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Basic Theory of Plates and Elastic Stability

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1.1 Introduction

This chapter is concerned with basic assumptions and equations of plates and basic concepts of elastic stability. Herein, we shall illustrate the concepts and the applications of these equations by means of relatively simple examples; more complex applications will be taken up in the following chapters.

1.2 Plates

1.2.1 Basic Assumptions

We consider a continuum shown in Figure 1.1. A feature of the body is that one dimension is much smaller than the other two dimensions:

$$t << L_x, L_y \tag{1.1}$$

where t, L_x, and L_y are representative dimensions in three directions (Figure 1.1). If the continuum has this geometrical characteristic of Equation 1.1 and is flat before loading, it is called a plate. Note that a shell possesses a similar geometrical characteristic but is curved even before loading.

The characteristic of Equation 1.1 lends itself to the following assumptions regarding some stress and strain components:

$$\sigma_z = 0 \tag{1.2}$$

$$\varepsilon_z = \varepsilon_{xz} = \varepsilon_{yz} = 0 \tag{1.3}$$

We can derive the following displacement field from Equation 1.3:

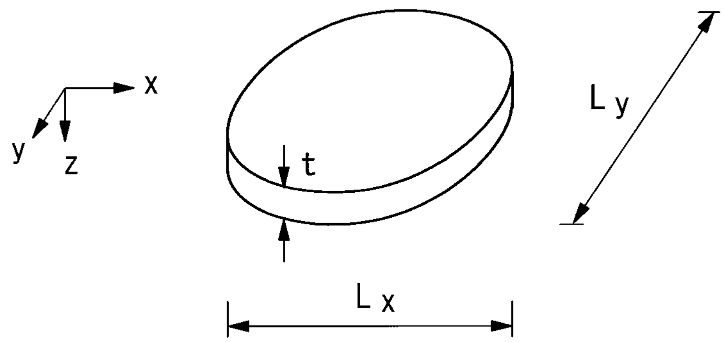

FIGURE 1.1: Plate.

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) - z\frac{\partial w_0}{\partial x} \\
v(x,y,z) &= v_0(x,y) - z\frac{\partial w_0}{\partial y} \\
w(x,y,z) &= w_0(x,y)
\end{aligned}
\tag{1.4}
$$

where u, v, and w are displacement components in the directions of x-, y-, and z-axes, respectively. As can be realized in Equation 1.4, u_0 and v_0 are displacement components associated with the plane of $z = 0$. Physically, Equation 1.4 implies that the linear filaments of the plate initially perpendicular to the middle surface remain straight and perpendicular to the deformed middle surface. This is known as the Kirchhoff hypothesis. Although we have derived Equation 1.4 from Equation 1.3 in the above, one can arrive at Equation 1.4 starting with the Kirchhoff hypothesis: the Kirchhoff hypothesis is equivalent to the assumptions of Equation 1.3.

1.2.2 Governing Equations

Strain-Displacement Relationships

Using the strain-displacement relationships in the continuum mechanics, we can obtain the following strain field associated with Equation 1.4:

$$
\begin{aligned}
\varepsilon_x &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2} \\
\varepsilon_y &= \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2} \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - z\frac{\partial^2 w_0}{\partial x \partial y}
\end{aligned}
\tag{1.5}
$$

This constitutes the strain-displacement relationships for the plate theory.

Equilibrium Equations

In the plate theory, equilibrium conditions are considered in terms of resultant forces and moments. This is derived by integrating the equilibrium equations over the thickness of a plate. Because of Equation 1.2, we obtain the equilibrium equations as follows:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q_x = 0 \tag{1.6a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + q_y = 0 \tag{1.6b}$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q_z = 0 \tag{1.6c}$$

where N_x, N_y, and N_{xy} are in-plane stress resultants; V_x and V_y are shearing forces; and q_x, q_y, and q_z are distributed loads per unit area. The terms associated with τ_{xz} and τ_{yz} vanish, since in the plate problems the top and the bottom surfaces of a plate are subjected to only vertical loads.

We must also consider the moment equilibrium of an infinitely small region of the plate, which leads to

$$\begin{aligned} \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - V_x &= 0 \\ \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - V_y &= 0 \end{aligned} \tag{1.7}$$

where M_x and M_y are bending moments and M_{xy} is a twisting moment.

The resultant forces and the moments are defined mathematically as

$$N_x = \int_z \sigma_x dz \tag{1.8a}$$

$$N_y = \int_z \sigma_y dz \tag{1.8b}$$

$$N_{xy} = N_{yx} = \int_z \tau_{xy} dz \tag{1.8c}$$

$$V_x = \int_z \tau_{xz} dz \tag{1.8d}$$

$$V_y = \int_z \tau_{yz} dz \tag{1.8e}$$

$$M_x = \int_z \sigma_x z dz \tag{1.8f}$$

$$M_y = \int_z \sigma_y z dz \tag{1.8g}$$

$$M_{xy} = M_{yx} = \int_z \tau_{xy} z dz \tag{1.8h}$$

The resultant forces and the moments are illustrated in Figure 1.2.

Constitutive Equations

Since the thickness of a plate is small in comparison with the other dimensions, it is usually accepted that the constitutive relations for a state of plane stress are applicable. Hence, the stress-strain relationships for an isotropic plate are given by

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = \frac{E}{1-\nu^2} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{array} \right] \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\} \tag{1.9}$$

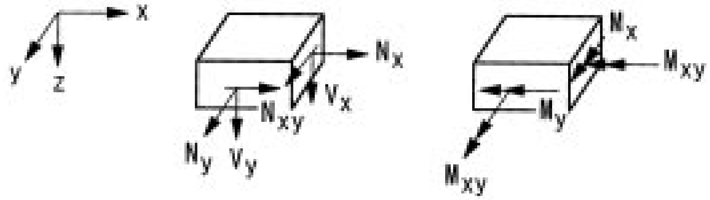

FIGURE 1.2: Resultant forces and moments.

where E and ν are Young's modulus and Poisson's ratio, respectively. Using Equations 1.5, 1.8, and 1.9, the constitutive relationships for an isotropic plate in terms of stress resultants and displacements are described by

$$N_x = \frac{Et}{1-\nu^2}\left(\frac{\partial u_0}{\partial x} + \nu\frac{\partial v_0}{\partial y}\right) \tag{1.10a}$$

$$N_y = \frac{Et}{1-\nu^2}\left(\frac{\partial v_0}{\partial y} + \nu\frac{\partial u_0}{\partial x}\right) \tag{1.10b}$$

$$N_{xy} = N_{yx}\frac{Et}{2(1+\nu)}\left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y}\right) \tag{1.10c}$$

$$M_x = -D\left(\frac{\partial^2 w_0}{\partial x^2} + \nu\frac{\partial^2 w_0}{\partial y^2}\right) \tag{1.10d}$$

$$M_y = -D\left(\frac{\partial^2 w_0}{\partial y^2} + \nu\frac{\partial^2 w_0}{\partial x^2}\right) \tag{1.10e}$$

$$M_{xy} = M_{yx} = -(1-\nu)D\frac{\partial^2 w_0}{\partial x \partial y} \tag{1.10f}$$

where t is the thickness of a plate and D is the flexural rigidity defined by

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{1.11}$$

In the derivation of Equation 1.10, we have assumed that the plate thickness t is constant and that the initial middle surface lies in the plane of $Z = 0$. Through Equation 1.7, we can relate the shearing forces to the displacement.

Equations 1.6, 1.7, and 1.10 constitute the framework of a plate problem: 11 equations for 11 unknowns, i.e., N_x, N_y, N_{xy}, M_x, M_y, M_{xy}, V_x, V_y, u_0, v_0, and w_0. In the subsequent sections, we shall drop the subscript 0 that has been associated with the displacements for the sake of brevity.

In-Plane and Out-Of-Plane Problems

As may be realized in the equations derived in the previous section, the problem can be decomposed into two sets of problems which are uncoupled with each other.

1. In-plane problems
   The problem may be also called a stretching problem of a plate and is governed by

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + q_x = 0$$
$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + q_y = 0 \qquad \text{(1.6a,b)}$$

$$\begin{aligned} N_x &= \frac{Et}{1-\nu^2}\left(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right) \\ N_y &= \frac{Et}{1-\nu^2}\left(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right) \\ N_{xy} &= N_{yx} = \frac{Et}{2(1+\nu)}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \end{aligned} \qquad \text{(1.10a}\sim\text{c)}$$

Here we have five equations for five unknowns. This problem can be viewed and treated in the same way as for a plane-stress problem in the theory of two-dimensional elasticity.

2. Out-of-plane problems
This problem is regarded as a bending problem and is governed by

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q_z = 0 \qquad \text{(1.6c)}$$
$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - V_x = 0$$
$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - V_y = 0 \qquad \text{(1.7)}$$

$$\begin{aligned} M_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \\ M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial x^2}\right) \\ M_{xy} &= M_{yx} = -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \qquad \text{(1.10d}\sim\text{f)}$$

Here are six equations for six unknowns.

Eliminating V_x and V_y from Equations 1.6c and 1.7, we obtain

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q_z = 0 \qquad \text{(1.12)}$$

Substituting Equations 1.10d∼f into the above, we obtain the governing equation in terms of displacement as

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = q_z \qquad \text{(1.13)}$$

or

$$\nabla^4 w = \frac{q_z}{D} \tag{1.14}$$

where the operator is defined as

$$\begin{aligned} \nabla^4 &= \nabla^2\nabla^2 \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned} \tag{1.15}$$

1.2.3 Boundary Conditions

Since the in-plane problem of a plate can be treated as a plane-stress problem in the theory of two-dimensional elasticity, the present section is focused solely on a bending problem.

Introducing the n-s-z coordinate system alongside boundaries as shown in Figure 1.3, we define the moments and the shearing force as

$$\begin{aligned} M_n &= \int_z \sigma_n z dz \\ M_{ns} &= M_{sn} = \int_z \tau_{ns} z dz \\ V_n &= \int_z \tau_{nz} dz \end{aligned} \tag{1.16}$$

In the plate theory, instead of considering these three quantities, we combine the twisting moment and the shearing force by replacing the action of the twisting moment M_{ns} with that of the shearing force, as can be seen in Figure 1.4. We then define the joint vertical as

$$S_n = V_n + \frac{\partial M_{ns}}{\partial s} \tag{1.17}$$

The boundary conditions are therefore given in general by

$$w = \overline{w} \text{ or } S_n = \overline{S}_n \tag{1.18}$$

$$-\frac{\partial w}{\partial n} = \overline{\lambda}_n \text{ or } M_n = \overline{M}_n \tag{1.19}$$

where the quantities with a bar are prescribed values and are illustrated in Figure 1.5. These two sets of boundary conditions ensure the unique solution of a bending problem of a plate.

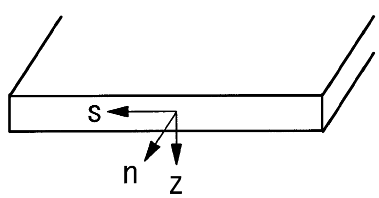

FIGURE 1.3: n-s-z coordinate system.

The boundary conditions for some practical cases are as follows:

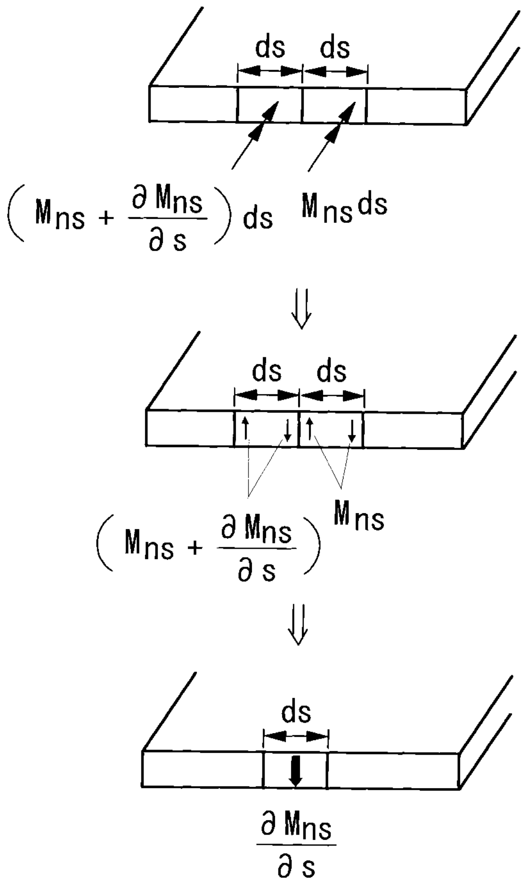

FIGURE 1.4: Shearing force due to twisting moment.

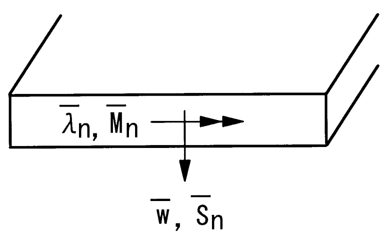

FIGURE 1.5: Prescribed quantities on the boundary.

1. Simply supported edge

$$w = 0, \quad M_n = \overline{M}_n \tag{1.20}$$

2. Built-in edge

$$w = 0, \quad \frac{\partial w}{\partial n} = 0 \tag{1.21}$$

3. Free edge

$$M_n = \overline{M}_n, \quad S_n = \overline{S}_n \tag{1.22}$$

4. Free corner (intersection of free edges)
   At the free corner, the twisting moments cause vertical action, as can be realized is Figure 1.6. Therefore, the following condition must be satisfied:

$$-2M_{xy} = \overline{P} \tag{1.23}$$

   where $\overline{P}$ is the external concentrated load acting in the Z direction at the corner.

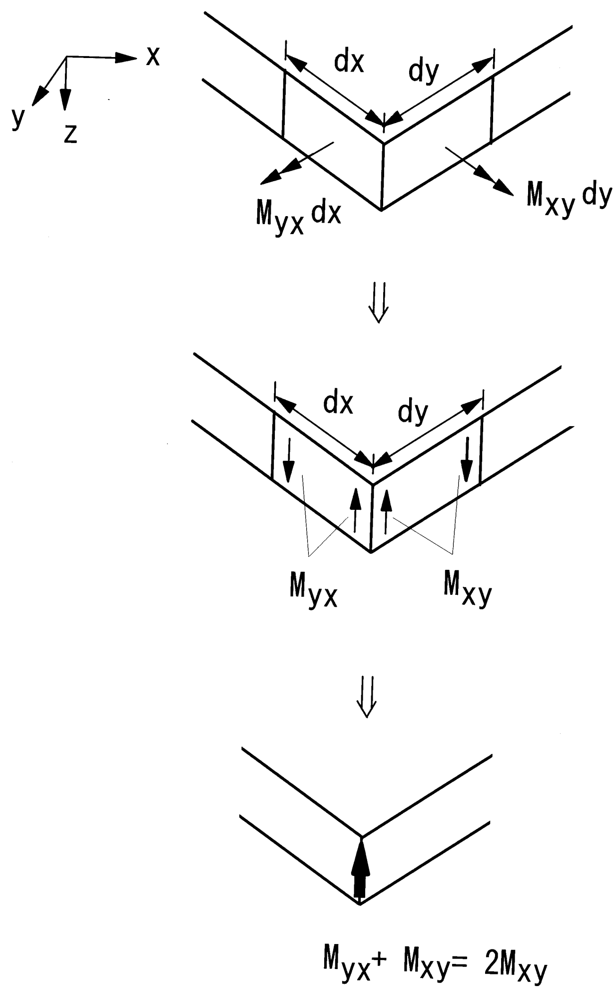

FIGURE 1.6: Vertical action at the corner due to twisting moment.

1.2.4 Circular Plate

Governing equations in the cylindrical coordinates are more convenient when circular plates are dealt with. Through the coordinate transformation, we can easily derive the Laplacian operator in the cylindrical coordinates and the equation that governs the behavior of the bending of a circular plate:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)w=\frac{q_z}{D} \tag{1.24}$$

The expressions of the resultants are given by

$$\begin{aligned}
M_r &= -D\left[(1-\nu)\frac{\partial^2 w}{\partial r^2}+\nu\nabla^2 w\right]\\
M_\theta &= -D\left[\nabla^2 w+(1-\nu)\frac{\partial^2 w}{\partial r^2}\right]\\
M_{r\theta} &= M_{\theta r}=-D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\\
S_r &= V_r+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta}\\
S_\theta &= V_\theta+\frac{\partial M_{r\theta}}{\partial r}
\end{aligned} \tag{1.25}$$

When the problem is axisymmetric, the problem can be simplified because all the variables are independent of θ. The governing equation for the bending behavior and the moment-deflection relationships then become

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right]=\frac{q_z}{D} \tag{1.26}$$

$$\begin{aligned}
M_r &= D\left(\frac{d^2 w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right)\\
M_\theta &= D\left(\frac{1}{r}\frac{dw}{dr}+\nu\frac{d^2 w}{dr^2}\right)\\
M_{r\theta} &= M_{\theta r}=0
\end{aligned} \tag{1.27}$$

Since the twisting moment does not exist, no particular care is needed about vertical actions.

1.2.5 Examples of Bending Problems

Simply Supported Rectangular Plate Subjected to Uniform Load

A plate shown in Figure 1.7 is considered here. The governing equation is given by

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q_0}{D} \tag{1.28}$$

in which q_0 represents the intensity of the load. The boundary conditions for the plate are

$$\begin{aligned}
w &= 0, \quad M_x=0 \;\text{ along }\; x=0,a\\
w &= 0, \quad M_y=0 \;\text{ along }\; y=0,b
\end{aligned} \tag{1.29}$$

Using Equation 1.10, we can rewrite the boundary conditions in terms of displacement. Furthermore, since $w = 0$ along the edges, we observe $\frac{\partial^2 w}{\partial x^2} = 0$ and $\frac{\partial^2 w}{\partial y^2} = 0$ for the edges parallel to the x and y axes, respectively, so that we may describe the boundary conditions as

$$\begin{aligned} w &= 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{along} \quad x = 0, a \\ w &= 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{along} \quad y = 0, b \end{aligned} \tag{1.30}$$

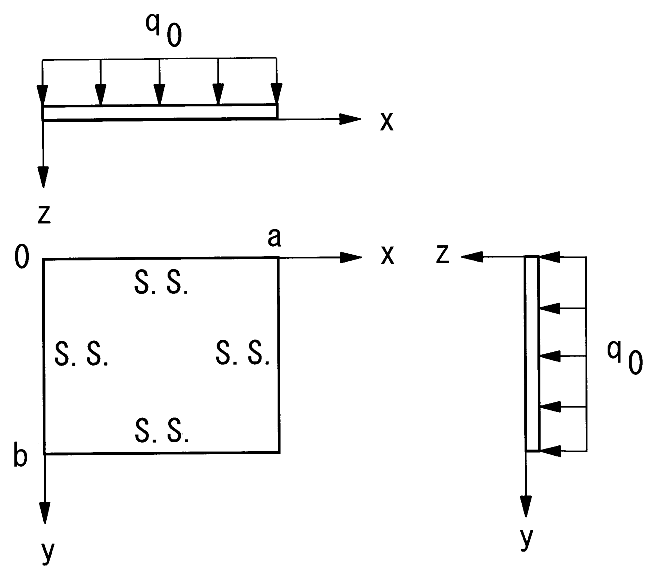

FIGURE 1.7: Simply supported rectangular plate subjected to uniform load.

We represent the deflection in the double trigonometric series as

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{1.31}$$

It is noted that this function satisfies all the boundary conditions of Equation 1.30. Similarly, we express the load intensity as

$$q_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{1.32}$$

where

$$B_{mn} = \frac{16 q_0}{\pi^2 mn} \tag{1.33}$$

Substituting Equations 1.31 and 1.32 into 1.28, we can obtain the expression of A_{mn} to yield

$$w = \frac{16 q_0}{\pi^6 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{1.34}$$

We can readily obtain the expressions for bending and twisting moments by differentiation.

Axisymmetric Circular Plate with Built-In Edge Subjected to Uniform Load

The governing equation of the plate shown in Figure 1.8 is

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right]=\frac{q_0}{D} \tag{1.35}$$

where q_0 is the intensity of the load. The boundary conditions for the plate are given by

$$w=\frac{dw}{dr}=0 \text{ at } r=a \tag{1.36}$$

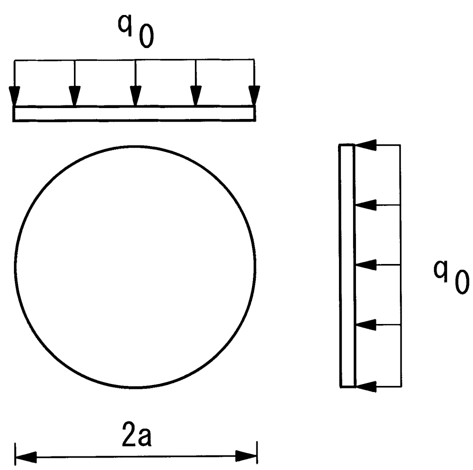

FIGURE 1.8: Circular plate with built-in edge subjected to uniform load.

We can solve Equation 1.35 without much difficulty to yield the following general solution:

$$w=\frac{q_0 r^4}{64D}+A_1 r^2\ln r+A_2\ln r+A_3 r^2+A_4 \tag{1.37}$$

We have four constants of integration in the above, while there are only two boundary conditions of Equation 1.36. Claiming that no singularities should occur in deflection and moments, however, we can eliminate A_1 and A_2, so that we determine the solution uniquely as

$$w=\frac{q_0 a^4}{64D}\left(\frac{r^2}{a^2}-1\right)^2 \tag{1.38}$$

Using Equation 1.27, we can readily compute the bending moments.

1.3 Stability

1.3.1 Basic Concepts

States of Equilibrium

To illustrate various forms of equilibrium, we consider three cases of equilibrium of the ball shown in Figure 1.9. We can easily see that if it is displaced slightly, the ball on the concave spherical surface will return to its original position upon the removal of the disturbance. On the other hand, the ball on the convex spherical surface will continue to move farther away from the original position if displaced slightly. A body that behaves in the former way is said to be in a state of stable equilibrium, while the latter is called unstable equilibrium. The ball on the horizontal plane shows yet another behavior: it remains at the position to which the small disturbance has taken it. This is called a state of neutral equilibrium.

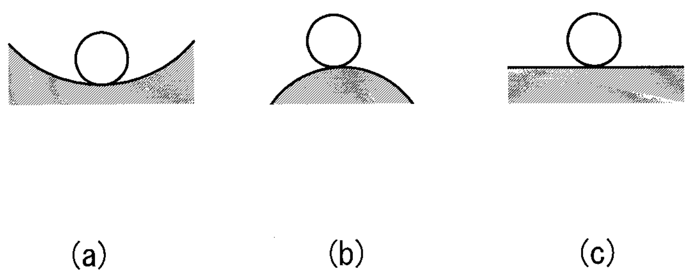

FIGURE 1.9: Three states of equilibrium.

For further illustration, we consider a system of a rigid bar and a linear spring. The vertical load P is applied at the top of the bar as depicted in Figure 1.10. When small disturbance θ is given, we can compute the moment about Point B M_B, yielding

$$\begin{aligned} M_B &= PL\sin\theta - (kL\sin\theta)(L\cos\theta) \\ &= L\sin\theta(P - kL\cos\theta) \end{aligned} \tag{1.39}$$

Using the fact that θ is infinitesimal, we can simplify Equation 1.39 as

$$\frac{M_B}{\theta} = L(P - kL) \tag{1.40}$$

We can claim that the system is stable when M_B acts in the opposite direction of the disturbance θ; that it is unstable when M_B and θ possess the same sign; and that it is in a state of neutral equilibrium when M_B vanishes. This classification obviously shares the same physical definition as that used in the first example (Figure 1.9). Mathematically, the classification is expressed as

$$(P - kL)\begin{cases} < 0 & : \text{stable} \\ = 0 & : \text{neutral} \\ > 0 & : \text{unstable} \end{cases} \tag{1.41}$$

Equation 1.41 implies that as P increases, the state of the system changes from stable equilibrium to unstable equilibrium. The critical load is kL, at which multiple equilibrium positions, i.e., $\theta = 0$ and $\theta \neq 0$, are possible. Thus, the critical load serves also as a bifurcation point of the equilibrium path. The load at such a bifurcation is called the buckling load.

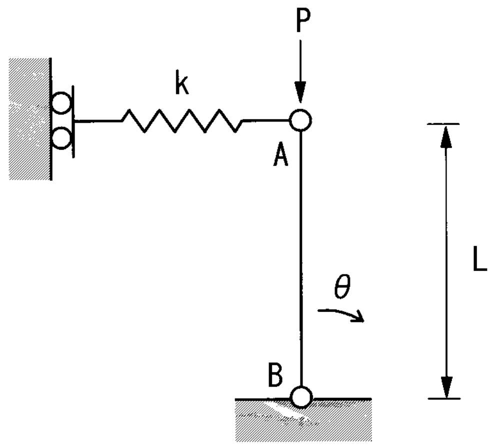

FIGURE 1.10: Rigid bar AB with a spring.

For the present system, the buckling load of kL is stability limit as well as neutral equilibrium. In general, the buckling load corresponds to a state of neutral equilibrium, but not necessarily to stability limit. Nevertheless, the buckling load is often associated with the characteristic change of structural behavior, and therefore can be regarded as the limit state of serviceability.

Linear Buckling Analysis

We can compute a buckling load by considering an equilibrium condition for a slightly deformed state. For the system of Figure 1.10, the moment equilibrium yields

$$PL\sin\theta - (kL\sin\theta)(L\cos\theta) = 0 \tag{1.42}$$

Since θ is infinitesimal, we obtain

$$L\theta(P - kL) = 0 \tag{1.43}$$

It is obvious that this equation is satisfied for any value of P if θ is zero: $\theta = 0$ is called the trivial solution. We are seeking the buckling load, at which the equilibrium condition is satisfied for $\theta \neq 0$. The trivial solution is apparently of no importance and from Equation 1.43 we can obtain the following buckling load P_C:

$$P_C = kL \tag{1.44}$$

A rigorous buckling analysis is quite involved, where we need to solve nonlinear equations even when elastic problems are dealt with. Consequently, the linear buckling analysis is frequently employed. The analysis can be justified, if deformation is negligible and structural behavior is linear before the buckling load is reached. The way we have obtained Equation 1.44 in the above is a typical application of the linear buckling analysis.

In mathematical terms, Equation 1.43 is called a characteristic equation and Equation 1.44 an eigenvalue. The linear buckling analysis is in fact regarded as an eigenvalue problem.

1.3.2 Structural Instability

Three classes of instability phenomenon are observed in structures: bifurcation, snap-through, and softening.

We have discussed a simple example of bifurcation in the previous section. Figure 1.11a depicts a schematic load-displacement relationship associated with the bifurcation: Point A is where the bifurcation takes place. In reality, due to imperfection such as the initial crookedness of a member and the eccentricity of loading, we can rarely observe the bifurcation. Instead, an actual structural behavior would be more like the one indicated in Figure 1.11a. However, the bifurcation load is still an important measure regarding structural stability and most instabilities of a column and a plate are indeed of this class. In many cases we can evaluate the bifurcation point by the linear buckling analysis.

In some structures, we observe that displacement increases abruptly at a certain load level. This can take place at Point A in Figure 1.11b; displacement increases from U_A to U_B at P_A, as illustrated by a broken arrow. The phenomenon is called snap-through. The equilibrium path of Figure 1.11b is typical of shell-like structures, including a shallow arch, and is traceable only by the finite displacement analysis.

The other instability phenomenon is the softening: as Figure 1.11c illustrates, there exists a peak load-carrying capacity, beyond which the structural strength deteriorates. We often observe this phenomenon when yielding takes place. To compute the associated equilibrium path, we need to resort to nonlinear structural analysis.

Since nonlinear analysis is complicated and costly, the information on stability limit and ultimate strength is deduced in practice from the bifurcation load, utilizing the linear buckling analysis. We shall therefore discuss the buckling loads (bifurcation points) of some structures in what follows.

1.3.3 Columns

Simply Supported Column

As a first example, we evaluate the buckling load of a simply supported column shown in Figure 1.12a. To this end, the moment equilibrium in a slightly deformed configuration is considered. Following the notation in Figure 1.12b, we can readily obtain

$$w'' + k^2 w = 0 \tag{1.45}$$

where

$$k^2 = \frac{P}{EI} \tag{1.46}$$

EI is the bending rigidity of the column. The general solution of Equation 1.45 is

$$w = A_1 \sin kx + A_2 \cos kx \tag{1.47}$$

The arbitrary constants A_1 and A_2 are to be determined by the following boundary conditions:

$$w = 0 \text{ at } x = 0 \tag{1.48a}$$
$$w = 0 \text{ at } x = L \tag{1.48b}$$

Equation 1.48a gives $A_2 = 0$ and from Equation 1.48b we reach

$$A_1 \sin kL = 0 \tag{1.49}$$

$A_1 = 0$ is a solution of the characteristic equation above, but this is the trivial solution corresponding to a perfectly straight column and is of no interest. Then we obtain the following buckling loads:

$$P_C = \frac{n^2 \pi^2 EI}{L^2} \tag{1.50}$$

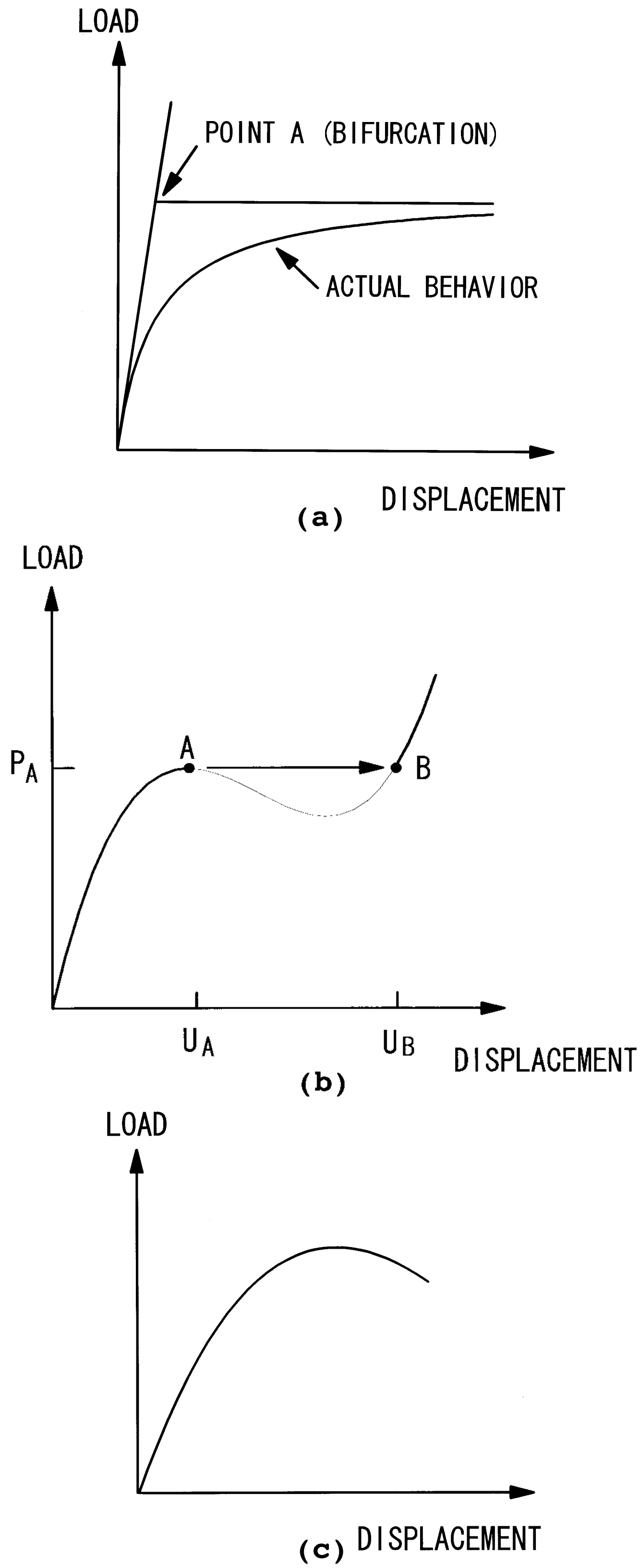

FIGURE 1.11: Unstable structural behaviors.

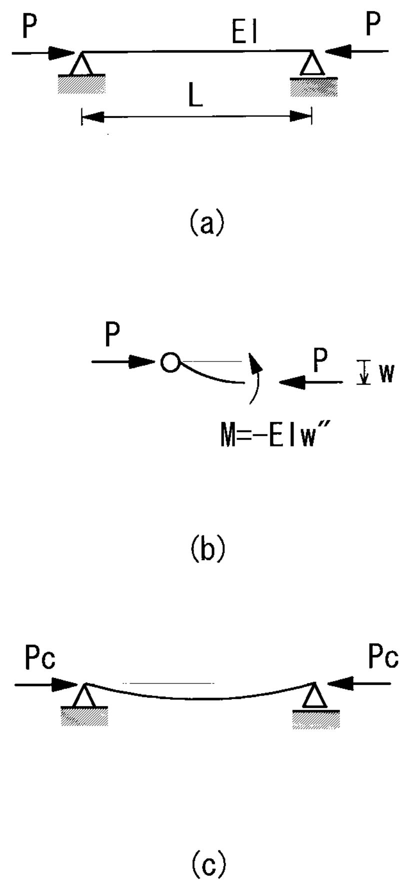

FIGURE 1.12: Simply-supported column.

Although n is any integer, our interest is in the lowest buckling load with $n = 1$ since it is the critical load from the practical point of view. The buckling load, thus, obtained is

$$P_C = \frac{\pi^2 EI}{L^2} \tag{1.51}$$

which is often referred to as the Euler load. From $A_2 = 0$ and Equation 1.51, Equation 1.47 indicates the following shape of the deformation:

$$w = A_1 \sin \frac{\pi x}{L} \tag{1.52}$$

This equation shows the buckled shape only, since A_1 represents the undetermined amplitude of the deflection and can have any value. The deflection curve is illustrated in Figure 1.12c.

The behavior of the simply supported column is summarized as follows: up to the Euler load the column remains straight; at the Euler load the state of the column becomes the neutral equilibrium and it can remain straight or it starts to bend in the mode expressed by Equation 1.52.

Cantilever Column

For the cantilever column of Figure 1.13a, by considering the equilibrium condition of the free body shown in Figure 1.13b, we can derive the following governing equation:

$$w'' + k^2 w = k^2 \delta \tag{1.53}$$

where δ is the deflection at the free tip. The boundary conditions are

$$\begin{aligned} w &= 0 \text{ at } x = 0 \\ w' &= 0 \text{ at } x = 0 \\ w &= \delta \text{ at } x = L \end{aligned} \tag{1.54}$$

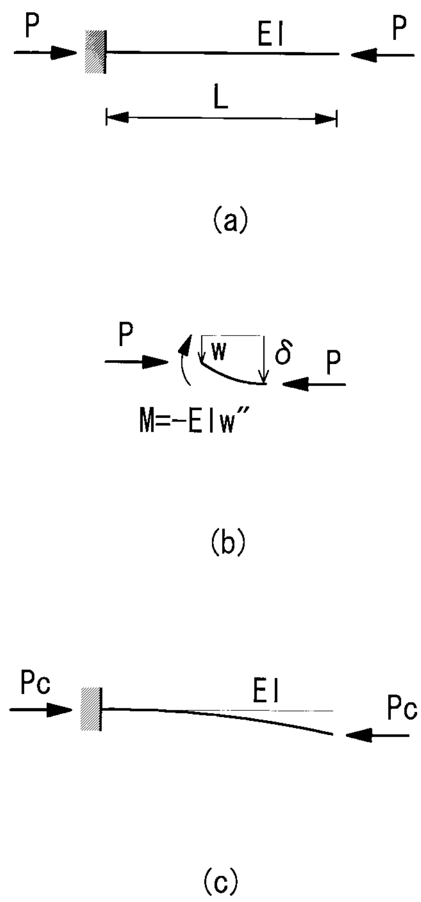

FIGURE 1.13: Cantilever column.

From these equations we can obtain the characteristic equation as

$$\delta \cos kL = 0 \tag{1.55}$$

which yields the following buckling load and deflection shape:

$$P_C = \frac{\pi^2 EI}{4L^2} \tag{1.56}$$

$$w = \delta\left(1 - \cos\frac{\pi x}{2L}\right) \tag{1.57}$$

The buckling mode is illustrated in Figure 1.13c. It is noted that the boundary conditions make much difference in the buckling load: the present buckling load is just a quarter of that for the simply supported column.

Higher-Order Differential Equation

We have thus far analyzed the two columns. In each problem, a second-order differential equation was derived and solved. This governing equation is problem-dependent and valid only for a particular problem. A more consistent approach is possible by making use of the governing equation for a beam-column with no laterally distributed load:

$$EIw^{IV} + Pw'' = q \tag{1.58}$$

Note that in this equation P is positive when compressive. This equation is applicable to any set of boundary conditions. The general solution of Equation 1.58 is given by

$$w = A_1 \sin kx + A_2 \cos kx + A_3 x + A_4 \tag{1.59}$$

where $A_1 \sim A_4$ are arbitrary constants and determined from boundary conditions.

We shall again solve the two column problems, using Equation 1.58.

1. Simply supported column (Figure 1.12a)
   Because of no deflection and no external moment at each end of the column, the boundary conditions are described as

$$\begin{aligned} w &= 0, \quad w'' = 0 \ \text{ at } x = 0 \\ w &= 0, \quad w'' = 0 \ \text{ at } x = L \end{aligned} \tag{1.60}$$

   From the conditions at $x = 0$, we can determine

$$A_2 = A_4 = 0 \tag{1.61}$$

   Using this result and the conditions at $x = L$, we obtain

$$\begin{bmatrix} \sin kL & L \\ -k^2 \sin kL & 0 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1.62}$$

   For the nontrivial solution to exist, the determinant of the coefficient matrix in Equation 1.62 must vanish, leading to the following characteristic equation:

$$k^2 L \sin kL = 0 \tag{1.63}$$

   from which we arrive at the same critical load as in Equation 1.51. By obtaining the corresponding eigenvector of Equation 1.62, we can get the buckled shape of Equation 1.52.

2. Cantilever column (Figure 1.13a)
   In this column, we observe no deflection and no slope at the fixed end; no external moment and no external shear force at the free end. Therefore, the boundary conditions are

$$\begin{array}{lll} w = 0, & w' = 0 & \text{at } x = 0 \\ w'' = 0, & w''' + k^2 w' = 0 & \text{at } x = L \end{array} \tag{1.64}$$

   Note that since we are dealing with a slightly deformed column in the linear buckling analysis, the axial force has a transverse component, which is why P comes in the boundary condition at $x = L$.

The latter condition at $x = L$ eliminates A_3. With this and the second condition at $x = 0$, we can claim $A_1 = 0$. The remaining two conditions then lead to

$$\begin{bmatrix} 1 & 1 \\ k^2 \cos kL & 0 \end{bmatrix} \begin{Bmatrix} A_2 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1.65}$$

The smallest eigenvalue and the corresponding eigenvector of Equation 1.65 coincide with the buckling load and the buckling mode that we have obtained previously in Section 1.3.3.

Effective Length

We have obtained the buckling loads for the simply supported and the cantilever columns. By either the second- or the fourth-order differential equation approach, we can compute buckling loads for a fixed-hinged column (Figure 1.14a) and a fixed-fixed column (Figure 1.14b) without much difficulty:

$$\begin{aligned} P_C &= \frac{\pi^2 EI}{(0.7L)^2} \quad \text{for a fixed - hinged column} \\ P_C &= \frac{\pi^2 EI}{(0.5L)^2} \quad \text{for a fixed - hinged column} \end{aligned} \tag{1.66}$$

For all the four columns considered thus far, and in fact for the columns with any other sets of boundary conditions, we can express the buckling load in the form of

$$P_C = \frac{\pi^2 EI}{(KL)^2} \tag{1.67}$$

where KL is called the effective length and represents presumably the length of the equivalent Euler column (the equivalent simply supported column).

For design purposes, Equation 1.67 is often transformed into

$$\sigma_C = \frac{\pi^2 E}{(KL/r)^2} \tag{1.68}$$

where r is the radius of gyration defined in terms of cross-sectional area A and the moment of inertia I by

$$r = \sqrt{\frac{I}{A}} \tag{1.69}$$

For an ideal elastic column, we can draw the curve of the critical stress σ_C vs. the slenderness ratio KL/r, as shown in Figure 1.15a.

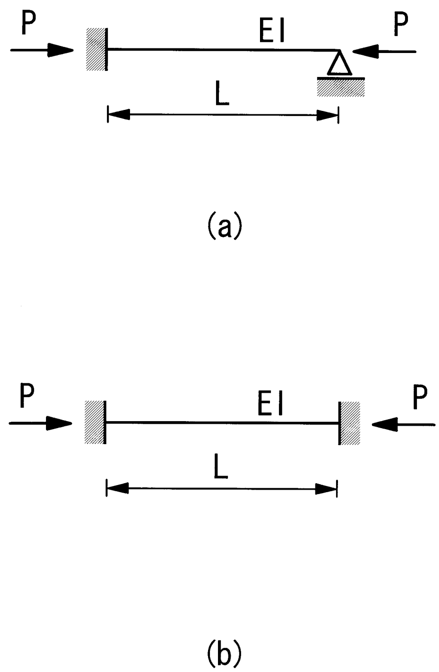

FIGURE 1.14: (a) Fixed-hinged column; (b) fixed-fixed column.

For a column of perfectly plastic material, stress never exceeds the yield stress σ_Y. For this class of column, we often employ a normalized form of Equation 1.68 as

$$\frac{\sigma_C}{\sigma_Y} = \frac{1}{\lambda^2} \tag{1.70}$$

where

$$\lambda = \frac{1}{\pi}\frac{KL}{r}\sqrt{\frac{\sigma_Y}{E}} \tag{1.71}$$

This equation is plotted in Figure 1.15b. For this column, with $\lambda < 1.0$, it collapses plastically; elastic buckling takes place for $\lambda > 1.0$.

Imperfect Columns

In the derivation of the buckling loads, we have dealt with the idealized columns; the member is perfectly straight and the loading is concentric at every cross-section. These idealizations help simplify the problem, but perfect members do not exist in the real world: minor crookedness of shape and small eccentricities of loading are always present. To this end, we shall investigate the behavior of an initially bent column in this section.

We consider a simply supported column shown in Figure 1.16. The column is initially bent and the initial crookedness w_i is assumed to be in the form of

$$w_i = a \sin\frac{\pi x}{L} \tag{1.72}$$

where a is a small value, representing the magnitude of the initial deflection at the midpoint. If we describe the additional deformation due to bending as w and consider the moment equilibrium in

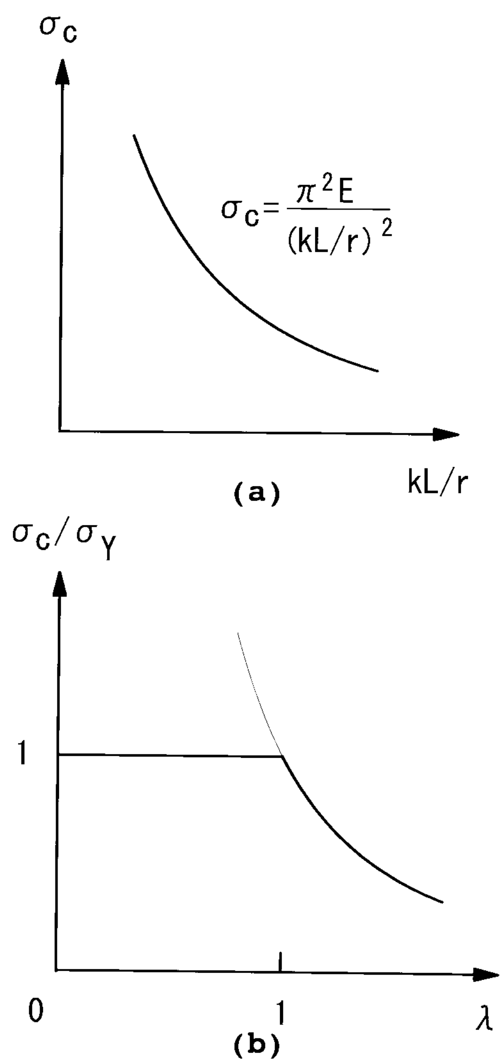

FIGURE 1.15: (a) Relationship between critical stress and slenderness ratio; (b) normalized relationship.

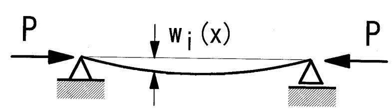

FIGURE 1.16: Initially bent column.

this configuration, we obtain

$$w'' + k^2 w = -k^2 a \sin \frac{\pi x}{L} \tag{1.73}$$

where k^2 is defined in Equation 1.46. The general solution of this differential equation is given by

$$w = A \sin \frac{\pi x}{L} + B \cos \frac{\pi x}{L} + \frac{P/P_E}{1 - P/P_E} a \sin \frac{\pi x}{L} \tag{1.74}$$

where P_E is the Euler load, i.e., $\pi^2 EI/L^2$. From the boundary conditions of Equation 1.48, we can determine the arbitrary constants A and B, yielding the following load-displacement relationship:

$$w = \frac{P/P_E}{1 - P/P_E} a \sin \frac{\pi x}{L} \tag{1.75}$$

By adding this expression to the initial deflection, we can obtain the total displacement w_t as

$$w_t = w_i + w = \frac{a}{1 - P/P_E} \sin \frac{\pi x}{L} \tag{1.76}$$

Figure 1.17 illustrates the variation of the deflection at the midpoint of this column w_m.

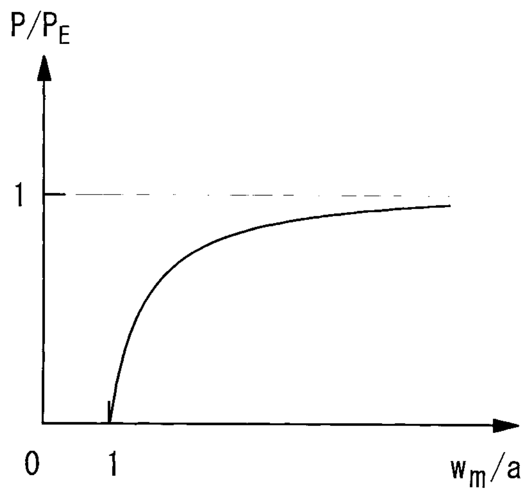

FIGURE 1.17: Load-displacement curve of the bent column.

Unlike the ideally perfect column, which remains straight up to the Euler load, we observe in this figure that the crooked column begins to bend at the onset of the loading. The deflection increases slowly at first, and as the applied load approaches the Euler load, the increase of the deflection is getting more and more rapid. Thus, although the behavior of the initially bent column is different from that of bifurcation, the buckling load still serves as an important measure of stability.

We have discussed the behavior of a column with geometrical imperfection in this section. However, the trend observed herein would be the same for general imperfect columns such as an eccentrically loaded column.

1.3.4 Thin-Walled Members

In the previous section, we assumed that a compressed column would buckle by bending. This type of buckling may be referred to as flexural buckling. However, a column may buckle by twisting or by a combination of twisting and bending. Such a mode of failure occurs when the torsional rigidity of the cross-section is low. Thin-walled open cross-sections have a low torsional rigidity in general and hence are susceptible of this type of buckling. In fact, a column of thin-walled open cross-section usually buckles by a combination of twisting and bending: this mode of buckling is often called the torsional-flexural buckling.

A bar subjected to bending in the plane of a major axis may buckle in yet another mode: at the critical load a compression side of the cross-section tends to bend sideways while the remainder is stable, resulting in the rotation and lateral movement of the entire cross-section. This type of buckling is referred to as lateral buckling. We need to use caution in particular, if a beam has no lateral supports and the flexural rigidity in the plane of bending is larger than the lateral flexural rigidity.

In the present section, we shall briefly discuss the two buckling modes mentioned above, both of which are of practical importance in the design of thin-walled members, particularly of open cross-section.

Torsional-Flexural Buckling

We consider a simply supported column subjected to compressive load P applied at the centroid of each end, as shown in Figure 1.18. Note that the x axis passes through the centroid of every cross-section. Taking into account that the cross-section undergoes translation and rotation as illustrated in Figure 1.19, we can derive the equilibrium conditions for the column deformed slightly by the torsional-flexural buckling

$$\begin{aligned} & EI_y v^{IV} + Pv'' + Pz_s\phi'' = 0 \\ & EI_z w^{IV} + Pw'' - Py_s\phi'' = 0 \\ & EI_w\phi^{IV} + \left(Pr_s^2\phi'' - GJ\right)\phi'' + Pz_s v'' - Py_s w'' = 0 \end{aligned} \tag{1.77}$$

where

$v,\ w$ = displacements in the $y,\ z$-directions, respectively
ϕ = rotation
EI_w = warping rigidity
GJ = torsional rigidity
$y_s,\ z_s$ = coordinates of the shear center

and

$$\begin{aligned} EI_y &= \int_A y^2 dA \\ EI_z &= \int_A z^2 dA \\ r_s^2 &= \frac{I_s}{A} \end{aligned} \tag{1.78}$$

where

I_s = polar moment of inertia about the shear center
A = cross-sectional area

We can obtain the buckling load by solving the eigenvalue problem governed by Equation 1.77 and the boundary conditions of

$$v = v'' = w = w'' = \phi = \phi'' = 0 \ \text{ at } x = 0, L \tag{1.79}$$

For doubly symmetric cross-section, the shear center coincides with the centroid. Therefore, $y_s,\ z_s$, and r_s vanish and the three equations in Equation 1.77 become independent of each other, if the cross-section of the column is doubly symmetric. In this case, we can compute three critical loads as follows:

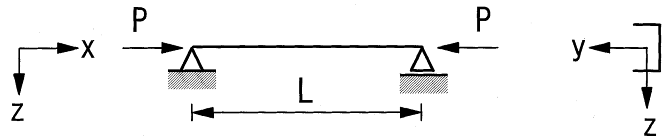

FIGURE 1.18: Simply-supported thin-walled column.

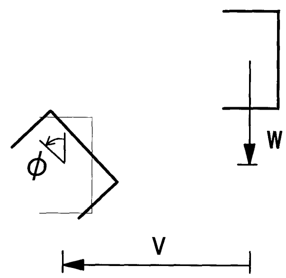

FIGURE 1.19: Translation and rotation of the cross-section.

$$P_{yC} = \frac{\pi^2 EI_y}{L^2} \tag{1.80a}$$

$$P_{zC} = \frac{\pi^2 EI_z}{L^2} \tag{1.80b}$$

$$P_{\phi C} = \frac{1}{r_s^2}\left(GJ + \frac{\pi^2 EI_w}{L^2}\right) \tag{1.80c}$$

The first two are associated with flexural buckling and the last one with torsional buckling. For all cases, the buckling mode is in the shape of $\sin \frac{\pi x}{L}$. The smallest of the three would be the critical load of practical importance: for a relatively short column with low GJ and EI_w, the torsional buckling may take place.

When the cross-section of a column is symmetric with respect only to the y axis, we rewrite Equation 1.77 as

$$EI_y v^{IV} + Pv'' = 0 \tag{1.81a}$$

$$EI_z w^{IV} + Pw'' - Py_s\phi'' = 0 \tag{1.81b}$$

$$EI_w \phi^{IV} + \left(Pr_s^2 - GJ\right)\phi'' - Py_s w'' = 0 \tag{1.81c}$$

The first equation indicates that the flexural buckling in the $x - y$ plane occurs independently and the corresponding critical load is given by P_{yC} of Equation 1.80a. The flexural buckling in the $x - z$ plane and the torsional buckling are coupled. By assuming that the buckling modes are described by $w = A_1 \sin \frac{\pi x}{L}$ and $\phi = A_2 \sin \frac{\pi x}{L}$, Equations 1.81b,c yields

$$\begin{bmatrix} P - P_{zC} & -Py_s \\ -Py_s & r_s^2\left(P - P_{\phi C}\right) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1.82}$$

This eigenvalue problem leads to

$$f(P) = r_s^2 \left(P - P_{\phi C}\right)(P - P_{zC}) - (Py_s)^2 = 0 \tag{1.83}$$

The solution of this quadratic equation is the critical load associated with torsional-flexural buckling. Since $f(0) = r_s^2 P_{\phi C} P_{zC} > 0$, $f(P_{\phi C} = -(Py_s)^2 < 0$, and $f(P_{zC}) = -(Py_s)^2 < 0$, it is obvious that the critical load is lower than P_{zC} and $P_{\phi C}$. If this load is smaller than P_{yC}, then the torsional-flexural buckling will take place.

If there is no axis of symmetry in the cross-section, all the three equations in Equation 1.77 are coupled. The torsional-flexural buckling occurs in this case, since the critical load for this buckling mode is lower than any of the three loads in Equation 1.80.

Lateral Buckling

The behavior of a simply supported beam in pure bending (Figure 1.20) is investigated. The equilibrium condition for a slightly translated and rotated configuration gives governing equations for the bifurcation. For a cross-section symmetric with respect to the y axis, we arrive at the following equations:

$$EI_y v^{IV} + M\phi'' = 0 \tag{1.84a}$$

$$EI_z w^{IV} = 0 \tag{1.84b}$$

$$EI_w \phi^{IV} - (GJ + M\beta)\phi'' + Mv'' = 0 \tag{1.84c}$$

where

$$\beta = \frac{1}{I_z} \int_A \left\{y^2 + (z - z_s)^2\right\} z dA \tag{1.85}$$

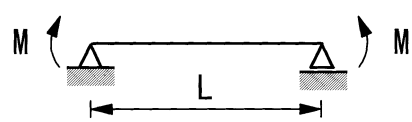

FIGURE 1.20: Simply supported beam in pure bending.

Equation 1.84b is a beam equation and has nothing to do with buckling. From the remaining two equations and the associated boundary conditions of Equation 1.79, we can evaluate the critical load for the lateral buckling. By assuming the bucking mode is in the shape of $\sin \frac{\pi x}{L}$ for both v and ϕ, we obtain the characteristic equation

$$M^2 - \beta P_{yC} M - r_s^2 P_{yC} P_{\phi C} = 0 \tag{1.86}$$

The smallest root of this quadratic equation is the critical load (moment) for the lateral buckling. For doubly symmetric sections where β is zero, the critical moment M_C is given by

$$M_C = \sqrt{r_s^2 P_{yC} P_{\phi C}} = \sqrt{\frac{\pi^2 EI_y}{L^2}\left(GJ + \frac{\pi^2 EI_w}{L^2}\right)} \tag{1.87}$$

1.3.5 Plates

Governing Equation

The buckling load of a plate is also obtained by the linear buckling analysis, i.e., by considering the equilibrium of a slightly deformed configuration. The plate counterpart of Equation 1.58, thus, derived is

$$D\nabla^4 w + \left(\overline{N}_x \frac{\partial^2 w}{\partial x^2} + 2\overline{N}_{xy} \frac{\partial^2 w}{\partial x \partial y} + \overline{N}_y \frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{1.88}$$

The definitions of $\overline{N}_x$, $\overline{N}_y$, and $\overline{N}_{xy}$ are the same as those of N_x, N_y, and N_{xy} given in Equations 1.8a through 1.8c, respectively, except the sign; $\overline{N}_x$, $\overline{N}_y$, and $\overline{N}_{xy}$ are positive when compressive. The boundary conditions are basically the same as discussed in Section 1.2.3 except the mechanical condition in the vertical direction: to include the effect of in-plane forces, we need to modify Equation 1.18 as

$$S_n + N_n \frac{\partial w}{\partial n} + N_{ns} \frac{\partial w}{\partial s} = \overline{S}_n \tag{1.89}$$

where

$$\begin{aligned} N_n &= \int_z \sigma_n dz \\ N_{ns} &= \int_z \tau_{ns} dz \end{aligned} \tag{1.90}$$

Simply Supported Plate

As an example, we shall discuss the buckling load of a simply supported plate under uniform compression shown in Figure 1.21. The governing equation for this plate is

$$D\nabla^4 w + \overline{N}_x \frac{\partial^2 w}{\partial x^2} = 0 \tag{1.91}$$

and the boundary conditions are

$$\begin{aligned} w &= 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{along } x = 0, a \\ w &= 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{along } y = 0, b \end{aligned} \tag{1.92}$$

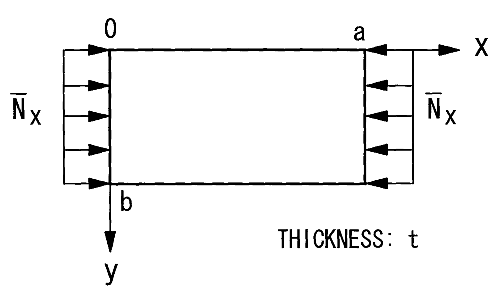

FIGURE 1.21: Simply supported plate subjected to uniform compression.

We assume that the solution is of the form

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{b} \tag{1.93}$$

where m and n are integers. Since this solution satisfies all the boundary conditions, we have only to ensure that it satisfies the governing equation. Substituting Equation 1.93 into 1.91, we obtain

$$A_{mn} \left[\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - \frac{\overline{N}_x}{D} \frac{m^2 \pi^2}{a^2} \right] = 0 \tag{1.94}$$

Since the trivial solution is of no interest, at least one of the coefficients a_{mn} must not be zero, the consideration of which leads to

$$\overline{N}_x = \frac{\pi^2 D}{b^2} \left(m \frac{b}{a} + \frac{n^2}{m} \frac{a}{b} \right)^2 \tag{1.95}$$

As the lowest $\overline{N}_x$ is crucial and $\overline{N}_x$ increases with n, we conclude $n = 1$: the buckling of this plate occurs in a single half-wave in the y direction and

$$\overline{N}_{xC} = \frac{k\pi^2 D}{b^2} \tag{1.96}$$

or

$$\sigma_{xC} = \frac{\overline{N}_{xC}}{t} = k \frac{\pi^2 E}{12(1 - \nu^2)} \frac{1}{(b/t)^2} \tag{1.97}$$

where

$$k = \left(m \frac{b}{a} + \frac{1}{m} \frac{a}{b} \right)^2 \tag{1.98}$$

Note that Equation 1.97 is comparable to Equation 1.68, and k is called the buckling stress coefficient.

The optimum value of m that gives the lowest $\overline{N}_{xC}$ depends on the aspect ratio a/b, as can be realized in Figure 1.22. For example, the optimum m is 1 for a square plate while it is 2 for a plate of $a/b = 2$. For a plate with a large aspect ratio, $k = 4.0$ serves as a good approximation. Since the aspect ratio of a component of a steel structural member such as a web plate is large in general, we can often assume k is simply equal to 4.0.

1.4 Defining Terms

The following is a list of terms as defined in the *Guide to Stability Design Criteria for Metal Structures*, 4th ed., Galambos, T.V., Structural Stability Research Council, John Wiley & Sons, New York, 1988.

Bifurcation: A term relating to the load-deflection behavior of a perfectly straight and perfectly centered compression element at critical load. Bifurcation can occur in the inelastic range only if the pattern of post-yield properties and/or residual stresses is symmetrically disposed so that no bending moment is developed at subcritical loads. At the critical load a member can be in equilibrium in either a straight or slightly deflected configuration, and a bifurcation results at a branch point in the plot of axial load vs. lateral deflection from which two alternative load-deflection plots are mathematically valid.

Braced frame: A frame in which the resistance to both lateral load and frame instability is provided by the combined action of floor diaphragms and structural core, shear walls, and/or a diagonal K brace, or other auxiliary system of bracing.

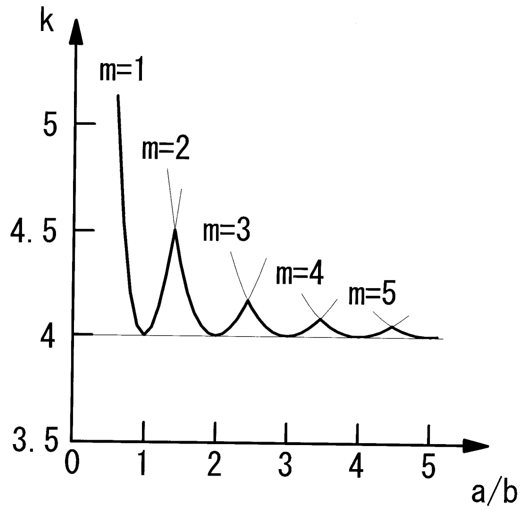

FIGURE 1.22: Variation of the buckling stress coefficient k with the aspect ratio a/b.

Effective length: The equivalent or effective length (KL) which, in the Euler formula for a hinged-end column, results in the same elastic critical load as for the framed member or other compression element under consideration at its theoretical critical load. The use of the effective length concept in the inelastic range implies that the ratio between elastic and inelastic critical loads for an equivalent hinged-end column is the same as the ratio between elastic and inelastic critical loads in the beam, frame, plate, or other structural element for which buckling equivalence has been assumed.

Instability: A condition reached during buckling under increasing load in a compression member, element, or frame at which the capacity for resistance to additional load is exhausted and continued deformation results in a decrease in load-resisting capacity.

Stability: The capacity of a compression member or element to remain in position and support load, even if forced slightly out of line or position by an added lateral force. In the elastic range, removal of the added lateral force would result in a return to the prior loaded position, unless the disturbance causes yielding to commence.

Unbraced frame: A frame in which the resistance to lateral loads is provided primarily by the bending of the frame members and their connections.

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