If the K-factor of the column under the load λP is desired, further modifications to K are necessary. Denoting K' as the effective length factor of the column with $I_c' = \alpha I_c$ subjected to the axial load, $P' = \lambda P$, as shown in Figure 17.9, then we have:

$$K' = K_{\text{adjusted}} \frac{L}{L'} \sqrt{\frac{\alpha}{\lambda}} \tag{17.66}$$

Equation 17.66 can be used to determine K-*factors* for columns in adjacent stories with different heights, L'.

17.5.5 Effects of Axial Forces in Restraining Members in Braced Frames

Bridge and Fraser [14] observed that K-factors of a column in a braced frame may be greater than unity due to "negative" restraining effects. Figure 17.10 shows the solutions obtained by considering both the "positive" and "negative" values of G-factors. The shaded portion of the graph corresponds to the alignment chart shown in Figure 17.4a when both G_A and G_B are positive.

To account for the effect of axial forces in the restraining members, Bridge and Fraser [14] proposed a more general expression for G-factor:

$$G = \frac{(I/L)}{\sum_n (I/L)_n \gamma_n m_n}$$

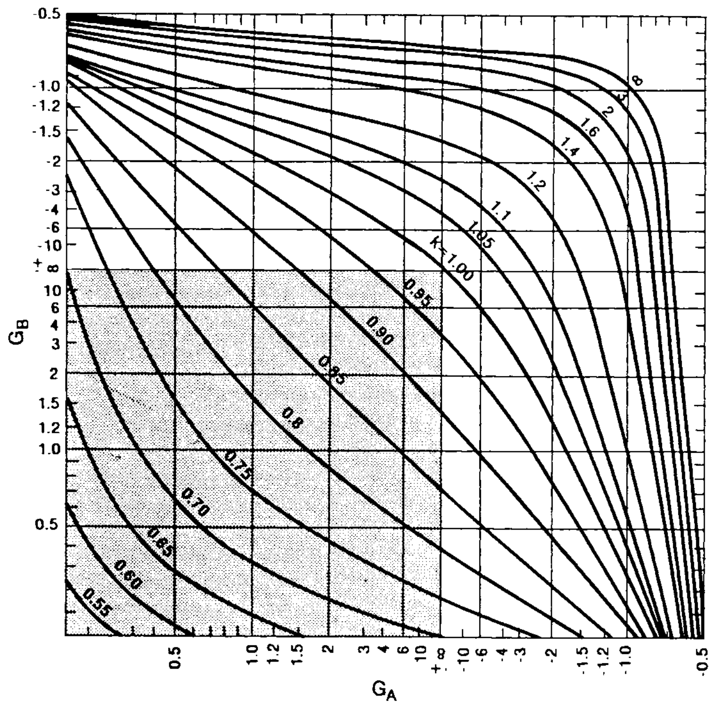


FIGURE 17.10: Effective length chart considering both positive and negative effects in braced frame. (From Bridge, R.Q. and Fraser, D.J., ASCE, *J. Struct. Eng.*, 113(6), 1341, 1987. With permission.)

$$= \frac{\text{stiffness of member } i \text{ under investigation}}{\text{stiffness of all rigidly connected members}} \tag{17.67}$$

where γ is a function of the stability functions S and C (Equations 17.3 and 17.4); m is a factor to account for the end conditions of the restraining member (see Figure 17.11); and subscript n represents the other members rigidly connected to member i. The summation in the denominator is for all members meeting at the joint.

Using Figures 17.10 and 17.11 and Equation 17.67, the effective length factor K_i for the i-th member can be determined by the following steps:

1. Sketch the buckled shape of the structure under consideration.
2. Assume a value of K_i for the member being investigated.
3. Calculate values of K_n for each of the other members that are rigidly connected to the i-th member using the equation

$$K_n = K_i \frac{L_i}{L_n} \sqrt{\left(\frac{P_i}{P_n}\right)\left(\frac{I_n}{I_i}\right)} \tag{17.68}$$

4. Calculate γ and obtain m from Figure 17.11 for each member.
5. Calculate G_i for the i-th member using Equation 17.67.
6. Obtain K_i from Figure 17.10 and compare with the assumed K_i at Step 2.
7. Repeat the procedure by using the calculated K_i as the assumed K_i until K_i calculated at the end the cycle is approximately (say 10%) equal to the K_i at the beginning of the cycle.

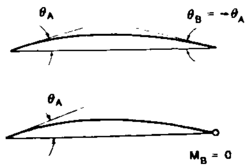
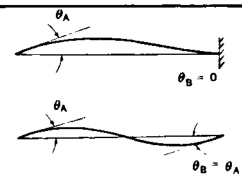
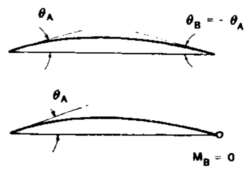
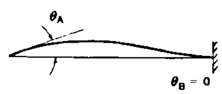
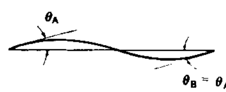
Axial Forces	Cases	Exact γ Formula	Approximate γ Formula	m
Tension		$\frac{C-S}{2}$	$1+\frac{1}{1.5K_n^2}$	1
		$\frac{C-S^2/C}{3}$		1.5
		$\frac{C}{4}$	$1+\frac{1}{4K_n^2}$	2
		$\frac{C+S}{6}$		3
Compression		$\frac{C-S}{2}$	$1-\frac{1}{K_n^2}$ *for* $K_n > 1.0$	1
		$\frac{C-S^2/C}{3}$	$2-\frac{2}{K_n^2}$ *for* $K_n \le 1.0$	1.5
		$\frac{C}{4}$	$1-\frac{1}{2K_n^2}$ *for* $K_n > 0.7$ $2-\frac{1}{K_n^2}$ *for* $K_n \le 0.7$	2
		$\frac{C+S}{6}$	$1-\frac{1}{4K_n^2}$	3
Note	C and S are stability Equations (18.3.2) and (18.3.3)			

FIGURE 17.11: Values of γ and m to account for the effect of axial forces in the restraining members.

8. Repeat steps 2 to 7 for other members of the frame.
9. The largest set of K values obtained is then used for design.

The above procedure has been illustrated [14] and verified [50] to provide a good elastic K-factor of columns in braced frames.

EXAMPLE 17.4:

Given: A braced column is shown in Figure 17.12. Consider axial force effects to determine K-factors for columns AB and BC.

Solutions

1. Sketch the buckled shape as shown in Figure 17.12b
2. Assume $K_{AB} = 0.94$.

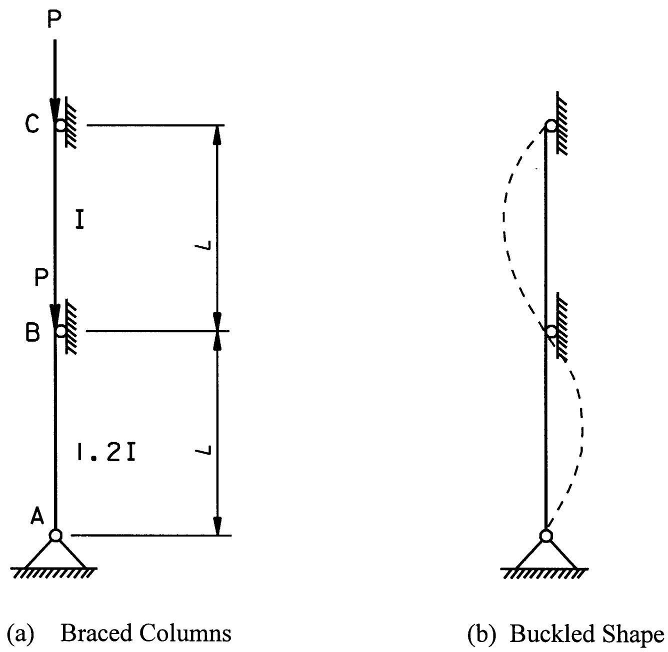


FIGURE 17.12: Braced columns.

3. Calculate K_{BC} by Equation 17.68.

$$
\begin{aligned}
K_{BC} &= K_{AB}\frac{L_{AB}}{L_{BC}} + \sqrt{\left(\frac{P_{AB}}{P_{BC}}\right)\left(\frac{I_{BC}}{I_{AB}}\right)} = 0.94\frac{L}{L}\sqrt{\frac{2PI}{P(1.2I)}} \\
&= 1.22
\end{aligned}
$$

4. Calculate γ and obtain m from Figure 17.11 for member BC.

   Since $K_{BC} > 1.0$

$$\gamma_{BC} = 1 - \frac{1}{K_{BC}^2} = 1 - \frac{1}{1.22^2} = 0.33$$

   Far end is pinned, $m_{BC} = 1.5$

5. Calculate G-factor for the member AB using Equation 17.67.

$$
\begin{aligned}
G_B &= \frac{(I/L)}{\sum_n (I/L)_n \gamma_n m_n} = \frac{(1.2I/L)}{(I/L)(0.33)(1.5)} = 2.42 \\
G_A &= \infty
\end{aligned}
$$

6. From Figure 17.10, $K_{AB} = 0.93$.
   Comparing with the assumed $K_{AB} = 0.94$ O.K.
7. Repeat the above procedure for member BC.

Assume $K_{BC} = 1.2$

Calculate K_{AB} by Equation 17.68

$$K_{AB} = K_{BC}\frac{L_{BC}}{L_{AB}}\sqrt{\left(\frac{P_{BC}}{P_{AB}}\right)\left(\frac{I_{AB}}{I_{BC}}\right)} = 1.2\frac{L}{L}\sqrt{\frac{P(1.2I)}{2PI}} = 0.93$$

Calculate γ and obtain m from Figure 17.11 for member AB

Since $K_{AB} < 1.0$

$$\gamma_{AB} = 2 - \frac{2}{K_{AB}^2} = 2 - \frac{2}{0.93^2} = -0.312$$

Far end is pinned, $m_{AB} = 1.5$

Calculate G-factor for the member BC using Equation 17.67.

$$\begin{aligned} G_B &= \frac{(I/L)}{\sum_n (I/L)_n \gamma_n m_n} = \frac{(I/L)}{(1.2I/L)(-0.312)(1.5)} = -1.78 \\ G_A &= \infty \end{aligned}$$

Read Figure 17.10, $K_{BC} = 1.18$

Comparing with the assumed $K_{AB} = 1.20$ O.K.

8. It is seen that the largest set of K-factors is
   $K_{AB} = 1.22$ and $K_{BC} = 0.93$

17.5.6 Consideration of Partial Column Base Fixity

In computing the effective length factor for monolithic connections, it is important to properly evaluate the degree of fixity in foundation. The following two approaches can be used to account for foundation fixity.

1. Fictitious Restraining Beam Approach
   Galambos [33] proposed that the effect of partial base fixity can be modelled as a fictitious beam. The approximate expression for the stiffness of the fictitious beam accounting for rotation of foundation in the soil has the form:

$$\frac{I_s}{L_B} = \frac{qBH^3}{72E_{\text{steel}}} \tag{17.69}$$

   where q is the modulus of subgrade reaction (varies from 50 to 400 lb/in.^{3}, 0.014 to 0.109 N/mm^3); B and H are the width and length (in bending plane) of the foundation; and E_{steel} is the modulus of elasticity of steel.

   Based on studies by Salmon, Schenker, and Johnston [65], the approximate expression for the stiffness of the fictitious beam accounting for the rotations between column ends and footing due to deformation of base plate, anchor bolts, and concrete can be written as:

$$\frac{I_s}{L_B} = \frac{bd^2}{72E_{\text{steel}}/E_{\text{concrete}}} \tag{17.70}$$

where b and d are the width and length of the base plate, and subscripts *concrete* and *steel* represent concrete and steel, respectively. Galambos [33] suggested that the smaller of the stiffness calculated by Equations 17.69 and 17.70 be used in determining K-factors.

2. AASHTO-LRFD Approach
   The following values are suggested by AASHTO-LRFD [1]:

G = 1.5	footing anchored on rock
G = 3.0	footing not anchored on rock
G = 5.0	footing on soil
G = 1.0	footing on multiple rows of end bearing piles

17.5.7 Inelastic *K*-factor

The effect of material inelasticity and end restrain on the K-factors has been studied during the last two decades [12, 15, 20, 44, 45, 58, 64, 67, 68, 69, 73] The inelastic K-factor developed originally by Yura [73] and expanded by Disque [20] makes use of the alignment charts with simple modifications. To consider inelasticity of material, the G values as defined by Equations 17.7 and 17.8 are replaced by G^* [20] as follows:

$$G^* = SRF(G) = \frac{E_t}{E}G \tag{17.71}$$

in which E_t is the tangent modulus of the material. For practical application, stiffness reduction factor $(SRF) = (E_t/E)$ can be taken as the ratio of the inelastic to elastic buckling stress of the column

$$SRF = \frac{E_t}{E} \approx \frac{(F_{cr})_{\text{inelastic}}}{(F_{cr})_{\text{elastic}}} \approx \frac{P_u/A_g}{(F_{cr})_{\text{elastic}}} \tag{17.72}$$

where P_u is the factored axial load and A_g is the cross-sectional area of the member. $(F_{cr})_{inelastic}$ and $(F_{cr})_{elastic}$ can be calculated by AISC-LRFD [4] column equations:

$$(F_{cr})_{\text{inelastic}} = (0.658)^{\lambda_c^2} F_y \tag{17.73}$$

$$(F_{cr})_{\text{elastic}} = \left[\frac{0.877}{\lambda_c^2}\right] F_y \tag{17.74}$$

$$\lambda_c = \frac{KL}{r\pi}\sqrt{\frac{F_y}{E}} \tag{17.75}$$

in which K is the elastic effective length factor and r is the radius of gyration about the plane of buckling. Table 17.3 gives SRF values for different stress levels and slenderness parameters.

EXAMPLE 17.5:

Given: A two-story steel frame is shown in Figure 17.5. Use the alignment chart to determine K-factor for inelastic column DE. $E = 29{,}000$ ksi (200 GPa) and $F_y = 36$ ksi (248 MPa).

Solution

1. Calculate the axial stress ratio:

$$\frac{P_u}{A_g F_y} = \frac{300}{13.3(36)} = 0.63$$

2. Obtain $SRF = 0.793$ from Table 17.3

TABLE 17.3 Stiffness Reduction Factor (SRF) for G values

	$\left(\frac{KL}{r}\right)_{\text{elastic}}$			
$\frac{P_u}{A_g F_y}$	36 ksi (248 MPa)	50 ksi (345 MPa)	λ_c	SRF (Eq. 18.72)
1.00	0.0	0.0	0.155	0.000
0.95	31.2	26.5	0.350	0.133
0.90	44.7	38.0	0.502	0.258
0.85	55.6	47.1	0.623	0.376
0.80	65.1	55.2	0.730	0.486
0.75	73.9	62.7	0.829	0.588
0.70	82.3	69.8	0.923	0.680
0.65	90.5	76.8	1.015	0.763
0.60	98.5	83.6	1.105	0.835
0.55	106.6	90.4	1.195	0.896
0.50	114.7	97.4	1.287	0.944
0.45	123.2	104.5	1.381	0.979
0.40	131.9	111.9	1.480	0.998
0.39	133.7	113.5	1.500	1.000

3. Calculate modified G-factor.

$$
\begin{aligned}
G_E &= 0.448 \quad \text{(Example 17.1)} \\
G_E^* &= SRF(G_E) = 0.794(0.448) = 0.355 \\
G_D &= 10 \qquad \text{(AISC-LRFD 1993)}
\end{aligned}
$$

4. From the alignment chart in Figure 17.4b , we have

$$(K_{DE})_{\text{inelastic}} = 1.75$$

17.6 Framed Columns—Alternative Methods

17.6.1 LeMessurier Method

Considering that all columns in a story buckle simultaneously and strong columns will brace weak columns (Figure 17.13), a more accurate approach to calculate K-factors for columns in a sidesway frame was developed by LeMessurier [52]. The K_i value for the i-th column in a story can be obtained by the following expression:

$$K_i = \sqrt{\frac{\pi^2 E I_i}{L_i^2 P_i}\left(\frac{\sum P + \sum C_L P}{\sum P L}\right)} \tag{17.76}$$

where P_i is the axial compressive force for member i, subscript i represents the i-th column, and $\sum P$ is the sum of the axial force of all columns in a story.

$$P_L = \frac{\beta E I}{L^2} \tag{17.77}$$

$$\beta = \frac{6\left(G_A + G_B\right) + 36}{2\left(G_A + G_B\right) + G_A G_B + 3} \tag{17.78}$$

$$C_L = \left(\beta \frac{K_o^2}{\pi^2} - 1\right) \tag{17.79}$$

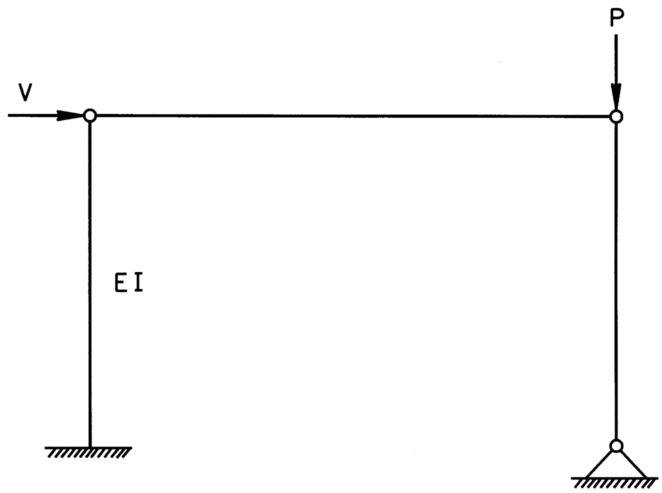


FIGURE 17.13: Subassemblage of the LeMessurier method.

in which K_o is the effective length factor obtained by the alignment chart for unbraced frames, and P_L is only for rigid columns which provide sidesway stiffness.

EXAMPLE 17.6:

Given: A sway frame with columns of unequal height is shown in Figure 17.14a. Determine elastic K-factors for columns by using the LeMessurier method. Member properties are:

Member	A in.^{2}	(mm^2)	I in.^{4}	(mm^4 $\times$ 10^8)	L in.	(mm)
AB	21.5	(13,871)	620	(2.58)	240	(6,096)
BD	21.5	(13,871)	620	(2.58)	240	(6,096)
CD	7.65	(4,935)	310	(1.29)	120	(3,048)

Solution

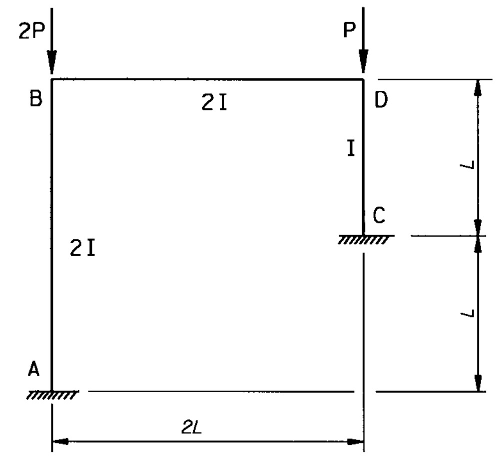
The detailed calculations are listed in Table 17.4.
Using Equation 17.76, we obtain:

$$K_{AB} = \sqrt{\frac{\pi^2 EI_{AB}}{L_{AB}^2 P_{AB}}\left(\frac{\sum P + \sum C_L P}{\sum P_L}\right)}$$

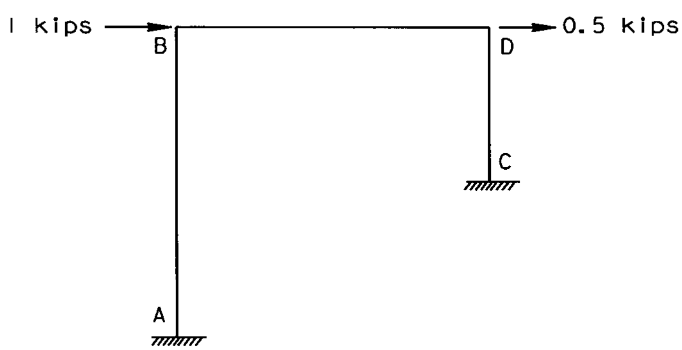
$$= \sqrt{\frac{\pi^2 E(620)}{(240)^2(2P)}\left(\frac{3P + 0.495P}{0.271E}\right)} = 0.83$$

$$K_{CD} = \sqrt{\frac{\pi^2 EI_{CD}}{L_{CD}^2 P_{CD}}\left(\frac{\sum P + \sum C_L P}{\sum P_L}\right)}$$

$$= \sqrt{\frac{\pi^2 E(310)}{(120)^2(P)}\left(\frac{3P + 0.495P}{0.271E}\right)} = 1.66$$



(a) Frame Dimensions and Loads



(b) Frame Subjected to Fictitious Lateral Loads

FIGURE 17.14: A frame with unequal columns.

17.6.2 Lui Method

A simple and straightforward approach for determining the effective length factors for framed columns without the use of alignment charts and other charts was proposed by Lui [57]. The formulas take into account both the member instability and frame instability effects explicitly. The K-factor for the i-th column in a story was obtained in a simple form:

$$K_i = \sqrt{\left(\frac{\pi^2 E I_i}{P_i L_i^2}\right)\left[\left(\sum \frac{P}{L}\right)\left(\frac{1}{5\sum \eta} + \frac{\Delta_1}{\sum H}\right)\right]} \tag{17.80}$$

where $\sum(P/L)$ represents the sum of the axial force-to-length ratio of all members in a story, $\sum H$ is the story lateral load producing Δ_1, Δ_1 is the first-order inter-story deflection, and η is the member

TABLE 17.4 Example 17.6—Detailed Calculation by the LeMessurier Method

Members	AB	CD	Sum	Notes
I in.^{4}	620	310	—	
(mm^4 $\times$ 10^8)	(2.58)	(1.29)		
L in.	240	120	—	
(mm)	(6,096)	(3,048)		
G_{top}	1.0	1.0	—	Eq. 18.7
G_{bottom}	0.0	0.0	—	Eq. 18.7
β	8.4	8.4	—	Eq. 18.78
K_{io}	1.17	1.17	—	Alignment Chart
C_L	0.165	0.165	—	Eq. 18.79
P_L	$0.09E$	$0.181E$	$0.271E$	Eq. 18.77
P	$2P$	P	$3P$	
$C_L P$	$0.33P$	$0.165P$	$0.495P$	

stiffness index and can be calculated by

$$\eta = \frac{\left(3 + 4.8m + 4.2m^2\right) EI}{L^3} \tag{17.81}$$

in which m is the ratio of the smaller to larger end moments of the member; it is taken as positive if the member bends in reverse curvature and negative for single curvature.

It is important to note that the term $\sum H$ used in Equation 17.80 is not the actual applied lateral load. Rather, it is a small disturbing or fictitious force (taken as a fraction of the story gravity loads) to be applied to each story of the frame. This fictitious force is applied in a direction such that the deformed configuration of the frame will resemble its buckled shape.

EXAMPLE 17.7:

Given: Determine K-factors by using the Lui method for the frame shown in Figure 17.14a.

$$E = 29{,}000 \;\text{ksi}\; (200 \;\text{GPa}).$$

Solution

Apply fictitious lateral forces at B and D (Figure 17.14b) and perform a first-order analysis. Detailed calculation is shown in Table 17.5.

Using Equation 17.80, we obtain:

$$\begin{aligned}
K_{AB} &= \sqrt{\left(\frac{\pi^2 EI_{AB}}{P_{AB}L_{AB}^2}\right)\left[\left(\sum \frac{P}{L}\right)\left(\frac{1}{5\sum\eta} + \frac{\Delta_1}{\sum H}\right)\right]} \\
&= \sqrt{\left(\frac{\pi^2(29{,}000)(620)}{(2P)(240)^2}\right)\left[\left(\frac{P}{60}\right)\left(\frac{1}{5(56.24)} + 0.019\right)\right]} = 0.76 \\
K_{CD} &= \sqrt{\left(\frac{\pi^2 EI_{CD}}{P_{CD}L_{CD}^2}\right)\left[\left(\sum \frac{P}{L}\right)\left(\frac{1}{5\sum\eta} + \frac{\Delta_1}{\sum H}\right)\right]} \\
&= \sqrt{\left(\frac{\pi^2(29{,}000)(310)}{(P)(120)^2}\right)\left[\left(\frac{P}{60}\right)\left(\frac{1}{5(56.24)} + 0.019\right)\right]} = 1.52
\end{aligned}$$

TABLE 17.5 Example 17.7—Detailed Calculation by the Lui Method

Members	AB	CD	Sum	Notes
I in.^{4}	620	310	—	
(mm^4 $\times$ 10^8)	(2.58)	(1.29)		
L in.	240	120	—	
(mm)	(6096)	(3048)		
H kips	1.0	0.5	1.5	
(kN)	(4.448)	(2.224)	(6.672)	
Δ_1 in.	0.0286	0.0283	—	
(mm)	(0.7264)	(0.7188)		
$\Delta_1/\sum H$ in./kips	—	—	0.019	Average
(mm/kN)			(0.108)	
M_{top} k-in.	−38.8	56.53	—	
(kN-m)	(−4.38)	(6.39)		
M_{bottom} k-in.	−46.2	81.18	—	
(kN-m)	(−5.22)	(9.17)		
m	0.84	0.69	—	
η kips/in.	13.00	43.24	56.24	Eq. 18.1
(kN/mm)	(2.28)	(7.57)	(9.85)	
P/L kips/in.	$P/120$	$P/120$	$P/60$	
(kN/mm)	$P/3048$	$P/3048$	$P/1524$	

17.6.3 Remarks

For comparison, Table 17.6 summarizes K-factors for the frame shown in Figure 17.14a obtained from the alignment chart, the LeMessurier and Lui methods, as well as an eigenvalue analysis. It is seen that errors in alignment chart results are rather significant in this case. Although K-factors predicted by Lui's and LeMessurier's formulas are identical in most cases, the simplicity and independence of any chart in the case of Lui's formulas make it more desirable for design office use [66].

TABLE 17.6 Comparison of K-Factors for the Frame in Figure 17.14a

Columns	Theoretical	Alignment chart	Lui Eq. 18.80	LeMessurier Eq. 18.76
AB	0.70	1.17	0.76	0.83
CD	1.40	1.17	1.52	1.67

17.7 Unbraced Frames With Leaning Columns

A column framed with simple connections is often called a leaning column. It has no lateral stiffness or sidesway resistance. A column framed with rigid moment-resisting connections is called a rigid column. It provides the lateral stiffness or sidesway resistance to the frame. When a frame system (Figure 17.15a) includes leaning columns, the effective length factors of rigid columns must be modified. Several approaches to account for the effect of "leaning columns" were reported in the literature [16, 52, 54, 73]. A detailed discussion about the leaning columns for practical applications was presented by Geschwindner [37].

17.7.1 Rigid Columns

1. Yura Method
   Yura [73] discussed frames with leaning columns and noted the behavior of stronger columns assisting weaker ones in resisting sidesway. He concluded that the alignment

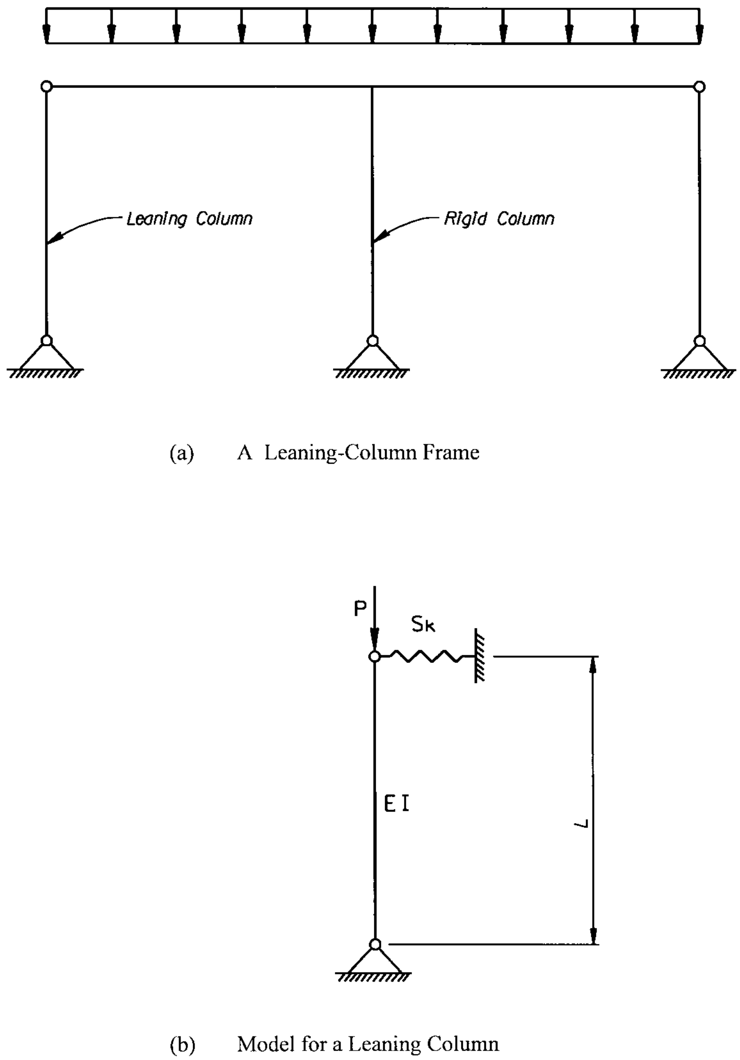


FIGURE 17.15: A frame with leaning columns.

chart gives valid sidesway buckling solutions if the columns are in the elastic range and all columns in a story reach their individual buckling loads simultaneously. For columns that do not satisfy these two conditions, the alignment chart is generally overly conservative. Yura states that

(a) The maximum load-carrying capacity of an individual column is limited to the load permitted on that column for braced case $K = 1.0$.

(b) The total gravity loads that produce sidesway are distributed among the columns, which provides lateral stiffness in a story.

2. Lim and McNamara Method
   Based on the story buckling concept and using the stability functions, Lim and McNamara [54] presented the following formula to account for the leaning column effect.

$$K_n = K_o \sqrt{1 + \frac{\sum Q}{\sum P}\left(\frac{F_o}{F_n}\right)} \tag{17.82}$$

where K_n is the effective length factor accounting for the leaning columns; K_o is the effective length factor determined by the alignment chart (Figure 17.3b) not accounting for the leaning columns; $\sum P$ and $\sum Q$ are the loads on the restraining columns and on the leaning columns in a story, respectively; and F_o and F_n are the eigenvalue solutions for a frame without and with leaning columns, respectively. For normal column end conditions that fall somewhere between fixed and pinned, $F_o/F_n = 1$ provides a K-factor on the conservative side by less than 2% [37]. Using $F_o/F_n = 1$, Equation 17.82 becomes:

$$K_n = K_o \sqrt{1 + \frac{\sum Q}{\sum P}} \tag{17.83}$$

Equation 17.83 gives the same K-factor as the modified Yura approach [37].

3. LeMessurier and Lui Methods
Equation 17.76 developed by LeMessurier [52] and Equation 17.80 proposed by Lui [57] can be used for frames with and without leaning columns. Since the K-factor expressions Equations 17.76 and 17.80 were derived for an entire story of the frame, they are applicable to frames with and without leaning columns.
4. AISC-LRFD Method
The current AISC-LRFD [4] commentary adopts the following modified effective length factor, K_i', for the i-th rigid column:

$$K_i' = \sqrt{\frac{\pi^2 E I_i}{L_i^2 P_{ui}}\left(\frac{\sum P_u}{\sum P_{e2}}\right)} \tag{17.84}$$

where $\sum P_{e2}$ is the Euler loads of all columns in a story providing lateral stiffness for the frame based on the effective length factor obtained from the alignment chart for an unbraced frame; P_{ui} is the required axial compressive strength for the i-th rigid column; and $\sum P_u$ is the required axial compressive strength of all columns in a story.
When E and L^2 are constant for all columns in a story, AISC [4] suggested that:

$$K_i' = \sqrt{\frac{\sum P_u}{P_{ui}} \times \left(\frac{I_i}{\sum \frac{I_i}{K_{io}^2}}\right)} \tag{17.85}$$

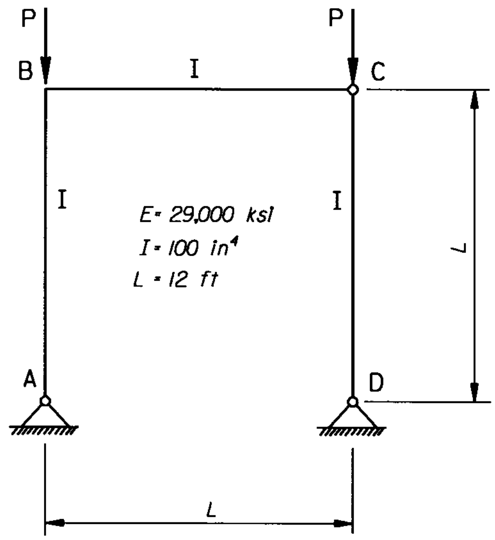
except

$$K_i' \geq \sqrt{\frac{5}{8}} K_{io} \tag{17.86}$$

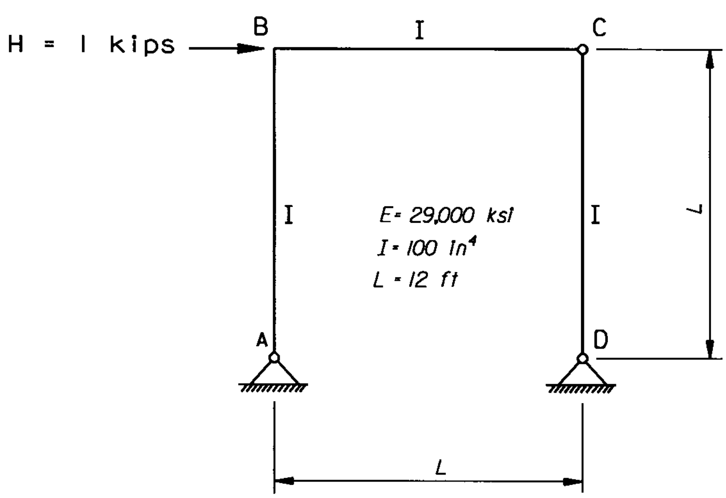
where K_{io} is the effective length factor of a rigid column based on the alignment chart for unbraced frames.

EXAMPLE 17.8:

Given: A frame with a leaning column is shown in Figure 17.16a [59]. Evaluate the K-factor for



(a) Frame Dimensions and Loads



(b) Frame Subjected to Fictitious Lateral Loads

FIGURE 17.16: A leaning column frame.

column AB using various methods. The bottom of column AB is assumed to be ideally pin-ended for comparison purposes. $E = 29{,}000$ ksi (200 GPa).

Solution

1. Alignment Chart Method

$$
\begin{aligned}
G_A &= \infty \\
G_B &= \frac{\sum E_c I_c / L_c}{\sum \alpha_k E_g I_g / L_g} = \frac{EI/L}{0.5EI/L} = 2.0
\end{aligned}
$$

From Figure 17.3b, we have $K_{AB} = 2.6$

2. Lim and McNamara Method

For this frame, $\sum P = \sum Q = P$ and $K_o = 2.6$. From Equation 17.83, we have

$$
K_{AB} = K_o \sqrt{1 + \frac{\sum Q}{\sum P}} = 2.6\sqrt{1+1} = 3.68
$$

3. LeMessurier Method

For column AB, $G_A = \infty$ and $G_B = 2.0$; from the alignment chart, $K_o = 2.6$. According to Equations 17.76 to 17.79 we have,

$$
\begin{aligned}
\beta \mid_{G_{A=\infty}} &= \frac{6(G_A + G_B) + 36}{2(G_A + G_B) + G_A G_B + 3} \mid_{G_{A=\infty}} = \frac{6}{2 + G_B} = \frac{6}{2+2} = 1.5 \\
\sum P_L &= (P_L)_{AB} = \frac{\beta EI}{L^2} = 1.5\frac{EI}{L^2} \\
C_L &= \left(\beta \frac{K_o^2}{\pi^2} - 1\right) = (1.5)\frac{2.6^2}{\pi^2} - 1 = 0.0274 \\
K_{AB} &= \sqrt{\frac{\pi^2 EI_{AB}}{L_{AB}^2 P_{AB}} \left(\frac{\sum P + \sum C_L P}{\sum P_L}\right)} = \sqrt{\frac{\pi^2 EI}{L^2 P}\left(\frac{2P + 0.0274P}{1.5EI/L^2}\right)} \\
&= \sqrt{13.34} = 3.65
\end{aligned}
$$

4. AISC-LRFD Method

Using Equation 17.85 for column AB:

$$
K_{AB} = \sqrt{\frac{\sum P_u}{P_{AB}} \times \left(\frac{I_{AB}}{\sum \frac{I}{K_{io}^2}}\right)} = K_{io}\sqrt{2} = 3.68
$$

5. Lui Method

   (a) Apply a small lateral force, $H = 1$ kip, as shown in Figure 17.16b.

   (b) Perform a first-order analysis and find $\Delta_1 = 0.687$ in. (17.45 mm).

   (c) Calculate η factors from Equation 17.81.

   Since column CD buckles in a single curvature, $m = -1$,

$$
\eta_{CD} = \frac{(3 + 4.8m + 4.2m^2)EI}{L^3} = \frac{(3 - 4.8 + 4.2)EI}{L^3} = \frac{2.4EI}{L^3}
$$

For column AB, $m = 0$,

$$\eta_{AB} = \frac{(3 + 4.8m + 4.2m^2)EI}{L^3} = \frac{3EI}{L^3}$$
$$\sum \eta = \frac{3EI}{L^3} + \frac{2.4EI}{L^3} = \frac{5.4(29{,}000)(100)}{(144)^3}$$
$$= 5.245 \text{ kips/in.} \quad (0.918 \text{ kN/mm})$$

(d) Calculate the K-factor from Equation 17.80.

$$K_{AB} = \sqrt{\left(\frac{\pi^2 EI_{AB}}{P_{AB}L_{AB}^2}\right)\left[\left(\sum \frac{P}{L}\right)\left(\frac{1}{5\sum\eta} + \frac{\Delta_1}{\sum H}\right)\right]}$$
$$= \sqrt{\left(\frac{\pi^2(29{,}000)(100)}{P(144)^2}\right)\left[\left(\frac{2P}{144}\right)\left(\frac{1}{5(5.245)} + \frac{0.687}{1}\right)\right]} = 3.73$$

From an eigenvalue analysis, $K_{AB} = 3.69$ is obtained. It is seen that a direct use of the alignment chart leads to a significant error for this frame, and other approaches give good results. However, the LeMessurier approach requires the use of the alignment chart, and the Lui approach requires a first-order analysis subjected to a fictitious lateral loading.

17.7.2 Leaning Columns

Recognizing that a leaning column is being braced by rigid columns, Lui [57] proposed a model for the leaning column, as shown in Figure 17.15b. Rigid columns provide lateral stability to the whole structure and are represented by a translation spring with a spring stiffness, S_K. The K-factor for a leaning column can be obtained as:

$$K = \text{larger of} \begin{cases} 1 \\ \sqrt{\frac{\pi^2 EI}{S_K L^3}} \end{cases} \tag{17.87}$$

For most commonly framed structures, the term $(\pi^2 EI/S_K L^3)$ normally does not exceed unity and so $K = 1$ often governs. AISC-LRFD [4] suggests that leaning columns with $K = 1$ may be used in unbraced frames provided that the lack of lateral stiffness from simple connections to the frame ($K = \infty$) is included in the design of moment frame columns.

17.7.3 Remarks

Numerical studies by Geschwindner [37] found that the Yura approach gives overly conservative results for some conditions; Lim and McNamara's approach provides sufficiently accurate results for design, and the LeMessurier approach is the most accurate of the three. The Lim and McNamara approach could be appropriate for preliminary design while the LeMessurier and Lui approaches would be appropriate for final design.

17.8 Cross Bracing Systems

Diagonal bracing or X-bracing is commonly used in steel structures to resist horizontal loads. In the current practice, the design of this type of bracing system is based on the assumptions that the compression diagonal has negligible capacity and the tension diagonal resists the total load. The assumption that compression diagonal has a negligible capacity usually results in an overdesign [62, 63].

Picard and Beaulieu [62, 63] reported theoretical and experimental studies on double diagonal cross bracings (Figure 17.6) and found that

1. A general effective length factor equation (Figure 17.17) is given as

$$K = \sqrt{0.523 - \frac{0.428}{C/T}} \geq 0.50 \tag{17.88}$$

where C and T represent compression and tension forces obtained from an elastic analysis, respectively.

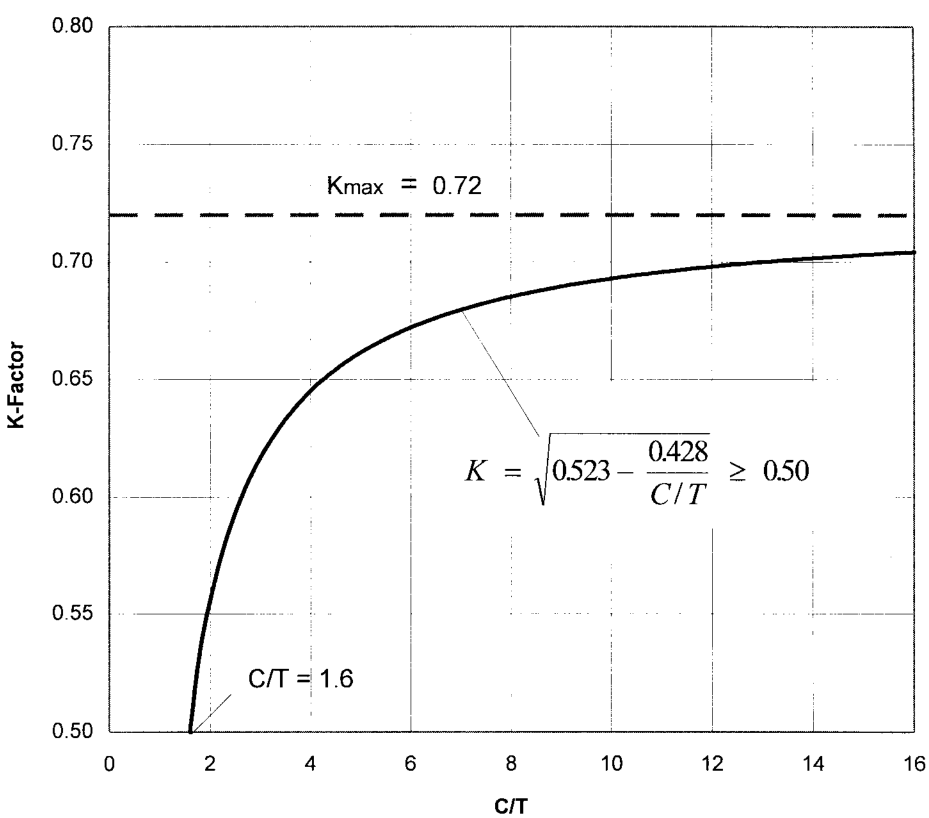


FIGURE 17.17: Effective length factor of compression diagonal. (From Picard, A. and Beaulieu, D., *AISC Eng. J.*, 24(3), 122, 1987. With permission.)

2. When the double diagonals are continuous and attached at their intersection point, the effective length of the compression diagonal is 0.5 times the diagonal length, i.e., $K = 0.5$, because the C/T ratio is usually smaller than 1.6.

EL-Tayem and Goel [27] reported a theoretical and experimental study about the X-bracing system made from single equal-leg angles. They concluded that:

1. Design of an X-bracing system should be based on an exclusive consideration of one half diagonal only.
2. For X-bracing systems made from single equal-leg angles, an effective length of 0.85 times the half diagonal length is reasonable, i.e., $K = 0.425$.

17.9 Latticed and Built-Up Members

The main difference of behavior between solid-webbed members, latticed members, and built-up members is the effect of shear deformation on their buckling strength. For solid-webbed members, shear deformation has a negligible effect on buckling strength. Whereas for latticed structural members using lacing bars and batten plates, shear deformation has a significant effect on buckling strength. It is a common practice that when a buckling model involves relative deformation produced by shear forces in the connectors, such as lacing bars and batten plates, between individual components, a modified effective length factor, K_m, is defined as follows:

$$K_m = \alpha_v K \tag{17.89}$$

in which K is the usual effective length factor of a latticed member acting as a unit obtained from a structural analysis, and α_v is the shear factor to account for shear deformation on the buckling strength, or the modified effective slenderness ratio, $(KL/r)_m$ should be used in the determination of the compressive strength. Details of the development of the shear factor, α_v, can be found in textbooks by Bleich [13] and Timoshenko and Gere [70]. The following section briefly summarizes α_v formulas for various latticed members.

17.9.1 Laced Columns

For laced members as shown in Figure 17.18, by considering shear deformation due to the lengthening of diagonal lacing bars in each panel and assuming hinges at joints, the shear factor, α_v, has the form:

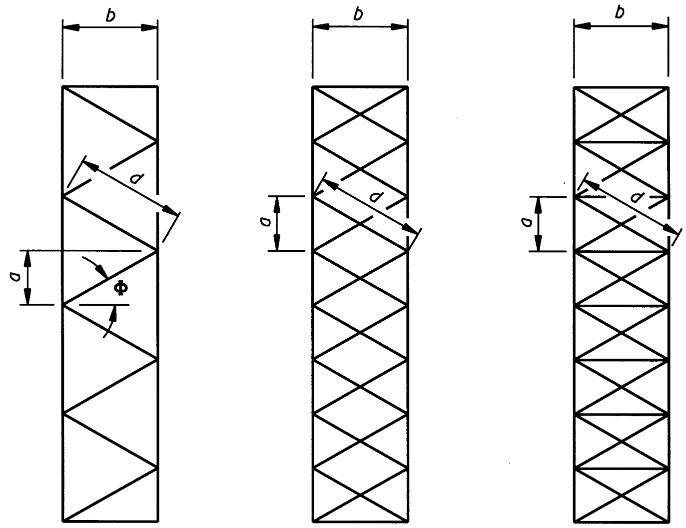
$$\alpha_v = \sqrt{1 + \frac{\pi^2 EI}{(KL^2)} \frac{1}{A_d E_d \sin\phi \cos^2\phi}} \tag{17.90}$$

where E_d is the modulus of elasticity of materials for the lacing bars, A_d is the cross-sectional area of all diagonals in one panel, and ϕ is the angle between the lacing diagonal and the axis that is perpendicular to the member axis.

If the length of the lacing bars is given (Figure 17.18), Equation 17.90 can be rewritten as:

$$\alpha_v = \sqrt{1 + \frac{\pi^2 EI}{(KL^2)} \frac{d^3}{A_d E_d a b^2}} \tag{17.91}$$

where a, b, and d are the height of panel, depth of member, and length of diagonal, respectively.



(a) Single Lacing (b) Double Lacing

FIGURE 17.18: Typical configurations of laced members.

The SSRC [36] suggested that a conservative estimate of the influence of 60 or 45° lacing, as generally specified in bridge design practice, can be made by modifying the overall effective length factor, K, by multiplying a factor, α_v, originally developed by Bleich [13] as follows:

For $\frac{KL}{r} > 40$,

$$\alpha_v = \sqrt{1 + 300/\left(KL/r\right)^2} \tag{17.92}$$

For $\frac{KL}{r} \leq 40$,

$$\alpha_v = 1.1 \tag{17.93}$$

EXAMPLE 17.9:

Given: A laced column with angles and cover plates is shown in Figure 17.19. $K_y = 1.25$, $L = 30$ ft (9144 mm). Determine the modified effective length factor, $(K_y)_m$, by considering the shear deformation effect.

Section properties:

$$\begin{aligned} I_y &= 2259 \text{ in.}^4 \ \ (9.4 \times 108 \ \ \text{mm}^4) \\ E &= E_d \qquad A_d = 1.69 \text{ in.}^2 \ \ (1090 \text{ mm}^2) \\ a &= 6 \text{ in.} \ \ (152 \text{ mm}) \qquad b = 11 \text{ in.} \ \ (279 \text{ mm}) \\ d &= 12.53 \ \ \text{in.} \ \ (318 \ \ \text{mm}) \end{aligned}$$

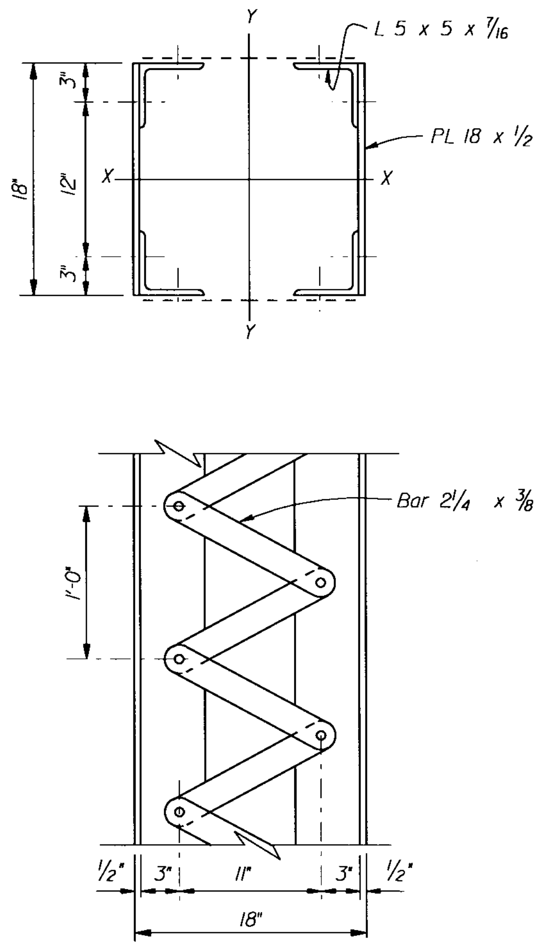


FIGURE 17.19: A laced column.

Solution

1. Calculate the shear factor, α_v, by Equation 17.91.

$$\begin{aligned}\alpha_v &= \sqrt{1+\frac{\pi^2 EI}{(KL)^2}\frac{d^3}{A_d E_d ab^2}} \\ &= \sqrt{1+\frac{\pi^2 E(2259)}{(1.25\times 30\times 12)^2}\frac{12.53^3}{1.69E(6)(11)^2}} = 1.09\end{aligned}$$

2. Calculate $(K_y)_m$ by Equation 17.89.

$$\left(K_y\right)_m = \alpha_v K_y = 1.09(1.25) = 1.36$$

17.9.2 Columns with Battens

The battened column has a greater shear flexibility than either the laced column or the column with perforated cover plates, hence the effect of shear distortion must be taken into account in calculating the effective length of a column [43]. For the battened members shown in Figure 17.20a, assuming that points of inflection in the battens are at the batten midpoints, and that points of inflection in the longitudinal element occur midway between the battens, the shear factor, α_v, is obtained as:

$$\alpha_v = \sqrt{1 + \frac{\pi^2 EI}{(KL)^2}\left(\frac{ab}{12E_b I_b} + \frac{a^2}{24EI_f}\right)} \tag{17.94}$$

where E_b is the modulus of elasticity of materials for the batten plates, I_b is the moment inertia of all the battens in one panel in the buckling plane, and I_f is the moment inertia of one side of the main components taken about the centroid axis of the flange in the buckling plane.

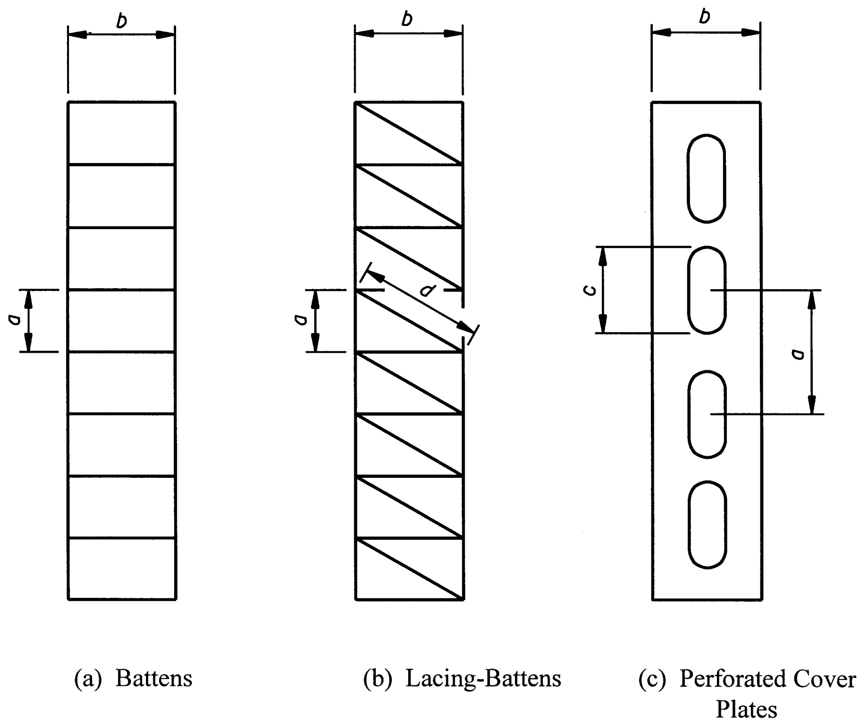


FIGURE 17.20: Typical configurations of members with battens and with perforated cover plates.

EXAMPLE 17.10:

Given: A battened column is shown in Figure 17.21. $K_y = 0.8$, $L = 30$ ft (9144 mm). Determine the modified effective length factor, $(K_y)_m$, by considering the shear deformation effect.

Section properties:

$$I_y = 144 \text{ in.}^4 \ (6.0 \times 10^7 \text{ mm}^4)$$

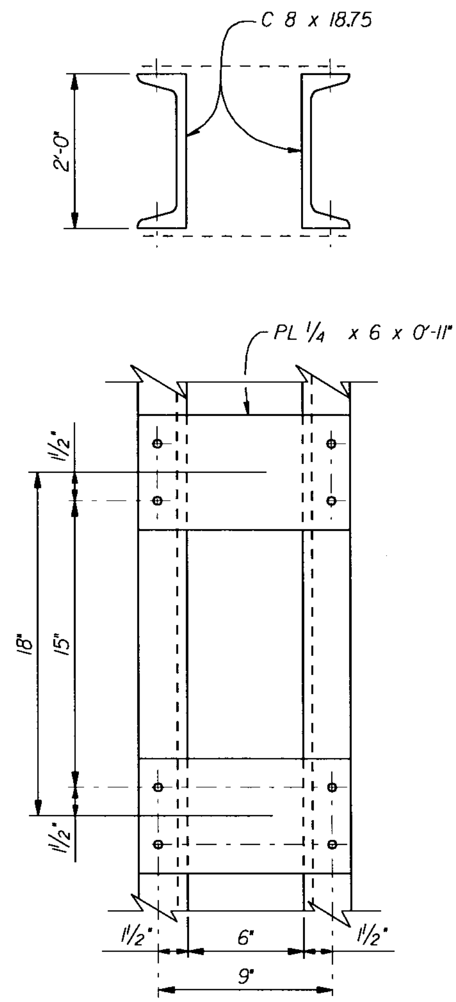


FIGURE 17.21: A battened column.

$$
\begin{aligned}
E &= E_b \\
I_f &= 1.98 \text{ in.}^4 \ (8.24 \times 10^5 \text{ mm}^4) \\
a &= 15 \text{ in. } (381 \text{ mm}) \\
b &= 9 \text{ in. } (229 \text{ mm}) \\
I_b &= 9 \text{ in.}^4 \ (3.75 \times 10^6 \text{ mm}^4)
\end{aligned}
$$

Solution

1. Calculate the shear factor, α_v, by Equation 17.94.

$$\begin{aligned}\alpha_v &= \sqrt{1+\frac{\pi^2 EI}{(KL)^2}\left(\frac{ab}{12EI_b}+\frac{a^2}{24EI_f}\right)} \\ &= \sqrt{1+\frac{\pi^2 E(144)}{(0.8\times 30\times 12)^2}\left(\frac{15(9)}{12E(9)}+\frac{15^2}{24E(1.98)}\right)} = 1.05\end{aligned}$$

2. Calculate $(K_y)_m$ by Equation 17.89.

$$(K_y)_m = \alpha_v K_y = 1.05(0.8) = 0.84$$

17.9.3 Laced-Battened Columns

For the laced-battened columns, as shown in Figure 17.20b, considering the shortening of the battens and the lengthening of the diagonal lacing bars in each panel, the shear factor, α_v, can be expressed as:

$$\alpha_v = \sqrt{1+\frac{\pi^2 EI}{(KL)^2}\left(\frac{d^3}{A_d E_d ab^2}+\frac{b}{aA_b E_b}\right)} \tag{17.95}$$

where E_b is the modulus of elasticity of the materials for battens and A_b is the cross-sectional area of all battens in one panel.

17.9.4 Columns with Perforated Cover Plates

For members with perforated cover plates, shown in Figure 17.20c, considering the horizontal cross member as infinitely rigid, the shear factor, α_v, has the form:

$$\alpha_v = \sqrt{1+\frac{\pi^2 EI}{(KL)^2}\left(\frac{9c^3}{64aEI_f}\right)} \tag{17.96}$$

where c is the length of a perforation.

It should be pointed out that the usual K-factor based on a solid member analysis is included in Equations 17.90 to 17.96. However, since the latticed members studied previously have pin-ended conditions, the K-factor of the member in the frame was not included in the second terms of the square root of the above equations in their original derivations [13, 70].

EXAMPLE 17.11:

Given: A column with perforated cover plates is shown in Figure 17.22. $K_y = 1.3$, $L = 25$ ft (7620 mm). Determine the modified effective length factor, $(K_y)_m$, by considering the shear deformation effect.

Section properties:

$$\begin{aligned} I_y &= 2467 \text{ in.}^4 \ (1.03\times 10^8 \text{ mm}^4) \\ I_f &= 35.5 \text{ in.}^4 \ (1.48\times 10^6 \text{ mm}^4) \\ a &= 30 \text{ in. } (762 \text{ mm}) \\ c &= 14 \text{ in. } (356 \text{ mm}) \end{aligned}$$

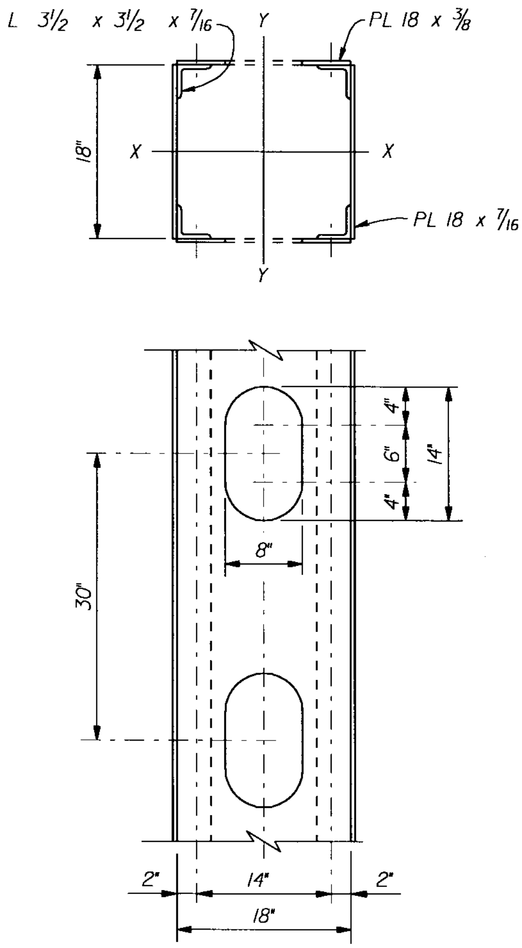


FIGURE 17.22: A column with perforate cover plates.

Solution

1. Calculate the shear factor, α_v, by Equation 17.96.

$$\begin{aligned}\alpha_v &= \sqrt{1 + \frac{\pi^2 EI}{(KL)^2}\left(\frac{9c^3}{64aEI_f}\right)} \\ &= \sqrt{1 + \frac{\pi^2 E(2467)}{(1.3 \times 25 \times 12)^2}\left(\frac{9(14)^3}{64(30)E(35.5)}\right)} = 1.03\end{aligned}$$

2. Calculate $(K_y)_m$ by Equation 17.89.

$$\left(K_y\right)_m = \alpha_v K_y = 1.03(1.3) = 1.34$$

17.9.5 Built-Up Members with Bolted and Welded Connectors

AISC-LRFD [4] specifies that if the buckling of a built-up member produces shear forces in the connectors between individual component members, the usual slenderness ratio, KL/r, for compression members must be replaced by the modified slenderness ratio, $\left(\frac{KL}{r}\right)_m$, in determining the compressive strength.

1. For snug-tight bolted connectors:

$$\left(\frac{KL}{r}\right)_m = \sqrt{\left(\frac{KL}{r}\right)_o^2 + \left(\frac{a}{r_i}\right)^2} \tag{17.97}$$

2. For welded connectors and for fully tightened bolted connectors:

$$\left(\frac{KL}{r}\right)_m = \sqrt{\left(\frac{KL}{r}\right)_o^2 + 0.82\frac{\alpha^2}{(1+\alpha^2)}\left(\frac{a}{r_{ib}}\right)^2} \tag{17.98}$$

where $\left(\frac{KL}{r}\right)_o$ is the slenderness ratio of the built-up member acting as a unit, $\left(\frac{KL}{r}\right)_m$ is the modified slenderness ratio of the built-up member, $\frac{a}{r_i}$ is the largest slenderness ratio of the individual components, $\frac{a}{r_{ib}}$ is the slenderness ratio of the individual components relative to its centroidal axis parallel to the axis of buckling, a is the distance between connectors, r_i is the minimum radius of gyration of individual components, r_{ib} is the radius of gyration of individual components relative to its centroidal axis parallel to the member axis of buckling, α is the separation ratio $= h/2r_{ib}$, and h is the distance between centroids of individual components perpendicular to the member axis of buckling.

Equation 17.97 is the same as that used in the current Italian code as well as other European specifications, based on test results [74]. In the equation, the bending effect is considered in the first term in square root, and shear force effect is taken into account in the second term. Equation 17.98 was derived from elastic stability theory and verified by test data [9]. In both cases the end connectors must be welded or slip-critical bolted [9].

EXAMPLE 17.12:

Given: A built-up member with two back-to-back angles is shown in Figure 17.23. Determine the modified slenderness ratio, $(KL/r)_m$, in accordance with AISC-LRFD [4] and Equation 17.98.

$$\begin{aligned} r_{ib} &= 0.735 \text{ in. } (19 \text{ mm}) \\ a &= 48 \text{ in. } (1219 \text{ mm}) \\ h &= 1.603 \text{ in. } (41 \text{ mm}) \\ (KL/r)_o &= 70 \end{aligned}$$

Solution

1. Calculate the separation factor α.

$$\alpha = \frac{h}{2r_{ib}} = \frac{1.603}{2(0.735)} = 1.09$$

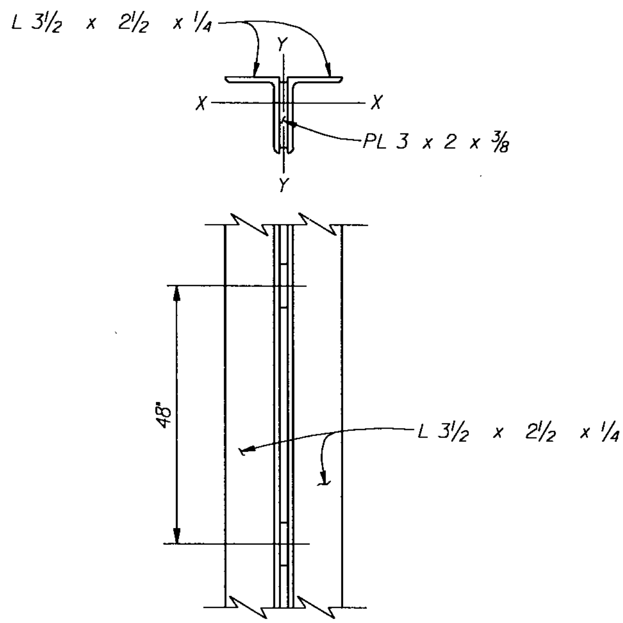


FIGURE 17.23: A built-up member with back-to-back angles.

2. Calculate the modified slenderness ratio, $(KL/r)_m$, by Equation 17.98.

$$\begin{aligned}\left(\frac{KL}{r}\right)_m &= \sqrt{\left(\frac{KL}{r}\right)_o^2 + 0.82\frac{\alpha^2}{(1+\alpha^2)}\left(\frac{a}{r_{ib}}\right)^2} \\ &= \sqrt{(70)^2 + 0.82\frac{1.09^2}{(1+1.09^2)}\left(\frac{48}{0.735}\right)^2} \\ &= 82.5\end{aligned}$$

17.10 Tapered Columns

The state-of-the-art design for tapered structural members was provided in the SSRC guide [36]. The charts shown in Figures 17.24 and 17.25 can be used to evaluate the effective length factors for tapered columns restrained by prismatic beams [36]. In these figures, I_T and I_B are the moment of inertia of the top and bottom beam, respectively; b and L are the length of beam and column, respectively; and γ is the tapering factor as defined by:

$$\gamma = \frac{d_1 - d_o}{d_o} \tag{17.99}$$

where d_o and d_1 are the section depth of column at the smaller and larger end, respectively.

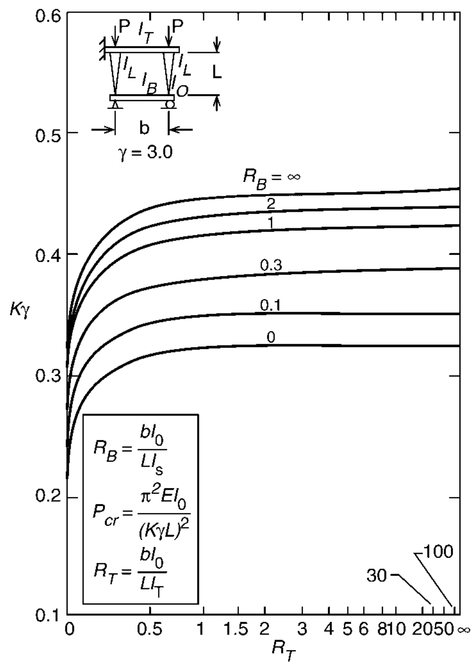


FIGURE 17.24: Effective length factor for tapered columns in braced frames.

17.11 Crane Columns

The columns in mill buildings and warehouses are designed to support overhead crane loads. The cross-section of a crane column may be uniform or stepped (see Figure 17.26). Over the past two decades, a number of simplified procedures have been developed for evaluating the K-factors for crane columns [5, 6, 7, 11, 29, 30, 41, 51, 61]. Those procedures have limitations in terms of column geometry, loading, and boundary conditions. Most importantly, most of these studies ignored the interaction effect between the left and right column of frames and were based on isolated member analyses [59]. Recently, a simple yet reasonably accurate procedure for calculating the K-factors for crane columns with any value of relative shaft length, moment of inertia, loading, and boundary conditions was developed by Lui and Sun [59]. Based on the story stiffness concept and accounting for both member and frame instability effects in the formulation, Lui and Sun [59] proposed the following procedure [see Figure 17.27]:

1. Apply the fictitious lateral loads, αP (α is an arbitrary factor; 0.001 may be used), in such a direction as to create a deflected geometry for the frame that closely approximates its actual buckled configuration.
2. Perform a first-order elastic analysis on the frame subjected to the fictitious lateral loads (Figure 17.27b). Calculate $\Delta_1/\sum H$, where Δ_1 is the average lateral deflection at the intermediate load points (i.e., points B and F) of columns, and $\sum H$ is the sum of all fictitious lateral loads that act at and above the intermediate load points.
3. Calculate η using results obtained from a first-order elastic analysis for lower shafts (i.e., segments AB and FG), according to Equation 17.81.
4. Calculate the K-factor for the lower shafts using Equation 17.80.
5. Calculate the K-factor for upper shafts using the following formula:

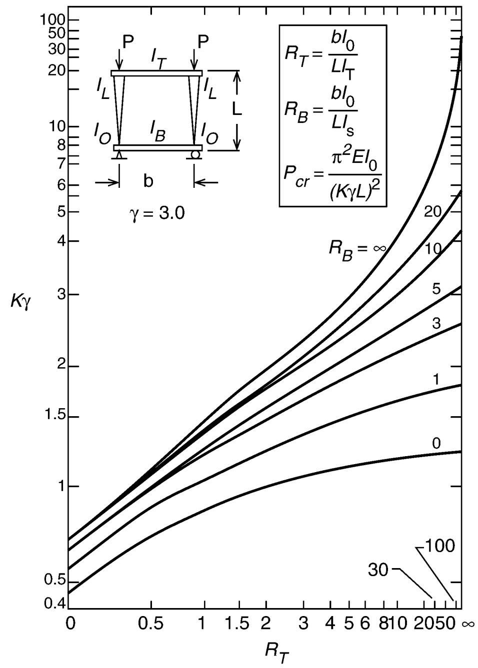


FIGURE 17.25: Effective length factor for tapered columns in unbraced frames.

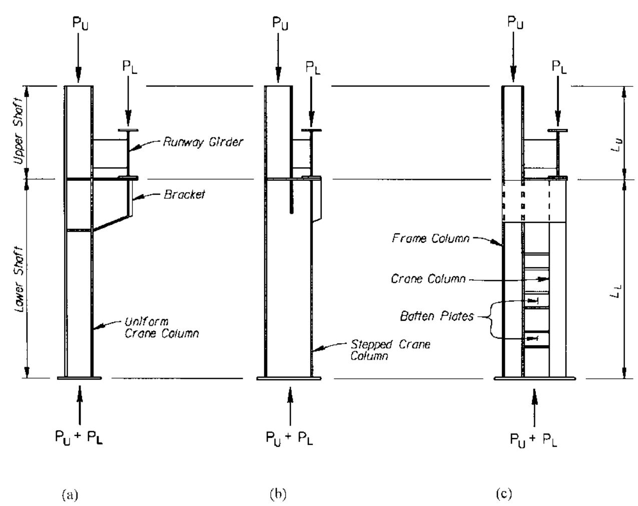
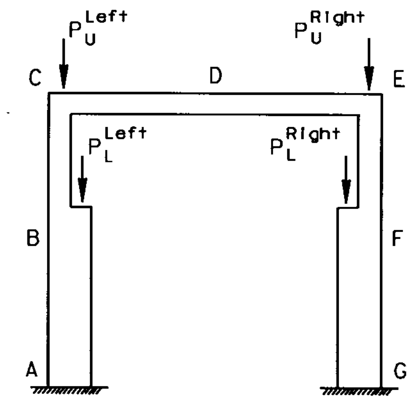
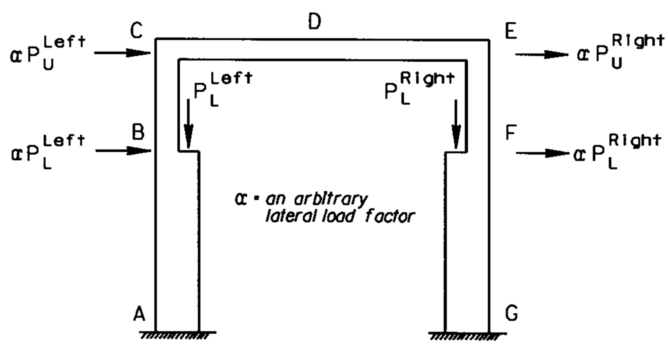


FIGURE 17.26: Typical crane columns. (From Lui, E.M. and Sun, M.Q., *AISC Eng. J.*, 32(2), 98, 1995. With permission.)



(a) Frame Subjected to Gravity Loads



(b) Frame Subjected to Fictitious Lateral Loads

FIGURE 17.27: Crane column model for effective length factor computation. (From Lui, E.M. and Sun, M.Q., *AISC Eng. J.*, 32(2), 98, 1995. With permission.)

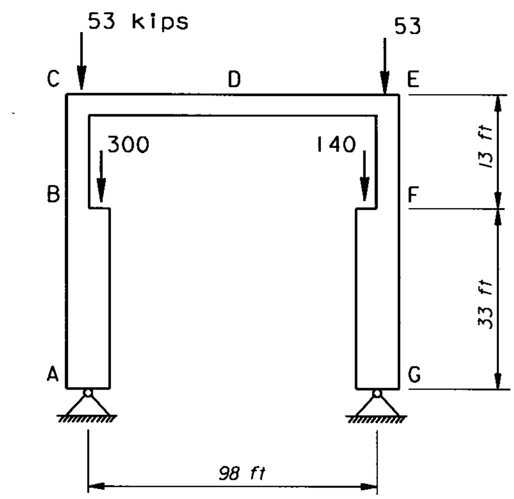
$$K_U = K_L \left(\frac{L_L}{L_U}\right)\sqrt{\left(\frac{P_L + P_U}{P_U}\right)\left(\frac{I_U}{I_L}\right)} \tag{17.100}$$

where P is the applied load and subscripts U and L represent upper and lower shafts, respectively.

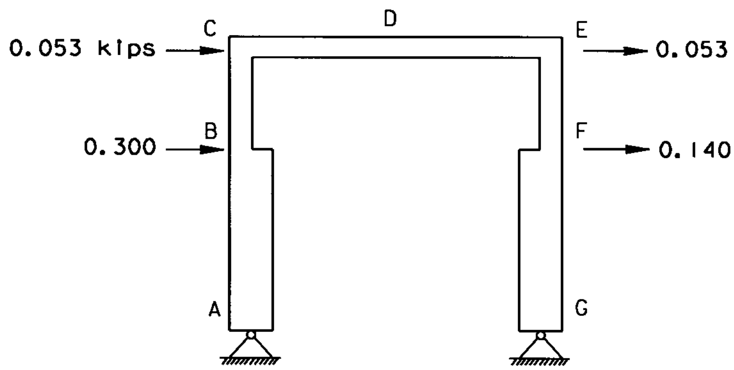
EXAMPLE 17.13:

Given: A stepped crane column is shown in Figure 17.28a. The example is the same frame as used by Fraser [30] and Lui and Sun [59]. Determine the effective length factors for all columns using the Lui approach. E = 29,000 ksi (200 GPa).

$$I_{AB} = I_{FG} = I_L = 30{,}000 \text{ in.}^4 (1.25 \times 10^{10} \text{ mm}^4)$$



(a) Frame Subjected to Gravity Loads



(b) Frame Subjected to Fictitious Lateral Loads

FIGURE 17.28: A pin-based stepped crane column. (From Lui, E.M. and Sun, M.Q., *AISC Eng. J.*, 32(2), 98, 1995. With permission.)

$$\begin{aligned} A_{AB} &= A_{FG} = A_L = 75 \text{ in.}^2 (48{,}387 \text{ mm}^2) \\ I_{BC} &= I_{EF} = I_{CE} = I_U = 5{,}420 \text{ in.}^4 (2.26 \times 10^9 \text{ mm}^4) \\ A_{BC} &= A_{EF} = A_{CE} = A_U = 34.14 \text{ in.}^2 (22{,}026 \text{ mm}^2) \end{aligned}$$

Solution

1. Apply a set of fictitious lateral forces with $\alpha = 0.001$ as shown in Figure 17.28b.

2. Perform a first-order analysis and find

$$(\Delta_1)_B = 0.1086 \text{ in.} (2.76 \text{ mm}) \text{ and } (\Delta_1)_F = 0.1077 \text{ in. } (2.74 \text{ mm})$$

so,

$$\frac{\Delta_1}{\sum H} = \frac{(0.1086 + 0.1077)/2}{0.053 + 0.3 + 0.053 + 0.14} = 0.198 \text{ in./kips } (1.131 \text{ mm/kN})$$

3. Calculate η factors from Equation 17.81.
Since the bottom of column AB and FG is pin-based, $m = 0$,

$$\begin{aligned}
\eta_{AB} &= \eta_{FG} = \frac{(3 + 4.8m + 4.2m^2)EI}{L^3} = \frac{3EI}{L^3} \\
&= \frac{(3)(29{,}000)(30{,}000)}{(396)^3} = 42.03 \text{ kips/in. } (7.36 \text{ mm/kN}) \\
\sum \eta &= 42.03 + 42.03 = 84.06 \text{ kips/in. } (14.72 \text{ mm/kN})
\end{aligned}$$

4. Calculate the K-factors for columns AB and FG using Equation 17.80.

$$\begin{aligned}
K_{AB} &= \sqrt{\left(\frac{\pi^2(29{,}000)(30{,}000)}{(353)(396)^2}\right)\left[\left(\frac{353 + 193}{396}\right)\left(\frac{1}{5(84.06)} + 0.198\right)\right]} \\
&= 6.55 \\
K_{FG} &= \sqrt{\left(\frac{\pi^2(29{,}000)(30{,}000)}{(193)(396)^2}\right)\left[\left(\frac{353 + 193}{396}\right)\left(\frac{1}{5(84.06)} + 0.198\right)\right]} \\
&= 8.85
\end{aligned}$$

5. Calculate the K-factors for columns BC and EF using Equation 17.100.

$$\begin{aligned}
K_{BC} &= K_{AB}\left(\frac{L_{AB}}{L_{BC}}\right)\sqrt{\left(\frac{P_{AB} + P_{BC}}{P_{BC}}\right)\left(\frac{I_{BC}}{I_{AB}}\right)} \\
&= 6.55\left(\frac{396}{156}\right)\sqrt{\left(\frac{353}{53}\right)\left(\frac{5420}{30{,}000}\right)} = 18.2 \\
K_{EF} &= K_{FG}\left(\frac{L_{FG}}{L_{EF}}\right)\sqrt{\left(\frac{P_{FG} + P_{EF}}{P_{EF}}\right)\left(\frac{I_{EF}}{I_{FG}}\right)} \\
&= 8.85\left(\frac{396}{156}\right)\sqrt{\left(\frac{193}{53}\right)\left(\frac{5420}{30{,}000}\right)} = 18.2
\end{aligned}$$

The K-factors calculated above are in good agreement with the theoretical values reported by Lui and Sun [59].

17.12 Columns in Gable Frames

For a pin-based gable frame subjected to a uniformly distributed load on the rafter, as shown in Figure 17.29a, Lu [56] presented a graph (Figure 17.29b) to determine the effective length factors of columns. For frames having different member sizes for rafter and columns with (L/h) of (f/h)

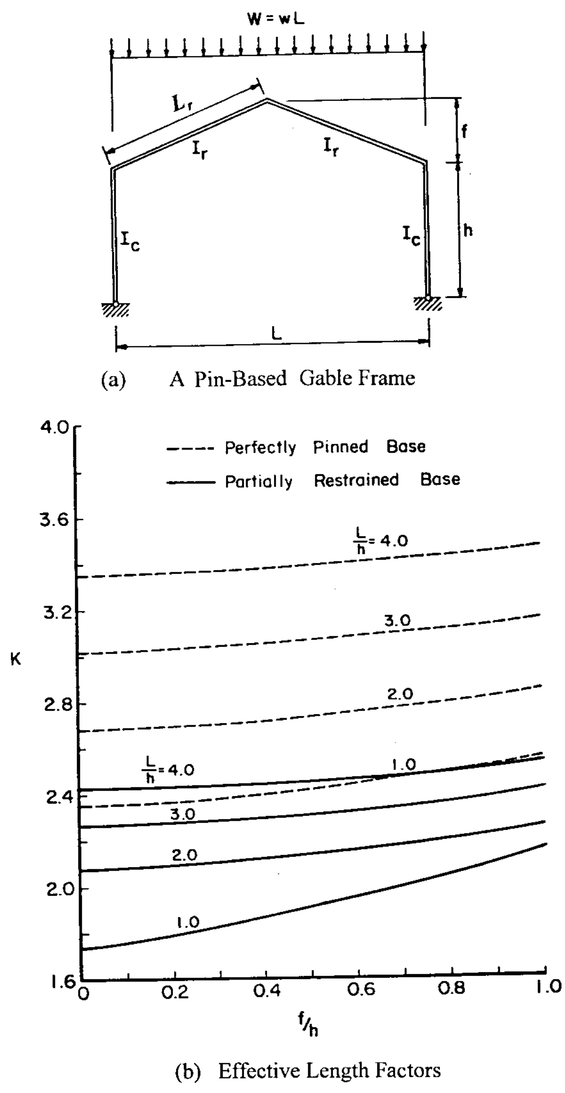


FIGURE 17.29: Effective length factors for columns in a pin-based gable frame. (From Lu, L.W., *AISC Eng. J.*, 2(2), 6, 1965. With permission.)

ratios not covered in Figure 17.29, an approximate method is available for determining K-factors of columns [39]. The method is to find an equivalent portal frame whose span length is equal to twice the rafter length, L_r (see Figure 17.29a). The K-factors can be determined form the alignment charts using $G_{\text{top}} = \frac{I_c/h}{I_r/2L_r}$ and corresponding G_{bottom}.

17.13 Summary

This chapter summarizes the state-of-the-art practice of the effective length factors for isolated columns, framed columns, diagonal bracing systems, latticed and built-up members, tapered columns, crane columns, and columns in gable frames. Design implementation with formulas, charts, tables, various modification factors adopted in current codes and specifications, as well as those used in engineering practice are described. Several examples illustrate the steps of practical applications of various methods.

17.14 Defining Terms

Alignment chart: A monograph for determining the effective length factor K for some types of compression members.

Braced frame: A frame in which the resistance to lateral load or frame instability is primarily provided by diagonal bracing, shear walls, or equivalent means.

Build-up member: A member made of structural metal elements that are welded, bolted, and riveted together.

Column: A structural member whose primary function is to carry loads parallel to its longitudinal axis.

Crane column: A column that is designed to support overhead crane loads.

Effective length factor K: A factor that when multiplied by actual length of the end-restrained column gives the length of an equivalent pin-ended column whose elastic buckling load is the same as that of the end-restrained column.

Framed column: A column in a framed structure.

Gable frame: A frame with a gabled roof.

Latticed member: A member made of two or more rolled-shapes that are connected to one another by means of lacing bars, batten plates, or perforated plates.

Leaning column: A column that is connected to a frame with simple connections and does not provide lateral stiffness or sidesway resistance.

LRFD (Load and Resistance Factor Design): A method of proportioning structural components (members, connectors, connecting elements, and assemblages) such that no applicable limit state is exceeded when the structure is subjected to all appropriate load combinations.

Tapered column: A column that has a continuous reduction in section from top to bottom.

Unbraced frame: A frame in which the resistance to lateral loads is provided by the bending stiffness of frame members and their connections.

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