CHAPTER 13

MICRO-TO-MACRO TRANSITION

Some fundamental aspects of the transition in the constitutive description of the material response from microlevel to macrolevel are discussed in this chapter. The analysis is aimed toward the derivation of the constitutive equations for polycrystalline aggregates based on the known constitutive equations for elastoplastic single crystals. The theoretical framework for this study was developed by Bishop and Hill (1951 a, b), Hill (1963, 1967, 1972), Mandel (1966), Bui (1970), Rice (1970, 1971, 1975), Hill and Rice (1973), Havner (1973, 1974), and others. The presentation in this chapter follows the large deformation formulation of Hill (1984, 1985). The representative macroelement is defined, and the macroscopic measures of stress and strain, and their rates, are introduced. The corresponding elastoplastic moduli and pseudomoduli tensors, the macroscopic normality and the macroscopic plastic potentials are then discussed.

13.1. Representative Macroelement

A polycrystalline aggregate is considered to be macroscopically homogeneous by assuming that local microscopic heterogeneities (due to different orientation and state of hardening of individual crystal grains) are distributed in such a way that the material elements beyond some minimum scale have essentially the same overall macroscopic properties. This minimum scale defines the size of the representative macroelement or representative cell (Fig. 13.1). The representative macroelement can be viewed as a material point in the continuum mechanics of macroscopic aggregate behavior. To be statistically representative of the local properties of its microconstituents, the representative macroelement must include a sufficiently large number of microelements (Kröner, 1971; Sanchez-Palencia, 1980; Kunin, 1982). For

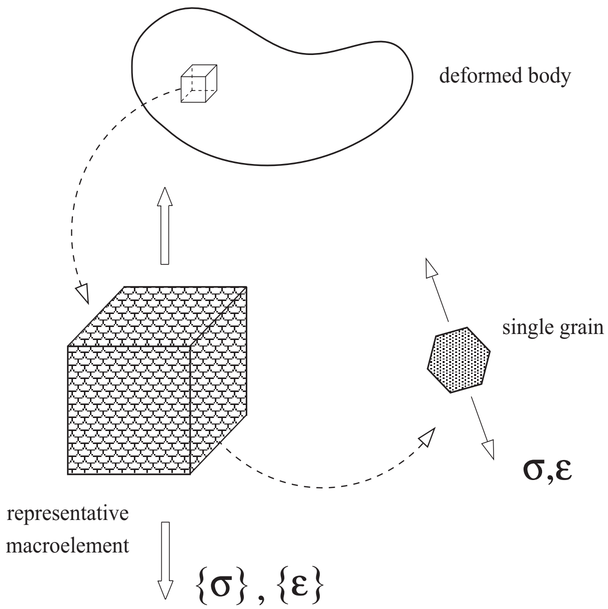

FIGURE 13.1. Representative macroelement of a deformed body consists of a large number of constituting microelements – single grains in the case of a polycrystalline aggregate (schematics adopted from Yang and Lee, 1993).

example, for relatively fine-grained metals, a representative macroelement of volume 1 mm^3 contains a minimum of 1000 crystal grains (Havner, 1992). The concept of the representative macroelement is used in various branches of the mechanics of heterogeneous materials, and is also referred to as the representative volume element (e.g., Mura, 1987; Suquet, 1987; Torquato, 1991; Maugin, 1992; Nemat-Nasser and Hori, 1993; Hori and Nemat-Nasser, 1999). See also Hashin (1964), Willis (1981), Sawicki (1983), Ortiz (1987), and Drugan and Willis (1996). For the linkage of atomistic and continuum models of the material response, the review by Ortiz and Phillips (1999) can be consulted.

13.2. Averages over a Macroelement

Experimental determination of the mechanical behavior of an aggregate is commonly based on the measured loads and displacements over its external surface. Consequently, the macrovariables introduced in the constitutive analysis should be expressible in terms of this surface data alone (Hill, 1972). Let

$$\mathbf{F}(\mathbf{X},t) = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}, \quad \det \mathbf{F} > 0, \tag{13.2.1}$$

be the deformation gradient at the microlevel of description, associated with a (continuous and piecewise continuously differentiable) microdeformation within a crystalline grain, $\mathbf{x} = \mathbf{x}(\mathbf{X},t)$. The reference position of the particle is $\mathbf{X}$, and its current position at time t (on some quasi-static scale, for rate-independent response) is $\mathbf{x}$. The volume average of the deformation gradient over the reference volume V^0 of the macroelement is

$$\langle \mathbf{F} \rangle = \frac{1}{V^0} \int_{V^0} \mathbf{F}\, \mathrm{d}V^0 = \frac{1}{V^0} \int_{S^0} \mathbf{x} \otimes \mathbf{n}^0\, \mathrm{d}S^0, \tag{13.2.2}$$

by the Gauss divergence theorem. The unit outward normal to the bounding surface S^0 of the macroelement volume is $\mathbf{n}^0$. In particular, with $\mathbf{F} = \mathbf{I}$ (unit tensor), Eq. (13.2.2) gives an identity

$$\frac{1}{V^0} \int_{S^0} \mathbf{X} \otimes \mathbf{n}^0\, \mathrm{d}S^0 = \mathbf{I}. \tag{13.2.3}$$

The volume average of the rate of deformation gradient,

$$\dot{\mathbf{F}}(\mathbf{X},t) = \frac{\partial \mathbf{v}}{\partial \mathbf{X}}, \quad \mathbf{v} = \dot{\mathbf{x}}(\mathbf{X},t), \tag{13.2.4}$$

where $\mathbf{v}$ is the velocity field, is

$$\langle \dot{\mathbf{F}} \rangle = \frac{1}{V^0} \int_{V^0} \dot{\mathbf{F}}\, \mathrm{d}V^0 = \frac{1}{V^0} \int_{S^0} \mathbf{v} \otimes \mathbf{n}^0\, \mathrm{d}S^0. \tag{13.2.5}$$

If the current configuration is taken as the reference configuration ($\mathbf{x} = \mathbf{X}$, $\mathbf{F} = \mathbf{I}$, $\dot{\mathbf{F}} = \mathbf{L} = \partial \mathbf{v}/\partial \mathbf{x}$), Eq. (13.2.2) gives

$$\frac{1}{V} \int_{S} \mathbf{x} \otimes \mathbf{n}\, \mathrm{d}S = \mathbf{I}. \tag{13.2.6}$$

The current volume of the deformed macroelement is V, and S is its bounding surface with the unit outward normal $\mathbf{n}$. With this choice of the reference configuration, the volume average of the velocity gradient $\mathbf{L}$ is, from Eq. (13.2.5),

$$\{\mathbf{L}\} = \frac{1}{V} \int_{V} \mathbf{L}\, \mathrm{d}V = \frac{1}{V} \int_{S} \mathbf{v} \otimes \mathbf{n}\, \mathrm{d}S. \tag{13.2.7}$$

Enclosure within $\{\,\}$ brackets is used to indicate that the average is taken over the deformed volume of the macroelement.

Let $\mathbf{P} = \mathbf{P}(\mathbf{X}, t)$ be a nonsymmetric nominal stress field within the macroelement. In the absence of body forces, equations of translational balance are

$$\boldsymbol{\nabla}^0 \cdot \mathbf{P} = \mathbf{0} \quad \text{in} \quad V^0, \qquad \mathbf{n}^0 \cdot \mathbf{P} = \mathbf{p}_n \quad \text{on} \quad S^0. \tag{13.2.8}$$

Here, $\boldsymbol{\nabla}^0 = \partial/\partial\mathbf{X}$ is the gradient operator with respect to reference coordinates, and $\mathbf{p}_n$ is the nominal traction (related to the true traction $\mathbf{t}_n$ by $\mathbf{p}_n\, \mathrm{d}S^0 = \mathbf{t}_n\, \mathrm{d}S$). The rotational balance requires $\mathbf{F} \cdot \mathbf{P} = \boldsymbol{\tau}$ to be a symmetric tensor, where $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$ is the Kirchhoff stress, and $\boldsymbol{\sigma}$ is the true or Cauchy stress.

Equations of the continuing translational balance are

$$\boldsymbol{\nabla}^0 \cdot \dot{\mathbf{P}} = \mathbf{0} \quad \text{in} \quad V^0, \qquad \mathbf{n}^0 \cdot \dot{\mathbf{P}} = \dot{\mathbf{p}}_n \quad \text{on} \quad S^0. \tag{13.2.9}$$

The rates of nominal and true traction are related by

$$\dot{\mathbf{p}}_n\, \mathrm{d}S^0 = \left[\dot{\mathbf{t}}_n + (\operatorname{tr} \mathbf{D} - \mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n})\, \mathbf{t}_n\right] \mathrm{d}S, \tag{13.2.10}$$

as in Eq. (3.8.16). The rate of deformation tensor is $\mathbf{D}$. By differentiating $\mathbf{F} \cdot \mathbf{P} = \mathbf{P}^T \cdot \mathbf{F}^T$ (expressing the symmetry of $\boldsymbol{\tau}$), we obtain the condition for the continuing rotational balance

$$\dot{\mathbf{F}} \cdot \mathbf{P} + \mathbf{F} \cdot \dot{\mathbf{P}} = \dot{\mathbf{P}}^T \cdot \mathbf{F}^T + \mathbf{P}^T \cdot \dot{\mathbf{F}}^T. \tag{13.2.11}$$

The volume averages of the nominal stress and its rate are (Hill, 1972)

$$\langle \mathbf{P} \rangle = \frac{1}{V^0} \int_{V^0} \mathbf{P}\, \mathrm{d}V^0 = \frac{1}{V^0} \int_{S^0} \mathbf{X} \otimes \mathbf{p}_n\, \mathrm{d}S^0, \tag{13.2.12}$$

$$\langle \dot{\mathbf{P}} \rangle = \frac{1}{V^0} \int_{V^0} \dot{\mathbf{P}}\, \mathrm{d}V^0 = \frac{1}{V^0} \int_{S^0} \mathbf{X} \otimes \dot{\mathbf{p}}_n\, \mathrm{d}S^0. \tag{13.2.13}$$

Both of these are expressed on the far right-hand sides solely in terms of the surface data $\mathbf{p}_n$ and $\dot{\mathbf{p}}_n$ over S^0. This follows from the divergence theorem and equilibrium equations (13.2.8) and (13.2.9). If current configuration is chosen as the reference ($\mathbf{P} = \boldsymbol{\sigma}$, $\mathbf{p}_n = \mathbf{t}_n$), Eq. (13.2.12) gives

$$\{\boldsymbol{\sigma}\} = \frac{1}{V} \int_{V} \boldsymbol{\sigma}\, \mathrm{d}V = \frac{1}{V} \int_{S} \mathbf{x} \otimes \mathbf{t}_n\, \mathrm{d}S. \tag{13.2.14}$$

With this choice of the reference configuration, the rate of nominal stress is from Eq. (3.9.10) equal to

$$\underline{\dot{\mathbf{P}}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \operatorname{tr} \mathbf{D} - \mathbf{L} \cdot \boldsymbol{\sigma}. \tag{13.2.15}$$

Thus, in view of Eq. (13.2.10), the average in Eq. (13.2.13) becomes

$$\{\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D} - \mathbf{L}\cdot\boldsymbol{\sigma}\} = \frac{1}{V}\int_S \mathbf{x}\otimes\left[\dot{\mathbf{t}}_n + (\mathrm{tr}\,\mathbf{D} - \mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n})\,\mathbf{t}_n\right]\mathrm{d}S. \quad (13.2.16)$$

Note that, from Eq. (13.2.14),

$$\int_{V^0} \boldsymbol{\tau}\,\mathrm{d}V^0 = \int_V \boldsymbol{\sigma}\,\mathrm{d}V = \int_S \mathbf{x}\otimes\mathbf{t}_n\,\mathrm{d}S = \int_{S^0} \mathbf{x}\otimes\mathbf{p}_n\,\mathrm{d}S^0, \quad (13.2.17)$$

so that

$$\langle\boldsymbol{\tau}\rangle = \frac{1}{V^0}\int_{V^0} \boldsymbol{\tau}\,\mathrm{d}V^0 = \frac{1}{V^0}\int_{S^0} \mathbf{x}\otimes\mathbf{p}_n\,\mathrm{d}S^0. \quad (13.2.18)$$

Since $\boldsymbol{\tau} = \mathbf{F}\cdot\mathbf{P}$, from Eq. (13.2.18) we have

$$\langle\mathbf{F}\cdot\mathbf{P}\rangle = \frac{1}{V^0}\int_{V^0} \mathbf{F}\cdot\mathbf{P}\,\mathrm{d}V^0 = \frac{1}{V^0}\int_{S^0} \mathbf{x}\otimes\mathbf{p}_n\,\mathrm{d}S^0. \quad (13.2.19)$$

This also follows directly by integration and application of the divergence theorem and equilibrium equations. Similarly,

$$\langle\mathbf{F}\cdot\dot{\mathbf{P}}\rangle = \frac{1}{V^0}\int_{V^0} \mathbf{F}\cdot\dot{\mathbf{P}}\,\mathrm{d}V^0 = \frac{1}{V^0}\int_{S^0} \mathbf{x}\otimes\dot{\mathbf{p}}_n\,\mathrm{d}S^0, \quad (13.2.20)$$

$$\langle\dot{\mathbf{F}}\cdot\mathbf{P}\rangle = \frac{1}{V^0}\int_{V^0} \dot{\mathbf{F}}\cdot\mathbf{P}\,\mathrm{d}V^0 = \frac{1}{V^0}\int_{S^0} \mathbf{v}\otimes\mathbf{p}_n\,\mathrm{d}S^0, \quad (13.2.21)$$

$$\langle\dot{\mathbf{F}}\cdot\dot{\mathbf{P}}\rangle = \frac{1}{V^0}\int_{V^0} \dot{\mathbf{F}}\cdot\dot{\mathbf{P}}\,\mathrm{d}V^0 = \frac{1}{V^0}\int_{S^0} \mathbf{v}\otimes\dot{\mathbf{p}}_n\,\mathrm{d}S^0. \quad (13.2.22)$$

In the last four expressions, the $\mathbf{F}$ and $\mathbf{P}$ fields, and their rates, need not be constitutively related to each other.

13.3. Theorem on Product Averages

In the mechanics of macroscopic aggregate behavior it is of fundamental importance to express the volume averages of various kinematic and kinetic quantities in terms of the basic macroscopic variables $\langle\mathbf{F}\rangle$ and $\langle\mathbf{P}\rangle$, and their rates. We begin with the evaluation of the product average $\langle\mathbf{F}\cdot\mathbf{P}\rangle$ in terms of $\langle\mathbf{F}\rangle$ and $\langle\mathbf{P}\rangle$. Following Hill (1984), consider the identity

$$\langle\mathbf{F}\cdot\mathbf{P}\rangle - \langle\mathbf{F}\rangle\cdot\langle\mathbf{P}\rangle = \langle(\mathbf{F} - \langle\mathbf{F}\rangle)\cdot(\mathbf{P} - \langle\mathbf{P}\rangle)\rangle. \quad (13.3.1)$$

This identity holds because, for example,

$$\langle\mathbf{F}\cdot\langle\mathbf{P}\rangle\rangle = \langle\langle\mathbf{F}\rangle\cdot\mathbf{P}\rangle = \langle\mathbf{F}\rangle\cdot\langle\mathbf{P}\rangle. \quad (13.3.2)$$

The right-hand side of Eq. (13.3.1) can be expressed as

$$\langle (\mathbf{F} - \langle \mathbf{F} \rangle) \cdot (\mathbf{P} - \langle \mathbf{P} \rangle) \rangle = \frac{1}{V^0} \int_{S^0} (\mathbf{x} - \langle \mathbf{F} \rangle \cdot \mathbf{X}) \otimes (\mathbf{P} - \langle \mathbf{P} \rangle)^T \cdot \mathbf{n}^0 \, \mathrm{d}S^0, \tag{13.3.3}$$

which can be verified by the Gauss divergence theorem. This leads to Hill's (1972, 1984) theorem on product averages: The product average decomposes into the product of averages,

$$\langle \mathbf{F} \cdot \mathbf{P} \rangle = \langle \mathbf{F} \rangle \cdot \langle \mathbf{P} \rangle, \tag{13.3.4}$$

provided that

$$\int_{S^0} (\mathbf{x} - \langle \mathbf{F} \rangle \cdot \mathbf{X}) \otimes (\mathbf{P} - \langle \mathbf{P} \rangle)^T \cdot \mathbf{n}^0 \, \mathrm{d}S^0 = 0. \tag{13.3.5}$$

The condition (13.3.5) is met, in particular, when the surface S^0 is deformed or loaded uniformly, i.e., when

$$\mathbf{x} = \overline{\mathbf{F}}(t) \cdot \mathbf{X} \quad \text{or} \quad \mathbf{p}_n = \mathbf{n}^0 \cdot \overline{\mathbf{P}}(t) \quad \text{on} \quad S^0, \tag{13.3.6}$$

since then

$$\langle \mathbf{F} \rangle = \overline{\mathbf{F}}(t) \quad \text{or} \quad \langle \mathbf{P} \rangle = \overline{\mathbf{P}}(t), \tag{13.3.7}$$

which makes the integral in (13.3.5) identically equal to zero.

An analog of Eqs. (13.3.4) and (13.3.5), involving the rate of $\mathbf{P}$, is

$$\langle \mathbf{F} \cdot \dot{\mathbf{P}} \rangle = \langle \mathbf{F} \rangle \cdot \langle \dot{\mathbf{P}} \rangle, \tag{13.3.8}$$

provided that

$$\int_{S^0} (\mathbf{x} - \langle \mathbf{F} \rangle \cdot \mathbf{X}) \otimes \left(\dot{\mathbf{P}} - \langle \dot{\mathbf{P}} \rangle \right)^T \cdot \mathbf{n}^0 \, \mathrm{d}S^0 = 0. \tag{13.3.9}$$

The condition (13.3.9) is, for example, met when

$$\mathbf{x} = \overline{\mathbf{F}}(t) \cdot \mathbf{X} \quad \text{or} \quad \dot{\mathbf{p}}_n = \mathbf{n}^0 \cdot \dot{\overline{\mathbf{P}}}(t) \quad \text{on} \quad S^0. \tag{13.3.10}$$

The other analogs are, evidently,

$$\langle \dot{\mathbf{F}} \cdot \mathbf{P} \rangle = \langle \dot{\mathbf{F}} \rangle \cdot \langle \mathbf{P} \rangle, \tag{13.3.11}$$

provided that

$$\int_{S^0} \left(\mathbf{v} - \langle \dot{\mathbf{F}} \rangle \cdot \mathbf{X} \right) \otimes (\mathbf{P} - \langle \mathbf{P} \rangle)^T \cdot \mathbf{n}^0 \, \mathrm{d}S^0 = 0, \tag{13.3.12}$$

and

$$\langle \dot{\mathbf{F}} \cdot \dot{\mathbf{P}} \rangle = \langle \dot{\mathbf{F}} \rangle \cdot \langle \dot{\mathbf{P}} \rangle, \tag{13.3.13}$$

provided that

$$\int_{S^0}\left(\mathbf{v}-\langle\dot{\mathbf{F}}\rangle\cdot\mathbf{X}\right)\otimes\left(\dot{\mathbf{P}}-\langle\dot{\mathbf{P}}\rangle\right)^T\cdot\mathbf{n}^0\,\mathrm{d}S^0=0. \tag{13.3.14}$$

For instance, the requirement (13.3.14) is met when

$$\mathbf{v}=\dot{\overline{\mathbf{F}}}(t)\cdot\mathbf{X}\quad\text{or}\quad\dot{\mathbf{p}}_n=\mathbf{n}^0\cdot\dot{\overline{\mathbf{P}}}(t)\quad\text{on}\quad S^0. \tag{13.3.15}$$

It is noted that, with the current configuration as the reference, Eq. (13.3.11) gives

$$\{\mathbf{L}\cdot\boldsymbol{\sigma}\}=\{\mathbf{L}\}\cdot\{\boldsymbol{\sigma}\}. \tag{13.3.16}$$

Under the prescribed uniform boundary conditions (13.3.6), the overall rotational balance, expressed in terms of the macrovariables, is

$$\langle\mathbf{F}\rangle\cdot\langle\mathbf{P}\rangle=\langle\mathbf{P}\rangle^T\cdot\langle\mathbf{F}\rangle^T. \tag{13.3.17}$$

This follows from Eq. (13.3.4) by applying the transpose operation to both sides, and by using the symmetry condition at microlevel $\mathbf{F}\cdot\mathbf{P}=\mathbf{P}^T\cdot\mathbf{F}^T$. Similarly, by differentiating Eq. (13.3.4), we have

$$\langle\dot{\mathbf{F}}\cdot\mathbf{P}+\mathbf{F}\cdot\dot{\mathbf{P}}\rangle=\langle\dot{\mathbf{F}}\rangle\cdot\langle\mathbf{P}\rangle+\langle\mathbf{F}\rangle\cdot\langle\dot{\mathbf{P}}\rangle. \tag{13.3.18}$$

By applying the transpose operation to both sides of this equation and by imposing (13.2.11), we establish the condition for the overall continuing rotational balance, in terms of the macrovariables, and under prescribed uniform boundary conditions. This is

$$\langle\dot{\mathbf{F}}\rangle\cdot\langle\mathbf{P}\rangle+\langle\mathbf{F}\rangle\cdot\langle\dot{\mathbf{P}}\rangle=\langle\mathbf{P}\rangle^T\cdot\langle\dot{\mathbf{F}}\rangle^T+\langle\dot{\mathbf{P}}\rangle^T\cdot\langle\mathbf{F}\rangle^T. \tag{13.3.19}$$

Upon contraction operation in Eq. (13.3.4), we obtain

$$\langle\mathbf{F}\cdot\cdot\,\mathbf{P}\rangle=\langle\mathbf{F}\rangle\cdot\cdot\,\langle\mathbf{P}\rangle. \tag{13.3.20}$$

Since the trace product is commutative, we also have

$$\langle\mathbf{P}\cdot\cdot\,\mathbf{F}\rangle=\langle\mathbf{P}\rangle\cdot\cdot\,\langle\mathbf{F}\rangle. \tag{13.3.21}$$

Likewise,

$$\langle\mathbf{P}\cdot\cdot\,\dot{\mathbf{F}}\rangle=\langle\mathbf{P}\rangle\cdot\cdot\,\langle\dot{\mathbf{F}}\rangle, \tag{13.3.22}$$

$$\langle\dot{\mathbf{P}}\cdot\cdot\,\mathbf{F}\rangle=\langle\dot{\mathbf{P}}\rangle\cdot\cdot\,\langle\mathbf{F}\rangle, \tag{13.3.23}$$

$$\langle\dot{\mathbf{P}}\cdot\cdot\,\dot{\mathbf{F}}\rangle=\langle\dot{\mathbf{P}}\rangle\cdot\cdot\,\langle\dot{\mathbf{F}}\rangle. \tag{13.3.24}$$

In these expressions, $\mathbf{P}$ and $\dot{\mathbf{P}}$ are statically admissible, while $\mathbf{F}$ and $\dot{\mathbf{F}}$ are kinematically admissible fields, but they are not necessarily constitutively related to each other. For example, if $\mathrm{d}\mathbf{P}$ and $\delta\mathbf{F}$ are two unrelated increments of $\mathbf{P}$ and $\mathbf{F}$, we can write

$$\langle \mathrm{d}\mathbf{P} \cdot\cdot\, \delta\mathbf{F} \rangle = \langle \mathrm{d}\mathbf{P} \rangle \cdot\cdot \langle \delta\mathbf{F} \rangle. \tag{13.3.25}$$

When the current configuration is the reference, Eq. (13.3.22) becomes

$$\{\boldsymbol{\sigma} : \mathbf{L}\} = \{\boldsymbol{\sigma}\} : \{\mathbf{L}\}, \quad \text{i.e.,} \quad \{\boldsymbol{\sigma} : \mathbf{D}\} = \{\boldsymbol{\sigma}\} : \{\mathbf{D}\}, \tag{13.3.26}$$

while Eq. (13.3.24) gives

$$\{(\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D} - \mathbf{L}\cdot\boldsymbol{\sigma}) \cdot\cdot\, \mathbf{L}\} = \{\,\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D} - \mathbf{L}\cdot\boldsymbol{\sigma}\,\} \cdot\cdot \{\mathbf{L}\}. \tag{13.3.27}$$

Additional analysis of the averaging theorems can be found in the paper by Nemat-Nasser (1999).

13.4. Macroscopic Measures of Stress and Strain

The macroscopic or aggregate measure of the symmetric Piola–Kirchhoff stress, denoted by $[\mathbf{T}]$, is defined such that

$$\langle \mathbf{P} \rangle = \langle \mathbf{T} \cdot \mathbf{F}^T \rangle = [\mathbf{T}] \cdot \langle \mathbf{F} \rangle^T. \tag{13.4.1}$$

Enclosure within square [ ] rather than $\langle\ \rangle$ brackets is used to indicate that the macroscopic measure of the Piola–Kirchhoff stress in Eq. (13.4.1) is not equal to the volume average of the microscopic Piola–Kirchhoff stress, i.e.,

$$[\mathbf{T}] \neq \frac{1}{V^0} \int_{V^0} \mathbf{T}\,\mathrm{d}V^0. \tag{13.4.2}$$

However, $[\mathbf{T}]$ is a symmetric tensor, because the tensor $\langle \mathbf{F} \rangle \cdot \langle \mathbf{P} \rangle$ is symmetric, by Eq. (13.3.17).

Although $[\mathbf{T}]$ is not a direct volume average of $\mathbf{T}$, it is defined in Eq. (13.4.1) in terms of the volume averages of $\langle \mathbf{F} \rangle$ and $\langle \mathbf{P} \rangle$, both of which are expressible in terms of the surface data alone. Thus, $[\mathbf{T}]$ is a suitable macroscopic variable for the constitutive analysis. (Since there is no explicit connection between $[\mathbf{T}]$ and $\langle \mathbf{T} \rangle$, the latter average is actually not suitable as a macrovariable at all). When the current configuration is taken for the reference ($\mathbf{P} = \mathbf{T} = \boldsymbol{\sigma}$), Eq. (13.4.1) gives

$$\{\boldsymbol{\sigma}\} = [\boldsymbol{\sigma}]. \tag{13.4.3}$$

This shows that the macroscopic measure of the Cauchy stress is the volume average of the microscopic Cauchy stress.

The macroscopic measure of the Lagrangian strain is defined by

$$[\mathbf{E}] = \frac{1}{2}\left(\langle\mathbf{F}\rangle^T \cdot \langle\mathbf{F}\rangle - \mathbf{I}\right), \tag{13.4.4}$$

for then $[\mathbf{T}]$ is generated from $[\mathbf{E}]$ by the work conjugency

$$\langle\dot{w}\rangle = \langle\mathbf{P}\cdot\cdot\,\dot{\mathbf{F}}\rangle = \langle\mathbf{T} : \dot{\mathbf{E}}\rangle = [\mathbf{T}] : [\dot{\mathbf{E}}]. \tag{13.4.5}$$

Indeed,

$$\langle\mathbf{P}\cdot\cdot\,\dot{\mathbf{F}}\rangle = \langle\mathbf{P}\rangle\cdot\cdot\,\langle\dot{\mathbf{F}}\rangle = [\mathbf{T}]\cdot\langle\mathbf{F}\rangle^T\cdot\cdot\,\langle\dot{\mathbf{F}}\rangle = [\mathbf{T}] : [\dot{\mathbf{E}}], \tag{13.4.6}$$

where

$$[\dot{\mathbf{E}}] = \frac{1}{2}\left(\langle\dot{\mathbf{F}}\rangle^T \cdot \langle\mathbf{F}\rangle + \langle\mathbf{F}\rangle^T \cdot \langle\dot{\mathbf{F}}\rangle\right). \tag{13.4.7}$$

The trace property $\mathbf{A}\cdot\mathbf{B}\cdot\cdot\,\mathbf{C} = \mathbf{A}\cdot\cdot\,\mathbf{B}\cdot\mathbf{C}$ was used for the second-order tensors, such as $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

The macroscopic measure of the Lagrangian strain $[\mathbf{E}]$ is not a direct volume average of the microscopic Lagrangian strain, i.e.,

$$[\mathbf{E}] \neq \frac{1}{V^0}\int_{V^0} \mathbf{E}\,\mathrm{d}V^0, \tag{13.4.8}$$

because

$$\langle\mathbf{F}^T\cdot\mathbf{F}\rangle \neq \langle\mathbf{F}\rangle^T\cdot\langle\mathbf{F}\rangle. \tag{13.4.9}$$

The rates of the macroscopic nominal and symmetric Piola–Kirchhoff stress tensors are related by

$$\langle\dot{\mathbf{P}}\rangle = [\dot{\mathbf{T}}]\cdot\langle\mathbf{F}\rangle^T + [\mathbf{T}]\cdot\langle\dot{\mathbf{F}}\rangle^T, \tag{13.4.10}$$

which follows from Eq. (13.4.1) by differentiation. When this is subjected to the trace product with $\langle\dot{\mathbf{F}}\rangle$, we obtain

$$\langle\dot{\mathbf{P}}\rangle\cdot\cdot\,\langle\dot{\mathbf{F}}\rangle = [\dot{\mathbf{T}}] : [\dot{\mathbf{E}}] + \mathbf{T} : \left(\langle\dot{\mathbf{F}}\rangle^T\cdot\langle\dot{\mathbf{F}}\rangle\right). \tag{13.4.11}$$

If the current configuration is selected for the reference, the stress rate $\dot{\mathbf{T}}$ is equal to (see Section 3.8)

$$\overset{\triangle}{\underline{\boldsymbol{\tau}}} = \overset{\triangle}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D}, \tag{13.4.12}$$

and Eq. (13.4.10) becomes

$$\{\,\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D} - \mathbf{L}\cdot\boldsymbol{\sigma}\,\} = [\,\overset{\triangle}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D}\,] + [\boldsymbol{\sigma}]\cdot\langle\mathbf{L}\rangle^T. \tag{13.4.13}$$

Since $[\boldsymbol{\sigma}] = \{\boldsymbol{\sigma}\}$, and since by direct integration

$$\{\boldsymbol{\sigma} \cdot \mathbf{L}^T\} = \{\boldsymbol{\sigma}\} \cdot \{\mathbf{L}\}^T, \tag{13.4.14}$$

we deduce from Eq. (13.4.13) that

$$\{\overset{\triangle}{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \operatorname{tr} \mathbf{D}\} = [\overset{\triangle}{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \operatorname{tr} \mathbf{D}], \tag{13.4.15}$$

i.e.,

$$[\overset{\triangle}{\underline{\boldsymbol{\tau}}}] = \{\overset{\triangle}{\underline{\boldsymbol{\tau}}}\}. \tag{13.4.16}$$

Furthermore, with the current configuration as the reference, Eq. (13.4.7) gives

$$[\mathbf{D}] = \{\mathbf{D}\}. \tag{13.4.17}$$

Thus, the macroscopic measure of the rate of deformation is the volume average of the microscopic rate of deformation.

The macroscopic infinitesimal deformation gradient and, thus, the macroscopic infinitesimal strain and rotation are also direct volume averages of the corresponding microscopic quantities. For the definition of the macroscopic measures of the rate of stress and deformation in the solids undergoing phase transformation, see Petryk (1998).

13.5. Influence Tensors of Elastic Heterogeneity

We consider materials for which the interior elastic fields depend uniquely and continuously on the surface data. Then, under uniform data on S^0, specified by (13.3.15), the fields $\dot{\mathbf{F}}$ and $\dot{\mathbf{P}}$ within V^0 depend uniquely on $\langle \dot{\mathbf{F}} \rangle$. For incrementally linear material response, this dependence is also linear. Thus, following Hill (1984), we introduce the influence tensors (functions) of elastic heterogeneity, denoted by $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{P}}$, such that

$$\dot{\mathbf{F}} = \boldsymbol{\mathcal{F}} \cdot\cdot \langle \dot{\mathbf{F}} \rangle = \langle \dot{\mathbf{F}} \rangle \cdot\cdot \boldsymbol{\mathcal{F}}^T, \tag{13.5.1}$$

$$\dot{\mathbf{P}} = \boldsymbol{\mathcal{P}} \cdot\cdot \langle \dot{\mathbf{P}} \rangle = \langle \dot{\mathbf{P}} \rangle \cdot\cdot \boldsymbol{\mathcal{P}}^T, \tag{13.5.2}$$

where

$$\langle \boldsymbol{\mathcal{F}} \rangle = \boldsymbol{I}, \quad \langle \boldsymbol{\mathcal{P}} \rangle = \boldsymbol{I}. \tag{13.5.3}$$

The rectangular components of the fourth-order unit tensor $\boldsymbol{I}$ are

$$I_{ijkl} = \delta_{il}\delta_{jk}, \quad I_{ijkl} = I_{klij}. \tag{13.5.4}$$

The influence tensors $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{P}}$ are functions of the current heterogeneities of stress and material properties within a macroelement. As pointed out by Hill (1984), kinematic data is never micro-uniform, since equivalent macroelements in a test specimen are constrained by one another, not by the apparatus. This results in fluctuations of $\dot{\mathbf{F}} \cdot \mathbf{X}$ on S^0 around $\langle \dot{\mathbf{F}} \rangle \cdot \mathbf{X}$, but the effect of these fluctuations decay rapidly with depth toward interior of the macroelement. Equations (13.5.1) and (13.5.2) can then be adopted for this macro-uniform surface data, as well, except within a negligible layer near the bounding surface of the macroelement. See also Mandel (1964) and Stolz (1997).

13.6. Macroscopic Free and Complementary Energy

The local free energy, per unit reference volume, is a potential for the local nominal stress, such that

$$\mathbf{P} = \frac{\partial \Psi}{\partial \mathbf{F}}, \quad \Psi = \Psi(\mathbf{F}, \mathcal{H}). \tag{13.6.1}$$

The pattern of internal rearrangement due to plastic deformation is designated by $\mathcal{H}$. The macroscopic free energy, per unit volume of the aggregate macroelement, is the volume average of Ψ,

$$\hat{\Psi} = \langle \Psi \rangle = \frac{1}{V^0} \int_{V^0} \Psi(\mathbf{F}, \mathcal{H}) \, \mathrm{d}V^0. \tag{13.6.2}$$

This acts as a potential for the macroscopic nominal stress, such that

$$\langle \mathbf{P} \rangle = \frac{\partial \hat{\Psi}}{\partial \langle \mathbf{F} \rangle}, \quad \hat{\Psi} = \hat{\Psi}(\langle \mathbf{F} \rangle, \mathcal{H}). \tag{13.6.3}$$

Indeed,

$$\frac{\partial \hat{\Psi}}{\partial \langle \mathbf{F} \rangle} = \frac{\partial}{\partial \langle \mathbf{F} \rangle} \langle \Psi \rangle = \langle \frac{\partial \Psi}{\partial \langle \mathbf{F} \rangle} \rangle = \langle \frac{\partial \Psi}{\partial \mathbf{F}} \cdot\cdot \frac{\partial \mathbf{F}}{\partial \langle \mathbf{F} \rangle} \rangle = \langle \mathbf{P} \cdot\cdot \boldsymbol{\mathcal{F}} \rangle = \langle \mathbf{P} \rangle. \tag{13.6.4}$$

It is noted that, at fixed $\mathcal{H}$, from Eq. (13.5.1) we have

$$\delta \langle \mathbf{F} \rangle = \boldsymbol{\mathcal{F}} \cdot\cdot \delta \langle \mathbf{F} \rangle, \quad \text{i.e.,} \quad \frac{\partial \mathbf{F}}{\partial \langle \mathbf{F} \rangle} = \boldsymbol{\mathcal{F}}, \tag{13.6.5}$$

which was used after partial differentiation in Eq. (13.6.4). Also, under uniform boundary data,

$$\langle \mathbf{P} \cdot\cdot \boldsymbol{\mathcal{F}} \rangle = \langle \mathbf{P} \rangle, \tag{13.6.6}$$

because

$$\langle \mathbf{P} \rangle \cdot\cdot \delta \langle \mathbf{F} \rangle = \langle \mathbf{P} \cdot\cdot \delta \mathbf{F} \rangle = \langle \mathbf{P} \cdot\cdot \boldsymbol{\mathcal{F}} \cdot\cdot \delta \langle \mathbf{F} \rangle \rangle = \langle \mathbf{P} \cdot\cdot \boldsymbol{\mathcal{F}} \rangle \cdot\cdot \delta \langle \mathbf{F} \rangle. \tag{13.6.7}$$

The local complementary energy Φ, per unit reference volume, is a potential for the local deformation gradient. This is a Legendre transform of Ψ, such that

$$\mathbf{F} = \frac{\partial \Phi}{\partial \mathbf{P}}, \quad \Phi(\mathbf{P}, \mathcal{H}) = \mathbf{P} \cdot\cdot\, \mathbf{F} - \Psi(\mathbf{F}, \mathcal{H}). \tag{13.6.8}$$

The macroscopic free energy, per unit volume of the aggregate macroelement, is a potential for the macroscopic deformation gradient,

$$\langle \mathbf{F} \rangle = \frac{\partial \hat{\Phi}}{\partial \langle \mathbf{P} \rangle}, \quad \hat{\Phi}(\langle \mathbf{P} \rangle, \mathcal{H}) = \langle \mathbf{P} \rangle \cdot\cdot\, \langle \mathbf{F} \rangle - \hat{\Psi}(\langle \mathbf{F} \rangle, \mathcal{H}). \tag{13.6.9}$$

Under conditions allowing the product theorem $\langle \mathbf{P} \cdot\cdot\, \delta \mathbf{F} \rangle = \langle \mathbf{P} \rangle \cdot\cdot\, \delta \langle \mathbf{F} \rangle$ to be used, $\hat{\Phi}$ is the volume average of Φ, i.e.,

$$\hat{\Phi} = \langle \Phi \rangle. \tag{13.6.10}$$

In this case, the potential property of $\hat{\Phi}$ can be demonstrated through

$$\frac{\partial \hat{\Phi}}{\partial \langle \mathbf{P} \rangle} = \frac{\partial}{\partial \langle \mathbf{P} \rangle} \langle \Phi \rangle = \langle \frac{\partial \Phi}{\partial \langle \mathbf{P} \rangle} \rangle = \langle \frac{\partial \Phi}{\partial \mathbf{P}} \cdot\cdot\, \frac{\partial \mathbf{P}}{\partial \langle \mathbf{P} \rangle} \rangle = \langle \mathbf{F} \cdot\cdot\, \boldsymbol{\mathcal{P}} \rangle = \langle \mathbf{F} \rangle. \tag{13.6.11}$$

Again, at fixed $\mathcal{H}$, from Eq. (13.5.2) we have

$$\delta \langle \mathbf{P} \rangle = \boldsymbol{\mathcal{P}} \cdot\cdot\, \delta \langle \mathbf{P} \rangle, \quad \text{i.e.,} \quad \frac{\partial \mathbf{P}}{\partial \langle \mathbf{P} \rangle} = \boldsymbol{\mathcal{P}}, \tag{13.6.12}$$

which was used after partial differentiation in Eq. (13.6.11). In addition, under uniform boundary data,

$$\langle \mathbf{F} \cdot\cdot\, \boldsymbol{\mathcal{P}} \rangle = \langle \mathbf{F} \rangle, \tag{13.6.13}$$

because

$$\langle \mathbf{F} \rangle \cdot\cdot\, \delta \langle \mathbf{P} \rangle = \langle \mathbf{F} \cdot\cdot\, \delta \mathbf{P} \rangle = \langle \mathbf{F} \cdot\cdot\, \boldsymbol{\mathcal{P}} \cdot\cdot\, \delta \langle \mathbf{P} \rangle \rangle = \langle \mathbf{F} \cdot\cdot\, \boldsymbol{\mathcal{P}} \rangle \cdot\cdot\, \delta \langle \mathbf{P} \rangle. \tag{13.6.14}$$

13.7. Macroscopic Elastic Pseudomoduli

The tensor of macroscopic elastic pseudomoduli is defined by

$$[\boldsymbol{\Lambda}] = \frac{\partial^2 \hat{\Psi}}{\partial \langle \mathbf{F} \rangle \otimes \partial \langle \mathbf{F} \rangle} = \frac{\partial \langle \mathbf{P} \rangle}{\partial \langle \mathbf{F} \rangle} = \langle \frac{\partial \mathbf{P}}{\partial \langle \mathbf{F} \rangle} \rangle = \langle \frac{\partial \mathbf{P}}{\partial \mathbf{F}} \cdot\cdot\, \frac{\partial \mathbf{F}}{\partial \langle \mathbf{F} \rangle} \rangle = \langle \boldsymbol{\Lambda} \cdot\cdot\, \boldsymbol{\mathcal{F}} \rangle. \tag{13.7.1}$$

The tensor of local elastic pseudomoduli is $\boldsymbol{\Lambda}$. Along an elastic branch of the material response at microlevel, the rates of $\mathbf{P}$ and $\mathbf{F}$ are related by

$$\dot{\mathbf{P}} = \boldsymbol{\Lambda} \cdot\cdot\, \dot{\mathbf{F}}, \quad \boldsymbol{\Lambda} = \frac{\partial \mathbf{P}}{\partial \mathbf{F}}. \tag{13.7.2}$$

The macroscopic tensor of elastic pseudomoduli $[\mathbf{\Lambda}]$ relates $\langle\dot{\mathbf{P}}\rangle$ and $\langle\dot{\mathbf{F}}\rangle$, such that

$$\langle\dot{\mathbf{P}}\rangle = \langle\mathbf{\Lambda}\cdot\cdot\,\dot{\mathbf{F}}\rangle = [\mathbf{\Lambda}]\cdot\cdot\,\langle\dot{\mathbf{F}}\rangle. \tag{13.7.3}$$

An alternative derivation of the relationship between the local and macroscopic pseudomoduli, given in Eq. (13.7.1) is as follows. First, by substituting Eq. (13.7.3) into Eq. (13.5.2), we have

$$\dot{\mathbf{P}} = \boldsymbol{\mathcal{P}}\cdot\cdot\,\langle\dot{\mathbf{P}}\rangle = \boldsymbol{\mathcal{P}}\cdot\cdot\,[\mathbf{\Lambda}]\cdot\cdot\,\langle\dot{\mathbf{F}}\rangle. \tag{13.7.4}$$

On the other hand, introducing (13.7.2), and then (13.5.1), into Eq. (13.5.2) gives

$$\dot{\mathbf{P}} = \boldsymbol{\mathcal{P}}\cdot\cdot\,\langle\dot{\mathbf{P}}\rangle = \boldsymbol{\mathcal{P}}\cdot\cdot\,\langle\mathbf{\Lambda}\cdot\cdot\,\dot{\mathbf{F}}\rangle = \boldsymbol{\mathcal{P}}\cdot\cdot\,\langle\mathbf{\Lambda}\cdot\cdot\,\boldsymbol{\mathcal{F}}\rangle\cdot\cdot\,\langle\dot{\mathbf{F}}\rangle. \tag{13.7.5}$$

Comparing Eqs. (13.7.4) and (13.7.5), we obtain

$$[\mathbf{\Lambda}] = \langle\mathbf{\Lambda}\cdot\cdot\,\boldsymbol{\mathcal{F}}\rangle. \tag{13.7.6}$$

This shows that the tensor of macroscopic elastic pseudomoduli is a weighted volume average of the tensor of local elastic pseudomoduli $\mathbf{\Lambda}$, the weight being the influence tensor $\boldsymbol{\mathcal{F}}$ of elastic heterogeneity within a representative macroelement. In addition, since

$$\dot{\mathbf{P}} = \mathbf{\Lambda}\cdot\cdot\,\dot{\mathbf{F}} = \mathbf{\Lambda}\cdot\cdot\,\boldsymbol{\mathcal{F}}\;\cdot\cdot\,\langle\dot{\mathbf{F}}\rangle, \tag{13.7.7}$$

by comparing with (13.7.4) we observe that

$$\boldsymbol{\mathcal{P}}\cdot\cdot\,[\mathbf{\Lambda}] = \mathbf{\Lambda} : \boldsymbol{\mathcal{F}}. \tag{13.7.8}$$

The symmetry of elastic response at the microlevel is transmitted to the macrolevel, i.e.,

$$\text{if}\quad \mathbf{\Lambda}^T = \mathbf{\Lambda},\quad \text{then}\quad [\mathbf{\Lambda}]^T = [\mathbf{\Lambda}]. \tag{13.7.9}$$

This does not appear to be evident at first from Eq. (13.7.6) or Eq. (13.7.8). However, since

$$\langle\dot{\mathbf{F}}\cdot\cdot\,\dot{\mathbf{P}}\rangle = \langle\dot{\mathbf{F}}\rangle\cdot\cdot\,\langle\dot{\mathbf{P}}\rangle, \tag{13.7.10}$$

and in view of Eqs. (13.5.1) and (13.5.2) giving

$$\langle\dot{\mathbf{F}}\cdot\cdot\,\dot{\mathbf{P}}\rangle = \langle\dot{\mathbf{F}}\rangle\cdot\cdot\,\langle\,\boldsymbol{\mathcal{F}}^T\cdot\cdot\,\boldsymbol{\mathcal{P}}\rangle\cdot\cdot\,\langle\dot{\mathbf{P}}\rangle, \tag{13.7.11}$$

the comparison with Eq. (13.7.10) establishes

$$\langle\,\boldsymbol{\mathcal{F}}^T\cdot\cdot\,\boldsymbol{\mathcal{P}}\rangle = \boldsymbol{I}. \tag{13.7.12}$$

Therefore, upon taking a trace product of Eq. (13.7.8) with $\boldsymbol{\mathcal{F}}^T$ from the left, and upon the volume averaging over V^0, there follows

$$[\boldsymbol{\Lambda}] = \langle \boldsymbol{\mathcal{F}}^T \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \boldsymbol{\mathcal{F}} \rangle. \tag{13.7.13}$$

This demonstrates that $[\boldsymbol{\Lambda}]$ is indeed symmetric whenever $\boldsymbol{\Lambda}$ is.

When the current configuration is the reference, the previous formulas reduce to

$$\{\dot{\underline{\mathbf{P}}}\} = [\,\underline{\boldsymbol{\Lambda}}\,] \cdot\cdot \{\mathbf{L}\}, \tag{13.7.14}$$

$$\mathbf{L} = \underline{\boldsymbol{\mathcal{F}}} \cdot\cdot \{\mathbf{L}\}, \quad \underline{\mathbf{P}} = \underline{\boldsymbol{\mathcal{P}}} \cdot\cdot \{\underline{\mathbf{P}}\}, \tag{13.7.15}$$

and

$$[\,\underline{\boldsymbol{\Lambda}}\,] = \{\,\underline{\boldsymbol{\mathcal{F}}}^T \cdot\cdot \underline{\boldsymbol{\Lambda}} \cdot\cdot \underline{\boldsymbol{\mathcal{F}}}\,\}. \tag{13.7.16}$$

The underlined symbol indicates that the current configuration is taken for the reference.

13.8. Macroscopic Elastic Pseudocompliances

Suppose that the local elastic pseudomoduli tensor $\boldsymbol{\Lambda}$ has its inverse, the local elastic pseudocompliances tensor $\mathbf{M} = \boldsymbol{\Lambda}^{-1}$ (except possibly at isolated singular points within each crystal grain, whose contribution to volume integrals over the macroelement can be ignored in the micro-to-macro transition; Hill, 1984). We then write

$$\dot{\mathbf{F}} = \mathbf{M} \cdot\cdot \dot{\mathbf{P}}, \tag{13.8.1}$$

where

$$\boldsymbol{\Lambda} \cdot\cdot \mathbf{M} = \mathbf{M} \cdot\cdot \boldsymbol{\Lambda}^{-1} = \boldsymbol{I}. \tag{13.8.2}$$

The macroscopic tensor of elastic pseudocompliances $[\mathbf{M}]$ is introduced by requiring that

$$\langle \dot{\mathbf{F}} \rangle = \langle \mathbf{M} \cdot\cdot \dot{\mathbf{P}} \rangle = [\mathbf{M}] \cdot\cdot \langle \dot{\mathbf{P}} \rangle. \tag{13.8.3}$$

By substituting Eq. (13.8.3) into (13.5.1), we obtain

$$\dot{\mathbf{F}} = \boldsymbol{\mathcal{F}} \cdot\cdot \langle \dot{\mathbf{F}} \rangle = \boldsymbol{\mathcal{F}} \cdot\cdot [\mathbf{M}] \cdot\cdot \langle \dot{\mathbf{P}} \rangle. \tag{13.8.4}$$

On the other hand, introducing (13.7.2), and then (13.5.2), into Eq. (13.5.1) gives

$$\dot{\mathbf{F}} = \boldsymbol{\mathcal{F}} \cdot\cdot \langle \dot{\mathbf{F}} \rangle = \boldsymbol{\mathcal{F}} \cdot\cdot \langle \mathbf{M} \cdot\cdot \dot{\mathbf{P}} \rangle = \boldsymbol{\mathcal{F}} \cdot\cdot \langle \mathbf{M} \cdot\cdot \boldsymbol{\mathcal{P}} \rangle \cdot\cdot \langle \dot{\mathbf{P}} \rangle. \tag{13.8.5}$$

Comparing Eqs. (13.8.4) and (13.8.5) yields

$$[\mathbf{M}] = \langle \mathbf{M} \cdot\cdot\, \boldsymbol{\mathcal{P}} \rangle. \tag{13.8.6}$$

This shows that the tensor of macroscopic elastic pseudocompliances is a weighted volume average of the tensor of local elastic pseudocompliances $\mathbf{M}$, the weight being the influence tensor $\boldsymbol{\mathcal{P}}$ of elastic heterogeneity within a representative macroelement. In addition, since

$$\dot{\mathbf{F}} = \mathbf{M} \cdot\cdot\, \dot{\mathbf{P}} = \mathbf{M} \cdot\cdot\, \boldsymbol{\mathcal{P}} \cdot\cdot\, \langle \dot{\mathbf{P}} \rangle, \tag{13.8.7}$$

by comparing with (13.8.4) there follows

$$\boldsymbol{\mathcal{F}} \cdot\cdot\, [\mathbf{M}] = \mathbf{M} : \boldsymbol{\mathcal{P}}. \tag{13.8.8}$$

We now demonstrate, independently of the proof from the previous section, that the symmetry of elastic response at the microlevel is transmitted to the macrolevel. First, we note that

$$\langle \dot{\mathbf{P}} \cdot\cdot\, \dot{\mathbf{F}} \rangle = \langle \dot{\mathbf{P}} \rangle \cdot\cdot\, \langle \dot{\mathbf{F}} \rangle. \tag{13.8.9}$$

Since, by (13.5.1) and (13.5.2), we have

$$\langle \dot{\mathbf{P}} \cdot\cdot\, \dot{\mathbf{F}} \rangle = \langle \dot{\mathbf{P}} \rangle \cdot\cdot\, \langle \boldsymbol{\mathcal{P}}^T \cdot\cdot\, \boldsymbol{\mathcal{F}} \rangle \cdot\cdot\, \langle \dot{\mathbf{F}} \rangle, \tag{13.8.10}$$

the comparison with Eq. (13.8.9) gives

$$\langle \boldsymbol{\mathcal{P}}^T \cdot\cdot\, \boldsymbol{\mathcal{F}} \rangle = \boldsymbol{I}. \tag{13.8.11}$$

Therefore, upon taking a trace product of Eq. (13.8.8) with $\boldsymbol{\mathcal{P}}^T$ from the left, and upon the volume averaging, we obtain

$$[\mathbf{M}] = \langle \boldsymbol{\mathcal{P}}^T \cdot\cdot\, \mathbf{M} \cdot\cdot\, \boldsymbol{\mathcal{P}} \rangle. \tag{13.8.12}$$

Consequently, if there is a symmetry of elastic response at the microlevel, it is transmitted to the macrolevel, i.e.,

$$\text{if} \quad \mathbf{M}^T = \mathbf{M}, \quad \text{then} \quad [\mathbf{M}]^T = [\mathbf{M}]. \tag{13.8.13}$$

When the macroscopic complementary energy is used to define the elastic pseudocompliances tensor, we can write

$$[\mathbf{M}] = \frac{\partial^2 \hat{\Phi}}{\partial \langle \mathbf{P} \rangle \otimes \partial \langle \mathbf{P} \rangle} = \frac{\partial \langle \mathbf{F} \rangle}{\partial \langle \mathbf{P} \rangle} = \Big\langle \frac{\partial \mathbf{F}}{\partial \langle \mathbf{P} \rangle} \Big\rangle = \Big\langle \frac{\partial \mathbf{F}}{\partial \mathbf{P}} \cdot\cdot\, \frac{\partial \mathbf{P}}{\partial \langle \mathbf{P} \rangle} \Big\rangle = \langle \mathbf{M} \cdot\cdot\, \boldsymbol{\mathcal{P}} \rangle. \tag{13.8.14}$$

13.9. Macroscopic Elastic Moduli

The macroscopic elastic moduli tensor $[\mathbf{\Lambda}_{(1)}]$, corresponding to the macroscopic Lagrangian strain and its conjugate stress, is defined by requiring that

$$[\dot{\mathbf{T}}] = [\mathbf{\Lambda}_{(1)}] : [\dot{\mathbf{E}}]. \tag{13.9.1}$$

To obtain the relationship between $[\mathbf{\Lambda}_{(1)}]$ and $[\mathbf{\Lambda}]$, we use Eq. (13.4.10), which is here conveniently rewritten as

$$\langle \dot{\mathbf{P}} \rangle = \langle \boldsymbol{\mathcal{K}} \rangle^T : [\dot{\mathbf{T}}] + [\boldsymbol{\mathcal{T}}] \cdot\cdot \langle \dot{\mathbf{F}} \rangle. \tag{13.9.2}$$

The rectangular components of the fourth-order tensors $\langle \boldsymbol{\mathcal{K}} \rangle$ and $[\boldsymbol{\mathcal{T}}]$ are

$$\langle \mathcal{K} \rangle_{ijkl} = \frac{1}{2} \left(\delta_{ik} \langle F \rangle_{lj} + \delta_{jk} \langle F \rangle_{li} \right), \quad [\mathcal{T}]_{ijkl} = [T]_{ik}\, \delta_{jl}. \tag{13.9.3}$$

Substitution of Eq. (13.7.3) into Eq. (13.9.2) gives

$$[\mathbf{\Lambda}] = \langle \boldsymbol{\mathcal{K}} \rangle^T : [\mathbf{\Lambda}_{(1)}] : \langle \boldsymbol{\mathcal{K}} \rangle + [\boldsymbol{\mathcal{T}}]. \tag{13.9.4}$$

Expressed in rectangular components, this is

$$[\Lambda]_{ijkl} = [\Lambda_{(1)}]_{ipkq} \langle F \rangle_{jp} \langle F \rangle_{lq} + [T]_{ik}\, \delta_{jl}. \tag{13.9.5}$$

Clearly, the symmetry $ij \leftrightarrow kl$ of the macroscopic pseudomoduli imposes the same symmetry for the macroscopic moduli, and *vice versa*. Also, recall the symmetry $\boldsymbol{\mathcal{T}}^T = \boldsymbol{\mathcal{T}}$.

When the current configuration is the reference, Eq. (13.9.4) reduces to

$$[\underline{\mathbf{\Lambda}}] = [\underline{\mathbf{\Lambda}}_{(1)}] + [\underline{\boldsymbol{\mathcal{T}}}], \tag{13.9.6}$$

with the component form

$$[\underline{\Lambda}]_{ijkl} = [\underline{\Lambda}_{(1)}]_{ijkl} + \{\boldsymbol{\sigma}\}_{ik}\, \delta_{jl}. \tag{13.9.7}$$

In addition, Eq. (13.9.1) becomes

$$\{\overset{\triangle}{\underline{\boldsymbol{\tau}}}\} = [\underline{\mathbf{\Lambda}}_{(1)}] : \{\mathbf{D}\}. \tag{13.9.8}$$

13.10. Plastic Increment of Macroscopic Nominal Stress

The increment of macroscopic nominal stress can be partitioned into elastic and plastic parts as

$$\mathrm{d}\langle \mathbf{P} \rangle = \mathrm{d}^{\mathrm{e}} \langle \mathbf{P} \rangle + \mathrm{d}^{\mathrm{p}} \langle \mathbf{P} \rangle. \tag{13.10.1}$$

The elastic part is defined by

$$\mathrm{d}^{\mathrm{e}} \langle \mathbf{P} \rangle = [\mathbf{\Lambda}] \cdot\cdot \mathrm{d} \langle \mathbf{F} \rangle. \tag{13.10.2}$$

The remaining part,

$$\mathrm{d}^{\mathrm{p}}\langle \mathbf{P} \rangle = \mathrm{d}\langle \mathbf{P} \rangle - [\boldsymbol{\Lambda}] \cdot\cdot\, \mathrm{d}\langle \mathbf{F} \rangle, \tag{13.10.3}$$

is the plastic part of the increment $\mathrm{d}\langle \mathbf{P} \rangle$. The macroscopic elastoplastic increment of the deformation gradient is $\mathrm{d}\langle \mathbf{F} \rangle$.

It is of interest to establish the relationship between the plastic increments of macroscopic and microscopic (local) nominal stress, $\mathrm{d}^{\mathrm{p}}\langle \mathbf{P} \rangle$ and $\mathrm{d}^{\mathrm{p}}\mathbf{P}$. To that goal, consider the volume average of the trace product between an elastic unloading increment of the local deformation gradient $\delta\mathbf{F}$ and the plastic increment of the local nominal stress $\mathrm{d}^{\mathrm{p}}\mathbf{P}$, i.e.,

$$\langle \delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P} \rangle = \langle \delta\mathbf{F} \cdot\cdot\, (\mathrm{d}\mathbf{P} - \boldsymbol{\Lambda} \cdot\cdot\, \mathrm{d}\mathbf{F}) \rangle = \langle \delta\mathbf{F} \cdot\cdot\, \mathrm{d}\mathbf{P} \rangle - \langle \delta\mathbf{F} \cdot\cdot\, \boldsymbol{\Lambda} \cdot\cdot\, \mathrm{d}\mathbf{F} \rangle. \tag{13.10.4}$$

Since $\mathrm{d}\mathbf{F}$ and $\delta\mathbf{F}$ are kinematically admissible, and $\mathrm{d}\mathbf{P}$ and $\delta\mathbf{F} \cdot\cdot\, \boldsymbol{\Lambda}$ are statically admissible fields, we can use the product theorem of Section 13.3 to write

$$\langle \delta\mathbf{F} \cdot\cdot\, \mathrm{d}\mathbf{P} \rangle = \langle \delta\mathbf{F} \rangle \cdot\cdot\, \langle \mathrm{d}\mathbf{P} \rangle = \delta\langle \mathbf{F} \rangle \cdot\cdot\, \mathrm{d}\langle \mathbf{P} \rangle, \tag{13.10.5}$$

$$\langle \delta\mathbf{F} \cdot\cdot\, \boldsymbol{\Lambda} \cdot\cdot\, \mathrm{d}\mathbf{F} \rangle = \langle \delta\mathbf{F} \cdot\cdot\, \boldsymbol{\Lambda} \rangle \cdot\cdot\, \mathrm{d}\langle \mathbf{F} \rangle = \delta\langle \mathbf{F} \rangle \cdot\cdot\, \langle \boldsymbol{\mathcal{F}}^T \cdot\cdot\, \boldsymbol{\Lambda} \rangle \cdot\cdot\, \mathrm{d}\langle \mathbf{F} \rangle. \tag{13.10.6}$$

Upon substitution into Eq. (13.10.4), there follows

$$\langle \delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P} \rangle = \delta\langle \mathbf{F} \rangle \cdot\cdot\, (\mathrm{d}\langle \mathbf{P} \rangle - [\boldsymbol{\Lambda}] \cdot\cdot\, \mathrm{d}\langle \mathbf{F} \rangle). \tag{13.10.7}$$

Recall that $[\boldsymbol{\Lambda}]$ is symmetric, and

$$\delta\mathbf{F} = \boldsymbol{\mathcal{F}} \cdot\cdot\, \delta\langle \mathbf{F} \rangle = \delta\langle \mathbf{F} \rangle \cdot\cdot\, \boldsymbol{\mathcal{F}}^T, \tag{13.10.8}$$

so that

$$[\boldsymbol{\Lambda}] = \langle \boldsymbol{\Lambda} \cdot\cdot\, \boldsymbol{\mathcal{F}} \rangle = \langle \boldsymbol{\mathcal{F}}^T \cdot\cdot\, \boldsymbol{\Lambda} \rangle. \tag{13.10.9}$$

Also note that

$$\langle \mathrm{d}\mathbf{P} \rangle = \mathrm{d}\langle \mathbf{P} \rangle, \quad \langle \mathrm{d}\mathbf{F} \rangle = \mathrm{d}\langle \mathbf{F} \rangle, \tag{13.10.10}$$

and likewise for δ increments. Consequently,

$$\langle \delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P} \rangle = \delta\langle \mathbf{F} \rangle \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\langle \mathbf{P} \rangle. \tag{13.10.11}$$

Furthermore,

$$\langle \delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P} \rangle = \delta\langle \mathbf{F} \rangle \cdot\cdot\, \mathrm{d}\langle \mathbf{P} \rangle - \delta\langle \mathbf{P} \rangle \cdot\cdot\, \mathrm{d}\langle \mathbf{F} \rangle, \tag{13.10.12}$$

which can be easily verified by substituting $\delta\langle\mathbf{P}\rangle = \delta\langle\mathbf{F}\rangle \cdot\cdot [\boldsymbol{\Lambda}]$, and by using Eq. (13.10.3).

On the other hand, from Eq. (13.5.1) we directly obtain

$$\langle \delta\mathbf{F} \cdot\cdot \mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle = \delta\langle\mathbf{F}\rangle \cdot\cdot \langle \boldsymbol{\mathcal{F}}^T \cdot\cdot \mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle. \tag{13.10.13}$$

The comparison of Eqs. (13.10.11) and (13.10.13) establishes

$$\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = \langle \boldsymbol{\mathcal{F}}^T \cdot\cdot \mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle. \tag{13.10.14}$$

Therefore, the plastic part of the increment of macroscopic nominal stress is a weighted volume average of the plastic part of the increment of local nominal stress (Hill, 1984; Havner, 1992).

13.10.1. Plastic Potential and Normality Rule

From Eq. (13.10.11) it follows, if the normality rule applies at the microlevel, it is transmitted to the macrolevel, i.e.,

$$\delta\mathbf{F} \cdot\cdot \mathrm{d}^{\mathrm{p}}\mathbf{P} > 0 \quad \text{implies} \quad \delta\langle\mathbf{F}\rangle \cdot\cdot \mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle > 0. \tag{13.10.15}$$

We recall from Section 12.7 that $-\sum(\tau^\alpha\,\mathrm{d}\gamma^\alpha)$ acts as the plastic potential for $\mathrm{d}^{\mathrm{p}}\mathbf{P}$ over an elastic domain in $\mathbf{F}$ space, such that

$$\mathrm{d}^{\mathrm{p}}\mathbf{P} = -\frac{\partial}{\partial\mathbf{F}} \sum_{\alpha=1}^{n} (\tau^\alpha\,\mathrm{d}\gamma^\alpha). \tag{13.10.16}$$

The partial differentiation is performed at the fixed slip and slip increments $\mathrm{d}\gamma^\alpha$. The local resolved shear stress on the α slip system is τ^α, and n is the number of active slip systems. Substitution into Eq. (13.10.14) gives

$$\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = -\langle \boldsymbol{\mathcal{F}}^T \cdot\cdot \frac{\partial}{\partial\mathbf{F}} \sum_{\alpha=1}^{n} (\tau^\alpha\,\mathrm{d}\gamma^\alpha)\rangle. \tag{13.10.17}$$

Since, at the fixed slip,

$$\frac{\partial}{\partial\langle\mathbf{F}\rangle} = \frac{\partial}{\partial\mathbf{F}} \cdot\cdot \frac{\partial\mathbf{F}}{\partial\langle\mathbf{F}\rangle} = \frac{\partial}{\partial\mathbf{F}} \cdot\cdot \boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}^T \cdot\cdot \frac{\partial}{\partial\mathbf{F}}\,, \tag{13.10.18}$$

Equation (13.10.17) becomes

$$\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = -\frac{\partial}{\partial\langle\mathbf{F}\rangle} \langle \sum_{\alpha=1}^{n} \tau^\alpha\,\mathrm{d}\gamma^\alpha\rangle. \tag{13.10.19}$$

This shows that $-\langle\sum\tau^\alpha\,\mathrm{d}\gamma^\alpha\rangle$ is a plastic potential for $\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle$ over an elastic domain in $\langle\mathbf{F}\rangle$ space (Hill and Rice, 1973; Havner, 1986). Since the number n of active slip systems changes from grain to grain, depending on its orientation and the state of hardening, the sum in Eq. (13.10.19) is kept within the $\langle\ \rangle$ brackets, i.e., within the volume integral appearing in the definition of the $\langle\ \rangle$ average.

13.10.2. Local Residual Increment of Nominal Stress

The plastic part of the increment of macroscopic nominal stress $\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle$ in Eq. (13.10.3) gives the macroscopic stress decrement after a cycle (application and removal) of the increment of macroscopic deformation gradient $\mathrm{d}\langle\mathbf{F}\rangle$. At the microlevel, however, the local decrement of stress $\mathrm{d}^{\mathrm{s}}\mathbf{P}$, after a cycle of the increment of macroscopic deformation gradient $\mathrm{d}\langle\mathbf{F}\rangle$, is obtained by subtracting from $\mathrm{d}\mathbf{P}$ the local stress increment associated with an imagined (conceptual) elastic removal of $\mathrm{d}\langle\mathbf{F}\rangle$. This is $\boldsymbol{\mathcal{P}}\cdot\cdot[\boldsymbol{\Lambda}]\cdot\cdot\mathrm{d}\langle\mathbf{F}\rangle$, so that (Hill, 1984; Havner, 1992)

$$\mathrm{d}^{\mathrm{s}}\mathbf{P}=\mathrm{d}\mathbf{P}-\boldsymbol{\mathcal{P}}\cdot\cdot[\boldsymbol{\Lambda}]\cdot\cdot\mathrm{d}\langle\mathbf{F}\rangle. \tag{13.10.20}$$

Upon a conceptual elastic removal of macroscopic $\mathrm{d}\langle\mathbf{F}\rangle$, the residual increment of the deformation gradient at microscopic level would be

$$\mathrm{d}^{\mathrm{s}}\mathbf{F}=\mathrm{d}\mathbf{F}-\boldsymbol{\mathcal{F}}\cdot\cdot\mathrm{d}\langle\mathbf{F}\rangle. \tag{13.10.21}$$

Recall from Eq. (13.7.8) that $\boldsymbol{\mathcal{P}}\cdot\cdot[\boldsymbol{\Lambda}]=\boldsymbol{\Lambda}:\boldsymbol{\mathcal{F}}$, so that

$$\mathrm{d}\mathbf{P}-\mathrm{d}^{\mathrm{s}}\mathbf{P}=\boldsymbol{\Lambda}\cdot\cdot(\mathrm{d}\mathbf{F}-\mathrm{d}^{\mathrm{s}}\mathbf{F}). \tag{13.10.22}$$

Note that $\mathrm{d}^{\mathrm{s}}\mathbf{F}$ is kinematically admissible field (because $\mathrm{d}\mathbf{F}$ and $\boldsymbol{\mathcal{F}}\cdot\cdot\mathrm{d}\langle\mathbf{F}\rangle$ are), while $\mathrm{d}^{\mathrm{s}}\mathbf{P}$ is statically admissible field (because $\mathrm{d}\mathbf{P}$ and $\boldsymbol{\Lambda}\cdot\cdot\boldsymbol{\mathcal{F}}\cdot\cdot\mathrm{d}\langle\mathbf{F}\rangle$ are).

The local increment of stress $\mathrm{d}^{\mathrm{s}}\mathbf{P}$ is different from the local plastic increment

$$\mathrm{d}^{\mathrm{p}}\mathbf{P}=\mathrm{d}\mathbf{P}-\boldsymbol{\Lambda}\cdot\cdot\mathrm{d}\mathbf{F}, \tag{13.10.23}$$

associated with a cycle of the increment of local deformation gradient $\mathrm{d}\mathbf{F}$. They are related by

$$\mathrm{d}^{\mathrm{s}}\mathbf{P}-\mathrm{d}^{\mathrm{p}}\mathbf{P}=\boldsymbol{\Lambda}\cdot\cdot\mathrm{d}^{\mathrm{s}}\mathbf{F}. \tag{13.10.24}$$

Also, it can be easily verified that

$$\mathrm{d}^{\mathrm{s}}\mathbf{F}-\mathrm{d}^{\mathrm{p}}\mathbf{F}=\mathbf{M}\cdot\cdot\mathrm{d}^{\mathrm{s}}\mathbf{P}. \tag{13.10.25}$$

On the other hand,

$$\langle\mathrm{d}^{\mathrm{s}}\mathbf{P}\rangle=\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle,\quad \langle\mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle=\mathbf{0}, \tag{13.10.26}$$

which follow from Eqs. (13.10.20) and (13.10.21), and $\langle \boldsymbol{\mathcal{F}} \rangle = \langle \boldsymbol{\mathcal{P}} \rangle = \boldsymbol{I}$.

Since $\mathrm{d}^{\mathrm{s}}\mathbf{F}$ is kinematically and $\mathrm{d}^{\mathrm{s}}\mathbf{P}$ is statically admissible field, by the theorem on product averages we obtain

$$\langle \mathrm{d}^{\mathrm{s}}\mathbf{P} \cdot\cdot\, \mathrm{d}^{\mathrm{s}}\mathbf{F} \rangle = \langle \mathrm{d}^{\mathrm{s}}\mathbf{P} \rangle \cdot\cdot \langle \mathrm{d}^{\mathrm{s}}\mathbf{F} \rangle = 0. \tag{13.10.27}$$

There is also an identity for the volume averages of the trace products

$$\langle \delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{s}}\mathbf{P} \rangle = \langle \delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P} \rangle, \tag{13.10.28}$$

where $\delta\mathbf{F}$ is an increment of the local deformation gradient along purely elastic branch of the response. Indeed,

$$\begin{aligned} \langle \delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{s}}\mathbf{P} \rangle &= \langle \delta\mathbf{F} \cdot\cdot (\mathrm{d}\mathbf{P} - \boldsymbol{\mathcal{P}} \cdot\cdot [\boldsymbol{\Lambda}] \cdot\cdot \mathrm{d}\langle \mathbf{F} \rangle) \rangle \\ &= \delta\langle \mathbf{F} \rangle \cdot\cdot\, \mathrm{d}\langle \mathbf{P} \rangle - \langle \delta\mathbf{F} \cdot\cdot \boldsymbol{\mathcal{P}} \rangle \cdot\cdot [\boldsymbol{\Lambda}] \cdot\cdot\, \mathrm{d}\langle \mathbf{F} \rangle. \end{aligned} \tag{13.10.29}$$

It is observed that

$$\langle \delta\mathbf{F} \cdot\cdot \boldsymbol{\mathcal{P}} \rangle = \langle \delta\langle \mathbf{F} \rangle \cdot\cdot \boldsymbol{\mathcal{F}}^T \cdot\cdot \boldsymbol{\mathcal{P}} \rangle = \delta\langle \mathbf{F} \rangle \cdot\cdot \langle \boldsymbol{\mathcal{F}}^T \cdot\cdot \boldsymbol{\mathcal{P}} \rangle = \delta\langle \mathbf{F} \rangle, \tag{13.10.30}$$

because $\langle \boldsymbol{\mathcal{F}}^T \cdot\cdot \boldsymbol{\mathcal{P}} \rangle = \boldsymbol{I}$, by (13.7.12). Thus, Eq. (13.10.29) becomes

$$\langle \delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{s}}\mathbf{P} \rangle = \delta\langle \mathbf{F} \rangle \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\langle \mathbf{P} \rangle. \tag{13.10.31}$$

In view of Eq. (13.10.11), this reduces to Eq. (13.10.28). Furthermore, since $\langle \mathrm{d}^{\mathrm{s}}\mathbf{P} \rangle = \mathrm{d}^{\mathrm{p}}\langle \mathbf{P} \rangle$, Eq. (13.10.31) gives

$$\langle \delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{s}}\mathbf{P} \rangle = \delta\langle \mathbf{F} \rangle \cdot\cdot \langle \mathrm{d}^{\mathrm{s}}\mathbf{P} \rangle. \tag{13.10.32}$$

This was anticipated from the theorem on product averages, because $\delta\mathbf{F}$ is kinematically admissible and $\mathrm{d}^{\mathrm{s}}\mathbf{P}$ is statically admissible field.

The following two identities are noted

$$\langle \mathrm{d}^{\mathrm{s}}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F} \rangle = \langle \mathrm{d}^{\mathrm{s}}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{s}}\mathbf{F} \rangle, \tag{13.10.33}$$

$$\langle \mathrm{d}^{\mathrm{s}}\mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P} \rangle = \langle \mathrm{d}^{\mathrm{s}}\mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot\, \mathrm{d}^{\mathrm{s}}\mathbf{P} \rangle. \tag{13.10.34}$$

They follow from Eqs. (13.10.24), (13.10.25), and (13.11.26).

13.11. Plastic Increment of Macroscopic Deformation Gradient

Dually to the analysis from the previous section, the increment of macroscopic deformation gradient can be partitioned into its elastic and plastic parts as

$$\mathrm{d}\langle \mathbf{F} \rangle = \mathrm{d}^{\mathrm{e}}\langle \mathbf{F} \rangle + \mathrm{d}^{\mathrm{p}}\langle \mathbf{F} \rangle. \tag{13.11.1}$$

The elastic part is defined by

$$\mathrm{d}^{\mathrm{e}}\langle\mathbf{F}\rangle = [\mathbf{M}]\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle, \tag{13.11.2}$$

while

$$\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle = \mathrm{d}\langle\mathbf{F}\rangle - [\mathbf{M}]\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle \tag{13.11.3}$$

is the plastic part of the increment $\mathrm{d}\langle\mathbf{F}\rangle$.

To establish the relationship between the plastic increments of macroscopic and microscopic deformation gradients, $\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle$ and $\mathrm{d}^{\mathrm{p}}\mathbf{F}$, consider the volume average of the trace product between an elastic unloading increment of the local nominal stress $\delta\mathbf{P}$ and the plastic increment of the local deformation gradient $\mathrm{d}^{\mathrm{p}}\mathbf{F}$, i.e.,

$$\langle\delta\mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{F}\rangle = \langle\delta\mathbf{P}\cdot\cdot(\mathrm{d}\mathbf{F} - \mathbf{M}\cdot\cdot\,\mathrm{d}\mathbf{P})\rangle = \langle\delta\mathbf{P}\cdot\cdot\,\mathrm{d}\mathbf{F}\rangle - \langle\delta\mathbf{P}\cdot\cdot\,\mathbf{M}\cdot\cdot\,\mathrm{d}\mathbf{P}\rangle. \tag{13.11.4}$$

Since $\mathrm{d}\mathbf{P}$ and $\delta\mathbf{P}$ are statically admissible, and $\mathrm{d}\mathbf{F}$ and $\delta\mathbf{P}\cdot\cdot\,\mathbf{M}$ are kinematically admissible fields, we can use the product theorem of Section 13.3 to write

$$\langle\delta\mathbf{P}\cdot\cdot\,\mathrm{d}\mathbf{F}\rangle = \delta\langle\mathbf{P}\rangle\cdot\cdot\,\mathrm{d}\langle\mathbf{F}\rangle, \tag{13.11.5}$$

$$\langle\delta\mathbf{P}\cdot\cdot\,\mathbf{M}\cdot\cdot\,\mathrm{d}\mathbf{P}\rangle = \langle\delta\mathbf{P}\cdot\cdot\,\mathbf{M}\rangle\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle = \delta\langle\mathbf{P}\rangle\cdot\cdot\langle\boldsymbol{\mathcal{P}}^T\cdot\cdot\,\mathbf{M}\rangle\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle. \tag{13.11.6}$$

Upon substitution into Eq. (13.11.4), we obtain

$$\langle\delta\mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{F}\rangle = \delta\langle\mathbf{P}\rangle\cdot\cdot\left(\mathrm{d}\langle\mathbf{F}\rangle - [\mathbf{M}]\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle\right). \tag{13.11.7}$$

Recall that $[\mathbf{M}]$ is symmetric, and

$$\delta\mathbf{P} = \boldsymbol{\mathcal{P}}\cdot\cdot\,\delta\langle\mathbf{P}\rangle = \delta\langle\mathbf{P}\rangle\cdot\cdot\,\boldsymbol{\mathcal{P}}^T, \tag{13.11.8}$$

so that

$$[\mathbf{M}] = \langle\mathbf{M}\cdot\cdot\,\boldsymbol{\mathcal{P}}\rangle = \langle\boldsymbol{\mathcal{P}}^T\cdot\cdot\,\mathbf{M}\rangle. \tag{13.11.9}$$

Consequently,

$$\langle\delta\mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{F}\rangle = \delta\langle\mathbf{P}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle. \tag{13.11.10}$$

Note that

$$\langle\delta\mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{F}\rangle = \delta\langle\mathbf{P}\rangle\cdot\cdot\,\mathrm{d}\langle\mathbf{F}\rangle - \delta\langle\mathbf{F}\rangle\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle, \tag{13.11.11}$$

which can be easily verified by substituting $\delta\langle\mathbf{F}\rangle = \delta\langle\mathbf{P}\rangle\cdot\cdot[\mathbf{M}]$, and by using Eq. (13.11.3).

On the other hand, from (13.5.2) we have

$$\langle \delta \mathbf{P} \cdot\cdot \mathrm{d}^{\mathrm{p}}\mathbf{F} \rangle = \delta \langle \mathbf{P} \rangle \cdot\cdot \langle \boldsymbol{\mathcal{P}}^T \cdot\cdot \mathrm{d}^{\mathrm{p}}\mathbf{F} \rangle. \tag{13.11.12}$$

Comparison of Eqs. (13.11.10) and (13.11.12) yields

$$\mathrm{d}^{\mathrm{p}}\langle \mathbf{F} \rangle = \langle \boldsymbol{\mathcal{P}}^T \cdot\cdot \mathrm{d}^{\mathrm{p}}\mathbf{F} \rangle. \tag{13.11.13}$$

Therefore, the plastic part of the increment of macroscopic deformation gradient is a weighted volume average of the plastic part of the increment of local deformation gradient.

13.11.1. Plastic Potential and Normality Rule

From Eq. (13.11.10) it follows, if the normality rule applies at the microlevel, it is transmitted to the macrolevel, i.e.,

$$\delta \mathbf{P} \cdot\cdot \mathrm{d}^{\mathrm{p}}\mathbf{F} < 0 \quad \text{implies} \quad \delta \langle \mathbf{P} \rangle \cdot\cdot \mathrm{d}^{\mathrm{p}}\langle \mathbf{F} \rangle < 0. \tag{13.11.14}$$

From Section 12.7 we recall that $\sum(\tau^\alpha \, \mathrm{d}\gamma^\alpha)$ acts as a plastic potential for $\mathrm{d}^{\mathrm{p}}\mathbf{F}$ over an elastic domain in $\mathbf{P}$ space, such that

$$\mathrm{d}^{\mathrm{p}}\mathbf{F} = \frac{\partial}{\partial \mathbf{P}} \sum_{\alpha=1}^{n} (\tau^\alpha \, \mathrm{d}\gamma^\alpha). \tag{13.11.15}$$

The partial differentiation is performed at the fixed slip and slip increments $\mathrm{d}\gamma^\alpha$. Substitution into Eq. (13.11.13) gives

$$\mathrm{d}^{\mathrm{p}}\langle \mathbf{F} \rangle = \langle \boldsymbol{\mathcal{P}}^T \cdot\cdot \frac{\partial}{\partial \mathbf{P}} \sum_{\alpha=1}^{n} (\tau^\alpha \, \mathrm{d}\gamma^\alpha) \rangle. \tag{13.11.16}$$

Since, at the fixed slip,

$$\frac{\partial}{\partial \langle \mathbf{P} \rangle} = \frac{\partial}{\partial \mathbf{P}} \cdot\cdot \frac{\partial \mathbf{P}}{\partial \langle \mathbf{P} \rangle} = \frac{\partial}{\partial \mathbf{P}} \cdot\cdot \boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}^T \cdot\cdot \frac{\partial}{\partial \mathbf{P}}, \tag{13.11.17}$$

Equation (13.11.16) becomes

$$\mathrm{d}^{\mathrm{p}}\langle \mathbf{F} \rangle = \frac{\partial}{\partial \langle \mathbf{P} \rangle} \langle \sum_{\alpha=1}^{n} \tau^\alpha \, \mathrm{d}\gamma^\alpha \rangle. \tag{13.11.18}$$

This shows that $\langle \sum \tau^\alpha \, \mathrm{d}\gamma^\alpha \rangle$ is a plastic potential for $\mathrm{d}^{\mathrm{p}}\langle \mathbf{F} \rangle$ over an elastic domain in $\langle \mathbf{P} \rangle$ space.

13.11.2. Local Residual Increment of Deformation Gradient

The plastic part of the increment of macroscopic deformation gradient $\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle$ in Eq. (13.11.3) represents a residual increment of macroscopic deformation gradient after a cycle of the increment of macroscopic nominal stress $\mathrm{d}\langle\mathbf{P}\rangle$. At the microlevel, however, the local residual increment of deformation gradient $\mathrm{d}^{\mathrm{s}}\mathbf{F}$, left upon a cycle of $\mathrm{d}\langle\mathbf{P}\rangle$, is obtained by subtracting from $\mathrm{d}\mathbf{F}$ the local deformation gradient increment associated with an imagined elastic removal of $\mathrm{d}\langle\mathbf{P}\rangle$. This is $\boldsymbol{\mathcal{F}}\cdot\cdot[\mathbf{M}]\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle$, so that

$$\mathrm{d}^{\mathrm{r}}\mathbf{F} = \mathrm{d}\mathbf{F} - \boldsymbol{\mathcal{F}}\cdot\cdot[\mathbf{M}]\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle. \tag{13.11.19}$$

Upon a conceptual elastic removal of macroscopic $\mathrm{d}\langle\mathbf{P}\rangle$, the residual change of the local nominal stress would be

$$\mathrm{d}^{\mathrm{r}}\mathbf{P} = \mathrm{d}\mathbf{P} - \boldsymbol{\mathcal{P}}\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle, \tag{13.11.20}$$

since $\boldsymbol{\mathcal{P}}\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle$ is the local stress due to $\mathrm{d}\langle\mathbf{P}\rangle$ in an imagined elastic response. Recall from Eq. (13.8.8) that $\boldsymbol{\mathcal{F}}\cdot\cdot[\mathbf{M}] = \mathbf{M} : \boldsymbol{\mathcal{P}}$, so that

$$\mathrm{d}\mathbf{F} - \mathrm{d}^{\mathrm{r}}\mathbf{F} = \mathbf{M}\cdot\cdot\left(\mathrm{d}\mathbf{P} - \mathrm{d}^{\mathrm{r}}\mathbf{P}\right). \tag{13.11.21}$$

Note that $\mathrm{d}^{\mathrm{r}}\mathbf{P}$ is statically admissible field (because $\mathrm{d}\mathbf{P}$ and $\boldsymbol{\mathcal{P}}\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle$ are), while $\mathrm{d}^{\mathrm{r}}\mathbf{F}$ is kinematically admissible field (because $\mathrm{d}\mathbf{F}$ and $\mathbf{M}\cdot\cdot\boldsymbol{\mathcal{P}}\cdot\cdot\,\mathrm{d}\langle\mathbf{P}\rangle$ are).

The local increment of deformation gradient $\mathrm{d}^{\mathrm{r}}\mathbf{F}$ is different from the local plastic increment

$$\mathrm{d}^{\mathrm{p}}\mathbf{F} = \mathrm{d}\mathbf{F} - \mathbf{M}\cdot\cdot\,\mathrm{d}\mathbf{P}, \tag{13.11.22}$$

associated with a cycle of the increment of local nominal stress $\mathrm{d}\mathbf{P}$. They are related by

$$\mathrm{d}^{\mathrm{r}}\mathbf{F} - \mathrm{d}^{\mathrm{p}}\mathbf{F} = \mathbf{M}\cdot\cdot\,\mathrm{d}^{\mathrm{r}}\mathbf{P}. \tag{13.11.23}$$

In addition, we have

$$\mathrm{d}^{\mathrm{r}}\mathbf{P} - \mathrm{d}^{\mathrm{p}}\mathbf{P} = \boldsymbol{\Lambda}\cdot\cdot\,\mathrm{d}^{\mathrm{r}}\mathbf{F}. \tag{13.11.24}$$

In general, neither $\mathrm{d}^{\mathrm{p}}\mathbf{F}$ is kinematically admissible, nor $\mathrm{d}^{\mathrm{p}}\mathbf{P}$ is statically admissible field. On the other hand,

$$\langle\mathrm{d}^{\mathrm{r}}\mathbf{F}\rangle = \mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle, \quad \langle\mathrm{d}^{\mathrm{r}}\mathbf{P}\rangle = \mathbf{0}, \tag{13.11.25}$$

which follow from Eqs. (13.11.19) and (13.11.20), and $\langle\boldsymbol{\mathcal{P}}\rangle = \langle\boldsymbol{\mathcal{F}}\rangle = \boldsymbol{I}$.

Since $\mathrm{d^r}\mathbf{F}$ is kinematically and $\mathrm{d^r}\mathbf{P}$ is statically admissible field, by the theorem on product averages we can write

$$\langle \mathrm{d^r}\mathbf{P} \cdot\cdot\, \mathrm{d^r}\mathbf{F}\rangle = \langle \mathrm{d^r}\mathbf{P}\rangle \cdot\cdot \langle \mathrm{d^r}\mathbf{F}\rangle = 0. \tag{13.11.26}$$

There is also an identity for the volume averages of the trace products

$$\langle \delta\mathbf{P} \cdot\cdot\, \mathrm{d^r}\mathbf{F}\rangle = \langle \delta\mathbf{P} \cdot\cdot\, \mathrm{d^p}\mathbf{F}\rangle, \tag{13.11.27}$$

where $\delta\mathbf{P}$ is an increment of the local nominal stress along purely elastic branch of the response. Indeed, by an analogous derivation as in Subsection 13.10.2, there follows

$$\begin{aligned}\langle \delta\mathbf{P} \cdot\cdot\, \mathrm{d^r}\mathbf{F}\rangle &= \langle \delta\mathbf{P} \cdot\cdot (\mathrm{d}\mathbf{F} - \boldsymbol{\mathcal{F}} \cdot\cdot [\mathbf{M}] \cdot\cdot\, \mathrm{d}\langle\mathbf{P}\rangle)\rangle \\ &= \delta\langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle - \langle \delta\mathbf{P} \cdot\cdot \boldsymbol{\mathcal{F}}\rangle \cdot\cdot [\mathbf{M}] \cdot\cdot\, \mathrm{d}\langle\mathbf{P}\rangle.\end{aligned} \tag{13.11.28}$$

Furthermore,

$$\langle \delta\mathbf{P} \cdot\cdot \boldsymbol{\mathcal{F}}\rangle = \langle \delta\langle\mathbf{P}\rangle \cdot\cdot \boldsymbol{\mathcal{P}}^T \cdot\cdot \boldsymbol{\mathcal{F}}\rangle = \delta\langle\mathbf{P}\rangle \cdot\cdot \langle \boldsymbol{\mathcal{P}}^T \cdot\cdot \boldsymbol{\mathcal{F}}\rangle = \delta\langle\mathbf{P}\rangle, \tag{13.11.29}$$

because $\langle \boldsymbol{\mathcal{P}}^T \cdot\cdot \boldsymbol{\mathcal{F}}\rangle = \boldsymbol{I}$, by Eq. (13.8.11). Thus, Eq. (13.11.28) becomes

$$\langle \delta\mathbf{P} \cdot\cdot\, \mathrm{d^r}\mathbf{F}\rangle = \delta\langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d^p}\langle\mathbf{F}\rangle. \tag{13.11.30}$$

In view of Eq. (13.11.10) this reduces to Eq. (13.11.27). Also, since $\langle \mathrm{d^r}\mathbf{F}\rangle = \mathrm{d^p}\langle\mathbf{F}\rangle$, Eq. (13.11.30) gives

$$\langle \delta\mathbf{P} \cdot\cdot\, \mathrm{d^r}\mathbf{F}\rangle = \delta\langle\mathbf{P}\rangle \cdot\cdot \langle \mathrm{d^r}\mathbf{F}\rangle. \tag{13.11.31}$$

This was anticipated from the theorem on product averages, because $\delta\mathbf{P}$ is statically admissible and $\mathrm{d^r}\mathbf{F}$ is kinematically admissible field.

The following two identities, which follow from Eqs. (13.11.23), (13.11.24), and (13.11.26), are noted

$$\langle \mathrm{d^r}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot\, \mathrm{d^p}\mathbf{F}\rangle = \langle \mathrm{d^r}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot\, \mathrm{d^r}\mathbf{F}\rangle, \tag{13.11.32}$$

$$\langle \mathrm{d^r}\mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot\, \mathrm{d^p}\mathbf{P}\rangle = \langle \mathrm{d^r}\mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot\, \mathrm{d^r}\mathbf{P}\rangle. \tag{13.11.33}$$

By comparing the results of this subsection with those from the Subsection 13.10.2, it can be easily verified that

$$\mathrm{d^r}\mathbf{P} - \mathrm{d^s}\mathbf{P} = \boldsymbol{\Lambda} \cdot\cdot (\mathrm{d^r}\mathbf{F} - \mathrm{d^s}\mathbf{F})\,. \tag{13.11.34}$$

The local residual quantities here discussed are of interest in the analysis of the work and energy-related macroscopic quantities considered in Section 13.14.

13.12. Plastic Increment of Macroscopic Piola–Kirchhoff Stress

The increment of the macroscopic symmetric Piola–Kirchhoff stress can be partitioned into its elastic and plastic parts, such that

$$\mathrm{d}[\mathbf{T}] = \mathrm{d}^{\mathrm{e}}[\mathbf{T}] + \mathrm{d}^{\mathrm{p}}[\mathbf{T}]. \tag{13.12.1}$$

The elastic part is defined by

$$\mathrm{d}^{\mathrm{e}}[\mathbf{T}] = [\mathbf{\Lambda}_{(1)}] : \mathrm{d}[\mathbf{E}]. \tag{13.12.2}$$

The remaining part,

$$\mathrm{d}^{\mathrm{p}}[\mathbf{T}] = \mathrm{d}[\mathbf{T}] - [\mathbf{\Lambda}_{(1)}] : \mathrm{d}[\mathbf{E}], \tag{13.12.3}$$

is the plastic part of the increment $\mathrm{d}[\mathbf{T}]$. The macroscopic elastoplastic increment of the Lagrangian strain is $\mathrm{d}[\mathbf{E}]$.

The plastic part $\mathrm{d}^{\mathrm{p}}[\mathbf{T}]$ can be related to $\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle$ by substituting Eq. (13.9.4), and

$$\mathrm{d}\langle\mathbf{P}\rangle = \langle\boldsymbol{\mathcal{K}}\rangle^T : \mathrm{d}[\mathbf{T}] + [\,\boldsymbol{\mathcal{T}}\,] \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle, \tag{13.12.4}$$

$$\mathrm{d}[\mathbf{E}] = \langle\boldsymbol{\mathcal{K}}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle, \tag{13.12.5}$$

into Eq. (13.10.3). The result is

$$\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = \langle\boldsymbol{\mathcal{K}}\rangle^T : \mathrm{d}^{\mathrm{p}}[\mathbf{T}]. \tag{13.12.6}$$

Normality Rules

To discuss the normality rules, we first observe that

$$\delta\langle\mathbf{F}\rangle \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = \delta\langle\mathbf{F}\rangle \cdot\cdot\, \langle\boldsymbol{\mathcal{K}}\rangle^T : \mathrm{d}^{\mathrm{p}}[\mathbf{T}] = \delta[\mathbf{E}] : \mathrm{d}^{\mathrm{p}}[\mathbf{T}]. \tag{13.12.7}$$

This shows, if the normality holds for the plastic part of the increment of macroscopic nominal stress, it also holds for the plastic part of the increment of macroscopic Piola–Kirchhoff stress, and *vice versa*, i.e.,

$$\delta\langle\mathbf{F}\rangle \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle > 0 \quad \Longleftrightarrow \quad \delta[\mathbf{E}] : \mathrm{d}^{\mathrm{p}}[\mathbf{T}] > 0. \tag{13.12.8}$$

Furthermore, we have

$$\langle\delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle = \langle\delta\mathbf{E} : \mathrm{d}^{\mathrm{p}}\mathbf{T}\rangle, \tag{13.12.9}$$

because locally $\delta\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P} = \delta\mathbf{E} : \mathrm{d}^{\mathrm{p}}\mathbf{T}$, as shown in Section 12.14. Thus, by comparing Eqs. (13.12.7) and (13.12.9), and having in mind Eq. (13.10.11), it follows that

$$\langle \delta\mathbf{E} : \mathrm{d}^{\mathrm{p}}\mathbf{T}\rangle = \delta[\mathbf{E}] : \mathrm{d}^{\mathrm{p}}[\mathbf{T}]. \tag{13.12.10}$$

Consequently, if the normality rule applies at the microlevel, it is transmitted to the macrolevel,

$$\delta\mathbf{E} : \mathrm{d}^{\mathrm{p}}\mathbf{T} > 0 \quad \Longrightarrow \quad \delta[\mathbf{E}] : \mathrm{d}^{\mathrm{p}}[\mathbf{T}] > 0. \tag{13.12.11}$$

We can derive an expression for $\mathrm{d}^{\mathrm{p}}\mathbf{T}$ in terms of the macroscopic plastic potential. To that goal, note that

$$\frac{\partial}{\partial\langle\mathbf{F}\rangle} = \langle\boldsymbol{\mathcal{K}}\rangle^T : \frac{\partial}{\partial[\mathbf{E}]}\,. \tag{13.12.12}$$

When this is substituted into Eq. (13.10.19), there follows

$$\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = -\frac{\partial}{\partial\langle\mathbf{F}\rangle}\,\langle\sum_{\alpha=1}^{n}\tau^{\alpha}\,\mathrm{d}\gamma^{\alpha}\rangle = -\langle\boldsymbol{\mathcal{K}}\rangle^T : \frac{\partial}{\partial[\mathbf{E}]}\,\langle\sum_{\alpha=1}^{n}\tau^{\alpha}\,\mathrm{d}\gamma^{\alpha}\rangle, \tag{13.12.13}$$

and the comparison with Eq. (13.12.6) establishes

$$\mathrm{d}^{\mathrm{p}}[\mathbf{T}] = -\frac{\partial}{\partial[\mathbf{E}]}\,\langle\sum_{\alpha=1}^{n}\tau^{\alpha}\,\mathrm{d}\gamma^{\alpha}\rangle. \tag{13.12.14}$$

This demonstrates that $-\langle\sum\tau^{\alpha}\,\mathrm{d}\gamma^{\alpha}\rangle$ is the plastic potential for $\mathrm{d}^{\mathrm{p}}[\mathbf{T}]$ over an elastic domain in $[\mathbf{E}]$ space . This result is originally due to Hill and Rice (1973).

13.13. Plastic Increment of Macroscopic Lagrangian Strain

The increment of the macroscopic Lagrangian strain is partitioned into its elastic and plastic parts as

$$\mathrm{d}[\mathbf{E}] = \mathrm{d}^{\mathrm{e}}[\mathbf{E}] + \mathrm{d}^{\mathrm{p}}[\mathbf{E}]. \tag{13.13.1}$$

The elastic part is

$$\mathrm{d}^{\mathrm{e}}[\mathbf{E}] = [\mathbf{M}_{(1)}] : \mathrm{d}[\mathbf{T}], \tag{13.13.2}$$

while

$$\mathrm{d}^{\mathrm{p}}[\mathbf{E}] = \mathrm{d}[\mathbf{E}] - [\mathbf{M}_{(1)}] : \mathrm{d}[\mathbf{T}] \tag{13.13.3}$$

represents the plastic part of the increment $\mathrm{d}[\mathbf{E}]$. The tensor of macroscopic elastic compliances is

$$[\mathbf{M}_{(1)}] = [\boldsymbol{\Lambda}_{(1)}]^{-1}. \tag{13.13.4}$$

From Eqs. (13.12.3) and (13.13.3), we observe the connections

$$\mathrm{d}^{\mathrm{p}}[\mathbf{T}] = -[\mathbf{\Lambda}_{(1)}] : \mathrm{d}[\mathbf{E}], \quad \mathrm{d}^{\mathrm{p}}[\mathbf{E}] = -[\mathbf{M}_{(1)}] : \mathrm{d}[\mathbf{T}]. \tag{13.13.5}$$

The plastic part $\mathrm{d}^{\mathrm{p}}[\mathbf{E}]$ can be related to $\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle$ by substituting

$$\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = -[\mathbf{\Lambda}] : \mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle, \quad \mathrm{d}^{\mathrm{p}}[\mathbf{T}] = -[\mathbf{\Lambda}_{(1)}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}] \tag{13.13.6}$$

into Eq. (13.12.6). The result is

$$[\mathbf{\Lambda}] \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle = \langle\boldsymbol{\mathcal{K}}\rangle^T : [\mathbf{\Lambda}_{(1)}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}], \tag{13.13.7}$$

i.e.,

$$\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle = [\mathbf{M}] \cdot\cdot\, \langle\boldsymbol{\mathcal{K}}\rangle^T : [\mathbf{\Lambda}_{(1)}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}]. \tag{13.13.8}$$

Normality Rules

First, it is noted that

$$\delta\langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle = \delta\langle\mathbf{P}\rangle \cdot\cdot\, [\mathbf{M}] \cdot\cdot\, \langle\boldsymbol{\mathcal{K}}\rangle^T : [\mathbf{\Lambda}_{(1)}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}]. \tag{13.13.9}$$

Since

$$\delta\langle\mathbf{P}\rangle \cdot\cdot\, [\mathbf{M}] \cdot\cdot\, \langle\boldsymbol{\mathcal{K}}\rangle^T = \delta\langle\mathbf{F}\rangle \cdot\cdot\, \langle\boldsymbol{\mathcal{K}}\rangle^T = \delta[\mathbf{E}], \tag{13.13.10}$$

and

$$\delta[\mathbf{E}] : [\mathbf{\Lambda}_{(1)}] = \delta[\mathbf{T}], \tag{13.13.11}$$

Equation (13.13.9) becomes

$$\delta\langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle = \delta[\mathbf{T}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}]. \tag{13.13.12}$$

Therefore, if the normality holds for the plastic part of the increment of macroscopic deformation gradient, it also holds for the plastic part of the increment of macroscopic Lagrangian strain, and *vice versa*, i.e.,

$$\delta\langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle < 0 \quad \Longleftrightarrow \quad \delta[\mathbf{T}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}] < 0. \tag{13.13.13}$$

Next, there is an identity

$$\langle\delta\mathbf{P} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F}\rangle = \langle\delta\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}\rangle, \tag{13.13.14}$$

because locally $\delta\mathbf{P} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F} = \delta\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}$, as can be inferred from the analysis in Section 12.14. Thus, by comparing Eqs. (13.13.12) and (13.13.14), and by recalling Eq. (13.11.10), it follows that

$$\langle\delta\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}\rangle = \delta[\mathbf{T}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}]. \tag{13.13.15}$$

Consequently, if the normality rule applies at the microlevel, it is transmitted to the macrolevel (Hill, 1972), i.e.,

$$\delta\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E} < 0 \quad \Longrightarrow \quad \delta[\mathbf{T}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}] < 0. \tag{13.13.16}$$

In the context of small deformation the result was originally obtained by Mandel (1966) and Hill (1967).

An expression for $\mathrm{d}^{\mathrm{p}}\mathbf{E}$ can be derived in terms of the macroscopic plastic potential by using the chain rule,

$$\frac{\partial}{\partial[\mathbf{E}]} = \frac{\partial}{\partial[\mathbf{T}]} : [\boldsymbol{\Lambda}_{(1)}], \tag{13.13.17}$$

in Eq. (13.12.14). This gives

$$\mathrm{d}^{\mathrm{p}}[\mathbf{T}] = -\frac{\partial}{\partial[\mathbf{T}]} : [\boldsymbol{\Lambda}_{(1)}]\,\langle\sum_{\alpha=1}^{n} \tau^\alpha\,\mathrm{d}\gamma^\alpha\rangle. \tag{13.13.18}$$

Upon the trace product with $[\mathbf{M}_{(1)}]$, we obtain

$$\mathrm{d}^{\mathrm{p}}[\mathbf{E}] = \frac{\partial}{\partial[\mathbf{T}]}\,\langle\sum_{\alpha=1}^{n} \tau^\alpha\,\mathrm{d}\gamma^\alpha\rangle, \tag{13.13.19}$$

having regard to (13.13.5). This shows that $\langle\sum \tau^\alpha\,\mathrm{d}\gamma^\alpha\rangle$ is a plastic potential for $\mathrm{d}^{\mathrm{p}}[\mathbf{E}]$ over an elastic domain in $[\mathbf{T}]$ space.

13.14. Macroscopic Increment of Plastic Work

The macroscopic increment of slip work, per unit volume of the macroelement, is the volume average

$$\langle \mathrm{d}w^{\mathrm{slip}}\rangle = \langle\sum_{\alpha=1}^{n} \tau^\alpha\,\mathrm{d}\gamma^\alpha\rangle = \frac{1}{V^0}\int_{V^0}\left(\sum_{\alpha=1}^{n} \tau^\alpha\,\mathrm{d}\gamma^\alpha\right)\mathrm{d}V^0. \tag{13.14.1}$$

The number n of active slip systems changes from grain to grain within the macroelement, depending on the grain orientation and the state of hardening.

Another quantity, which will be referred to as the macroscopic increment of plastic work, can be introduced as follows. Consider a cycle of the application and removal of the macroscopic increment of nominal stress $\mathrm{d}\langle\mathbf{P}\rangle$. The corresponding macroscopic work can be determined by considering the volume average of the first-order work quantity

$$\mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{F} = \mathbf{P}\cdot\cdot\left(\mathrm{d}^{\mathrm{r}}\mathbf{F} - \mathbf{M}\cdot\cdot\,\mathrm{d}^{\mathrm{r}}\mathbf{P}\right), \tag{13.14.2}$$

which is

$$\langle \mathbf{P} \cdot\cdot \, \mathrm{d}^{\mathrm{p}}\mathbf{F} \rangle = \langle \mathbf{P} \rangle \cdot\cdot \, \mathrm{d}^{\mathrm{p}} \langle \mathbf{F} \rangle - \langle \mathbf{P} \cdot\cdot \, \mathbf{M} \cdot\cdot \, \mathrm{d}^{\mathrm{r}}\mathbf{P} \rangle. \tag{13.14.3}$$

This follows because $\mathbf{P}$ is statically admissible and $\mathrm{d}^{\mathrm{r}}\mathbf{F}$ is kinematically admissible, so that

$$\langle \mathbf{P} \cdot\cdot \, \mathrm{d}^{\mathrm{r}}\mathbf{F} \rangle = \langle \mathbf{P} \rangle \cdot\cdot \, \langle \mathrm{d}^{\mathrm{r}}\mathbf{F} \rangle = \langle \mathbf{P} \rangle \cdot\cdot \, \mathrm{d}^{\mathrm{p}} \langle \mathbf{F} \rangle. \tag{13.14.4}$$

Thus,

$$\langle \mathbf{P} \rangle \cdot\cdot \, \mathrm{d}^{\mathrm{p}} \langle \mathbf{F} \rangle = \langle \mathbf{P} \cdot\cdot \, \mathrm{d}^{\mathrm{p}}\mathbf{F} \rangle + \langle \mathbf{P} \cdot\cdot \, \mathbf{M} \cdot\cdot \, \mathrm{d}^{\mathrm{r}}\mathbf{P} \rangle. \tag{13.14.5}$$

The result shows that the macroscopic first-order work quantity in the cycle of $\mathrm{d}\langle \mathbf{P} \rangle$ is not equal to the volume average of the local work quantity $\mathbf{P} \cdot\cdot \, \mathrm{d}^{\mathrm{p}}\mathbf{F}$. This was expected on physical grounds, because cycling $\mathrm{d}\langle \mathbf{P} \rangle$ macroscopically does not simultaneously cycle every $\mathrm{d}\mathbf{P}$ locally. In fact, the residual increment of stress left locally upon the cycle of $\mathrm{d}\langle \mathbf{P} \rangle$ is $\mathrm{d}^{\mathrm{r}}\langle \mathbf{P} \rangle$ of Eq. (13.11.20).

To analyze the increment of macroscopic plastic work with an accuracy to the second order, consider

$$\langle (\mathbf{P} + \frac{1}{2}\,\mathrm{d}\mathbf{P}) \cdot\cdot \, \mathrm{d}^{\mathrm{p}}\mathbf{F} \rangle = \langle \mathbf{P} \cdot\cdot \, \mathrm{d}^{\mathrm{p}}\mathbf{F} \rangle + \frac{1}{2}\, \langle \mathrm{d}\mathbf{P} \cdot\cdot \, \mathrm{d}^{\mathrm{p}}\mathbf{F} \rangle. \tag{13.14.6}$$

The second-order contribution can be expressed by using the identity

$$\mathrm{d}\mathbf{P} \cdot\cdot \, \mathrm{d}^{\mathrm{p}}\mathbf{F} = \mathrm{d}\mathbf{P} \cdot\cdot \, (\mathrm{d}^{\mathrm{r}}\mathbf{F} - \mathbf{M} \cdot\cdot \, \mathrm{d}^{\mathrm{r}}\mathbf{P})\,. \tag{13.14.7}$$

In view of (13.11.20), this can be rewritten as

$$\mathrm{d}\mathbf{P} \cdot\cdot \, \mathrm{d}^{\mathrm{p}}\mathbf{F} = \mathrm{d}\mathbf{P} \cdot\cdot \, \mathrm{d}^{\mathrm{r}}\mathbf{F} - \left(\mathrm{d}^{\mathrm{r}}\mathbf{P} + \mathrm{d}\langle \mathbf{P} \rangle \cdot\cdot \, \boldsymbol{\mathcal{P}}^T \right) \cdot\cdot \, \mathbf{M} \cdot\cdot \, \mathrm{d}^{\mathrm{r}}\mathbf{P}. \tag{13.14.8}$$

Since $\mathrm{d}^{\mathrm{r}}\mathbf{F}$ and $\mathrm{d}\langle \mathbf{P} \rangle \cdot\cdot \boldsymbol{\mathcal{P}}^T \cdot\cdot \, \mathbf{M} = \mathbf{M} \cdot\cdot \boldsymbol{\mathcal{P}} \cdot\cdot \, \mathrm{d}\langle \mathbf{P} \rangle$ are kinematically admissible fields, and since $\langle \mathrm{d}^{\mathrm{r}}\mathbf{F} \rangle = \mathrm{d}^{\mathrm{p}}\langle \mathbf{F} \rangle$ and $\langle \mathrm{d}^{\mathrm{r}}\mathbf{P} \rangle = \mathbf{0}$, upon the averaging of Eq. (13.14.8) we obtain

$$\langle \mathrm{d}\mathbf{P} \cdot\cdot \, \mathrm{d}^{\mathrm{p}}\mathbf{F} \rangle = \mathrm{d}\langle \mathbf{P} \rangle \cdot\cdot \, \mathrm{d}^{\mathrm{p}} \langle \mathbf{F} \rangle - \langle \mathrm{d}^{\mathrm{r}}\mathbf{P} \cdot\cdot \, \mathbf{M} \cdot\cdot \, \mathrm{d}^{\mathrm{r}}\mathbf{P} \rangle, \tag{13.14.9}$$

i.e.,

$$\mathrm{d}\langle \mathbf{P} \rangle \cdot\cdot \, \mathrm{d}^{\mathrm{p}} \langle \mathbf{F} \rangle = \langle \mathrm{d}\mathbf{P} \cdot\cdot \, \mathrm{d}^{\mathrm{p}}\mathbf{F} \rangle + \langle \mathrm{d}^{\mathrm{r}}\mathbf{P} \cdot\cdot \, \mathbf{M} \cdot\cdot \, \mathrm{d}^{\mathrm{r}}\mathbf{P} \rangle. \tag{13.14.10}$$

Combining Eqs. (13.14.4), (13.14.6), and (13.14.9), the increment of macroscopic plastic work, to second order, can be expressed as

$$\begin{aligned}(\langle \mathbf{P}\rangle + \frac{1}{2}\,\mathrm{d}\langle \mathbf{P}\rangle)\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle \mathbf{F}\rangle &= \langle(\mathbf{P} + \frac{1}{2}\,\mathrm{d}\mathbf{P})\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{F}\rangle \\ &\quad + \langle(\mathbf{P} + \frac{1}{2}\,\mathrm{d}^{\mathrm{r}}\mathbf{P})\cdot\cdot\,\mathbf{M}\cdot\cdot\,\mathrm{d}^{\mathrm{r}}\mathbf{P}\rangle.\end{aligned} \tag{13.14.11}$$

The first- and second-order plastic work quantities, defined by $\mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{F}$ and $\mathrm{d}\mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{F}$, are not equal to $\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}$ and $\mathrm{d}\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}$, as discussed in Section 12.8. The latter quantities are actually not measure invariant, but change their values with the change of the strain and conjugate stress measure.

Related Work Expressions

When the Lagrangian strain and Piola–Kirchhoff stress are used, we have from Eqs. (12.8.13) and (12.8.17),

$$\mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{F} = \mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E} + \mathbf{T} : \mathbf{M}_{(1)} : \mathrm{d}\mathbf{T} - \mathbf{P}\cdot\cdot\,\mathbf{M}\cdot\cdot\,\mathrm{d}\mathbf{P}, \tag{13.14.12}$$

$$\mathrm{d}\mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{F} = \mathrm{d}\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E} + \mathrm{d}\mathbf{T} : \mathbf{M}_{(1)} : \mathrm{d}\mathbf{T} - \mathrm{d}\mathbf{P}\cdot\cdot\,\mathbf{M}\cdot\cdot\,\mathrm{d}\mathbf{P} + \mathrm{d}\mathbf{F}\cdot\cdot\,\boldsymbol{\mathcal{T}}\cdot\cdot\,\mathrm{d}\mathbf{F}. \tag{13.14.13}$$

The corresponding expressions for the macroscopic quantities are readily obtained. The first one is

$$\begin{aligned}\langle \mathbf{P}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle \mathbf{F}\rangle &= \langle \mathbf{P}\rangle\cdot\cdot\,(\mathrm{d}\langle \mathbf{F}\rangle - [\mathbf{M}]\cdot\cdot\,\mathrm{d}\langle \mathbf{P}\rangle) \\ &= [\mathbf{T}] : \mathrm{d}[\mathbf{E}] - \langle \mathbf{P}\rangle\cdot\cdot\,[\mathbf{M}]\cdot\cdot\,\mathrm{d}\langle \mathbf{P}\rangle,\end{aligned} \tag{13.14.14}$$

i.e.,

$$\langle \mathbf{P}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle \mathbf{F}\rangle = [\mathbf{T}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}] + [\mathbf{T}] : [\mathbf{M}_{(1)}] : \mathrm{d}[\mathbf{T}] - \langle \mathbf{P}\rangle\cdot\cdot\,[\mathbf{M}]\cdot\cdot\,\mathrm{d}\langle \mathbf{P}\rangle. \tag{13.14.15}$$

Similarly,

$$\mathrm{d}\langle \mathbf{P}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle \mathbf{F}\rangle = \mathrm{d}[\mathbf{T}] : \mathrm{d}[\mathbf{E}] - \mathrm{d}\langle \mathbf{P}\rangle\cdot\cdot\,[\mathbf{M}]\cdot\cdot\,\mathrm{d}\langle \mathbf{P}\rangle + \mathrm{d}\langle \mathbf{F}\rangle\cdot\cdot\,[\boldsymbol{\mathcal{T}}]\cdot\cdot\,\mathrm{d}\langle \mathbf{F}\rangle, \tag{13.14.16}$$

and

$$\begin{aligned}\mathrm{d}\langle \mathbf{P}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle \mathbf{F}\rangle &= \mathrm{d}[\mathbf{T}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}] + \mathrm{d}[\mathbf{T}] : [\mathbf{M}_{(1)}] : \mathrm{d}[\mathbf{T}] \\ &\quad - \mathrm{d}\langle \mathbf{P}\rangle\cdot\cdot\,[\mathbf{M}]\cdot\cdot\,\mathrm{d}\langle \mathbf{P}\rangle + \mathrm{d}\langle \mathbf{F}\rangle\cdot\cdot\,[\boldsymbol{\mathcal{T}}]\cdot\cdot\,\mathrm{d}\langle \mathbf{F}\rangle.\end{aligned} \tag{13.14.17}$$

We now proceed to establish the relationships between the macroscopic quantities $[\mathbf{T}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}]$ and $\mathrm{d}[\mathbf{T}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}]$, and the volume averages $\langle \mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}\rangle$ and $\langle \mathrm{d}\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}\rangle$. First, since from Eq. (13.4.5),

$$[\mathbf{T}] : \mathrm{d}[\mathbf{E}] = \langle \mathbf{T} : \mathrm{d}\mathbf{E}\rangle, \tag{13.14.18}$$

we obtain

$$[\mathbf{T}] : \big(\mathrm{d}^{\mathrm{p}}[\mathbf{E}] + [\mathbf{M}_{(1)}] : \mathrm{d}[\mathbf{T}]\big) = \langle \mathbf{T} : \big(\mathrm{d}^{\mathrm{p}}\mathbf{E} + \mathbf{M}_{(1)} : \mathrm{d}\mathbf{T}\big)\rangle. \tag{13.14.19}$$

Therefore,

$$[\mathbf{T}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}] = \langle \mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}\rangle + \langle \mathbf{T} : \mathbf{M}_{(1)} : \mathrm{d}\mathbf{T}\rangle - [\mathbf{T}] : [\mathbf{M}_{(1)}] : \mathrm{d}[\mathbf{T}]. \tag{13.14.20}$$

To derive the formula for the second-order work quantity, we begin by volume averaging of (13.14.13), i.e.,

$$\begin{aligned} \langle \mathrm{d}\mathbf{P} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F}\rangle &= \langle \mathrm{d}\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}\rangle + \langle \mathrm{d}\mathbf{T} : \mathbf{M}_{(1)} : \mathrm{d}\mathbf{T}\rangle \\ &\quad - \langle \mathrm{d}\mathbf{P} \cdot\cdot\, \mathbf{M} \cdot\cdot\, \mathrm{d}\mathbf{P}\rangle + \langle \mathrm{d}\mathbf{F} \cdot\cdot\, \boldsymbol{\mathcal{T}} \cdot\cdot\, \mathrm{d}\mathbf{F}\rangle. \end{aligned} \tag{13.14.21}$$

On the other hand, there is a relationship

$$\langle \mathrm{d}\mathbf{P} \cdot\cdot\, \mathbf{M} \cdot\cdot\, \mathrm{d}\mathbf{P}\rangle - \langle \mathrm{d}^{\mathrm{r}}\mathbf{P} \cdot\cdot\, \mathbf{M} \cdot\cdot\, \mathrm{d}^{\mathrm{r}}\mathbf{P}\rangle = \mathrm{d}\langle \mathbf{P}\rangle \cdot\cdot\, [\mathbf{M}] \cdot\cdot\, \mathrm{d}\langle \mathbf{P}\rangle. \tag{13.14.22}$$

The latter can be verified by subtracting

$$\langle \mathrm{d}^{\mathrm{r}}\mathbf{P} \cdot\cdot\, \mathbf{M} \cdot\cdot\, \mathrm{d}^{\mathrm{r}}\mathbf{P}\rangle = \langle \mathrm{d}^{\mathrm{r}}\mathbf{P} \cdot\cdot\, \mathbf{M} \cdot\cdot\, (\mathrm{d}\mathbf{P} - \boldsymbol{\mathcal{P}} \cdot\cdot\, \mathrm{d}\langle \mathbf{P}\rangle)\rangle \tag{13.14.23}$$

from

$$\langle \mathrm{d}\mathbf{P} \cdot\cdot\, \mathbf{M} \cdot\cdot\, \mathrm{d}\mathbf{P}\rangle = \langle \big(\mathrm{d}^{\mathrm{r}}\mathbf{P} + \mathrm{d}\langle \mathbf{P}\rangle \cdot\cdot\, \boldsymbol{\mathcal{P}}^T\big) \cdot\cdot\, \mathbf{M} \cdot\cdot\, \mathrm{d}\mathbf{P}\rangle, \tag{13.14.24}$$

and by using the theorem on product averages for the appropriate admissible fields. The results $\langle \boldsymbol{\mathcal{P}}^T \cdot\cdot\, \mathbf{M}\rangle = [\mathbf{M}]$ and $\langle \mathrm{d}^{\mathrm{r}}\mathbf{P}\rangle = \mathbf{0}$, from Eqs. (13.8.6) and (13.11.25), were also used. Substitution of Eq. (13.14.22) into (13.14.21) then gives

$$\begin{aligned} &\mathrm{d}\langle \mathbf{P}\rangle \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\langle \mathbf{F}\rangle + \mathrm{d}\langle \mathbf{P}\rangle \cdot\cdot\, [\mathbf{M}] \cdot\cdot\, \mathrm{d}\langle \mathbf{P}\rangle \\ &\quad = \langle \mathrm{d}\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}\rangle + \langle \mathrm{d}\mathbf{T} : \mathbf{M}_{(1)} : \mathrm{d}\mathbf{T}\rangle + \langle \mathrm{d}\mathbf{F} \cdot\cdot\, \boldsymbol{\mathcal{T}} \cdot\cdot\, \mathrm{d}\mathbf{F}\rangle. \end{aligned} \tag{13.14.25}$$

Equation (13.2.9) was used to eliminate $\langle \mathrm{d}\mathbf{P} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F}\rangle$ in terms of $\mathrm{d}\langle \mathbf{P}\rangle \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\langle \mathbf{F}\rangle$. By combining Eq. (13.14.25) with Eq. (13.14.17), we finally obtain

$$\begin{aligned} \mathrm{d}[\mathbf{T}] : \mathrm{d}^{\mathrm{p}}[\mathbf{E}] &= \langle \mathrm{d}\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}\rangle + \langle \mathrm{d}\mathbf{T} : \mathbf{M}_{(1)} : \mathrm{d}\mathbf{T}\rangle + \langle \mathrm{d}\mathbf{F} \cdot\cdot\, \boldsymbol{\mathcal{T}} \cdot\cdot\, \mathrm{d}\mathbf{F}\rangle \\ &\quad - \mathrm{d}[\mathbf{T}] : [\mathbf{M}_{(1)}] : \mathrm{d}[\mathbf{T}] - \mathrm{d}\langle \mathbf{F}\rangle \cdot\cdot\, [\boldsymbol{\mathcal{T}}] \cdot\cdot\, \mathrm{d}\langle \mathbf{F}\rangle, \end{aligned} \tag{13.14.26}$$

which was originally derived by Hill (1985).

In the infinitesimal ($\boldsymbol{\varepsilon}$) strain theory, there is no distinction between various stress and strain measures, and both (13.14.10) and (13.14.26) reduce to

$$\mathrm{d}\langle \boldsymbol{\sigma}\rangle : \mathrm{d}^{\mathrm{p}}\langle \boldsymbol{\varepsilon}\rangle = \langle \mathrm{d}\boldsymbol{\sigma} : \mathrm{d}^{\mathrm{p}}\boldsymbol{\varepsilon}\rangle + \langle \mathrm{d}^{\mathrm{r}}\boldsymbol{\sigma} : \mathbf{M} : \mathrm{d}^{\mathrm{r}}\boldsymbol{\sigma}\rangle. \tag{13.14.27}$$

The rotational effects on the stress rate are neglected if Eq. (13.14.27) is deduced from Eq. (13.14.26), and the Cauchy stress $\boldsymbol{\sigma}$ is used in place of $\mathbf{P}$ in Eq. (13.14.22). All elastic compliances are given by the tensor $\mathbf{M}$. Equation (13.14.27) was originally derived by Mandel (1966). With the positive definite $\mathbf{M}$, it follows that

$$\mathrm{d}\langle\boldsymbol{\sigma}\rangle : \mathrm{d}^{\mathrm{p}}\langle\boldsymbol{\varepsilon}\rangle > \langle\mathrm{d}\boldsymbol{\sigma} : \mathrm{d}^{\mathrm{p}}\boldsymbol{\varepsilon}\rangle. \tag{13.14.28}$$

Thus, within infinitesimal range, the stability at microlevel, $\mathrm{d}\boldsymbol{\sigma} : \mathrm{d}^{\mathrm{p}}\boldsymbol{\varepsilon} > 0$, ensures the stability at macrolevel, $\mathrm{d}\langle\boldsymbol{\sigma}\rangle : \mathrm{d}^{\mathrm{p}}\langle\boldsymbol{\varepsilon}\rangle > 0$.

13.15. Nontransmissibility of Basic Crystal Inequality

Consider a cycle of the application and removal of the macroscopic increment of deformation gradient $\mathrm{d}\langle\mathbf{F}\rangle$. Since

$$\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P} = \mathbf{F}\cdot\cdot\left(\mathrm{d}^{\mathrm{s}}\mathbf{P} - \boldsymbol{\Lambda}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\right), \tag{13.15.1}$$

the volume average is

$$\langle\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle = \langle\mathbf{F}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle - \langle\mathbf{F}\cdot\cdot\boldsymbol{\Lambda}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle. \tag{13.15.2}$$

This follows because $\mathbf{F}$ is kinematically admissible and $\mathrm{d}^{\mathrm{s}}\mathbf{P}$ is statically admissible, so that

$$\langle\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{P}\rangle = \langle\mathbf{F}\rangle\cdot\cdot\langle\mathrm{d}^{\mathrm{s}}\mathbf{P}\rangle = \langle\mathbf{F}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle. \tag{13.15.3}$$

Thus, dually to Eq. (13.14.5), we have

$$\langle\mathbf{F}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = \langle\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle + \langle\mathbf{F}\cdot\cdot\boldsymbol{\Lambda}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle. \tag{13.15.4}$$

This was expected on physical grounds, because cycling $\mathrm{d}\langle\mathbf{F}\rangle$ macroscopically does not simultaneously cycle every $\mathrm{d}\mathbf{F}$ locally. In fact, the residual increment of deformation left locally upon the cycle of $\mathrm{d}\langle\mathbf{F}\rangle$ is $\mathrm{d}^{\mathrm{s}}\langle\mathbf{F}\rangle$, given by Eq. (13.10.21).

Consider next the net expenditure of work in a cycle of $\mathrm{d}\langle\mathbf{F}\rangle$. By the trapezoidal rule of quadrature, the net work expended locally is

$$-\frac{1}{2}\,\mathrm{d}\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P}, \tag{13.15.5}$$

to second-order. The quantity

$$\mathrm{d}\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P} = \mathrm{d}\mathbf{F}\cdot\cdot\left(\mathrm{d}^{\mathrm{s}}\mathbf{P} - \boldsymbol{\Lambda}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\right) \tag{13.15.6}$$

can be rewritten, by using Eq. (13.10.21), as

$$\mathrm{d}\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P} = \mathrm{d}\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{P} - \left(\mathrm{d}^{\mathrm{s}}\mathbf{F} + \mathrm{d}\langle\mathbf{F}\rangle\cdot\cdot\,\boldsymbol{\mathcal{F}}^{T}\right)\cdot\cdot\,\boldsymbol{\Lambda}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}. \tag{13.15.7}$$

Since $\mathrm{d}^{\mathrm{s}}\mathbf{P}$ and $\mathrm{d}\langle\mathbf{F}\rangle\cdot\cdot\,\boldsymbol{\mathcal{F}}^{T}\cdot\cdot\,\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\cdot\cdot\,\boldsymbol{\mathcal{F}}\cdot\cdot\,\mathrm{d}\langle\mathbf{F}\rangle$ are statically admissible fields, and since $\langle\mathrm{d}^{\mathrm{s}}\mathbf{P}\rangle = \mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle$ and $\langle\mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle = \mathbf{0}$, upon the averaging of Eq. (13.15.7) we obtain

$$\langle\mathrm{d}\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle = \mathrm{d}\langle\mathbf{F}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle - \langle\mathrm{d}^{\mathrm{s}}\mathbf{F}\cdot\cdot\,\boldsymbol{\Lambda}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle, \tag{13.15.8}$$

i.e.,

$$\mathrm{d}\langle\mathbf{F}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = \langle\mathrm{d}\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle + \langle\mathrm{d}^{\mathrm{s}}\mathbf{F}\cdot\cdot\,\boldsymbol{\Lambda}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle. \tag{13.15.9}$$

This shows that $\mathrm{d}\langle\mathbf{F}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle$ is not equal to the volume average of the local quantity $\mathrm{d}\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P}$, because cycling $\mathrm{d}\langle\mathbf{F}\rangle$ macroscopically does not simultaneously cycle every $\mathrm{d}\mathbf{F}$ locally.

The second-order work quantity $\mathrm{d}\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P}$ is equal to the measure invariant quantity $\mathrm{d}\mathbf{E}:\mathrm{d}^{\mathrm{p}}\mathbf{T}$, as discussed in Section 12.8. Thus,

$$\langle\mathrm{d}\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle = \langle\mathrm{d}\mathbf{E}:\mathrm{d}^{\mathrm{p}}\mathbf{T}\rangle. \tag{13.15.10}$$

Furthermore, from Eq. (13.12.6), we have

$$\mathrm{d}\langle\mathbf{F}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = \mathrm{d}\langle\mathbf{F}\rangle\cdot\cdot\,\langle\boldsymbol{\mathcal{K}}\rangle^{T}:\mathrm{d}^{\mathrm{p}}[\mathbf{T}] = \mathrm{d}[\mathbf{E}]:\mathrm{d}^{\mathrm{p}}[\mathbf{T}]. \tag{13.15.11}$$

Substitution of Eqs. (13.15.10) and (13.15.11) into Eq. (13.15.9) gives

$$\mathrm{d}[\mathbf{E}]:\mathrm{d}^{\mathrm{p}}[\mathbf{T}] = \langle\mathrm{d}\mathbf{E}:\mathrm{d}^{\mathrm{p}}\mathbf{T}\rangle + \langle\mathrm{d}^{\mathrm{s}}\mathbf{F}\cdot\cdot\,\boldsymbol{\Lambda}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle. \tag{13.15.12}$$

The second-order quantity $\mathrm{d}\langle\mathbf{E}\rangle:\mathrm{d}^{\mathrm{p}}\langle\mathbf{T}\rangle$ is not equal to the volume average of the local quantity $\mathrm{d}\mathbf{E}:\mathrm{d}^{\mathrm{p}}\mathbf{T}$, because cycling $\mathrm{d}\langle\mathbf{E}\rangle$ macroscopically does not simultaneously cycle every $\mathrm{d}\mathbf{E}$ locally. We conclude that the macroscopic inequality $\mathrm{d}[\mathbf{E}]:\mathrm{d}^{\mathrm{p}}[\mathbf{T}] < 0$ is not guaranteed by the basic single crystal inequality at the local level $\mathrm{d}\mathbf{E}:\mathrm{d}^{\mathrm{p}}\mathbf{T} < 0$. However, since $\langle\mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle = \mathbf{0}$, it is reasonable to expect that $\langle\mathrm{d}^{\mathrm{s}}\mathbf{F}\cdot\cdot\,\boldsymbol{\Lambda}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle$ is small (being either positive or negative, since $\boldsymbol{\Lambda}$ is not necessarily positive definite); see Havner (1992).

In the infinitesimal strain theory, Eqs. (13.15.9) and (13.15.12) reduce to

$$\mathrm{d}\langle\boldsymbol{\varepsilon}\rangle:\mathrm{d}^{\mathrm{p}}\langle\boldsymbol{\sigma}\rangle = \langle\mathrm{d}\boldsymbol{\varepsilon}:\mathrm{d}^{\mathrm{p}}\boldsymbol{\sigma}\rangle - \langle\mathrm{d}^{\mathrm{s}}\boldsymbol{\varepsilon}:\boldsymbol{\Lambda}:\mathrm{d}^{\mathrm{s}}\boldsymbol{\varepsilon}\rangle. \tag{13.15.13}$$

Equation (13.15.13) was originally derived by Hill (1972). With the positive definite $\boldsymbol{\Lambda}$, it only implies that

$$\mathrm{d}\langle\boldsymbol{\varepsilon}\rangle:\mathrm{d}^{\mathrm{p}}\langle\boldsymbol{\sigma}\rangle > \langle\mathrm{d}\boldsymbol{\varepsilon}:\mathrm{d}^{\mathrm{p}}\boldsymbol{\sigma}\rangle. \tag{13.15.14}$$

Evidently, the stability at the microlevel, $\mathrm{d}\boldsymbol{\varepsilon} : \mathrm{d}^{\mathrm{p}}\boldsymbol{\sigma} < 0$, does not ensure the stability at the macrolevel, $\mathrm{d}\langle\boldsymbol{\varepsilon}\rangle : \mathrm{d}^{\mathrm{p}}\langle\boldsymbol{\sigma}\rangle < 0$.

It is noted that, dually to relation (13.14.22), we have

$$\langle \mathrm{d}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}\mathbf{F}\rangle - \langle \mathrm{d}^{\mathrm{s}}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle = \mathrm{d}\langle\mathbf{F}\rangle \cdot\cdot [\boldsymbol{\Lambda}] \cdot\cdot \mathrm{d}\langle\mathbf{F}\rangle. \tag{13.15.15}$$

This can be verified by subtracting

$$\langle \mathrm{d}^{\mathrm{s}}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle = \langle \mathrm{d}^{\mathrm{s}}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot (\mathrm{d}\mathbf{F} - \boldsymbol{\mathcal{F}} \cdot\cdot \mathrm{d}\langle\mathbf{F}\rangle)\rangle \tag{13.15.16}$$

from

$$\langle \mathrm{d}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}\mathbf{F}\rangle = \langle \left(\mathrm{d}^{\mathrm{s}}\mathbf{F} + \mathrm{d}\langle\mathbf{F}\rangle \cdot\cdot \boldsymbol{\mathcal{F}}^{T}\right) \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}\mathbf{F}\rangle, \tag{13.15.17}$$

and by using the theorem on product averages for appropriate admissible fields. The results $\langle \boldsymbol{\mathcal{F}}^{T} \cdot\cdot \boldsymbol{\Lambda}\rangle = [\boldsymbol{\Lambda}]$ and $\langle \mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle = \mathbf{0}$, from Eqs. (13.7.6) and (13.10.26), were also used.

We record an additional result. From Eq. (12.8.18) we have

$$\langle \mathbf{F} \cdot\cdot \mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle = \langle \mathbf{C} : \mathrm{d}^{\mathrm{p}}\mathbf{T}\rangle, \tag{13.15.18}$$

where $\mathbf{C} = \mathbf{F}^{T} \cdot \mathbf{F}$ is the right Cauchy–Green deformation tensor. Thus, in conjunction with (13.3.4), we conclude that

$$[\mathbf{C}] : \mathrm{d}^{\mathrm{p}}[\mathbf{T}] = \langle \mathbf{C} : \mathrm{d}^{\mathrm{p}}\mathbf{T}\rangle + \langle \mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle. \tag{13.15.19}$$

13.16. Analysis of Second-Order Work Quantities

Since $\mathrm{d}\mathbf{P}$ is statically and $\mathrm{d}\mathbf{F}$ is kinematically admissible, by the theorem on product averages, we can write for the volume average of the second-order work quantity

$$\langle \mathrm{d}\mathbf{P} \cdot\cdot \mathrm{d}\mathbf{F}\rangle = \langle \mathrm{d}\mathbf{P}\rangle \cdot\cdot \langle \mathrm{d}\mathbf{F}\rangle. \tag{13.16.1}$$

Recalling the definitions of plastic increments, we further have

$$\mathrm{d}\langle\mathbf{P}\rangle \cdot\cdot \mathrm{d}\langle\mathbf{F}\rangle = \mathrm{d}\langle\mathbf{P}\rangle \cdot\cdot \mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle + \mathrm{d}\langle\mathbf{P}\rangle \cdot\cdot [\mathbf{M}] \cdot\cdot \mathrm{d}\langle\mathbf{P}\rangle, \tag{13.16.2}$$

$$\langle \mathrm{d}\mathbf{P} \cdot\cdot \mathrm{d}\mathbf{F}\rangle = \langle \mathrm{d}^{\mathrm{p}}\mathbf{P} \cdot\cdot \mathrm{d}\mathbf{F}\rangle + \langle \mathrm{d}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}\mathbf{F}\rangle. \tag{13.16.3}$$

Since $\mathrm{d}\mathbf{F} = \mathrm{d}^{\mathrm{r}}\mathbf{F} + \mathbf{M} \cdot\cdot \boldsymbol{\mathcal{P}} \cdot\cdot \mathrm{d}\langle\mathbf{P}\rangle$, from Eq. (13.11.19), by expansion and the use of the product theorem, the last term on the right-hand side of Eq. (13.16.3) becomes

$$\langle \mathrm{d}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}\mathbf{F}\rangle = 2\,\mathrm{d}\langle\mathbf{P}\rangle \cdot\cdot \mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle + \mathrm{d}\langle\mathbf{P}\rangle \cdot\cdot [\mathbf{M}] \cdot\cdot \mathrm{d}\langle\mathbf{P}\rangle + \langle \mathrm{d}^{\mathrm{r}}\mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}^{\mathrm{r}}\mathbf{F}\rangle. \tag{13.16.4}$$

The relationship $\langle \boldsymbol{\mathcal{P}}^T \cdot\cdot \mathbf{M} \cdot\cdot \boldsymbol{\mathcal{P}} \rangle = [\mathbf{M}]$ from Eq. (13.8.12) was also used. The substitution of Eqs. (13.16.2)–(13.16.4) into Eq. (13.16.1) gives

$$\mathrm{d}\langle \mathbf{P} \rangle \cdot\cdot \mathrm{d}^{\mathrm{p}} \langle \mathbf{F} \rangle = -\langle \mathrm{d}^{\mathrm{p}} \mathbf{P} \cdot\cdot \mathrm{d}\mathbf{F} \rangle - \langle \mathrm{d}^{\mathrm{r}} \mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}^{\mathrm{r}} \mathbf{F} \rangle. \tag{13.16.5}$$

Furthermore, by summing the expressions in Eqs. (13.16.5) and (13.15.9), there follows

$$\mathrm{d}\langle \mathbf{P} \rangle \cdot\cdot \mathrm{d}^{\mathrm{p}} \langle \mathbf{F} \rangle + \mathrm{d}\langle \mathbf{F} \rangle \cdot\cdot \mathrm{d}^{\mathrm{p}} \langle \mathbf{P} \rangle = \langle \mathrm{d}^{\mathrm{s}} \mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}^{\mathrm{s}} \mathbf{F} \rangle - \langle \mathrm{d}^{\mathrm{r}} \mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}^{\mathrm{r}} \mathbf{F} \rangle. \tag{13.16.6}$$

The right-hand side can be recast as

$$\langle \mathrm{d}^{\mathrm{s}} \mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}^{\mathrm{p}} \mathbf{F} \rangle - \langle \mathrm{d}^{\mathrm{r}} \mathbf{F} \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \mathrm{d}^{\mathrm{p}} \mathbf{F} \rangle = \langle (\mathrm{d}^{\mathrm{r}} \mathbf{F} - \mathrm{d}^{\mathrm{s}} \mathbf{F}) \cdot\cdot \mathrm{d}^{\mathrm{p}} \mathbf{P} \rangle, \tag{13.16.7}$$

recalling Eqs. (13.10.33) and (13.11.32), and $\mathrm{d}^{\mathrm{p}} \mathbf{P} = -\boldsymbol{\Lambda} : \mathrm{d}^{\mathrm{p}} \mathbf{F}$.

Expressions dual to (13.16.5)–(13.16.7) can also be derived. We start from

$$\mathrm{d}\langle \mathbf{F} \rangle \cdot\cdot \mathrm{d}\langle \mathbf{P} \rangle = \mathrm{d}\langle \mathbf{F} \rangle \cdot\cdot \mathrm{d}^{\mathrm{p}} \langle \mathbf{P} \rangle + \mathrm{d}\langle \mathbf{F} \rangle \cdot\cdot [\boldsymbol{\Lambda}] \cdot\cdot \mathrm{d}\langle \mathbf{F} \rangle, \tag{13.16.8}$$

$$\langle \mathrm{d}\mathbf{F} \cdot\cdot \mathrm{d}\mathbf{P} \rangle = \langle \mathrm{d}^{\mathrm{p}} \mathbf{F} \cdot\cdot \mathrm{d}\mathbf{P} \rangle + \langle \mathrm{d}\mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot \mathrm{d}\mathbf{P} \rangle. \tag{13.16.9}$$

Since $\mathrm{d}\mathbf{P} = \mathrm{d}^{\mathrm{s}} \mathbf{P} + \boldsymbol{\Lambda} \cdot\cdot \boldsymbol{\mathcal{F}} \cdot\cdot \mathrm{d}\langle \mathbf{F} \rangle$, according to Eq. (13.10.20), by expansion and the use of the product theorem, the last term on the right-hand side of Eq. (13.16.9) becomes

$$\langle \mathrm{d}\mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot \mathrm{d}\mathbf{F} \rangle = 2\, \mathrm{d}\langle \mathbf{F} \rangle \cdot\cdot \mathrm{d}^{\mathrm{p}} \langle \mathbf{P} \rangle + \mathrm{d}\langle \mathbf{F} \rangle \cdot\cdot [\boldsymbol{\Lambda}] \cdot\cdot \mathrm{d}\langle \mathbf{F} \rangle + \langle \mathrm{d}^{\mathrm{s}} \mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot \mathrm{d}^{\mathrm{s}} \mathbf{P} \rangle. \tag{13.16.10}$$

The relationship $\langle \boldsymbol{\mathcal{F}}^T \cdot\cdot \boldsymbol{\Lambda} \cdot\cdot \boldsymbol{\mathcal{F}} \rangle = [\boldsymbol{\Lambda}]$ from (13.7.16) was used. Substituting Eqs. (13.16.8)–(13.16.10) into Eq. (13.16.1) then gives

$$\mathrm{d}\langle \mathbf{F} \rangle \cdot\cdot \mathrm{d}^{\mathrm{p}} \langle \mathbf{P} \rangle = -\langle \mathrm{d}^{\mathrm{p}} \mathbf{F} \cdot\cdot \mathrm{d}\mathbf{P} \rangle - \langle \mathrm{d}^{\mathrm{s}} \mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot \mathrm{d}^{\mathrm{s}} \mathbf{P} \rangle, \tag{13.16.11}$$

which is dual to Eq. (13.16.5).

On the other hand, by summing expressions in Eqs. (13.16.11) and (13.14.10), there follows

$$\mathrm{d}\langle \mathbf{F} \rangle \cdot\cdot \mathrm{d}^{\mathrm{p}} \langle \mathbf{P} \rangle + \mathrm{d}\langle \mathbf{P} \rangle \cdot\cdot \mathrm{d}^{\mathrm{p}} \langle \mathbf{F} \rangle = \langle \mathrm{d}^{\mathrm{r}} \mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot \mathrm{d}^{\mathrm{r}} \mathbf{P} \rangle - \langle \mathrm{d}^{\mathrm{s}} \mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot \mathrm{d}^{\mathrm{s}} \mathbf{P} \rangle. \tag{13.16.12}$$

The right-hand side is also equal to

$$\langle \mathrm{d}^{\mathrm{r}} \mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot \mathrm{d}^{\mathrm{p}} \mathbf{P} \rangle - \langle \mathrm{d}^{\mathrm{s}} \mathbf{P} \cdot\cdot \mathbf{M} \cdot\cdot \mathrm{d}^{\mathrm{p}} \mathbf{P} \rangle = \langle (\mathrm{d}^{\mathrm{s}} \mathbf{P} - \mathrm{d}^{\mathrm{r}} \mathbf{P}) \cdot\cdot \mathrm{d}^{\mathrm{p}} \mathbf{F} \rangle, \tag{13.16.13}$$

by Eqs. (13.10.34) and (13.11.33), and because $\mathrm{d}^{\mathrm{p}}\mathbf{F} = -\mathbf{M} : \mathrm{d}^{\mathrm{p}}\mathbf{M}$. It is easily verified that Eqs. (13.16.6) and (13.16.12) are in accord, since

$$\langle(\mathrm{d}^{\mathrm{r}}\mathbf{F} - \mathrm{d}^{\mathrm{s}}\mathbf{F}) \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle = \langle(\mathrm{d}^{\mathrm{s}}\mathbf{P} - \mathrm{d}^{\mathrm{r}}\mathbf{P}) \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F}\rangle, \tag{13.16.14}$$

by Eq. (13.11.34).

We end this section by listing two additional identities. They are

$$\langle\, \mathrm{d}^{\mathrm{p}}\mathbf{F} \cdot\cdot\, \mathbf{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F}\,\rangle = \langle\, \mathrm{d}^{\mathrm{s}}\mathbf{F} \cdot\cdot\, \mathbf{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{s}}\mathbf{F}\,\rangle + \langle\, \mathrm{d}^{\mathrm{s}}\mathbf{P} \cdot\cdot\, \mathbf{M} \cdot\cdot\, \mathrm{d}^{\mathrm{s}}\mathbf{P}\,\rangle, \tag{13.16.15}$$

and

$$\langle\, \mathrm{d}^{\mathrm{p}}\mathbf{P} \cdot\cdot\, \mathbf{M} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P}\,\rangle = \langle\, \mathrm{d}^{\mathrm{r}}\mathbf{F} \cdot\cdot\, \mathbf{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{r}}\mathbf{F}\,\rangle + \langle\, \mathrm{d}^{\mathrm{r}}\mathbf{P} \cdot\cdot\, \mathbf{M} \cdot\cdot\, \mathrm{d}^{\mathrm{r}}\mathbf{P}\,\rangle. \tag{13.16.16}$$

For example, the first one follows from

$$\begin{aligned}\mathrm{d}^{\mathrm{p}}\mathbf{F} \cdot\cdot\, \mathbf{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F} &= (\mathrm{d}^{\mathrm{s}}\mathbf{F} - \mathrm{d}^{\mathrm{s}}\mathbf{P} \cdot\cdot\, \mathbf{M}) \cdot\cdot\, \mathbf{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F} \\ &= \mathrm{d}^{\mathrm{s}}\mathbf{F} \cdot\cdot\, \mathbf{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F} + \mathrm{d}^{\mathrm{s}}\mathbf{P} \cdot\cdot\, \mathbf{M} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P},\end{aligned} \tag{13.16.17}$$

by taking the volume average and by using Eqs. (13.10.33) and (13.10.34). Note that the left-hand sides in Eqs. (13.16.15) and (13.16.16) are actually equal to each other, both being equal to $-\langle\, \mathrm{d}^{\mathrm{p}}\mathbf{P} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F}\,\rangle$.

13.17. General Analysis of Macroscopic Plastic Potentials

A general study of the transmissibility of plastic potentials and normality rules from micro-to-macrolevel is presented in this section. The analysis is originally due to Hill and Rice (1973), who used the framework of general conjugate stress and strain measures in their formulation. Here, the formulation is conveniently cast by using the deformation gradient and the nominal stress. The plastic part of the free energy increment at the microlevel,

$$\mathrm{d}^{\mathrm{p}}\Psi = \Psi\left(\mathbf{F}, \mathcal{H} + d\mathcal{H}\right) - \Psi\left(\mathbf{F}, \mathcal{H}\right), \tag{13.17.1}$$

is a potential for the plastic part of the nominal stress increment,

$$\mathrm{d}^{\mathrm{p}}\mathbf{P} = \mathbf{P}\left(\mathbf{F}, \mathcal{H} + d\mathcal{H}\right) - \mathbf{P}\left(\mathbf{F}, \mathcal{H}\right), \tag{13.17.2}$$

such that

$$\mathrm{d}^{\mathrm{p}}\mathbf{P} = \frac{\partial}{\partial \mathbf{F}}\left(\mathrm{d}^{\mathrm{p}}\Psi\right). \tag{13.17.3}$$

If the trace product of $\mathrm{d}^{\mathrm{p}}\mathbf{P}$ with an elastic increment $\delta\mathbf{F}$ is positive,

$$\delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P} = \delta\mathbf{F} \cdot\cdot\, \frac{\partial}{\partial \mathbf{F}}\left(\mathrm{d}^{\mathrm{p}}\Psi\right) = \delta\left(\mathrm{d}^{\mathrm{p}}\Psi\right) > 0, \tag{13.17.4}$$

we say that the material response complies with the normality rule at microlevel in the deformation space.

Dually, the plastic part of the increment of complementary energy at the microlevel,

$$\mathrm{d}^{\mathrm{p}}\Phi = \Psi\left(\mathbf{P}, \mathcal{H} + d\mathcal{H}\right) - \Phi\left(\mathbf{P}, \mathcal{H}\right), \tag{13.17.5}$$

is a potential for the plastic part of the deformation gradient increment,

$$\mathrm{d}^{\mathrm{p}}\mathbf{F} = \mathbf{F}\left(\mathbf{P}, \mathcal{H} + d\mathcal{H}\right) - \mathbf{F}\left(\mathbf{P}, \mathcal{H}\right), \tag{13.17.6}$$

such that

$$\mathrm{d}^{\mathrm{p}}\mathbf{F} = \frac{\partial}{\partial \mathbf{P}}\left(\mathrm{d}^{\mathrm{p}}\Phi\right). \tag{13.17.7}$$

If the trace product of $\mathrm{d}^{\mathrm{p}}\mathbf{F}$ with an elastic increment $\delta\mathbf{P}$ is negative,

$$\delta\mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\mathbf{F} = \delta\mathbf{P}\cdot\cdot\,\frac{\partial}{\partial \mathbf{P}}\left(\mathrm{d}^{\mathrm{p}}\Phi\right) = \delta\left(\mathrm{d}^{\mathrm{p}}\Phi\right) < 0, \tag{13.17.8}$$

the material response complies with the normality rule at microlevel in the stress space. With these preliminaries from the microlevel, we now examine the macroscopic potentials and macroscopic normality rules.

13.17.1. Deformation Space Formulation

The plastic part of the increment of macroscopic free energy, associated with a cycle of the application and removal of an elastoplastic increment of the macroscopic deformation gradient $\mathrm{d}\langle\mathbf{F}\rangle$, is defined by

$$\mathrm{d}^{\mathrm{p}}\hat{\Psi} = \hat{\Psi}\left(\langle\mathbf{F}\rangle, \mathcal{H} + d\mathcal{H}\right) - \hat{\Psi}\left(\langle\mathbf{F}\rangle, \mathcal{H}\right). \tag{13.17.9}$$

The macroscopic free energy before the cycle is

$$\hat{\Psi}\left(\langle\mathbf{F}\rangle, \mathcal{H}\right) = \frac{1}{V^0}\int_{V^0} \Psi\left(\mathbf{F}, \mathcal{H}\right)\mathrm{d}V^0, \tag{13.17.10}$$

where $\mathbf{F}$ is the local deformation gradient field within the macroelement. After a cycle of $\mathrm{d}\langle\mathbf{F}\rangle$, the local deformation gradients within V^0 are in general not restored, so that

$$\begin{aligned}\hat{\Psi}\left(\langle\mathbf{F}\rangle, \mathcal{H} + d\mathcal{H}\right) &= \frac{1}{V^0}\int_{V^0} \Psi\left(\mathbf{F} + \mathrm{d}^{\mathrm{s}}\mathbf{F}, \mathcal{H} + d\mathcal{H}\right)\mathrm{d}V^0 \\ &= \frac{1}{V^0}\int_{V^0}\left[\Psi\left(\mathbf{F}, \mathcal{H} + d\mathcal{H}\right) + \frac{\partial \Psi}{\partial \mathbf{F}}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\right]\mathrm{d}V^0.\end{aligned} \tag{13.17.11}$$

Here, $\mathrm{d}^{\mathrm{s}}\mathbf{F}$ represents a residual increment of the deformation gradient that remains at the microlevel after macroscopic cycle of $\mathrm{d}\langle\mathbf{F}\rangle$. Upon substitution of Eqs. (13.17.10) and (13.17.11) into Eq. (13.17.9), there follows

$$\mathrm{d}^{\mathrm{p}}\hat{\Psi} = \frac{1}{V^0}\int_{V^0}\left[\Psi\left(\mathbf{F}, \mathcal{H} + d\mathcal{H}\right) - \Psi\left(\mathbf{F}, \mathcal{H}\right)\right]\mathrm{d}V^0 + \frac{1}{V^0}\int_{V^0}\mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\,\mathrm{d}V^0, \tag{13.17.12}$$

i.e.,

$$\mathrm{d}^{\mathrm{p}}\hat{\Psi} = \langle \mathrm{d}^{\mathrm{p}}\Psi\rangle + \langle \mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle. \tag{13.17.13}$$

Recalling that $\mathbf{P}$ is statically admissible, while $\mathrm{d}^{\mathrm{s}}\mathbf{F}$ is kinematically admissible field, and since $\langle \mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle = \mathbf{0}$ by Eq. (13.10.26), we have

$$\langle \mathbf{P}\cdot\cdot\,\mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle = \langle \mathbf{P}\rangle\cdot\cdot\langle \mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle = 0. \tag{13.17.14}$$

Equation (13.17.13) consequently reduces to

$$\mathrm{d}^{\mathrm{p}}\hat{\Psi} = \langle \mathrm{d}^{\mathrm{p}}\Psi\rangle. \tag{13.17.15}$$

Thus, the plastic increment of macroscopic free energy is a direct volume average of the plastic increment of microscopic free energy.

The potential property is established through

$$\begin{aligned}\frac{\partial}{\partial\langle \mathbf{F}\rangle}\,(\mathrm{d}^{\mathrm{p}}\hat{\Psi}) &= \frac{\partial}{\partial\langle \mathbf{F}\rangle}\,\langle \mathrm{d}^{\mathrm{p}}\Psi\rangle = \Big\langle\,\frac{\partial\,(\mathrm{d}^{\mathrm{p}}\Psi)}{\partial\langle \mathbf{F}\rangle}\,\Big\rangle \\ &= \Big\langle\,\frac{\partial\,(\mathrm{d}^{\mathrm{p}}\Psi)}{\partial\mathbf{F}}\cdot\cdot\,\frac{\partial\mathbf{F}}{\partial\langle \mathbf{F}\rangle}\,\Big\rangle = \langle \mathrm{d}^{\mathrm{p}}\mathbf{P}\cdot\cdot\,\boldsymbol{\mathcal{F}}\,\rangle.\end{aligned} \tag{13.17.16}$$

Since the plastic part of the increment of macroscopic nominal stress is a weighted volume average of the plastic part of the increment of local nominal stress, as seen from Eq. (13.10.14), we deduce that $\mathrm{d}^{\mathrm{p}}\hat{\Psi}$ is indeed a plastic potential for $\mathrm{d}^{\mathrm{p}}\langle \mathbf{P}\rangle$, i.e.,

$$\mathrm{d}^{\mathrm{p}}\langle \mathbf{P}\rangle = \frac{\partial}{\partial\langle \mathbf{F}\rangle}\,(\mathrm{d}^{\mathrm{p}}\hat{\Psi}). \tag{13.17.17}$$

If Eq. (13.17.17) is subjected to the trace product with an elastic increment $\delta\langle \mathbf{F}\rangle$, there follows

$$\delta\langle \mathbf{F}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle \mathbf{P}\rangle = \delta\langle \mathbf{F}\rangle\cdot\cdot\,\frac{\partial}{\partial\langle \mathbf{F}\rangle}\,(\mathrm{d}^{\mathrm{p}}\hat{\Psi}) = \delta(\mathrm{d}^{\mathrm{p}}\hat{\Psi}). \tag{13.17.18}$$

Substitution of (13.17.15) gives

$$\delta(\mathrm{d}^{\mathrm{p}}\hat{\Psi}) = \delta\,\langle \mathrm{d}^{\mathrm{p}}\Psi\,\rangle = \langle\,\delta(\mathrm{d}^{\mathrm{p}}\Psi)\,\rangle. \tag{13.17.19}$$

Thus, the normality at the microlevel ensures the normality at the macrolevel, i.e.,

$$\text{if}\quad \delta\,(\mathrm{d}^{\mathrm{p}}\Psi) > 0,\quad \text{then}\quad \delta\,(\mathrm{d}^{\mathrm{p}}\hat{\Psi}) > 0. \tag{13.17.20}$$

If the conjugate stress and strain measures $\mathbf{T}$ and $\mathbf{E}$ are utilized, Eq. (13.17.17) becomes

$$\mathrm{d}^{\mathrm{p}}[\mathbf{T}] = \frac{\partial}{\partial[\mathbf{E}]}\,(\mathrm{d}^{\mathrm{p}}\hat{\Psi}). \tag{13.17.21}$$

This follows because the relationships from Section 13.12 hold,

$$\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = \langle\boldsymbol{\mathcal{K}}\rangle^T : \mathrm{d}^{\mathrm{p}}[\mathbf{T}], \quad \frac{\partial}{\partial\langle\mathbf{F}\rangle} = \langle\boldsymbol{\mathcal{K}}\rangle^T : \frac{\partial}{\partial[\mathbf{E}]}\,. \tag{13.17.22}$$

13.17.2. Stress Space Formulation

In a dual analysis, we introduce the plastic part of the increment of macroscopic complementary energy, associated with a cycle of the application and removal of an elastoplastic increment of macroscopic stress $\mathrm{d}\langle\mathbf{P}\rangle$, such that

$$\mathrm{d}^{\mathrm{p}}\hat{\Phi} = \hat{\Phi}\left(\langle\mathbf{P}\rangle, \mathcal{H} + d\mathcal{H}\right) - \hat{\Phi}\left(\langle\mathbf{P}\rangle, \mathcal{H}\right). \tag{13.17.23}$$

The macroscopic complementary energy before the cycle is

$$\hat{\Phi}\left(\langle\mathbf{P}\rangle, \mathcal{H}\right) = \frac{1}{V^0}\int_{V^0} \Phi\left(\mathbf{P}, \mathcal{H}\right)\mathrm{d}V^0, \tag{13.17.24}$$

where $\mathbf{P}$ is the local stress field within the macroelement. After a cycle of $\mathrm{d}\langle\mathbf{P}\rangle$, the local stresses within V^0 are in general not restored, so that

$$\begin{aligned}\hat{\Phi}\left(\langle\mathbf{P}\rangle, \mathcal{H} + d\mathcal{H}\right) &= \frac{1}{V^0}\int_{V^0} \Phi\left(\mathbf{P} + \mathrm{d}^{\mathrm{r}}\mathbf{P}, \mathcal{H} + d\mathcal{H}\right)\mathrm{d}V^0 \\ &= \frac{1}{V^0}\int_{V^0}\left[\Phi\left(\mathbf{P}, \mathcal{H} + d\mathcal{H}\right) + \frac{\partial\Phi}{\partial\mathbf{P}}\cdot\cdot\,\mathrm{d}^{\mathrm{r}}\mathbf{P}\right]\mathrm{d}V^0,\end{aligned} \tag{13.17.25}$$

where $\mathrm{d}^{\mathrm{r}}\mathbf{P}$ represents a residual increment of stress that remains at the microlevel upon macroscopic cycle of $\mathrm{d}\langle\mathbf{P}\rangle$. Substitution of Eqs. (13.17.24) and (13.17.25) into Eq. (13.17.23) yields

$$\mathrm{d}^{\mathrm{p}}\hat{\Phi} = \frac{1}{V^0}\int_{V^0}\left[\Phi\left(\mathbf{P}, \mathcal{H} + d\mathcal{H}\right) - \Phi\left(\mathbf{P}, \mathcal{H}\right)\right]\mathrm{d}V^0 + \frac{1}{V^0}\int_{V^0}\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{r}}\mathbf{P}\,\mathrm{d}V^0, \tag{13.17.26}$$

i.e.,

$$\mathrm{d}^{\mathrm{p}}\hat{\Phi} = \langle\mathrm{d}^{\mathrm{p}}\Phi\rangle + \langle\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{r}}\mathbf{P}\rangle. \tag{13.17.27}$$

Since $\mathbf{F}$ is kinematically admissible, while $\mathrm{d}^{\mathrm{r}}\mathbf{P}$ is statically admissible field, and since $\langle\mathrm{d}^{\mathrm{r}}\mathbf{P}\rangle = 0$ by Eq. (13.11.25), we have

$$\langle\mathbf{F}\cdot\cdot\,\mathrm{d}^{\mathrm{r}}\mathbf{P}\rangle = \langle\mathbf{F}\rangle\cdot\cdot\langle\mathrm{d}^{\mathrm{r}}\mathbf{P}\rangle = 0. \tag{13.17.28}$$

Consequently, Eq. (13.17.27) reduces to

$$\mathrm{d}^{\mathrm{p}}\hat{\Phi} = \langle\mathrm{d}^{\mathrm{p}}\Phi\rangle. \tag{13.17.29}$$

This shows that the plastic increment of macroscopic complementary energy is a direct volume average of the plastic increment of microscopic complementary energy.

The potential property follows from

$$
\begin{aligned}
\frac{\partial}{\partial\langle\mathbf{P}\rangle}\,(\mathrm{d}^{\mathrm{p}}\hat{\Phi}) &= \frac{\partial}{\partial\langle\mathbf{P}\rangle}\,\langle\mathrm{d}^{\mathrm{p}}\Phi\rangle = \langle\,\frac{\partial\,(\mathrm{d}^{\mathrm{p}}\Phi)}{\partial\langle\mathbf{P}\rangle}\,\rangle \\
&= \langle\,\frac{\partial\,(\mathrm{d}^{\mathrm{p}}\Phi)}{\partial\mathbf{P}}\cdot\cdot\,\frac{\partial\mathbf{P}}{\partial\langle\mathbf{P}\rangle}\,\rangle = \langle\mathrm{d}^{\mathrm{p}}\mathbf{F}\cdot\cdot\boldsymbol{\mathcal{P}}\,\rangle.
\end{aligned}
\tag{13.17.30}
$$

Since the plastic part of the increment of macroscopic deformation gradient is a weighted volume average of the plastic part of the increment of local deformation gradient, as shown in Eq. (13.11.13), we deduce that $\mathrm{d}^{\mathrm{p}}\hat{\Phi}$ is indeed a plastic potential for $\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle$, i.e.,

$$
\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle = \frac{\partial}{\partial\langle\mathbf{P}\rangle}\,(\mathrm{d}^{\mathrm{p}}\hat{\Phi}). \tag{13.17.31}
$$

Furthermore, the trace product of Eq. (13.17.31) with an elastic increment $\delta\langle\mathbf{P}\rangle$ gives

$$
\delta\langle\mathbf{P}\rangle\cdot\cdot\,\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle = \delta\langle\mathbf{P}\rangle\cdot\cdot\,\frac{\partial}{\partial\langle\mathbf{P}\rangle}\,(\mathrm{d}^{\mathrm{p}}\hat{\Phi}) = \delta(\mathrm{d}^{\mathrm{p}}\hat{\Phi}). \tag{13.17.32}
$$

In view of Eq. (13.17.29), therefore,

$$
\delta(\mathrm{d}^{\mathrm{p}}\hat{\Phi}) = \delta\,\langle\mathrm{d}^{\mathrm{p}}\Phi\,\rangle = \langle\,\delta(\mathrm{d}^{\mathrm{p}}\Phi)\,\rangle. \tag{13.17.33}
$$

From this we conclude that the normality at the microlevel, ensures the normality at the macrolevel, i.e.,

$$
\text{if}\quad \delta\,(\mathrm{d}^{\mathrm{p}}\Phi) < 0,\quad \text{then}\quad \delta\,(\mathrm{d}^{\mathrm{p}}\hat{\Phi}) < 0. \tag{13.17.34}
$$

It is observed that

$$
\mathrm{d}^{\mathrm{p}}\hat{\Psi} + \mathrm{d}^{\mathrm{p}}\hat{\Phi} = 0, \tag{13.17.35}
$$

since locally $\mathrm{d}^{\mathrm{p}}\Psi + \mathrm{d}^{\mathrm{p}}\Phi = 0$, as well. Thus, having in mind that

$$
\frac{\partial}{\partial[\mathbf{E}]} = [\boldsymbol{\Lambda}_{(1)}] : \frac{\partial}{\partial[\mathbf{T}]}\,, \tag{13.17.36}
$$

we can rewrite Eq. (13.17.21) as

$$
\mathrm{d}^{\mathrm{p}}[\mathbf{T}] = [\boldsymbol{\Lambda}_{(1)}] : \frac{\partial}{\partial[\mathbf{T}]}\,(-\mathrm{d}^{\mathrm{p}}\hat{\Phi}). \tag{13.17.37}
$$

Upon taking the trace product with $[\mathbf{M}_{(1)}] = [\mathbf{\Lambda}_{(1)}]^{-1}$, and recalling that $\mathrm{d}^{\mathrm{p}}[\mathbf{E}] = -[\mathbf{M}_{(1)}] : \mathrm{d}^{\mathrm{p}}[\mathbf{T}]$, Eq. (13.17.37) gives

$$\mathrm{d}^{\mathrm{p}}[\mathbf{E}] = \frac{\partial}{\partial[\mathbf{T}]}\,(\mathrm{d}^{\mathrm{p}}\hat{\Phi}). \tag{13.17.38}$$

This shows that $\mathrm{d}^{\mathrm{p}}\hat{\Phi}$, when expressed in terms of $[\mathbf{T}]$, is a potential for the plastic increment $\mathrm{d}^{\mathrm{p}}[\mathbf{E}]$.

13.18. Transmissibility of Ilyushin's Postulate

Suppose that the aggregate is taken through the deformation cycle which, at some stage, involves plastic deformation. Following an analogous analysis as in Section 8.5, the cycle emanates from the state $A^0\left(\langle\mathbf{F}\rangle^0, \mathcal{H}\right)$ within the macroscopic yield surface, it includes an elastic segment from A^0 to $A\left(\langle\mathbf{F}\rangle, \mathcal{H}\right)$ on the current yield surface, followed by an infinitesimal elastoplastic segment from A to $B\left(\langle\mathbf{F}\rangle + \mathrm{d}\langle\mathbf{F}\rangle, \mathcal{H} + d\mathcal{H}\right)$, and the elastic unloading segments from B to $C(\langle\mathbf{F}\rangle, \mathcal{H}+d\mathcal{H})$, and from C to $C^0\left(\langle\mathbf{F}\rangle^0, \mathcal{H} + d\mathcal{H}\right)$. The work done along the segments A^0A and CC^0 is

$$\begin{aligned}\int_{A^0}^{A} \langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle &= \int_{A^0}^{A} \frac{\partial\hat{\Psi}}{\partial\langle\mathbf{F}\rangle} \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle \\ &= \hat{\Psi}\left(\langle\mathbf{F}\rangle, \mathcal{H}\right) - \hat{\Psi}\left(\langle\mathbf{F}\rangle^0, \mathcal{H}\right),\end{aligned} \tag{13.18.1}$$

$$\begin{aligned}\int_{C}^{C^0} \langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle &= \int_{C}^{C^0} \frac{\partial\hat{\Psi}}{\partial\langle\mathbf{F}\rangle} \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle \\ &= \hat{\Psi}\left(\langle\mathbf{F}\rangle^0, \mathcal{H} + d\mathcal{H}\right) - \hat{\Psi}\left(\langle\mathbf{F}\rangle, \mathcal{H} + d\mathcal{H}\right).\end{aligned} \tag{13.18.2}$$

The work done along the segments AB and BC is, by the trapezoidal rule of quadrature,

$$\int_{A}^{B} \langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle = \langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle + \frac{1}{2}\,\mathrm{d}\langle\mathbf{P}\rangle : \mathrm{d}\langle\mathbf{F}\rangle, \tag{13.18.3}$$

$$\int_{B}^{C} \langle\mathbf{P}\rangle : \mathrm{d}\langle\mathbf{F}\rangle = -\langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle - \frac{1}{2}\left(\mathrm{d}\langle\mathbf{P}\rangle + \mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle\right) \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle, \tag{13.18.4}$$

to second-order terms. Consequently,

$$\oint_{\langle F\rangle} \langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle = -\frac{1}{2}\,\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle + (\mathrm{d}^{\mathrm{p}}\hat{\Psi})^0 - \mathrm{d}^{\mathrm{p}}\hat{\Psi}, \tag{13.18.5}$$

where

$$\mathrm{d}^{\mathrm{p}}\hat{\Psi} = \hat{\Psi}\left(\langle\mathbf{F}\rangle, \mathcal{H} + d\mathcal{H}\right) - \hat{\Psi}\left(\langle\mathbf{F}\rangle, \mathcal{H}\right),$$
$$(\mathrm{d}^{\mathrm{p}}\hat{\Psi})^0 = \hat{\Psi}\left(\langle\mathbf{F}\rangle^0, \mathcal{H} + d\mathcal{H}\right) - \hat{\Psi}\left(\langle\mathbf{F}\rangle^0, \mathcal{H}\right). \tag{13.18.6}$$

For the cycle with a sufficiently small segment along which the plastic deformation takes place, Eq. (13.18.5) becomes, to first order,

$$\oint_{\langle F\rangle} \langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle = (\mathrm{d}^{\mathrm{p}}\hat{\Psi})^0 - \mathrm{d}^{\mathrm{p}}\hat{\Psi}. \tag{13.18.7}$$

Since the plastic increment of macroscopic free energy is the volume average of the plastic increment of microscopic free energy, $\mathrm{d}^{\mathrm{p}}\hat{\Psi} = \langle \mathrm{d}^{\mathrm{p}}\Psi\rangle$, as shown in Eq. (13.17.15), we can rewrite Eq. (13.18.7) as

$$\oint_{\langle F\rangle} \langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle = \langle\, (\mathrm{d}^{\mathrm{p}}\Psi)^0 - \mathrm{d}^{\mathrm{p}}\Psi\rangle. \tag{13.18.8}$$

This holds even though the local $\mathbf{F}$ field is generally not restored in the macroscopic cycle of $\mathrm{d}\langle\mathbf{F}\rangle$. Equation (13.18.8) evidently implies, if

$$(\mathrm{d}^{\mathrm{p}}\Psi)^0 - \mathrm{d}^{\mathrm{p}}\Psi > 0 \tag{13.18.9}$$

at the microlevel, then

$$\langle\, (\mathrm{d}^{\mathrm{p}}\Psi)^0 - \mathrm{d}^{\mathrm{p}}\Psi\,\rangle > 0 \tag{13.18.10}$$

at the macrolevel. In other words, the restricted Ilyushin's postulate (for the specified deformation cycles with sufficiently small plastic segments) is transmitted from the microlevel to the macrolevel (Hill and Rice, 1973).

If the cycle begins from the point on the yield surface, i.e., if $A^0 = A$ and $\langle\mathbf{F}\rangle^0 = \langle\mathbf{F}\rangle$, Eq. (13.18.5) reduces to

$$\oint_{\langle F\rangle} \langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle = -\frac{1}{2}\,\mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle \cdot\cdot\, \mathrm{d}\langle\mathbf{F}\rangle. \tag{13.18.11}$$

On the other hand, from Eq. (13.15.9) we have

$$\mathrm{d}\langle\mathbf{F}\rangle \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\langle\mathbf{P}\rangle = \langle \mathrm{d}\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P}\rangle + \langle \mathrm{d}^{\mathrm{s}}\mathbf{F} \cdot\cdot\, \boldsymbol{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{s}}\mathbf{F}\rangle. \tag{13.18.12}$$

Since $\boldsymbol{\Lambda}$ is not necessarily positive definite, we conclude that the compliance with the restricted Ilyushin's postulate (for infinitesimal cycles emanating from the yield surface) at the microlevel,

$$\oint_{F} \mathbf{P} \cdot\cdot\, \mathrm{d}\mathbf{F} = -\frac{1}{2}\,\mathrm{d}^{\mathrm{p}}\mathbf{P} \cdot\cdot\, \mathrm{d}\mathbf{F} > 0, \tag{13.18.13}$$

is not necessarily transmitted to the macrolevel.

13.19. Aggregate Minimum Shear and Maximum Work Principle

Consider an aggregate macroelement in the deformed equilibrium configuration. The local deformation gradient and the nominal stress fields are $\mathbf{F}$ and $\mathbf{P}$. Let $\mathrm{d}\mathbf{F}$ be the actual increment of deformation gradient that physically occurs under prescribed increment of displacement $\mathrm{d}\mathbf{u}$ on the bounding surface S^0 of the aggregate macroelement. Furthermore, let $\mathrm{d}\bar{\mathbf{F}}$ be any kinematically admissible field of the increment of deformation gradient that is associated with the same prescribed increment of displacement $\mathrm{d}\mathbf{u}$ over S^0. By the Gauss divergence theorem, the volume averages of $\mathrm{d}\mathbf{F}$ and $\mathrm{d}\bar{\mathbf{F}}$, over the macroelement volume, are equal to each other,

$$\langle\, \mathrm{d}\mathbf{F}\,\rangle = \langle\, \mathrm{d}\bar{\mathbf{F}}\,\rangle = \int_{S^0} \mathrm{d}\mathbf{u} \otimes \mathbf{n}^0 \,\mathrm{d}S^0. \tag{13.19.1}$$

In addition, there is an equality

$$\langle\, \mathbf{P}\cdot\cdot\,\mathrm{d}\mathbf{F}\,\rangle = \langle\, \mathbf{P}\cdot\cdot\,\mathrm{d}\bar{\mathbf{F}}\,\rangle = \int_{S^0} \mathbf{p}_n \otimes \mathrm{d}\mathbf{u}\,\mathrm{d}S^0. \tag{13.19.2}$$

Suppose that simple shearing on active slip systems is the only mechanism of deformation in a rigid-plastic aggregate. Let n shears $\mathrm{d}\gamma^\alpha$ be a set of local slip increments which give rise to local strain increment $\mathrm{d}\mathbf{E}$. These are actual, physically operative slips, so that on each slip system of this set

$$|\tau^\alpha| = \tau^\alpha_{\mathrm{cr}}\,, \quad (\alpha = 1, 2, \ldots, n). \tag{13.19.3}$$

The slip in the opposite sense along the same slip direction is not considered as an independent slip system. The Bauschinger effect is assumed to be absent, so that $\tau^\alpha_{\mathrm{cr}}$ is equal in both senses along the same slip direction. In view of Eqs. (12.1.22) and (12.1.24), we can write

$$\mathrm{d}\mathbf{E} = \sum_{\alpha=1}^{n} \mathbf{P}^\alpha_0 \,\mathrm{d}\gamma^\alpha, \quad \mathbf{P}^\alpha_0 = \mathbf{F}^T \cdot \mathbf{P}^\alpha \cdot \mathbf{F} = \mathbf{F}^T \cdot (\mathbf{s}^\alpha \otimes \mathbf{m}^\alpha)_{\mathrm{s}} \cdot \mathbf{F}. \tag{13.19.4}$$

Further, let $\bar{n}$ shears $\mathrm{d}\bar{\gamma}^\alpha$ be a set of local slip increments which give rise to local strain increment $\mathrm{d}\bar{\mathbf{E}}$, but which are not necessarily physically operative, so that

$$|\bar{\tau}^\alpha| \le \bar{\tau}^\alpha_{\mathrm{cr}}\,, \quad (\alpha = 1, 2, \ldots, \bar{n}). \tag{13.19.5}$$

For this set we can write

$$\mathrm{d}\bar{\mathbf{E}} = \sum_{\alpha=1}^{\bar{n}} \bar{\mathbf{P}}^\alpha_0 \,\mathrm{d}\bar{\gamma}^\alpha, \quad \bar{\mathbf{P}}^\alpha_0 = \mathbf{F}^T \cdot \bar{\mathbf{P}}^\alpha \cdot \mathbf{F} = \mathbf{F}^T \cdot (\bar{\mathbf{s}}^\alpha \otimes \bar{\mathbf{m}}^\alpha)_{\mathrm{s}} \cdot \mathbf{F}. \tag{13.19.6}$$

The slip system vectors of the second set are denoted by $\bar{\mathbf{s}}^\alpha$ and $\bar{\mathbf{m}}^\alpha$. (Even if it happens that $\mathrm{d}\bar{\mathbf{E}} = \mathrm{d}\mathbf{E}$ at some point or the subelement, there still may be different sets of shears corresponding to that same $\mathrm{d}\mathbf{E}$. These are geometrically equivalent sets of shears, which were the main concern of the single crystal consideration in Section 12.19). Consequently,

$$\langle \mathbf{P} \cdot\cdot\, \mathrm{d}\mathbf{F} \rangle = \langle \mathbf{T} : \mathrm{d}\mathbf{E} \rangle = \langle \sum_{\alpha=1}^{n} \tau^\alpha \, \mathrm{d}\gamma^\alpha \rangle , \quad \tau^\alpha = \boldsymbol{\tau} : \mathbf{P}^\alpha, \tag{13.19.7}$$

$$\langle \mathbf{P} \cdot\cdot\, \mathrm{d}\bar{\mathbf{F}} \rangle = \langle \mathbf{T} : \mathrm{d}\bar{\mathbf{E}} \rangle = \langle \sum_{\alpha=1}^{\bar{n}} \bar{\tau}^\alpha \, \mathrm{d}\bar{\gamma}^\alpha \rangle , \quad \bar{\tau}^\alpha = \boldsymbol{\tau} : \bar{\mathbf{P}}^\alpha, \tag{13.19.8}$$

where $\boldsymbol{\tau} = \mathbf{F} \cdot \mathbf{P} = \mathbf{F} \cdot \mathbf{T} \cdot \mathbf{T}^T$ is the Kirchhoff stress (equal here to the Cauchy stress $\boldsymbol{\sigma}$, because the deformation of rigid-plastic polycrystalline aggregate is isochoric, $\det \mathbf{F} = 1$). Since slip in the opposite sense along the same slip direction is not considered as an independent slip system, $\mathrm{d}\gamma^\alpha < 0$ when $\tau^\alpha < 0$, and the above equations can be recast as

$$\langle \sum_{\alpha=1}^{n} \tau^\alpha \, \mathrm{d}\gamma^\alpha \rangle = \langle \sum_{\alpha=1}^{n} |\tau^\alpha| \, |\mathrm{d}\gamma^\alpha| \rangle = \langle \sum_{\alpha=1}^{n} \tau_{\mathrm{cr}}^\alpha \, |\mathrm{d}\gamma^\alpha| \rangle , \tag{13.19.9}$$

$$\langle \sum_{\alpha=1}^{\bar{n}} \bar{\tau}^\alpha \, \mathrm{d}\bar{\gamma}^\alpha \rangle = \langle \sum_{\alpha=1}^{\bar{n}} |\bar{\tau}^\alpha| \, |\mathrm{d}\bar{\gamma}^\alpha| \rangle \leq \langle \sum_{\alpha=1}^{\bar{n}} \bar{\tau}_{\mathrm{cr}}^\alpha \, |\mathrm{d}\bar{\gamma}^\alpha| \rangle . \tag{13.19.10}$$

Recall that $|\tau^\alpha| = \tau_{\mathrm{cr}}^\alpha$ and $|\bar{\tau}^\alpha| \leq \bar{\tau}_{\mathrm{cr}}^\alpha$. Thus, we conclude from Eqs. (13.19.2), (13.19.9), and (13.19.10) that

$$\langle \sum_{\alpha=1}^{n} \tau_{\mathrm{cr}}^\alpha \, |\mathrm{d}\gamma^\alpha| \rangle \leq \langle \sum_{\alpha=1}^{\bar{n}} \bar{\tau}_{\mathrm{cr}}^\alpha \, |\mathrm{d}\bar{\gamma}^\alpha| \rangle . \tag{13.19.11}$$

If the hardening in each grain is isotropic, we have

$$\langle \tau_{\mathrm{cr}}^\alpha \sum_{\alpha=1}^{n} |\mathrm{d}\gamma^\alpha| \rangle \leq \langle \bar{\tau}_{\mathrm{cr}}^\alpha \sum_{\alpha=1}^{\bar{n}} |\mathrm{d}\bar{\gamma}^\alpha| \rangle . \tag{13.19.12}$$

Assuming, in addition, that all grains harden equally, the critical resolved shear stress is uniform throughout the aggregate, and (13.19.12) reduces to

$$\langle \sum_{\alpha=1}^{n} |\mathrm{d}\gamma^\alpha| \rangle \leq \langle \sum_{\alpha=1}^{\bar{n}} |\mathrm{d}\bar{\gamma}^\alpha| \rangle . \tag{13.19.13}$$

This is the minimum shear principle for an aggregate macroelement. In the context of infinitesimal strain, the original proof was given by Bishop and Hill (1951a).

Bishop and Hill (*op. cit.*) also proved the maximum work principle for an aggregate of rigid-plastic crystals. Let $\dot{\mathbf{F}}$ be the rate of deformation gradient that takes place at the state of stress $\mathbf{P}$, and let $\mathbf{P}_*$ be any other state of stress which does not violate the yield condition on any slip system. The difference of the corresponding local rates of work per unit volume is, from Eq. (12.19.14),

$$(\boldsymbol{\tau} - \boldsymbol{\tau}_*) : \mathbf{D} = (\mathbf{P} - \mathbf{P}_*) \cdot\cdot \dot{\mathbf{F}} = (\mathbf{T} - \mathbf{T}_*) : \dot{\mathbf{E}} \geq 0. \tag{13.19.14}$$

Upon integration over the representative macroelement volume, there follows

$$\langle(\mathbf{P} - \mathbf{P}_*) \cdot\cdot \dot{\mathbf{F}}\rangle = (\langle\mathbf{P}\rangle - \langle\mathbf{P}_*\rangle) \cdot\cdot \langle\dot{\mathbf{F}}\rangle = ([\,\mathbf{T}\,] - [\,\mathbf{T}_*\,]) : [\dot{\mathbf{E}}\,] \geq 0. \tag{13.19.15}$$

If the current configuration is taken for the reference, we can write

$$(\{\boldsymbol{\sigma}\} - \{\boldsymbol{\sigma}_*\}) : \{\mathbf{D}\} \geq 0. \tag{13.19.16}$$

The last two expressions are the alternative statements of the maximum work principle for an aggregate.

13.20. Macroscopic Flow Potential for Rate-Dependent Plasticity

In a rate-dependent plastic aggregate, which exhibits the instantaneous elastic response to rapid loading or straining, the plastic part of the rate of macroscopic deformation gradient is defined by

$$\frac{\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle}{\mathrm{d}t} = \frac{\mathrm{d}\langle\mathbf{F}\rangle}{\mathrm{d}t} - [\mathbf{M}] \cdot\cdot \frac{\mathrm{d}\langle\mathbf{P}\rangle}{\mathrm{d}t}\,, \tag{13.20.1}$$

where t stands for the physical time. By an analogous expression to (13.11.13), this is related to the local rate of deformation gradient by

$$\frac{\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle}{\mathrm{d}t} = \langle\, \frac{\mathrm{d}^{\mathrm{p}}\mathbf{F}}{\mathrm{d}t} \cdot\cdot \boldsymbol{\mathcal{P}}\,\rangle. \tag{13.20.2}$$

The fourth-order tensor $\boldsymbol{\mathcal{P}}$ is the influence tensor of elastic heterogeneity, which relates the elastic increments of the local and macroscopic nominal stress, $\delta\mathbf{P} = \boldsymbol{\mathcal{P}} \cdot\cdot \delta\langle\mathbf{P}\rangle$.

Suppose that the flow potential exists at the microlevel, such that (see Section 8.4)

$$\frac{\mathrm{d}^{\mathrm{p}}\mathbf{F}}{\mathrm{d}t} = \frac{\partial\Omega\,(\mathbf{P}, \mathcal{H})}{\partial\mathbf{P}}\,. \tag{13.20.3}$$

Substitution of Eq. (13.20.3) into Eq. (13.20.2) gives

$$\frac{\mathrm{d}^{\mathrm{p}}\langle\mathbf{F}\rangle}{\mathrm{d}t} = \langle\, \frac{\partial\Omega}{\partial\mathbf{P}} \cdot\cdot \boldsymbol{\mathcal{P}}\,\rangle = \langle\, \frac{\partial\Omega}{\partial\langle\mathbf{P}\rangle}\,\rangle = \frac{\partial}{\partial\langle\mathbf{P}\rangle}\,\langle\,\Omega\,\rangle. \tag{13.20.4}$$

In the derivation, the partial differentiation enables the transition

$$\frac{\partial \Omega}{\partial \langle \mathbf{P} \rangle} = \frac{\partial \Omega}{\partial \mathbf{P}} \cdot\cdot \frac{\partial \mathbf{P}}{\partial \langle \mathbf{P} \rangle} = \frac{\partial \Omega}{\partial \mathbf{P}} \cdot\cdot \boldsymbol{\mathcal{P}}. \tag{13.20.5}$$

From Eq. (13.20.4) we conclude that the existence of the flow potential Ω at the microlevel implies the existence of the flow potential at the macrolevel. The macroscopic flow potential is equal to the volume average $\langle\, \Omega \,\rangle$ of the microscopic flow potentials.

Since

$$\frac{\mathrm{d}^{\mathrm{p}} \langle \mathbf{P} \rangle}{\mathrm{d}t} = -[\boldsymbol{\Lambda}] \cdot\cdot \frac{\mathrm{d}^{\mathrm{p}} \langle \mathbf{F} \rangle}{\mathrm{d}t}\,, \tag{13.20.6}$$

and since at fixed $\mathcal{H}$,

$$\frac{\partial}{\partial \langle \mathbf{F} \rangle} = [\boldsymbol{\Lambda}] \cdot\cdot \frac{\partial}{\partial \langle \mathbf{P} \rangle}\,, \tag{13.20.7}$$

we have, dually to Eq. (13.20.4),

$$\frac{\mathrm{d}^{\mathrm{p}} \langle \mathbf{P} \rangle}{\mathrm{d}t} = -\frac{\partial}{\partial \langle \mathbf{F} \rangle} \langle\, \Omega \,\rangle. \tag{13.20.8}$$

If the stress and strain measures $\mathbf{T}$ and $\mathbf{E}$ are used, there follows

$$\frac{\mathrm{d}^{\mathrm{p}} [\mathbf{E}]}{\mathrm{d}t} = \frac{\partial}{\partial [\mathbf{T}]} \langle\, \Omega \,\rangle, \tag{13.20.9}$$

$$\frac{\mathrm{d}^{\mathrm{p}} [\mathbf{T}]}{\mathrm{d}t} = -\frac{\partial}{\partial [\mathbf{E}]} \langle\, \Omega \,\rangle. \tag{13.20.10}$$

The original proof for the transmissibility of the flow potential from the local (subelement) to the macroscopic (aggregate) level is due to Hill and Rice (1973). See also Zarka (1972), Hutchinson (1976), and Ponter and Leckie (1976).

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