CHAPTER 11

MULTIPLICATIVE DECOMPOSITION

This chapter deals with the formulation of the constitutive theory for large elastoplastic deformations within the framework of Lee's multiplicative decomposition of the deformation gradient. Kinematic and kinetic aspects of the theory are presented, with a particular accent given to the partition of the rate of deformation tensor into its elastic and plastic parts. The significance of plastic spin in the phenomenological theory is discussed. Isotropic and orthotropic materials are considered, and an introductory treatment of the damage-elastoplasticity is given.

11.1. Multiplicative Decomposition $\mathbf{F} = \mathbf{F}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}}$

Consider the current elastoplastically deformed configuration of the material sample $\mathcal{B}$, whose initial undeformed configuration was $\mathcal{B}^0$. Let $\mathbf{F}$ be the deformation gradient that maps an infinitesimal material element $\mathrm{d}\mathbf{X}$ from $\mathcal{B}^0$ to $\mathrm{d}\mathbf{x}$ in $\mathcal{B}$, such that

$$\mathrm{d}\mathbf{x} = \mathbf{F} \cdot \mathrm{d}\mathbf{X}. \tag{11.1.1}$$

The initial and current location of the material particle are both referred to the same, fixed set of Cartesian coordinate axes. Introduce an intermediate configuration $\mathcal{B}^{\mathrm{p}}$ by elastically destressing the current configuration $\mathcal{B}$ to zero stress (Fig. 11.1). Such configuration differs from the initial configuration by residual (plastic) deformation, and from the current configuration by reversible (elastic) deformation. If $\mathrm{d}\mathbf{x}^{\mathrm{p}}$ is the material element in $\mathcal{B}^{\mathrm{p}}$, corresponding to $\mathrm{d}\mathbf{x}$ in $\mathcal{B}$, then

$$\mathrm{d}\mathbf{x} = \mathbf{F}^{\mathrm{e}} \cdot \mathrm{d}\mathbf{x}^{\mathrm{p}}, \tag{11.1.2}$$

where $\mathbf{F}^{\mathrm{e}}$ represents the deformation gradient associated with the elastic loading from $\mathcal{B}^{\mathrm{p}}$ to $\mathcal{B}$. If the deformation gradient of the transformation

$\mathcal{B}^0 \to \mathcal{B}^{\mathrm{p}}$ is $\mathbf{F}^{\mathrm{p}}$, such that

$$\mathrm{d}\mathbf{x}^{\mathrm{p}} = \mathbf{F}^{\mathrm{p}} \cdot \mathrm{d}\mathbf{X}, \tag{11.1.3}$$

the multiplicative decomposition of the total deformation gradient into its elastic and plastic parts follows

$$\mathbf{F} = \mathbf{F}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}}. \tag{11.1.4}$$

The decomposition was introduced in the phenomenological theory of plasticity by Lee and Liu (1967), and Lee (1969). Early contributions also include Fox (1968), Willis (1969), Mandel (1971,1973), and Kröner and Teodosiu (1973). For inhomogeneous deformations only $\mathbf{F}$ is a true deformation gradient, whose components are the partial derivatives $\partial\mathbf{x}/\partial\mathbf{X}$. The mappings $\mathcal{B}^{\mathrm{p}} \to \mathcal{B}$ and $\mathcal{B}^0 \to \mathcal{B}^{\mathrm{p}}$ are not, in general, continuous one-to-one mappings, so that $\mathbf{F}^{\mathrm{e}}$ and $\mathbf{F}^{\mathrm{p}}$ are not defined as the gradients of the respective mappings (which may not exist), but as the point functions (local deformation gradients). In the case when elastic destressing to zero stress ($\mathcal{B} \to \mathcal{B}^{\mathrm{p}}$) is not physically achievable due to possible onset of reverse inelastic deformation before the state of zero stress is reached (which could occur at advanced stages of deformation due to anisotropic hardening and strong Bauschinger effect), the intermediate configuration can be conceptually introduced by virtual destressing to zero stress, locking all inelastic structural changes that would take place during the actual destressing.

There is a similar decomposition of the deformation gradient in thermoelasticity, where the total deformation gradient is expressed as the product of the elastic and thermal part. This has been studied by many, with a recent contribution given by Imam and Johnson (1998).

11.1.1. Nonuniqueness of Decomposition

The deformation gradients $\mathbf{F}^{\mathrm{e}}$ and $\mathbf{F}^{\mathrm{p}}$ are not uniquely defined because the intermediate unstressed configuration is not unique. Arbitrary local material rotations can be superposed to the intermediate configuration, preserving it unstressed. Thus, we can write

$$\mathbf{F} = \mathbf{F}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}} = \hat{\mathbf{F}}^{\mathrm{e}} \cdot \hat{\mathbf{F}}^{\mathrm{p}}, \tag{11.1.5}$$

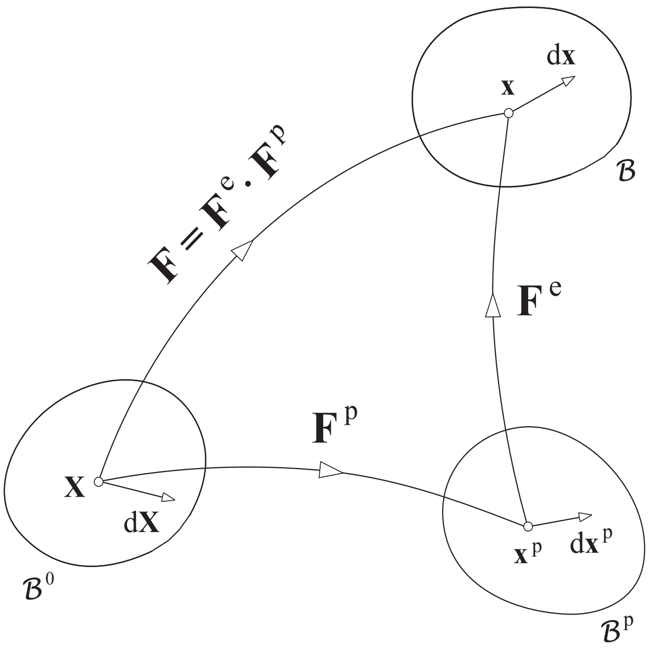

FIGURE 11.1. Schematic representation of the multiplicative decomposition of deformation gradient into its elastic and plastic parts. The intermediate configuration $\mathcal{B}^{\mathrm{p}}$ is obtained from the current configuration $\mathcal{B}$ by elastic destressing to zero stress.

where

$$\hat{\mathbf{F}}^{\mathrm{e}} = \mathbf{F}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^{T}, \quad \hat{\mathbf{F}}^{\mathrm{p}} = \hat{\mathbf{Q}} \cdot \mathbf{F}^{\mathrm{p}}. \tag{11.1.6}$$

A local rotation is represented by an orthogonal tensor $\hat{\mathbf{Q}}$. If polar decompositions of deformation gradients are used,

$$\mathbf{F}^{\mathrm{e}} = \mathbf{V}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{e}}, \quad \mathbf{F}^{\mathrm{p}} = \mathbf{R}^{\mathrm{p}} \cdot \mathbf{U}^{\mathrm{p}}, \tag{11.1.7}$$

it follows that only $\mathbf{R}^{\mathrm{e}}$ and $\mathbf{R}^{\mathrm{p}}$, not $\mathbf{V}^{\mathrm{e}}$ and $\mathbf{U}^{\mathrm{p}}$, are affected by the rotation of the intermediate state, i.e.,

$$\hat{\mathbf{R}}^{\mathrm{e}} = \mathbf{R}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^{T}, \quad \hat{\mathbf{R}}^{\mathrm{p}} = \hat{\mathbf{Q}} \cdot \mathbf{R}^{\mathrm{p}}, \tag{11.1.8}$$

while

$$\hat{\mathbf{V}}^{\mathrm{e}} = \mathbf{V}^{\mathrm{e}}, \quad \hat{\mathbf{U}}^{\mathrm{p}} = \mathbf{U}^{\mathrm{p}}. \tag{11.1.9}$$

Further discussion of the nonuniqueness of the decomposition can be found in the articles by Green and Naghdi (1971), Casey and Naghdi (1980), and

Naghdi (1990). Note that there is a unique decomposition

$$\mathbf{F} = \mathbf{V}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{ep}} \cdot \mathbf{U}^{\mathrm{p}}, \tag{11.1.10}$$

since the rotation tensor

$$\mathbf{R}^{\mathrm{ep}} = \mathbf{R}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{p}} = \hat{\mathbf{R}}^{\mathrm{e}} \cdot \hat{\mathbf{R}}^{\mathrm{p}} \tag{11.1.11}$$

is a unique tensor ($\hat{\mathbf{R}}^{\mathrm{ep}} = \mathbf{R}^{\mathrm{ep}}$).

In applications, the decomposition (11.1.4) can be made unique by additional requirements or specifications, dictated by the nature of the considered material model. For example, for elastically isotropic materials the stress response from $\mathcal{B}^{\mathrm{p}}$ to $\mathcal{B}$ depends only on the elastic stretch $\mathbf{V}^{\mathrm{e}}$, and not on the rotation $\mathbf{R}^{\mathrm{e}}$. Consequently, the intermediate configuration can be specified uniquely by requiring that the elastic unloading takes place without rotation,

$$\mathbf{F}^{\mathrm{e}} = \mathbf{V}^{\mathrm{e}}. \tag{11.1.12}$$

On the other hand, in single crystal plasticity (see Chapter 12), the orientation of the intermediate configuration is specified by a fixed orientation of the crystalline lattice, through which the material flows by crystallographic slip in the mapping from $\mathcal{B}^0$ to $\mathcal{B}^{\mathrm{p}}$. In Mandel's (1973, 1983) model, if the triad of orthogonal (director) vectors is attached to the initial configuration, and if this triad remains unaltered by plastic deformation, the intermediate configuration is referred to as isoclinic. Such configuration is unique at a given stage of elastoplastic deformation, because a superposed rotation $\hat{\mathbf{Q}} \neq \mathbf{I}$ would change the orientation of the director vectors, and the intermediate configuration would not remain isoclinic. For additional discussion, see the papers by Sidoroff (1975), and Kleiber and Raniecki (1985).

11.2. Decomposition of Strain Tensors

The left and right Cauchy–Green deformation tensors,

$$\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T, \quad \mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}, \tag{11.2.1}$$

can be decomposed as

$$\mathbf{B} = \mathbf{F}^{\mathrm{e}} \cdot \mathbf{B}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}T}, \quad \mathbf{C} = \mathbf{F}^{\mathrm{p}T} \cdot \mathbf{C}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}}. \tag{11.2.2}$$

If the Lagrangian strains corresponding to deformation gradients $\mathbf{F}^{\mathrm{e}}$ and $\mathbf{F}^{\mathrm{p}}$ are defined by

$$\mathbf{E}^{\mathrm{e}} = \frac{1}{2}\left(\mathbf{C}^{\mathrm{e}} - \mathbf{I}\right), \quad \mathbf{E}^{\mathrm{p}} = \frac{1}{2}\left(\mathbf{C}^{\mathrm{p}} - \mathbf{I}\right), \tag{11.2.3}$$

where

$$\mathbf{C}^{\mathrm{e}} = \mathbf{F}^{\mathrm{e}T} \cdot \mathbf{F}^{\mathrm{e}}, \quad \mathbf{C}^{\mathrm{p}} = \mathbf{F}^{\mathrm{p}T} \cdot \mathbf{F}^{\mathrm{p}}, \tag{11.2.4}$$

the total Lagrangian strain (the strain measure from the family of material strain tensors (2.3.1) corresponding to $n = 1$) can be expressed as

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{I}\right) = \mathbf{E}^{\mathrm{p}} + \mathbf{F}^{\mathrm{p}T} \cdot \mathbf{E}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}}. \tag{11.2.5}$$

The elastic and plastic strains $\mathbf{E}^{\mathrm{e}}$ and $\mathbf{E}^{\mathrm{p}}$ do not sum to give the total strain $\mathbf{E}$, because $\mathbf{E}$ and $\mathbf{E}^{\mathrm{p}}$ are defined relative to the initial configuration $\mathcal{B}^0$ as the reference configuration, while $\mathbf{E}^{\mathrm{e}}$ is defined relative to the intermediate configuration $\mathcal{B}^{\mathrm{p}}$ as the reference configuration. Consequently, it is the strain $\mathbf{F}^{\mathrm{p}T} \cdot \mathbf{E}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}}$, induced from the elastic strain $\mathbf{E}^{\mathrm{e}}$ by plastic deformation $\mathbf{F}^{\mathrm{p}}$, that sums up with the plastic strain $\mathbf{E}^{\mathrm{p}}$ to give the total strain $\mathbf{E}$.

If Eulerian strains corresponding to deformation gradients $\mathbf{F}^{\mathrm{e}}$ and $\mathbf{F}^{\mathrm{p}}$ are introduced as

$$\boldsymbol{\mathcal{E}}^{\mathrm{e}} = \frac{1}{2}\left(\mathbf{I} - \mathbf{B}^{\mathrm{e}-1}\right), \quad \boldsymbol{\mathcal{E}}^{\mathrm{p}} = \frac{1}{2}\left(\mathbf{I} - \mathbf{B}^{\mathrm{p}-1}\right), \tag{11.2.6}$$

where

$$\mathbf{B}^{\mathrm{e}} = \mathbf{F}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}T}, \quad \mathbf{B}^{\mathrm{p}} = \mathbf{F}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}T}, \tag{11.2.7}$$

the total Eulerian strain (the strain measure from the family of spatial strain tensors (2.3.14) corresponding to $n = -1$) can be written as

$$\boldsymbol{\mathcal{E}} = \frac{1}{2}\left(\mathbf{I} - \mathbf{B}^{-1}\right) = \boldsymbol{\mathcal{E}}^{\mathrm{e}} + \mathbf{F}^{\mathrm{e}-T} \cdot \boldsymbol{\mathcal{E}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}. \tag{11.2.8}$$

The additive decomposition does not hold for the Eulerian strains either, because the elastic and total strain measures, $\boldsymbol{\mathcal{E}}^{\mathrm{e}}$ and $\boldsymbol{\mathcal{E}}$, are defined relative to the current configuration $\mathcal{B}$, while plastic strain $\boldsymbol{\mathcal{E}}^{\mathrm{p}}$ is defined relative to the intermediate configuration $\mathcal{B}^{\mathrm{p}}$. Since

$$\mathbf{E} = \mathbf{F}^{T} \cdot \boldsymbol{\mathcal{E}} \cdot \mathbf{F}, \tag{11.2.9}$$

there is a useful relationship

$$\mathbf{E} - \mathbf{E}^{\mathrm{p}} = \mathbf{F}^{T} \cdot \boldsymbol{\mathcal{E}}^{\mathrm{e}} \cdot \mathbf{F}, \tag{11.2.10}$$

which shows that the difference between the total and plastic Lagrangian strain tensors is equal to the strain tensor induced from the Eulerian elastic strain $\boldsymbol{\mathcal{E}}^{\mathrm{e}}$ by the deformation $\mathbf{F}$. Dually, we have

$$\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{E}}^{\mathrm{e}} = \mathbf{F}^{-T} \cdot \mathbf{E}^{\mathrm{p}} \cdot \mathbf{F}^{-1}. \tag{11.2.11}$$

11.3. Velocity Gradient and Strain Rates

Consider the velocity gradient in the current configuration at time t, defined by

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}. \tag{11.3.1}$$

The superposed dot designates the material time derivative. By introducing the multiplicative decomposition of deformation gradient (11.1.4), the velocity gradient becomes

$$\mathbf{L} = \dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} + \mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}. \tag{11.3.2}$$

The rate of deformation $\mathbf{D}$ and the spin $\mathbf{W}$ are, respectively, the symmetric and antisymmetric part of $\mathbf{L}$,

$$\mathbf{D} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} + \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{s}}, \tag{11.3.3}$$

$$\mathbf{W} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}} + \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{a}}. \tag{11.3.4}$$

For later purposes, it is convenient to identify the spin

$$\boldsymbol{\omega}^{\mathrm{p}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{a}}. \tag{11.3.5}$$

Since

$$\dot{\mathbf{E}} = \mathbf{F}^{T} \cdot \mathbf{D} \cdot \mathbf{F}, \tag{11.3.6}$$

the following expressions hold for the rates of the introduced Lagrangian strains

$$\dot{\mathbf{E}}^{\mathrm{e}} = \mathbf{F}^{\mathrm{p}-T} \cdot \dot{\mathbf{E}} \cdot \mathbf{F}^{\mathrm{p}-1} - \left[\mathbf{C}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right)\right]_{\mathrm{s}}, \tag{11.3.7}$$

$$\dot{\mathbf{E}}^{\mathrm{p}} = \mathbf{F}^{\mathrm{p}T} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right)_{\mathrm{s}} \cdot \mathbf{F}^{\mathrm{p}}. \tag{11.3.8}$$

These are here cast in terms of the strain rate $\dot{\mathbf{E}}$ and the velocity gradient $\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}$.

In Section 11.13 it will be shown that the elastic part of the rate of Lagrangian strain,

$$(\dot{\mathbf{E}})^{\mathrm{e}} = \boldsymbol{\Lambda}_{(1)}^{-1} : \dot{\mathbf{T}}, \tag{11.3.9}$$

where $\mathbf{T}$ is the symmetric Piola–Kirchhoff stress tensor, is in general different from the rate of strain $\dot{\mathbf{E}}^{\mathrm{e}}$. Similarly,

$$(\dot{\mathbf{E}})^{\mathrm{p}} = \dot{\mathbf{E}} - (\dot{\mathbf{E}})^{\mathrm{e}} \neq \dot{\mathbf{E}}^{\mathrm{p}}. \tag{11.3.10}$$

While $(\dot{\mathbf{E}})^{\mathrm{e}}$ and $(\dot{\mathbf{E}})^{\mathrm{p}}$ sum up to give $\dot{\mathbf{E}}$, in general

$$\dot{\mathbf{E}}^{\mathrm{e}} + \dot{\mathbf{E}}^{\mathrm{p}} \neq \dot{\mathbf{E}}. \tag{11.3.11}$$

For the rates of Eulerian strains we have

$$\dot{\boldsymbol{\mathcal{E}}}^{\mathrm{e}} = \left[\mathbf{B}^{\mathrm{e}-1} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)\right]_{\mathrm{s}}, \tag{11.3.12}$$

$$\dot{\boldsymbol{\mathcal{E}}}^{\mathrm{p}} = \mathbf{F}^{\mathrm{e}T} \cdot \dot{\boldsymbol{\mathcal{E}}} \cdot \mathbf{F}^{\mathrm{e}} - \left[\mathbf{B}^{-1} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)\right]_{\mathrm{s}}, \tag{11.3.13}$$

expressed in terms of the strain rate $\dot{\boldsymbol{\mathcal{E}}}$ and the velocity gradient $\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}$.

11.4. Objectivity Requirements

Upon superimposing a time-dependent rigid-body rotation $\mathbf{Q}$ to the current configuration $\mathcal{B}$, the deformation gradient $\mathbf{F}$ becomes

$$\mathbf{F}^{*} = \mathbf{Q} \cdot \mathbf{F}, \tag{11.4.1}$$

while the elastic and plastic parts $\mathbf{F}^{\mathrm{e}}$ and $\mathbf{F}^{\mathrm{p}}$ change to

$$\mathbf{F}^{\mathrm{e}*} = \mathbf{Q} \cdot \mathbf{F}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^{T}, \quad \mathbf{F}^{\mathrm{p}*} = \hat{\mathbf{Q}} \cdot \mathbf{F}^{\mathrm{p}}. \tag{11.4.2}$$

The rotation tensor $\hat{\mathbf{Q}}$ is imposed on the intermediate configuration $\mathcal{B}^{\mathrm{p}}$. This rotation depends on the rotation $\mathbf{Q}$ of the current configuration, and on the definition of the intermediate configuration used in the particular constitutive model (Lubarda, 1991a). For example, if the intermediate configuration is defined to be isoclinic, then necessarily

$$\hat{\mathbf{Q}} = \mathbf{I}. \tag{11.4.3}$$

If the intermediate configuration is obtained from the current configuration by destressing without rotation $\mathbf{F}^{\mathrm{e}} = \mathbf{V}^{\mathrm{e}}$, then

$$\hat{\mathbf{Q}} = \mathbf{Q}, \tag{11.4.4}$$

in order that $\mathbf{F}^{\mathrm{e}}$ remains symmetric. Thus, with $\hat{\mathbf{Q}}$ appropriately specified in any particular case, the following transformation rules apply

$$\mathbf{V}^{\mathrm{e}*} = \mathbf{Q} \cdot \mathbf{V}^{\mathrm{e}} \cdot \mathbf{Q}^T, \quad \mathbf{V}^{\mathrm{p}*} = \hat{\mathbf{Q}} \cdot \mathbf{V}^{\mathrm{p}} \cdot \hat{\mathbf{Q}}^T, \tag{11.4.5}$$

$$\mathbf{R}^{\mathrm{e}*} = \mathbf{Q} \cdot \mathbf{R}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^T, \quad \mathbf{R}^{\mathrm{p}*} = \hat{\mathbf{Q}} \cdot \mathbf{R}^{\mathrm{p}}, \tag{11.4.6}$$

$$\mathbf{U}^{\mathrm{e}*} = \hat{\mathbf{Q}} \cdot \mathbf{U}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^T, \quad \mathbf{U}^{\mathrm{p}*} = \mathbf{U}^{\mathrm{p}}, \tag{11.4.7}$$

$$\mathbf{B}^{\mathrm{e}*} = \mathbf{Q} \cdot \mathbf{B}^{\mathrm{e}} \cdot \mathbf{Q}^T, \quad \mathbf{B}^{\mathrm{p}*} = \hat{\mathbf{Q}} \cdot \mathbf{B}^{\mathrm{p}} \cdot \hat{\mathbf{Q}}^T, \tag{11.4.8}$$

$$\mathbf{C}^{\mathrm{e}*} = \hat{\mathbf{Q}} \cdot \mathbf{C}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^T, \quad \mathbf{C}^{\mathrm{p}*} = \mathbf{C}^{\mathrm{p}}, \tag{11.4.9}$$

$$\mathbf{E}^{\mathrm{e}*} = \hat{\mathbf{Q}} \cdot \mathbf{E}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^T, \quad \mathbf{E}^{\mathrm{p}*} = \mathbf{E}^{\mathrm{p}}, \tag{11.4.10}$$

$$\boldsymbol{\mathcal{E}}^{\mathrm{e}*} = \mathbf{Q} \cdot \boldsymbol{\mathcal{E}}^{\mathrm{e}} \cdot \mathbf{Q}^T, \quad \boldsymbol{\mathcal{E}}^{\mathrm{p}*} = \hat{\mathbf{Q}} \cdot \boldsymbol{\mathcal{E}}^{\mathrm{p}} \cdot \hat{\mathbf{Q}}^T. \tag{11.4.11}$$

The transformation rules for the velocity gradients associated with $\mathbf{F}^{\mathrm{e}}$ and $\mathbf{F}^{\mathrm{p}}$ are

$$\dot{\mathbf{F}}^{\mathrm{e}*} \cdot \mathbf{F}^{\mathrm{e}*-1} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1} + \mathbf{Q} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right) \cdot \mathbf{Q}^T - \mathbf{F}^{\mathrm{e}*} \cdot \left(\dot{\hat{\mathbf{Q}}} \cdot \hat{\mathbf{Q}}^{-1}\right) \cdot \mathbf{F}^{\mathrm{e}*-1}, \tag{11.4.12}$$

$$\dot{\mathbf{F}}^{\mathrm{p}*} \cdot \mathbf{F}^{\mathrm{p}*-1} = \dot{\hat{\mathbf{Q}}} \cdot \hat{\mathbf{Q}}^{-1} + \hat{\mathbf{Q}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \hat{\mathbf{Q}}^T. \tag{11.4.13}$$

The corresponding Lagrangian and Eulerian strain rates transform according to

$$\dot{\mathbf{E}}^{\mathrm{e}*} = \hat{\mathbf{Q}} \cdot \dot{\mathbf{E}}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^T + \left(\dot{\hat{\mathbf{Q}}} \cdot \hat{\mathbf{Q}}^{-1}\right) \cdot \mathbf{E}^{\mathrm{e}*} - \mathbf{E}^{\mathrm{e}*} \cdot \left(\dot{\hat{\mathbf{Q}}} \cdot \hat{\mathbf{Q}}^{-1}\right), \tag{11.4.14}$$

$$\dot{\mathbf{E}}^{\mathrm{p}*} = \dot{\mathbf{E}}^{\mathrm{p}}, \tag{11.4.15}$$

$$\dot{\boldsymbol{\mathcal{E}}}^{\mathrm{e}*} = \mathbf{Q} \cdot \dot{\boldsymbol{\mathcal{E}}}^{\mathrm{e}} \cdot \mathbf{Q}^T + \left(\dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1}\right) \cdot \boldsymbol{\mathcal{E}}^{\mathrm{e}*} - \boldsymbol{\mathcal{E}}^{\mathrm{e}*} \cdot \left(\dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1}\right), \tag{11.4.16}$$

$$\dot{\boldsymbol{\mathcal{E}}}^{\mathrm{p}*} = \hat{\mathbf{Q}} \cdot \dot{\boldsymbol{\mathcal{E}}}^{\mathrm{p}} \cdot \hat{\mathbf{Q}}^T + \left(\dot{\hat{\mathbf{Q}}} \cdot \hat{\mathbf{Q}}^{-1}\right) \cdot \boldsymbol{\mathcal{E}}^{\mathrm{p}*} - \boldsymbol{\mathcal{E}}^{\mathrm{p}*} \cdot \left(\dot{\hat{\mathbf{Q}}} \cdot \hat{\mathbf{Q}}^{-1}\right). \tag{11.4.17}$$

Finally, the transformation rules for the total velocity gradient, and the total strain rates are

$$\mathbf{L}^* = \dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1} + \mathbf{Q} \cdot \mathbf{L} \cdot \mathbf{Q}^T, \tag{11.4.18}$$

$$\dot{\mathbf{E}}^* = \dot{\mathbf{E}}, \tag{11.4.19}$$

$$\dot{\boldsymbol{\mathcal{E}}}^* = \mathbf{Q} \cdot \dot{\boldsymbol{\mathcal{E}}} \cdot \mathbf{Q}^T + \left(\dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1}\right) \cdot \boldsymbol{\mathcal{E}}^* - \boldsymbol{\mathcal{E}}^* \cdot \left(\dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1}\right), \tag{11.4.20}$$

as previously discussed in Section 2.9.

The objectivity requirements that need to be imposed in the theory of elastoplasticity based on the multiplicative decomposition of deformation gradient have been extensively discussed in the literature. Some of the representative references include Naghdi and Trapp (1974), Lubarda and Lee (1981), Casey and Naghdi (1981), Dashner (1986a, b), Casey (1987), Dafalias (1987, 1988), Naghdi (1990), and Xiao, Bruhns, and Meyers (2000).

11.5. Jaumann Derivative of Elastic Deformation Gradient

In the context of the multiplicative decomposition of deformation gradient based on the intermediate configuration, it is convenient to introduce a particular type of the Jaumann derivative of elastic deformation gradient $\mathbf{F}^{\mathrm{e}}$. This is defined as the time derivative observed in two rotating coordinate systems, one rotating with the spin $\boldsymbol{\Omega}$ in the current configuration $\mathcal{B}$ and the other rotating with the spin $\boldsymbol{\Omega}^{\mathrm{p}}$ in the intermediate configuration $\mathcal{B}^{\mathrm{p}}$, such that (Lubarda, 1991a; Lubarda and Shih, 1994)

$$\overset{\bullet}{\mathbf{F}}{}^{\mathrm{e}} = \dot{\mathbf{F}}^{\mathrm{e}} - \boldsymbol{\Omega} \cdot \mathbf{F}^{\mathrm{e}} + \mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}}. \tag{11.5.1}$$

The spin tensors $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^{\mathrm{p}}$ are at this point unspecified. They can be different or equal to each other, depending on the selected intermediate configuration and the intended application (see also Section 2.8). In any case, they transform under rigid-body rotations $\mathbf{Q}$ and $\hat{\mathbf{Q}}$ of the current and intermediate configurations according to

$$\boldsymbol{\Omega}^* = \dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1} + \mathbf{Q} \cdot \boldsymbol{\Omega} \cdot \mathbf{Q}^T, \quad \boldsymbol{\Omega}^{\mathrm{p}*} = \dot{\hat{\mathbf{Q}}} \cdot \hat{\mathbf{Q}}^{-1} + \hat{\mathbf{Q}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \hat{\mathbf{Q}}^T. \tag{11.5.2}$$

The Jaumann derivatives of $\mathbf{V}^{\mathrm{e}}$ and $\mathbf{R}^{\mathrm{e}}$, corresponding to Eq. (11.5.1), are

$$\overset{\bullet}{\mathbf{V}}{}^{\mathrm{e}} = \dot{\mathbf{V}}^{\mathrm{e}} - \boldsymbol{\Omega} \cdot \mathbf{V}^{\mathrm{e}} + \mathbf{V}^{\mathrm{e}} \cdot \boldsymbol{\Omega}, \quad \overset{\bullet}{\mathbf{R}}{}^{\mathrm{e}} = \dot{\mathbf{R}}^{\mathrm{e}} - \boldsymbol{\Omega} \cdot \mathbf{R}^{\mathrm{e}} + \mathbf{R}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}}, \tag{11.5.3}$$

while those of $\mathbf{B}^{\mathrm{e}}$ and $\mathbf{C}^{\mathrm{e}}$ are

$$\overset{\bullet}{\mathbf{B}}{}^{\mathrm{e}} = \dot{\mathbf{B}}^{\mathrm{e}} - \boldsymbol{\Omega} \cdot \mathbf{B}^{\mathrm{e}} + \mathbf{B}^{\mathrm{e}} \cdot \boldsymbol{\Omega}, \quad \overset{\bullet}{\mathbf{C}}{}^{\mathrm{e}} = \dot{\mathbf{C}}^{\mathrm{e}} - \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{C}^{\mathrm{e}} + \mathbf{C}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}}. \tag{11.5.4}$$

It is easily verified that

$$\overset{\bullet}{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} = \dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} + \mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1} - \boldsymbol{\Omega}, \tag{11.5.5}$$

$$\overset{\bullet}{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} = \dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} + \mathbf{V}^{\mathrm{e}} \cdot \boldsymbol{\Omega} \cdot \mathbf{V}^{\mathrm{e}-1} - \boldsymbol{\Omega}. \tag{11.5.6}$$

Under rigid-body rotations $\mathbf{Q}$ and $\hat{\mathbf{Q}}$, the introduced Jaumann derivatives transform as

$$\overset{\bullet}{\mathbf{F}}^{\mathrm{e}*} = \mathbf{Q} \cdot \overset{\bullet}{\mathbf{F}}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^{T}, \quad \overset{\bullet}{\mathbf{V}}^{\mathrm{e}*} = \mathbf{Q} \cdot \overset{\bullet}{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{Q}^{T}, \quad \overset{\bullet}{\mathbf{R}}^{\mathrm{e}*} = \mathbf{Q} \cdot \overset{\bullet}{\mathbf{R}}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^{T}, \tag{11.5.7}$$

$$\overset{\bullet}{\mathbf{B}}^{\mathrm{e}*} = \mathbf{Q} \cdot \overset{\bullet}{\mathbf{B}}^{\mathrm{e}} \cdot \mathbf{Q}^{T}, \quad \overset{\bullet}{\mathbf{C}}^{\mathrm{e}*} = \hat{\mathbf{Q}} \cdot \overset{\bullet}{\mathbf{C}}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^{T}. \tag{11.5.8}$$

Consequently,

$$\overset{\bullet}{\mathbf{F}}^{\mathrm{e}*} \cdot \mathbf{F}^{\mathrm{e}*-1} = \mathbf{Q} \cdot \left(\overset{\bullet}{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} \right) \cdot \mathbf{Q}^{T}, \tag{11.5.9}$$

and likewise for the corresponding quantities associated with the Jaumann derivatives of $\mathbf{V}^{\mathrm{e}}$ and $\mathbf{R}^{\mathrm{e}}$.

11.6. Partition of Elastoplastic Rate of Deformation

In this section it is assumed that the material is elastically isotropic in its initial undeformed state, and that plastic deformation does not affect its elastic properties. The elastic response from $\mathcal{B}^{\mathrm{p}}$ to $\mathcal{B}$ is then independent of the rotation superposed to the intermediate configuration, and is given by

$$\boldsymbol{\tau} = \mathbf{F}^{\mathrm{e}} \cdot \frac{\partial \Psi^{\mathrm{e}}(\mathbf{E}^{\mathrm{e}})}{\partial \mathbf{E}^{\mathrm{e}}} \cdot \mathbf{F}^{\mathrm{e}T}. \tag{11.6.1}$$

The elastic strain energy per unit unstressed volume, Ψ^{e}, is an isotropic function of the Lagrangian strain $\mathbf{E}^{\mathrm{e}} = \left(\mathbf{F}^{\mathrm{e}T} \cdot \mathbf{F}^{\mathrm{e}} - \mathbf{I}\right)/2$. Plastic deformation is assumed to be incompressible ($\det \mathbf{F}^{\mathrm{e}} = \det \mathbf{F}$), so that $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$ is the Kirchhoff stress (the Cauchy stress $\boldsymbol{\sigma}$ weighted by $\det \mathbf{F}$).

By differentiating Eq. (11.6.1), we obtain

$$\dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} \right) \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} \right)^{T} = \boldsymbol{\mathcal{L}}_{(1)} : \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} \right)_{\mathrm{s}}. \tag{11.6.2}$$

The subscript (1) is attached to the elastic moduli tensor $\boldsymbol{\mathcal{L}}_{(1)}$ to make the contact with the notation used in Section 6.2, e.g., Eq. (6.2.4). The rectangular components of $\boldsymbol{\mathcal{L}}_{(1)}$ are

$$\mathcal{L}^{(1)}_{ijkl} = F^{\mathrm{e}}_{iM} F^{\mathrm{e}}_{jN} \, \frac{\partial^2 \Psi^{\mathrm{e}}}{\partial E^{\mathrm{e}}_{MN} \partial E^{\mathrm{e}}_{PQ}} \, F^{\mathrm{e}}_{kP} F^{\mathrm{e}}_{lQ}. \tag{11.6.3}$$

Equation (11.6.2) can be equivalently written, in terms of the Jaumann derivative of $\boldsymbol{\tau}$ with respect to spin $\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}}$, as

$$\dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}} = \boldsymbol{\mathcal{L}}_{(0)} : \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}. \tag{11.6.4}$$

The elastic moduli tensor $\boldsymbol{\mathcal{L}}_{(0)}$ has the components

$$\mathcal{L}^{(0)}_{ijkl} = \mathcal{L}^{(1)}_{ijkl} + \frac{1}{2}\left(\tau_{ik}\delta_{jl} + \tau_{jk}\delta_{il} + \tau_{il}\delta_{jk} + \tau_{jl}\delta_{ik}\right), \tag{11.6.5}$$

as in Eq. (6.2.15).

The elastic deformation gradient $\mathbf{F}^{\mathrm{e}}$ is defined relative to the intermediate configuration, which is changing during the ongoing elastoplastic deformation. This causes two difficulties in the identification of the elastic rate of deformation $\mathbf{D}^{\mathrm{e}}$. First, since $\mathbf{F}^{\mathrm{e}}$ and $\mathbf{F}^{\mathrm{p}}$ are specified only to within an arbitrary rotation $\hat{\mathbf{Q}}$, the velocity gradient $\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}$ and its symmetric and antisymmetric parts are not unique. Secondly, the deforming intermediate configuration also makes contribution to the elastic rate of deformation, so that this is not in general given only by $\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}$. To overcome these difficulties, we resort to kinetic definition of the elastic strain increment $\mathbf{D}^{\mathrm{e}}\,\mathrm{d}t$, which is a reversible part of the total strain increment $\mathbf{D}\,\mathrm{d}t$ recovered upon loading–unloading cycle of the stress increment $\overset{\circ}{\boldsymbol{\tau}}\mathrm{d}t$. The Jaumann derivative of the Kirchhoff stress relative to material spin $\mathbf{W}$ is $\overset{\circ}{\boldsymbol{\tau}}$. Thus, we define

$$\mathbf{D}^{\mathrm{e}} = \boldsymbol{\mathcal{L}}^{-1}_{(0)} : \overset{\circ}{\boldsymbol{\tau}}, \quad \overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{W}. \tag{11.6.6}$$

The remaining part of the total rate of deformation,

$$\mathbf{D}^{\mathrm{p}} = \mathbf{D} - \mathbf{D}^{\mathrm{e}}, \tag{11.6.7}$$

is the plastic part, which gives a residual strain increment left upon considered infinitesimal cycle of stress. When the material obeys Ilyushin's postulate, so defined plastic rate of deformation $\mathbf{D}^{\mathrm{p}}$ is codirectional with the outward normal to a locally smooth yield surface in the Cauchy stress space.

Therefore, to identify in Eq. (11.6.4) the elastic strain rate, according to the kinetic definition (11.6.6), we use Eq. (11.3.4) to eliminate $\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}}$ and obtain

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_{(0)} : \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} - \boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}}. \tag{11.6.8}$$

The spin $\boldsymbol{\omega}^{\mathrm{p}}$ is defined by Eq. (11.3.5). Consequently, the elastic rate of deformation is

$$\mathbf{D}^{\mathrm{e}} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} - \boldsymbol{\mathcal{L}}_{(0)}^{-1} : \left(\boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}}\right). \tag{11.6.9}$$

From Eq. (11.6.7), the corresponding plastic rate of deformation is given by

$$\mathbf{D}^{\mathrm{p}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{s}} + \boldsymbol{\mathcal{L}}_{(0)}^{-1} : \left(\boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}}\right). \tag{11.6.10}$$

Since $\boldsymbol{\mathcal{L}}_{(0)}^{-1}$ and $\overset{\circ}{\boldsymbol{\tau}}$ in (11.6.6) are independent of a superposed rotation to the intermediate configuration, Eq. (11.6.9) specifies $\mathbf{D}^{\mathrm{e}}$ uniquely. In contrast, its constituents, $\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}$ and the term associated with the spin $\boldsymbol{\omega}^{\mathrm{p}}$, do depend on the choice of the intermediate configuration. Similar remarks apply to plastic rate of deformation $\mathbf{D}^{\mathrm{p}}$ in its representation (11.6.10).

As we have shown, the right hand side of (11.6.9) is the correct expression for the elastic rate of deformation, and not $\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}$ alone. Only if the intermediate configuration (i.e., the rotation $\mathbf{R}^{\mathrm{e}}$ during destressing program) is chosen such that the spin $\boldsymbol{\omega}^{\mathrm{p}}$ vanishes,

$$\boldsymbol{\omega}^{\mathrm{p}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{a}} = \mathbf{0}, \tag{11.6.11}$$

the rate of deformation $\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}$ is exactly equal to $\mathbf{D}^{\mathrm{e}}$. Within the framework under discussion, this choice of the spin represents purely geometric (kinematic) specification of the intermediate configuration. It is not a constitutive assumption and has no consequences on Eq. (11.6.9). We could just as well define the intermediate configuration by requiring that the spin $\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}}$ vanishes identically. In this case,

$$\boldsymbol{\omega}^{\mathrm{p}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{a}} = \mathbf{W}. \tag{11.6.12}$$

The end result is still Eq. (11.6.9), as can be checked by inspection.

The partition of $\mathbf{D}$ into its elastic and plastic parts within the framework of the multiplicative decomposition has been a topic of active research and discussion. Some of the representative references include Lee (1969), Freund (1970), Kratochvil (1973), Kleiber (1975), Nemat-Nasser (1979,1982), Lubarda and Lee (1981), Lee (1981,1985), Sidoroff (1982), Dafalias (1987), and Lubarda and Shih (1994).

We note that the second part of the rate of deformation $\mathbf{D}^{\mathrm{e}}$, in its representation (11.6.9), makes no contribution to elastic work. This follows by observing that, in view of elastic isotropy, the part of the rate of deformation

$$\boldsymbol{\mathcal{L}}_{(0)}^{-1} : (\boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}}) \tag{11.6.13}$$

has its principal directions parallel to those of the associated stress rate $(\boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}})$. Since direction of this stress rate is normal to $\boldsymbol{\tau}$, their trace is zero, hence

$$\dot{\Psi}^{\mathrm{e}} = \boldsymbol{\tau} : \mathbf{D}^{\mathrm{e}} = \boldsymbol{\tau} : \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}. \tag{11.6.14}$$

11.7. Analysis of Elastic Rate of Deformation

We present an alternative derivation of the expression for the elastic rate of deformation of elastically isotropic materials, which gives additional insight in the kinematics of elastoplastic deformation and the partitioning of the rate of deformation. We show that $\mathbf{D}^{\mathrm{e}}$ can be expressed as

$$\mathbf{D}^{\mathrm{e}} = \left(\overset{\bullet}{\mathbf{F}}{}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} + \left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}. \tag{11.7.1}$$

The Jaumann derivative $\overset{\bullet}{\mathbf{F}}{}^{\mathrm{e}}$ is defined by Eq. (11.5.1) with $\boldsymbol{\Omega} = \boldsymbol{\Omega}^{\mathrm{p}}$, i.e.,

$$\overset{\bullet}{\mathbf{F}}{}^{\mathrm{e}} = \dot{\mathbf{F}}^{\mathrm{e}} - \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}} + \mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}}. \tag{11.7.2}$$

This represents the rate of $\mathbf{F}^{\mathrm{e}}$ observed in the coordinate systems that rotate with the spin $\boldsymbol{\Omega}^{\mathrm{p}}$ in both current and intermediate configurations. The spin $\boldsymbol{\Omega}^{\mathrm{p}}$ is defined as the solution of the matrix equation

$$\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}} + \left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}} = \mathbf{W}. \tag{11.7.3}$$

In proof, the application of the Jaumann derivative with respect to spin $\boldsymbol{\Omega}^{\mathrm{p}}$ to Eq. (11.6.1) gives

$$\overset{\bullet}{\boldsymbol{\tau}} = \left(\overset{\bullet}{\mathbf{F}}{}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right) \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left(\overset{\bullet}{\mathbf{F}}{}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)^{T} + \mathbf{F}^{\mathrm{e}} \cdot \left(\frac{\partial^2 \Psi^{\mathrm{e}}}{\partial \mathbf{E}^{\mathrm{e}} \otimes \partial \mathbf{E}^{\mathrm{e}}} : \overset{\bullet}{\mathbf{E}}{}^{\mathrm{e}}\right) \cdot \mathbf{F}^{\mathrm{e}T}, \tag{11.7.4}$$

where

$$\overset{\bullet}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \boldsymbol{\Omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\Omega}^{\mathrm{p}}, \quad \overset{\bullet}{\mathbf{E}}{}^{\mathrm{e}} = \mathbf{F}^{\mathrm{e}T} \cdot \left(\overset{\bullet}{\mathbf{F}}{}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} \cdot \mathbf{F}^{\mathrm{e}}. \tag{11.7.5}$$

Therefore, if Eqs. (11.7.1) and (11.7.3) hold, so that

$$\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} + \mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1} = \mathbf{D}^{\mathrm{e}} + \mathbf{W}, \tag{11.7.6}$$

$$\overset{\bullet}{\mathbf{F}}{}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} = \mathbf{D}^{\mathrm{e}} + \mathbf{W} - \boldsymbol{\Omega}^{\mathrm{p}}, \tag{11.7.7}$$

the substitution into Eq. (11.7.4) yields

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_{(0)} : \mathbf{D}^{\mathrm{e}}, \quad \mathbf{D}^{\mathrm{e}} = \left(\overset{\bullet}{\mathbf{F}}{}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}. \tag{11.7.8}$$

The two contributions to the elastic rate of deformation $\mathbf{D}^{\mathrm{e}}$ in Eq. (11.7.1) both depend on the choice of the intermediate configuration, i.e., the elastic rotation $\mathbf{R}^{\mathrm{e}}$ of the destressing program, but their sum giving $\mathbf{D}^{\mathrm{e}}$ does not. If elastic destressing is performed without rotation ($\mathbf{R}^{\mathrm{e}} = \mathbf{I}$), the spin $\boldsymbol{\Omega}^{\mathrm{p}} = \boldsymbol{\Omega}_I^{\mathrm{p}}$ is the solution of

$$\left(\dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1}\right)_{\mathrm{a}} + \left(\mathbf{V}^{\mathrm{e}} \cdot \boldsymbol{\Omega}_I^{\mathrm{p}} \cdot \mathbf{V}^{\mathrm{e}-1}\right)_{\mathrm{a}} = \mathbf{W}. \tag{11.7.9}$$

This defines the spin $\boldsymbol{\Omega}_I^{\mathrm{p}}$ uniquely in terms of $\mathbf{W}$, $\mathbf{V}^{\mathrm{e}}$ and $\dot{\mathbf{V}}^{\mathrm{e}}$. The expression for the elastic rate of deformation (11.7.1) is in this case

$$\mathbf{D}^{\mathrm{e}} = \left(\overset{\bullet}{\mathbf{V}}{}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1}\right)_{\mathrm{s}} = \left(\dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1}\right)_{\mathrm{s}} + \left(\mathbf{V}^{\mathrm{e}} \cdot \boldsymbol{\Omega}_I^{\mathrm{p}} \cdot \mathbf{V}^{\mathrm{e}-1}\right)_{\mathrm{s}}. \tag{11.7.10}$$

The first term on the right-hand side represents the contribution to $\mathbf{D}^{\mathrm{e}}$ from the elastic stretching rate $\left(\dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1}\right)_{\mathrm{s}}$, while the second depends on the spin $\boldsymbol{\Omega}_I^{\mathrm{p}}$ and accounts for the effects of the deforming and rotating intermediate configuration.

Since

$$\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} + \mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1} = \dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} + \mathbf{V}^{\mathrm{e}} \cdot \boldsymbol{\Omega}_I^{\mathrm{p}} \cdot \mathbf{V}^{\mathrm{e}-1}, \tag{11.7.11}$$

and

$$\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} = \dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} + \mathbf{V}^{\mathrm{e}} \cdot \left(\dot{\mathbf{R}}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{e}-1}\right) \cdot \mathbf{V}^{\mathrm{e}-1}, \tag{11.7.12}$$

it follows that, for any other choice of the rotation $\mathbf{R}^{\mathrm{e}}$, the corresponding spin in the expression for the elastic rate of deformation (11.7.1) is

$$\boldsymbol{\Omega}^{\mathrm{p}} = \mathbf{R}^{\mathrm{e}T} \cdot \left(\boldsymbol{\Omega}_I^{\mathrm{p}} - \dot{\mathbf{R}}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{e}-1}\right) \cdot \mathbf{R}^{\mathrm{e}}. \tag{11.7.13}$$

The expression for the elastic rate of deformation in Eq. (11.7.1) involves only kinematic quantities ($\mathbf{F}^{\mathrm{e}}$ and $\boldsymbol{\Omega}^{\mathrm{p}}$), while the previously derived expression (11.6.9) involves both kinematic and kinetic quantities. Clearly, there is a connection

$$\left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} = -\boldsymbol{\mathcal{L}}_{(0)}^{-1} : \left(\boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}}\right). \tag{11.7.14}$$

Note also that Eq. (11.7.1) can be recast in the form

$$\mathbf{D}^{\mathrm{e}} = \frac{1}{2}\, \mathbf{F}^{\mathrm{e}-T} \cdot \overset{\bullet}{\mathbf{C}}{}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}, \quad \overset{\bullet}{\mathbf{C}}{}^{\mathrm{e}} = \dot{\mathbf{C}}^{\mathrm{e}} - \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{C}^{\mathrm{e}} + \mathbf{C}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}}. \tag{11.7.15}$$

This expression, as well as (11.7.1), holds for elastoplastic deformation of elastically isotropic materials, regardless of whether the material hardens isotropically or anisotropically in the course of plastic deformation.

11.7.1. Analysis of Spin $\boldsymbol{\Omega}^{\mathrm{p}}$

The spin $\boldsymbol{\Omega}_I^{\mathrm{p}}$, obtained as the solution of Eq. (11.7.9), depends on $\mathbf{W}$, $\mathbf{V}^{\mathrm{e}}$, and $\dot{\mathbf{V}}^{\mathrm{e}}$. It is possible to derive an expression for this spin in terms of $\mathbf{W}$, $\mathbf{V}^{\mathrm{e}}$, and $\mathbf{D}^{\mathrm{e}}$. Proceeding as in Section 2.7, we first observe the identity

$$\mathbf{V}^{\mathrm{e}-1} \cdot \left(\dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} \right) = \left(\dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} \right)^T \cdot \mathbf{V}^{\mathrm{e}-1}, \tag{11.7.16}$$

which can be rewritten in the form

$$\mathbf{V}^{\mathrm{e}-1} \cdot \left(\dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} \right)_{\mathrm{a}} + \left(\dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} \right)_{\mathrm{a}} \cdot \mathbf{V}^{\mathrm{e}-1} = \mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}}. \tag{11.7.17}$$

This equation can be solved for $\left(\dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} \right)_{\mathrm{a}}$ as

$$\begin{aligned} \left(\dot{\mathbf{V}}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} \right)_{\mathrm{a}} &= K_1 \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}} \right) \\ &- \left(J_1 \mathbf{I} - \mathbf{V}^{\mathrm{e}-1} \right)^{-1} \cdot \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}} \right) \\ &- \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}} \right) \cdot \left(J_1 \mathbf{I} - \mathbf{V}^{\mathrm{e}-1} \right)^{-1}, \end{aligned} \tag{11.7.18}$$

where

$$J_1 = \mathrm{tr}\, \mathbf{V}^{\mathrm{e}-1}, \quad K_1 = \mathrm{tr} \left(J_1 \mathbf{I} - \mathbf{V}^{\mathrm{e}-1} \right)^{-1}. \tag{11.7.19}$$

The left-hand side of Eq. (11.7.18) is also equal to $\mathbf{W} - \boldsymbol{\Omega}_I^{\mathrm{p}}$, by Eq. (11.7.7). Therefore,

$$\begin{aligned} \boldsymbol{\Omega}_I^{\mathrm{p}} = \mathbf{W} &- K_1 \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}} \right) \\ &+ \left(J_1 \mathbf{I} - \mathbf{V}^{\mathrm{e}-1} \right)^{-1} \cdot \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}} \right) \\ &+ \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}} \right) \cdot \left(J_1 \mathbf{I} - \mathbf{V}^{\mathrm{e}-1} \right)^{-1}. \end{aligned} \tag{11.7.20}$$

The expression for $\boldsymbol{\Omega}^{\mathrm{p}}$ is obtained by substituting Eq. (11.7.20) into Eq. (11.7.13). The result is

$$\begin{aligned} \boldsymbol{\Omega}^{\mathrm{p}} = \mathbf{R}^{\mathrm{e}T} \cdot \left(\mathbf{W} - \dot{\mathbf{R}}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{e}-1} \right) \cdot \mathbf{R}^{\mathrm{e}} &- K_1 \left(\hat{\mathbf{D}}^{\mathrm{e}} \cdot \mathbf{U}^{\mathrm{e}-1} - \mathbf{U}^{\mathrm{e}-1} \cdot \hat{\mathbf{D}}^{\mathrm{e}} \right) \\ &+ \left(J_1 \mathbf{I} - \mathbf{U}^{\mathrm{e}-1} \right)^{-1} \cdot \left(\hat{\mathbf{D}}^{\mathrm{e}} \cdot \mathbf{U}^{\mathrm{e}-1} - \mathbf{U}^{\mathrm{e}-1} \cdot \hat{\mathbf{D}}^{\mathrm{e}} \right) \\ &+ \left(\hat{\mathbf{D}}^{\mathrm{e}} \cdot \mathbf{U}^{\mathrm{e}-1} - \mathbf{U}^{\mathrm{e}-1} \cdot \hat{\mathbf{D}}^{\mathrm{e}} \right) \cdot \left(J_1 \mathbf{I} - \mathbf{U}^{\mathrm{e}-1} \right)^{-1}, \end{aligned} \tag{11.7.21}$$

where

$$\hat{\mathbf{D}}^{\mathrm{e}} = \mathbf{R}^{\mathrm{e}T} \cdot \mathbf{D}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{e}}, \quad \mathbf{U}^{\mathrm{e}-1} = \mathbf{R}^{\mathrm{e}T} \cdot \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{R}^{\mathrm{e}}. \tag{11.7.22}$$

With a specified rotation $\mathbf{R}^{\mathrm{e}}$ of the destressing program, Eq. (11.7.21) determines the corresponding spin $\boldsymbol{\Omega}^{\mathrm{p}}$.

11.8. Analysis of Plastic Rate of Deformation

Having defined the elastic rate of deformation by Eq. (11.7.1), the remaining plastic rate of deformation is

$$\mathbf{D}^{\mathrm{p}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{s}} - \left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}. \tag{11.8.1}$$

In view of Eq. (11.7.7), we also have

$$\mathbf{D}^{\mathrm{p}} = \mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1} - \mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}, \tag{11.8.2}$$

since

$$\mathbf{D}^{\mathrm{e}} + \mathbf{D}^{\mathrm{p}} + \mathbf{W} = \mathbf{L}, \tag{11.8.3}$$

as given by Eq. (11.3.2). Alternatively, Eq. (11.8.2) can be written as

$$\mathbf{D}^{\mathrm{p}} = \mathbf{F}^{\mathrm{e}} \cdot \left(\overset{\bullet}{\mathbf{F}}{}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}, \quad \overset{\bullet}{\mathbf{F}}{}^{\mathrm{p}} = \dot{\mathbf{F}}^{\mathrm{p}} - \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}}. \tag{11.8.4}$$

By taking the antisymmetric part of Eq. (11.8.2), therefore,

$$\left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{a}}. \tag{11.8.5}$$

Furthermore, from Eq. (11.8.2) we have

$$\boldsymbol{\mathcal{D}}^{\mathrm{p}} = \left(\mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}\right)_{\mathrm{s}}, \tag{11.8.6}$$

$$\boldsymbol{\mathcal{W}}^{\mathrm{p}} = \boldsymbol{\Omega}^{\mathrm{p}} + \left(\mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}\right)_{\mathrm{a}}. \tag{11.8.7}$$

For convenience, the rate of deformation and the spin of the intermediate configuration are denoted by

$$\boldsymbol{\mathcal{D}}^{\mathrm{p}} = \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right)_{\mathrm{s}}, \quad \boldsymbol{\mathcal{W}}^{\mathrm{p}} = \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right)_{\mathrm{a}}. \tag{11.8.8}$$

These quantities, of course, depend on the choice of the intermediate configuration.

To elaborate, we start from the identity

$$\mathbf{C}^{\mathrm{e}} \cdot \left(\mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}\right) = \left(\mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}\right)^{T} \cdot \mathbf{C}^{\mathrm{e}}, \tag{11.8.9}$$

which can be recast as

$$\mathbf{C}^{\mathrm{e}} \cdot \left(\mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}\right)_{\mathrm{a}} + \left(\mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}\right)_{\mathrm{a}} \cdot \mathbf{C}^{\mathrm{e}} = \boldsymbol{\mathcal{D}}^{\mathrm{p}} \cdot \mathbf{C}^{\mathrm{e}} - \mathbf{C}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{D}}^{\mathrm{p}}. \tag{11.8.10}$$

The last equation can be solved for $\left(\mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}\right)_{\mathrm{a}}$ in terms of $\mathbf{C}^{\mathrm{e}}$ and $\boldsymbol{\mathcal{D}}^{\mathrm{p}}$. The result is

$$\begin{aligned} \left(\mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}\right)_{\mathrm{a}} = k_1 & \left(\boldsymbol{\mathcal{D}}^{\mathrm{p}} \cdot \mathbf{C}^{\mathrm{e}} - \mathbf{C}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{D}}^{\mathrm{p}}\right) \\ & - \left(j_1 \mathbf{I} - \mathbf{C}^{\mathrm{e}}\right)^{-1} \cdot \left(\boldsymbol{\mathcal{D}}^{\mathrm{p}} \cdot \mathbf{C}^{\mathrm{e}} - \mathbf{C}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{D}}^{\mathrm{p}}\right) \\ & - \left(\boldsymbol{\mathcal{D}}^{\mathrm{p}} \cdot \mathbf{C}^{\mathrm{e}} - \mathbf{C}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{D}}^{\mathrm{p}}\right) \cdot \left(j_1 \mathbf{I} - \mathbf{C}^{\mathrm{e}}\right)^{-1}, \end{aligned} \tag{11.8.11}$$

where

$$j_1 = \operatorname{tr} \mathbf{C}^{\mathrm{e}}, \quad k_1 = \operatorname{tr} \left(j_1 \mathbf{I} - \mathbf{C}^{\mathrm{e}}\right)^{-1}. \tag{11.8.12}$$

The substitution of Eqs. (11.8.11) and (11.7.21) into Eq. (11.8.7) gives

$$\begin{aligned} \dot{\mathbf{R}}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{e}-1} + \mathbf{R}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{W}}^{\mathrm{p}} \cdot \mathbf{R}^{\mathrm{e}T} = \mathbf{W} & - K_1 \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}}\right) \\ & + \left(J_1 \mathbf{I} - \mathbf{V}^{\mathrm{e}-1}\right)^{-1} \cdot \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}}\right) \\ & + \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}}\right) \cdot \left(J_1 \mathbf{I} - \mathbf{V}^{\mathrm{e}-1}\right)^{-1} \\ & + k_1 \left(\bar{\boldsymbol{\mathcal{D}}}^{\mathrm{p}} \cdot \mathbf{B}^{\mathrm{e}} - \mathbf{B}^{\mathrm{e}} \cdot \bar{\boldsymbol{\mathcal{D}}}^{\mathrm{p}}\right) \\ & - \left(j_1 \mathbf{I} - \mathbf{B}^{\mathrm{e}}\right)^{-1} \cdot \left(\bar{\boldsymbol{\mathcal{D}}}^{\mathrm{p}} \cdot \mathbf{B}^{\mathrm{e}} - \mathbf{B}^{\mathrm{e}} \cdot \bar{\boldsymbol{\mathcal{D}}}^{\mathrm{p}}\right) \\ & - \left(\bar{\boldsymbol{\mathcal{D}}}^{\mathrm{p}} \cdot \mathbf{B}^{\mathrm{e}} - \mathbf{B}^{\mathrm{e}} \cdot \bar{\boldsymbol{\mathcal{D}}}^{\mathrm{p}}\right) \cdot \left(j_1 \mathbf{I} - \mathbf{B}^{\mathrm{e}}\right)^{-1}. \end{aligned} \tag{11.8.13}$$

The tensor

$$\bar{\boldsymbol{\mathcal{D}}}^{\mathrm{p}} = \mathbf{R}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{D}}^{\mathrm{p}} \cdot \mathbf{R}^{\mathrm{e}T} \tag{11.8.14}$$

is actually independent of the rotation $\mathbf{R}^{\mathrm{e}}$, since it can be expressed from Eq. (11.8.2) as

$$\bar{\boldsymbol{\mathcal{D}}}^{\mathrm{p}} = \left(\mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{V}^{\mathrm{e}}\right)_{\mathrm{s}}. \tag{11.8.15}$$

Note that

$$\operatorname{tr} \mathbf{C}^{\mathrm{e}} = \operatorname{tr} \mathbf{B}^{\mathrm{e}}, \quad \operatorname{tr} \left(j_1 \mathbf{I} - \mathbf{C}^{\mathrm{e}}\right)^{-1} = \operatorname{tr} \left(j_1 \mathbf{I} - \mathbf{B}^{\mathrm{e}}\right)^{-1}. \tag{11.8.16}$$

Therefore, the spin

$$\dot{\mathbf{R}}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{e}-1} + \mathbf{R}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{W}}^{\mathrm{p}} \cdot \mathbf{R}^{\mathrm{e}T} \tag{11.8.17}$$

in Eq. (11.8.13) is expressed in terms of $\mathbf{V}^{\mathrm{e}}$, $\mathbf{W}$, $\mathbf{D}^{\mathrm{e}}$, and $\mathbf{D}^{\mathrm{p}}$. For example, if destressing is without rotation ($\mathbf{R}^{\mathrm{e}} = \mathbf{I}$), Eq. (11.8.13) defines the corresponding spin $\boldsymbol{\mathcal{W}}^{\mathrm{p}}$ of the intermediate configuration. On the other hand, if destressing program is defined such that the spin of intermediate configuration vanishes ($\boldsymbol{\mathcal{W}}^{\mathrm{p}} = \mathbf{0}$), Eq. (11.8.13) defines the corresponding rotation $\mathbf{R}^{\mathrm{e}}$

of the destressing program. These (different) choices, however, do not affect the end result and the values of the components of the elastic and plastic rates of deformation $\mathbf{D}^{\rm e}$ and $\mathbf{D}^{\rm p}$.

11.8.1. Relationship between $\mathbf{D}^{\rm p}$ and $\boldsymbol{\mathcal{D}}^{\rm p}$

Equation (11.8.6), which expresses $\boldsymbol{\mathcal{D}}^{\rm p}$ in terms of $\mathbf{D}^{\rm p}$, can be rewritten as

$$\boldsymbol{\mathcal{D}}^{\rm p} = \frac{1}{2}\,\mathbf{F}^{{\rm e}-1}\cdot\left(\mathbf{D}^{\rm p}\cdot\mathbf{B}^{\rm e}+\mathbf{B}^{\rm e}\cdot\mathbf{D}^{\rm p}\right)\cdot\mathbf{F}^{{\rm e}-T}, \tag{11.8.18}$$

or,

$$\mathbf{D}^{\rm p}\cdot\mathbf{B}^{\rm e}+\mathbf{B}^{\rm e}\cdot\mathbf{D}^{\rm p} = 2\,\mathbf{F}^{\rm e}\cdot\boldsymbol{\mathcal{D}}^{\rm p}\cdot\mathbf{F}^{{\rm e}T}. \tag{11.8.19}$$

The solution for $\mathbf{D}^{\rm p}$ in terms of $\boldsymbol{\mathcal{D}}^{\rm p}$ is

$$\begin{aligned}\mathbf{D}^{\rm p} = 2k_1\left(\mathbf{F}^{\rm e}\cdot\boldsymbol{\mathcal{D}}^{\rm p}\cdot\mathbf{F}^{{\rm e}T}\right) &- 2\left(j_1\mathbf{I}-\mathbf{B}^{\rm e}\right)^{-1}\cdot\left(\mathbf{F}^{\rm e}\cdot\boldsymbol{\mathcal{D}}^{\rm p}\cdot\mathbf{F}^{{\rm e}T}\right)\\ &- 2\left(\mathbf{F}^{\rm e}\cdot\boldsymbol{\mathcal{D}}^{\rm p}\cdot\mathbf{F}^{{\rm e}T}\right)\cdot\left(j_1\mathbf{I}-\mathbf{B}^{\rm e}\right)^{-1}.\end{aligned} \tag{11.8.20}$$

Alternatively, we can start from Eqs. (11.8.2) and (11.8.7), i.e.,

$$\mathbf{D}^{\rm p} = \mathbf{F}^{\rm e}\cdot\left(\boldsymbol{\mathcal{D}}^{\rm p}+\boldsymbol{\mathcal{W}}^{\rm p}-\boldsymbol{\Omega}^{\rm p}\right)\cdot\mathbf{F}^{{\rm e}-1} = \mathbf{F}^{\rm e}\cdot\left[\boldsymbol{\mathcal{D}}^{\rm p}+\left(\mathbf{F}^{{\rm e}-1}\cdot\mathbf{D}^{\rm p}\cdot\mathbf{F}^{\rm e}\right)_{\rm a}\right]\cdot\mathbf{F}^{{\rm e}-1}. \tag{11.8.21}$$

The substitution of Eq. (11.8.11) gives

$$\begin{aligned}\mathbf{D}^{\rm p} = \mathbf{F}^{\rm e}\cdot\big[\boldsymbol{\mathcal{D}}^{\rm p} &+ (\operatorname{tr}\mathbf{A}^{\rm e})\left(\boldsymbol{\mathcal{D}}^{\rm p}\cdot\mathbf{C}^{\rm e}-\mathbf{C}^{\rm e}\cdot\boldsymbol{\mathcal{D}}^{\rm p}\right)\\ &- \mathbf{A}^{\rm e}\cdot\left(\boldsymbol{\mathcal{D}}^{\rm p}\cdot\mathbf{C}^{\rm e}-\mathbf{C}^{\rm e}\cdot\boldsymbol{\mathcal{D}}^{\rm p}\right)-\left(\boldsymbol{\mathcal{D}}^{\rm p}\cdot\mathbf{C}^{\rm e}-\mathbf{C}^{\rm e}\cdot\boldsymbol{\mathcal{D}}^{\rm p}\right)\cdot\mathbf{A}^{\rm e}\big]\cdot\mathbf{F}^{{\rm e}-1},\end{aligned} \tag{11.8.22}$$

where

$$\mathbf{A}^{\rm e} = \left(j_1\mathbf{I}-\mathbf{C}^{\rm e}\right)^{-1}. \tag{11.8.23}$$

The antisymmetric part of Eq. (11.8.22) vanishes identically.

11.9. Expression for $\mathbf{D}^{\rm e}$ in Terms of $\mathbf{F}^{\rm e}$, $\mathbf{F}^{\rm p}$, and Their Rates

In Eq. (11.7.1) the elastic rate of deformation $\mathbf{D}^{\rm e}$ was the sum of two terms, the second term being dependent on the spin $\boldsymbol{\Omega}^{\rm p}$. It is possible to express this term as an explicit function of $\mathbf{F}^{\rm e}$ and $\mathbf{F}^{\rm p}$, and their rates. To that goal, consider the identity

$$\mathbf{B}^{{\rm e}-1}\cdot\left(\mathbf{F}^{\rm e}\cdot\boldsymbol{\Omega}^{\rm p}\cdot\mathbf{F}^{{\rm e}-1}\right) = -\left(\mathbf{F}^{\rm e}\cdot\boldsymbol{\Omega}^{\rm p}\cdot\mathbf{F}^{{\rm e}-1}\right)^{T}\cdot\mathbf{B}^{{\rm e}-1}, \tag{11.9.1}$$

which can be rewritten as

$$
\begin{aligned}
&\mathbf{B}^{\mathrm{e}-1} \cdot \left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} + \left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} \cdot \mathbf{B}^{\mathrm{e}-1} \\
&= \left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}} \cdot \mathbf{B}^{\mathrm{e}-1} - \mathbf{B}^{\mathrm{e}-1} \cdot \left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}} \\
&= \boldsymbol{\omega}^{\mathrm{p}} \cdot \mathbf{B}^{\mathrm{e}-1} - \mathbf{B}^{\mathrm{e}-1} \cdot \boldsymbol{\omega}^{\mathrm{p}}.
\end{aligned}
\tag{11.9.2}
$$

Expression (11.8.5) was used in the last step. Equation (11.9.2) can be solved for $\left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}$ in terms of $\mathbf{B}^{\mathrm{e}-1}$ and the spin $\boldsymbol{\omega}^{\mathrm{p}}$, with the result

$$
\begin{aligned}
\left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} &= k_1' \left(\boldsymbol{\omega}^{\mathrm{p}} \cdot \mathbf{B}^{\mathrm{e}-1} - \mathbf{B}^{\mathrm{e}-1} \cdot \boldsymbol{\omega}^{\mathrm{p}}\right) \\
&\quad - \left(j_1' \mathbf{I} - \mathbf{B}^{\mathrm{e}-1}\right)^{-1} \cdot \left(\boldsymbol{\omega}^{\mathrm{p}} \cdot \mathbf{B}^{\mathrm{e}-1} - \mathbf{B}^{\mathrm{e}-1} \cdot \boldsymbol{\omega}^{\mathrm{p}}\right) \\
&\quad - \left(\boldsymbol{\omega}^{\mathrm{p}} \cdot \mathbf{B}^{\mathrm{e}-1} - \mathbf{B}^{\mathrm{e}-1} \cdot \boldsymbol{\omega}^{\mathrm{p}}\right) \cdot \left(j_1' \mathbf{I} - \mathbf{B}^{\mathrm{e}-1}\right)^{-1},
\end{aligned}
\tag{11.9.3}
$$

where

$$
j_1' = \operatorname{tr} \mathbf{B}^{\mathrm{e}-1}, \quad k_1' = \operatorname{tr} \left(j_1 \mathbf{I} - \mathbf{B}^{\mathrm{e}-1}\right)^{-1}. \tag{11.9.4}
$$

Consequently, incorporating Eq. (11.9.3) into Eq. (11.7.1) gives an expression for the elastic rate of deformation, solely in terms of $\mathbf{F}^{\mathrm{e}}$ and $\mathbf{F}^{\mathrm{p}}$, and their rates. This is

$$
\begin{aligned}
\mathbf{D}^{\mathrm{e}} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} &+ k_1' \left(\boldsymbol{\omega}^{\mathrm{p}} \cdot \mathbf{B}^{\mathrm{e}-1} - \mathbf{B}^{\mathrm{e}-1} \cdot \boldsymbol{\omega}^{\mathrm{p}}\right) \\
&- \left(j_1' \mathbf{I} - \mathbf{B}^{\mathrm{e}-1}\right)^{-1} \cdot \left(\boldsymbol{\omega}^{\mathrm{p}} \cdot \mathbf{B}^{\mathrm{e}-1} - \mathbf{B}^{\mathrm{e}-1} \cdot \boldsymbol{\omega}^{\mathrm{p}}\right) \\
&- \left(\boldsymbol{\omega}^{\mathrm{p}} \cdot \mathbf{B}^{\mathrm{e}-1} - \mathbf{B}^{\mathrm{e}-1} \cdot \boldsymbol{\omega}^{\mathrm{p}}\right) \cdot \left(j_1' \mathbf{I} - \mathbf{B}^{\mathrm{e}-1}\right)^{-1}.
\end{aligned}
\tag{11.9.5}
$$

11.9.1. Intermediate Configuration with $\boldsymbol{\omega}^{\mathrm{p}} = \mathbf{0}$

The three most appealing choices of the intermediate configuration correspond to

$$
\begin{aligned}
\mathbf{R}^{\mathrm{e}} &= \mathbf{I}, \\
\boldsymbol{\mathcal{W}}^{\mathrm{p}} &= \mathbf{0}, \\
\boldsymbol{\omega}^{\mathrm{p}} &= \mathbf{0}.
\end{aligned}
\tag{11.9.6}
$$

We discuss here the last choice, i.e., we consider the intermediate configuration obtained by the destressing program such that

$$
\boldsymbol{\omega}^{\mathrm{p}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{a}} = \mathbf{0}. \tag{11.9.7}
$$

From Eqs. (11.8.5) and (11.9.3) it follows that

$$
\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1} = \mathbf{0}, \quad \text{i.e.,} \quad \boldsymbol{\Omega}^{\mathrm{p}} = \mathbf{0}. \tag{11.9.8}
$$

The corresponding rotation is, from Eq. (11.7.13),

$$
\dot{\mathbf{R}}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{e}-1} = \boldsymbol{\Omega}_I^{\mathrm{p}}, \tag{11.9.9}
$$

where $\boldsymbol{\Omega}_I^{\mathrm{p}}$ is defined by Eq. (11.7.20). Furthermore, from Eqs. (11.8.7) and (11.8.11), the spin of the intermediate configuration is

$$\begin{aligned}\boldsymbol{\mathcal{W}}^{\mathrm{p}} = \left(\mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}\right)_{\mathrm{a}} &= k_1 \left(\boldsymbol{\mathcal{D}}^{\mathrm{p}} \cdot \mathbf{C}^{\mathrm{e}} - \mathbf{C}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{D}}^{\mathrm{p}}\right) \\ &- \left(j_1 \mathbf{I} - \mathbf{C}^{\mathrm{e}}\right)^{-1} \cdot \left(\boldsymbol{\mathcal{D}}^{\mathrm{p}} \cdot \mathbf{C}^{\mathrm{e}} - \mathbf{C}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{D}}^{\mathrm{p}}\right) \\ &- \left(\boldsymbol{\mathcal{D}}^{\mathrm{p}} \cdot \mathbf{C}^{\mathrm{e}} - \mathbf{C}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{D}}^{\mathrm{p}}\right) \cdot \left(j_1 \mathbf{I} - \mathbf{C}^{\mathrm{e}}\right)^{-1}.\end{aligned} \tag{11.9.10}$$

The elastic and plastic rates of deformation are

$$\mathbf{D}^{\mathrm{e}} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}, \tag{11.9.11}$$

$$\mathbf{D}^{\mathrm{p}} = \mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}. \tag{11.9.12}$$

For any other choice of the intermediate configuration, not associated with the choice (11.9.7), the symmetric part of $\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}$ is not all, but only a portion of the elastic rate of deformation $\mathbf{D}^{\mathrm{e}}$.

11.10. Isotropic Hardening

In the case of isotropic hardening the yield function is an isotropic function of the Cauchy stress $\boldsymbol{\sigma}$. Thus, if the normality rule applies, the plastic rate of deformation $\mathbf{D}^{\mathrm{p}}$ is codirectional with the outward normal to a locally smooth yield surface in stress space, and its principal directions are parallel to those of the stress $\boldsymbol{\sigma}$. Since for elastically isotropic material $\mathbf{V}^{\mathrm{e}}$ and $\mathbf{B}^{\mathrm{e}}$ are also coaxial with $\boldsymbol{\sigma}$, their matrix products commute, and Eqs. (11.8.6) and (11.8.7) become

$$\boldsymbol{\mathcal{D}}^{\mathrm{p}} = \mathbf{R}^{\mathrm{e}T} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{R}^{\mathrm{e}}, \quad \boldsymbol{\mathcal{W}}^{\mathrm{p}} = \boldsymbol{\Omega}^{\mathrm{p}}, \tag{11.10.1}$$

because

$$\left(\mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}\right)_{\mathrm{a}} = \left(\mathbf{R}^{\mathrm{e}T} \cdot \mathbf{D}^{\mathrm{p}} \cdot \mathbf{R}^{\mathrm{e}}\right)_{\mathrm{a}} = \mathbf{0}. \tag{11.10.2}$$

Furthermore, since from Eq. (11.8.15) in the case of isotropic hardening,

$$\bar{\boldsymbol{\mathcal{D}}}^{\mathrm{p}} = \mathbf{D}^{\mathrm{p}}, \tag{11.10.3}$$

Equation (11.8.13) reduces to

$$\begin{aligned}\dot{\mathbf{R}}^{\mathrm{e}} \cdot \mathbf{R}^{\mathrm{e}-1} + \mathbf{R}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{W}}^{\mathrm{p}} \cdot \mathbf{R}^{\mathrm{e}T} &= \mathbf{W} - K_1 \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}}\right) \\ &+ \left(J_1 \mathbf{I} - \mathbf{V}^{\mathrm{e}-1}\right)^{-1} \cdot \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}}\right) \\ &+ \left(\mathbf{D}^{\mathrm{e}} \cdot \mathbf{V}^{\mathrm{e}-1} - \mathbf{V}^{\mathrm{e}-1} \cdot \mathbf{D}^{\mathrm{e}}\right) \cdot \left(J_1 \mathbf{I} - \mathbf{V}^{\mathrm{e}-1}\right)^{-1}.\end{aligned} \tag{11.10.4}$$

This is precisely the spin $\boldsymbol{\Omega}_I^{\rm p}$ of Eq. (11.7.20). In addition, we have

$$\boldsymbol{\omega}^{\rm p} = \left[\mathbf{F}^{\rm e} \cdot \left(\dot{\mathbf{F}}^{\rm p} \cdot \mathbf{F}^{\rm p-1}\right) \cdot \mathbf{F}^{\rm e-1}\right]_{\rm a} = \left(\mathbf{F}^{\rm e} \cdot \boldsymbol{\mathcal{W}}^{\rm p} \cdot \mathbf{F}^{\rm e-1}\right)_{\rm a}. \tag{11.10.5}$$

If the intermediate configuration is selected so that $\mathbf{R}^{\rm e} = \mathbf{I}$, Eq. (11.10.4) specifies the corresponding spin (Lubarda and Lee, 1981), as

$$\boldsymbol{\mathcal{W}}^{\rm p} = \boldsymbol{\Omega}_I^{\rm p}. \tag{11.10.6}$$

If the intermediate configuration is selected so that $\boldsymbol{\omega}^{\rm p} = \mathbf{0}$, then

$$\boldsymbol{\mathcal{W}}^{\rm p} = \mathbf{0} \tag{11.10.7}$$

(and *vice versa*, for isotropic hardening). The right-hand side of Eq. (11.10.4) defines the spin due to $\mathbf{R}^{\rm e}$, i.e.,

$$\dot{\mathbf{R}}^{\rm e} \cdot \mathbf{R}^{\rm e-1} = \boldsymbol{\Omega}_I^{\rm p}. \tag{11.10.8}$$

11.11. Kinematic Hardening

To approximately account for the Bauschinger effect and anisotropic hardening, the kinematic hardening model was introduced in Subsection 9.4.2. Translation of the yield surface in stress space is prescribed by the evolution equation for the back stress $\boldsymbol{\alpha}$ (center of the yield surface). A fairly general objective equation for this evolution is

$$\overset{\circ}{\boldsymbol{\alpha}} = \dot{\boldsymbol{\alpha}} - \mathbf{W} \cdot \boldsymbol{\alpha} + \boldsymbol{\alpha} \cdot \mathbf{W} = \mathbf{A}(\boldsymbol{\alpha}, \mathbf{D}^{\rm p}), \tag{11.11.1}$$

where $\mathbf{A}$ is an isotropic function of both $\boldsymbol{\alpha}$ and $\mathbf{D}^{\rm p}$. Its polynomial representation is given by Eq. (1.11.10). Assuming that $\boldsymbol{\alpha}$ is deviatoric and that the material response is rate-independent, the function $\mathbf{A}$ can be written as

$$\mathbf{A}(\boldsymbol{\alpha}, \mathbf{D}^{\rm p}) = \mathbf{G}(\boldsymbol{\alpha}, \mathbf{D}^{\rm p}) + \boldsymbol{\alpha} \cdot \hat{\mathbf{W}} - \hat{\mathbf{W}} \cdot \boldsymbol{\alpha}. \tag{11.11.2}$$

The tensor function $\mathbf{G}$ is

$$\begin{aligned}\mathbf{G}(\boldsymbol{\alpha}, \mathbf{D}^{\rm p}) = {} & \eta_1 \,\mathbf{D}^{\rm p} + \eta_2 \,D^{\rm p}\, \boldsymbol{\alpha} + \eta_3 \,D^{\rm p} \left[\boldsymbol{\alpha}^2 - \frac{1}{3}\,\mathrm{tr}\left(\boldsymbol{\alpha}^2\right)\mathbf{I}\right] \\ & + \eta_4 \left[\boldsymbol{\alpha} \cdot \mathbf{D}^{\rm p} + \mathbf{D}^{\rm p} \cdot \boldsymbol{\alpha} - \frac{2}{3}\,\mathrm{tr}\,(\boldsymbol{\alpha} \cdot \mathbf{D}^{\rm p})\,\mathbf{I}\right] \\ & + \eta_5 \left[\boldsymbol{\alpha}^2 \cdot \mathbf{D}^{\rm p} + \mathbf{D}^{\rm p} \cdot \boldsymbol{\alpha}^2 - \frac{2}{3}\,\mathrm{tr}\left(\boldsymbol{\alpha}^2 \cdot \mathbf{D}^{\rm p}\right)\mathbf{I}\right],\end{aligned} \tag{11.11.3}$$

and the spin

$$\begin{aligned}\hat{\mathbf{W}} = {} & \vartheta_1\,(\boldsymbol{\alpha} \cdot \mathbf{D}^{\rm p} - \mathbf{D}^{\rm p} \cdot \boldsymbol{\alpha}) + \vartheta_2\left(\boldsymbol{\alpha}^2 \cdot \mathbf{D}^{\rm p} - \mathbf{D}^{\rm p} \cdot \boldsymbol{\alpha}^2\right) + \\ & + \vartheta_3\left(\boldsymbol{\alpha}^2 \cdot \mathbf{D}^{\rm p} \cdot \boldsymbol{\alpha} - \boldsymbol{\alpha} \cdot \mathbf{D}^{\rm p} \cdot \boldsymbol{\alpha}^2\right).\end{aligned} \tag{11.11.4}$$

The scalar

$$D^{\mathrm{p}} = \left(2\,\mathbf{D}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}}\right)^{1/2} \tag{11.11.5}$$

is a homogeneous function of degree one in the components of plastic rate of deformation, while $\eta_i\,(i = 1, 2, \ldots, 5)$ and $\vartheta_i\,(i = 1, 2, 3)$ are scalar functions of the invariants of $\boldsymbol{\alpha}$. The representation of antisymmetric function $\hat{\mathbf{W}}$ in terms of $\boldsymbol{\alpha}$ and $\mathbf{D}^{\mathrm{p}}$ is constructed according to Eq. (1.11.11). The combination of terms $(\boldsymbol{\alpha}\cdot\hat{\mathbf{W}} - \hat{\mathbf{W}}\cdot\boldsymbol{\alpha})$, which is an isotropic symmetric function of $\boldsymbol{\alpha}$ and $\mathbf{D}^{\mathrm{p}}$, is given separately in the representation (11.11.2), so that the function $\mathbf{G}$ incorporates direct influence of the rate of deformation on the evolution of $\boldsymbol{\alpha}$, while $(\boldsymbol{\alpha}\cdot\hat{\mathbf{W}} - \hat{\mathbf{W}}\cdot\boldsymbol{\alpha})$ incorporates the influence of deformation imposed rotation of the lines of material elements considered to carry the embedded back stress (Agah-Tehrani, Lee, Mallett, and Onat, 1987). Such rotation can have a significant effect on the evolution, quite independently of the overall material spin $\mathbf{W}$. An example in the case of straining in simple shear is given by Lee, Mallett, and Wertheimer (1983). See also Dafalias (1983), Atluri (1984), Johnson and Bammann (1984), and Van der Giessen (1989).

The following relationships are further observed

$$\boldsymbol{\alpha}^2\cdot\mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha}^2 = \boldsymbol{\alpha}\cdot(\boldsymbol{\alpha}\cdot\mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha}) + (\boldsymbol{\alpha}\cdot\mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha})\cdot\boldsymbol{\alpha}, \tag{11.11.6}$$

and

$$\begin{aligned}\boldsymbol{\alpha}^2\cdot\mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha} - \boldsymbol{\alpha}\cdot\mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha}^2 &= \boldsymbol{\alpha}\cdot\left(\boldsymbol{\alpha}^2\cdot\mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha}^2\right)\\ &+ \left(\boldsymbol{\alpha}^2\cdot\mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha}^2\right)\cdot\boldsymbol{\alpha} - \frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\alpha}^2\right)(\boldsymbol{\alpha}\cdot\mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha})\,.\end{aligned} \tag{11.11.7}$$

The second of these can be expressed as

$$\begin{aligned}\boldsymbol{\alpha}^2\cdot\mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha} - \boldsymbol{\alpha}\cdot\mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha}^2 &= -\boldsymbol{\alpha}^2\cdot(\boldsymbol{\alpha}\cdot\mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha})\\ &- (\boldsymbol{\alpha}\cdot\mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha})\cdot\boldsymbol{\alpha}^2 + \frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\alpha}^2\right)(\boldsymbol{\alpha}\cdot\mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}}\cdot\boldsymbol{\alpha})\,.\end{aligned} \tag{11.11.8}$$

This is easily verified by recalling that $\boldsymbol{\alpha}$ is deviatoric ($\mathrm{tr}\,\boldsymbol{\alpha} = 0$) and that, from the Cayley–Hamilton theorem (1.4.1),

$$\boldsymbol{\alpha}^3 = \frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\alpha}^2\right)\boldsymbol{\alpha} + (\det\boldsymbol{\alpha})\,\mathbf{I}. \tag{11.11.9}$$

Substitution of Eqs. (11.11.6) and (11.11.8) into Eq. (11.11.4) thus yields

$$\hat{\mathbf{W}} = -\mathbf{H} \cdot (\boldsymbol{\alpha} \cdot \mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}} \cdot \boldsymbol{\alpha}) - (\boldsymbol{\alpha} \cdot \mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}} \cdot \boldsymbol{\alpha}) \cdot \mathbf{H} + (\mathrm{tr}\,\mathbf{H})\,(\boldsymbol{\alpha} \cdot \mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}} \cdot \boldsymbol{\alpha})\,. \tag{11.11.10}$$

This expresses $\hat{\mathbf{W}}$ in terms of a basic antisymmetric tensor $(\boldsymbol{\alpha} \cdot \mathbf{D}^{\mathrm{p}} - \mathbf{D}^{\mathrm{p}} \cdot \boldsymbol{\alpha})$ and an isotropic tensor function $\mathbf{H}(\boldsymbol{\alpha})$, defined by

$$\mathbf{H} = \vartheta_1\,\mathbf{I} - \vartheta_2\,\boldsymbol{\alpha} + \vartheta_3 \left[\boldsymbol{\alpha}^2 - \frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\alpha}^2\right)\mathbf{I}\right]. \tag{11.11.11}$$

The evolution equation for the back stress (11.11.1) consequently becomes

$$\overset{\bullet}{\boldsymbol{\alpha}} = \mathbf{G}(\boldsymbol{\alpha}, \mathbf{D}^{\mathrm{p}}), \tag{11.11.12}$$

where, in view of Eq. (11.11.2),

$$\overset{\bullet}{\boldsymbol{\alpha}} = \overset{\circ}{\boldsymbol{\alpha}} + \hat{\mathbf{W}} \cdot \boldsymbol{\alpha} - \boldsymbol{\alpha} \cdot \hat{\mathbf{W}} = \dot{\boldsymbol{\alpha}} - \hat{\boldsymbol{\omega}} \cdot \boldsymbol{\alpha} + \boldsymbol{\alpha} \cdot \hat{\boldsymbol{\omega}}. \tag{11.11.13}$$

The spin used to define the Jaumann derivative $\overset{\bullet}{\boldsymbol{\alpha}}$ is

$$\hat{\boldsymbol{\omega}} = \mathbf{W} - \hat{\mathbf{W}}. \tag{11.11.14}$$

Either the spin $\hat{\mathbf{W}}$, associated with the angular velocity of the embedded back stress, or the relative spin $\hat{\boldsymbol{\omega}}$ (relative to the deforming material), can be referred to as the plastic spin. The constitutive equation for $\hat{\mathbf{W}}$ is given by Eqs. (11.11.10) and (11.11.11), with the appropriately specified parameters $\vartheta_i\ (i = 1, 2, 3)$.

The introduction of the plastic spin as an ingredient of the phenomenological theory of plasticity was motivated by the attempts to eliminate spurious oscillations of shear stress, obtained under monotonically increasing straining in simple shear, within the model of kinematic hardening and simple evolution equation for the back stress $\overset{\circ}{\boldsymbol{\alpha}} \propto \mathbf{D}^{\mathrm{p}}$ (Nagtegaal and de Jong, 1982; Lee, Mallett, and Wertheimer, 1983). Further research on plastic spin was subsequently stimulated by the work of Loret (1983) and Dafalias (1983,1985). Various aspects of this work have been discussed or reviewed by Aifantis (1987), Zbib and Aifantis (1988), Van der Giessen (1991), Nemat-Nasser (1992), Lubarda and Shih (1994), Besseling and Van der Giessen (1994), and Dafalias (1999). The survey paper by Dafalias (1999) contains additional references. Research on the plastic spin in crystal plasticity is discussed in Chapter 12. An analysis of plastic spin in the corner theory of plasticity was presented by Kuroda (1995).

The elastoplastic behavior of amorphous polymers was studied within the framework of multiplicative decomposition by Boyce, Parks, and Argon (1988), and Boyce, Weber, and Parks (1990). Other viscoplastic solids were considered by Weber and Anand (1990). See also Anand (1980) for an application to pressure sensitive dilatant materials. Computational aspects of finite deformation elastoplasticity based on the multiplicative decomposition were examined by Needleman (1985), Simo and Ortiz (1985), Moran, Ortiz, and Shih (1990), Simo (1998), Simo and Hughes (1998), and Belytschko, Liu, and Moran (2000).

11.12. Rates of Deformation Due to Convected Stress Rate

The rate of deformation tensor was partitioned in Section 11.6 into its elastic and plastic parts by using the Jaumann rate of the Kirchhoff stress, such that

$$\mathbf{D}^{\mathrm{e}}_{(0)} = \boldsymbol{\mathcal{L}}^{-1}_{(0)} : \overset{\circ}{\boldsymbol{\tau}}, \quad \mathbf{D}^{\mathrm{p}}_{(0)} = \mathbf{D} - \mathbf{D}^{\mathrm{e}}_{(0)}. \tag{11.12.1}$$

The subscript (0) is added to indicate that the partition was with respect to the stress rate $\overset{\circ}{\boldsymbol{\tau}}$. In terms of $\mathbf{F}^{\mathrm{e}}$ and $\mathbf{F}^{\mathrm{p}}$, and their rates, it was found that

$$\begin{aligned}\mathbf{D}^{\mathrm{e}}_{(0)} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} - \boldsymbol{\mathcal{L}}^{-1}_{(0)} : \Big\{ \Big[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\Big]_{\mathrm{a}} \cdot \boldsymbol{\tau} \\ - \boldsymbol{\tau} \cdot \Big[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\Big]_{\mathrm{a}} \Big\},\end{aligned} \tag{11.12.2}$$

$$\begin{aligned}\mathbf{D}^{\mathrm{p}}_{(0)} = \Big[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\Big]_{\mathrm{s}} + \boldsymbol{\mathcal{L}}^{-1}_{(0)} : \Big\{ \Big[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\Big]_{\mathrm{a}} \cdot \boldsymbol{\tau} \\ - \boldsymbol{\tau} \cdot \Big[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\Big]_{\mathrm{a}} \Big\}.\end{aligned} \tag{11.12.3}$$

The corresponding elastic and plastic parts of the stress rate $\overset{\circ}{\boldsymbol{\tau}}$ are

$$\overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{e}} = \boldsymbol{\mathcal{L}}_{(0)} : \mathbf{D}, \quad \overset{\circ}{\boldsymbol{\tau}}{}^{\mathrm{p}} = -\boldsymbol{\mathcal{L}}_{(0)} : \mathbf{D}^{\mathrm{p}}_{(0)}. \tag{11.12.4}$$

An alternative partition of the rate of deformation tensor can be obtained by using the convected rate of the Kirchhoff stress $\overset{\triangle}{\boldsymbol{\tau}}$, such that

$$\mathbf{D}^{\mathrm{e}}_{(1)} = \boldsymbol{\mathcal{L}}^{-1}_{(1)} : \overset{\triangle}{\boldsymbol{\tau}}, \quad \mathbf{D}^{\mathrm{p}}_{(1)} = \mathbf{D} - \mathbf{D}^{\mathrm{e}}_{(1)}. \tag{11.12.5}$$

Indeed, from Eq. (11.6.2) it follows that

$$\overset{\triangle}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_{(1)} : \left(\dot{\mathbf{F}}^{\rm e} \cdot \mathbf{F}^{\rm e-1}\right)_{\rm s} - \left\{ \left[\mathbf{F}^{\rm e} \cdot \left(\dot{\mathbf{F}}^{\rm p} \cdot \mathbf{F}^{\rm p-1}\right) \cdot \mathbf{F}^{\rm e-1}\right] \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left[\mathbf{F}^{\rm e} \cdot \left(\dot{\mathbf{F}}^{\rm p} \cdot \mathbf{F}^{\rm p-1}\right) \cdot \mathbf{F}^{\rm e-1}\right]^T \right\}. \tag{11.12.6}$$

This defines the elastic part of the rate of deformation corresponding to the stress rate $\overset{\triangle}{\boldsymbol{\tau}}$, which is

$$\mathbf{D}^{\rm e}_{(1)} = \left(\dot{\mathbf{F}}^{\rm e} \cdot \mathbf{F}^{\rm e-1}\right)_{\rm s} - \boldsymbol{\mathcal{L}}^{-1}_{(1)} : \left\{ \left[\mathbf{F}^{\rm e} \cdot \left(\dot{\mathbf{F}}^{\rm p} \cdot \mathbf{F}^{\rm p-1}\right) \cdot \mathbf{F}^{\rm e-1}\right] \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left[\mathbf{F}^{\rm e} \cdot \left(\dot{\mathbf{F}}^{\rm p} \cdot \mathbf{F}^{\rm p-1}\right) \cdot \mathbf{F}^{\rm e-1}\right]^T \right\}. \tag{11.12.7}$$

The remaining part of the rate of deformation is the plastic part,

$$\mathbf{D}^{\rm p}_{(1)} = \left[\mathbf{F}^{\rm e} \cdot \left(\dot{\mathbf{F}}^{\rm p} \cdot \mathbf{F}^{\rm p-1}\right) \cdot \mathbf{F}^{\rm e-1}\right]_{\rm s} + \boldsymbol{\mathcal{L}}^{-1}_{(1)} : \left\{ \left[\mathbf{F}^{\rm e} \cdot \left(\dot{\mathbf{F}}^{\rm p} \cdot \mathbf{F}^{\rm p-1}\right) \cdot \mathbf{F}^{\rm e-1}\right] \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left[\mathbf{F}^{\rm e} \cdot \left(\dot{\mathbf{F}}^{\rm p} \cdot \mathbf{F}^{\rm p-1}\right) \cdot \mathbf{F}^{\rm e-1}\right]^T \right\}. \tag{11.12.8}$$

The corresponding elastic and plastic parts of the stress rate $\overset{\triangle}{\boldsymbol{\tau}}$ are

$$\overset{\triangle}{\boldsymbol{\tau}}{}^{\rm e} = \boldsymbol{\mathcal{L}}_{(1)} : \mathbf{D}, \quad \overset{\triangle}{\boldsymbol{\tau}}{}^{\rm p} = -\boldsymbol{\mathcal{L}}_{(1)} : \mathbf{D}^{\rm p}_{(1)}. \tag{11.12.9}$$

It is readily verified that

$$\overset{\circ}{\boldsymbol{\tau}}{}^{\rm p} = \overset{\triangle}{\boldsymbol{\tau}}{}^{\rm p}. \tag{11.12.10}$$

The partition of the rate of deformation based on the convected rate of the Kirchhoff stress, which involves in its definition both the spin and the rate of deformation, may appear less appealing than the partition based on the Jaumann rate, which involves only the spin part of the velocity gradient. However, the partition based on the convected rate is inherent in the constitutive formulation based on the Lagrangian strain and its conjugate, symmetric Piola–Kirchhoff stress. Since $\dot{\mathbf{E}} = \mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F}$, the elastic and plastic parts of the rate of Lagrangian strain are (Lubarda, 1994a)

$$(\dot{\mathbf{E}})^{\rm e} = \mathbf{F}^T \cdot \mathbf{D}^{\rm e}_{(1)} \cdot \mathbf{F}, \quad (\dot{\mathbf{E}})^{\rm p} = \mathbf{F}^T \cdot \mathbf{D}^{\rm p}_{(1)} \cdot \mathbf{F}. \tag{11.12.11}$$

These are defined such that

$$(\dot{\mathbf{E}})^{\rm e} = \boldsymbol{\Lambda}^{-1}_{(1)} : \dot{\mathbf{T}}, \quad (\dot{\mathbf{E}})^{\rm p} = \dot{\mathbf{E}} - (\dot{\mathbf{E}})^{\rm e}, \tag{11.12.12}$$

where $\mathbf{T}$ is the symmetric Piola–Kirchhoff stress tensor, conjugate to the Lagrangian strain $\mathbf{E}$ (the conjugate measures from Chapters 2 and 3 corresponding to $n = 1$; for simplicity we omit here the subscript (1) in the notation for $\mathbf{T}_{(1)}$ and $\mathbf{E}_{(1)}$). The plastic part of the rate of Lagrangian strain is normal to a locally smooth yield surface in the Piola–Kirchhoff stress space, and is within the cone of outward normals at the vertex of the yield surface. An independent derivation of the partition of the rate of Lagrangian strain into its elastic and plastic parts is presented in the following section.

11.13. Partition of the Rate of Lagrangian Strain

If elastic strain energy per unit unstressed volume is an isotropic function of the Lagrangian strain $\mathbf{E}^{\mathrm{e}}$, it can be expressed, with the help of Eq. (11.2.5), as

$$\Psi^{\mathrm{e}} = \Psi^{\mathrm{e}}\left(\mathbf{E}^{\mathrm{e}}\right) = \Psi^{\mathrm{e}}\left[\mathbf{F}^{\mathrm{p}-T} \cdot \left(\mathbf{E} - \mathbf{E}^{\mathrm{p}}\right) \cdot \mathbf{F}^{\mathrm{p}-1}\right]. \tag{11.13.1}$$

From this we deduce that

$$\mathbf{T}^{\mathrm{e}} = \frac{\partial \Psi^{\mathrm{e}}}{\partial \mathbf{E}^{\mathrm{e}}}, \quad \mathbf{T} = \frac{\partial \Psi^{\mathrm{e}}}{\partial \mathbf{E}}, \tag{11.13.2}$$

with a connection between the two stress tensors

$$\mathbf{T}^{\mathrm{e}} = \mathbf{F}^{\mathrm{p}} \cdot \mathbf{T} \cdot \mathbf{F}^{\mathrm{p}T}. \tag{11.13.3}$$

The stress tensors $\mathbf{T}^{\mathrm{e}}$ and $\mathbf{T}$ are related to the Kirchhoff stress $\boldsymbol{\tau}$ by

$$\mathbf{T}^{\mathrm{e}} = \mathbf{F}^{\mathrm{e}-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}^{\mathrm{e}-T}, \quad \mathbf{T} = \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}^{-T}. \tag{11.13.4}$$

The plastic incompressibility is assumed, so that

$$\det \mathbf{F}^{\mathrm{e}} = \det \mathbf{F}. \tag{11.13.5}$$

The two moduli tensors are defined by

$$\boldsymbol{\Lambda}^{\mathrm{e}}_{(1)} = \frac{\partial^2 \Psi^{\mathrm{e}}}{\partial \mathbf{E}^{\mathrm{e}} \otimes \partial \mathbf{E}^{\mathrm{e}}}, \quad \boldsymbol{\Lambda}_{(1)} = \frac{\partial^2 \Psi^{\mathrm{e}}}{\partial \mathbf{E} \otimes \partial \mathbf{E}}, \tag{11.13.6}$$

such that

$$\boldsymbol{\Lambda}_{(1)} = \mathbf{F}^{\mathrm{p}-1}\,\mathbf{F}^{\mathrm{p}-1}\,\boldsymbol{\Lambda}^{\mathrm{e}}_{(1)}\,\mathbf{F}^{\mathrm{p}-T}\,\mathbf{F}^{\mathrm{p}-T}. \tag{11.13.7}$$

In addition, the moduli tensor $\boldsymbol{\mathcal{L}}_{(1)}$ is

$$\boldsymbol{\mathcal{L}}_{(1)} = \mathbf{F}^{\mathrm{e}}\,\mathbf{F}^{\mathrm{e}}\,\boldsymbol{\Lambda}^{\mathrm{e}}_{(1)}\,\mathbf{F}^{\mathrm{e}T}\,\mathbf{F}^{\mathrm{e}T} = \mathbf{F}\,\mathbf{F}\,\boldsymbol{\Lambda}_{(1)}\,\mathbf{F}^{T}\,\mathbf{F}^{T}. \tag{11.13.8}$$

The tensor products are here defined as in Eq. (11.6.3).

By differentiating the first expression in (11.13.2), there follows

$$\dot{\mathbf{T}}^{\mathrm{e}} = \mathbf{\Lambda}^{\mathrm{e}}_{(1)} : \dot{\mathbf{E}}^{\mathrm{e}}, \tag{11.13.9}$$

while differentiation of Eq. (11.13.3) gives

$$\dot{\mathbf{T}}^{\mathrm{e}} = \mathbf{F}^{\mathrm{p}} \cdot \left(\dot{\mathbf{T}} + \mathbf{Z}^{\mathrm{p}} \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{Z}^{\mathrm{p}T}\right) \cdot \mathbf{F}^{\mathrm{p}T}. \tag{11.13.10}$$

The second-order tensor $\mathbf{Z}^{\mathrm{p}}$ is

$$\mathbf{Z}^{\mathrm{p}} = \mathbf{F}^{\mathrm{p}-1} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \mathbf{F}^{\mathrm{p}}. \tag{11.13.11}$$

Since, from Eq. (11.3.7),

$$\dot{\mathbf{E}}^{\mathrm{e}} = \mathbf{F}^{\mathrm{p}-T} \cdot \left\{\dot{\mathbf{E}} - \mathbf{F}^{\mathrm{p}T} \cdot \left[\mathbf{C}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right)\right]_{\mathrm{s}} \cdot \mathbf{F}^{\mathrm{p}}\right\} \cdot \mathbf{F}^{\mathrm{p}-1}, \tag{11.13.12}$$

the substitution of Eqs. (11.13.10) and (11.13.12) into Eq. (11.13.9) yields

$$\dot{\mathbf{T}} = \mathbf{\Lambda}_{(1)} : \left\{\dot{\mathbf{E}} - \mathbf{F}^{\mathrm{p}T} \cdot \left[\mathbf{C}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right)\right]_{\mathrm{s}} \cdot \mathbf{F}^{\mathrm{p}}\right\} - \left(\mathbf{Z}^{\mathrm{p}} \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{Z}^{\mathrm{p}T}\right). \tag{11.13.13}$$

The elastic part of the rate of Lagrangian strain is defined by

$$(\dot{\mathbf{E}})^{\mathrm{e}} = \mathbf{\Lambda}^{-1}_{(1)} : \dot{\mathbf{T}}. \tag{11.13.14}$$

Consequently, upon partitioning the total rate of strain as (Fig. 11.2)

$$\dot{\mathbf{E}} = (\dot{\mathbf{E}})^{\mathrm{e}} + (\dot{\mathbf{E}})^{\mathrm{p}}, \tag{11.13.15}$$

we identify from Eq. (11.13.13) the plastic part of the rate of Lagrangian strain as

$$(\dot{\mathbf{E}})^{\mathrm{p}} = \mathbf{F}^{\mathrm{p}T} \cdot \left[\mathbf{C}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right)\right]_{\mathrm{s}} \cdot \mathbf{F}^{\mathrm{p}} + \mathbf{\Lambda}^{-1}_{(1)} : \left(\mathbf{Z}^{\mathrm{p}} \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{Z}^{\mathrm{p}T}\right). \tag{11.13.16}$$

The elastic part is then

$$(\dot{\mathbf{E}})^{\mathrm{e}} = \mathbf{F}^{\mathrm{p}T} \cdot \dot{\mathbf{E}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}} - \mathbf{\Lambda}^{-1}_{(1)} : \left(\mathbf{Z}^{\mathrm{p}} \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{Z}^{\mathrm{p}T}\right), \tag{11.13.17}$$

where

$$\dot{\mathbf{E}}^{\mathrm{e}} = \mathbf{F}^{\mathrm{e}T} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} \cdot \mathbf{F}^{\mathrm{e}}. \tag{11.13.18}$$

It can be easily verified that the expressions (11.13.16) and(11.13.17) agree with the expressions (11.12.11), provided that $\mathbf{D}^{\mathrm{e}}_{(1)}$ and $\mathbf{D}^{\mathrm{p}}_{(1)}$ are defined by Eqs. (11.12.7) and (11.12.8).

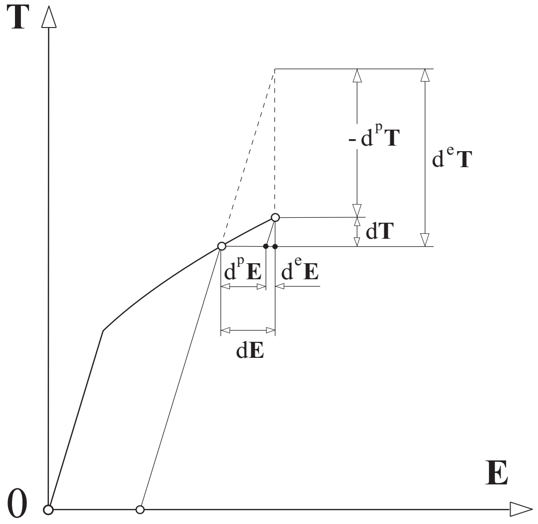

FIGURE 11.2. Geometric interpretation of the partition of the stress and strain increments into their elastic and plastic parts.

11.14. Partition of the Rate of Deformation Gradient

The rate of deformation gradient $\dot{\mathbf{F}}$ can also be partitioned into its elastic and plastic parts,

$$\dot{\mathbf{F}} = (\dot{\mathbf{F}})^{\mathrm{e}} + (\dot{\mathbf{F}})^{\mathrm{p}}. \tag{11.14.1}$$

The elastic part is defined by

$$(\dot{\mathbf{F}})^{\mathrm{e}} = \boldsymbol{\Lambda}^{-1} \cdot\cdot\, \dot{\mathbf{P}}. \tag{11.14.2}$$

It is assumed that the elastic pseudomoduli tensor $\boldsymbol{\Lambda}$ has its inverse, the elastic pseudocompliances tensor $\boldsymbol{\Lambda}^{-1}$, such that

$$\boldsymbol{\Lambda} \cdot\cdot\, \boldsymbol{\Lambda}^{-1} = \boldsymbol{\Lambda}^{-1} \cdot\cdot\, \boldsymbol{\Lambda} = \boldsymbol{I}, \tag{11.14.3}$$

where $I_{ijkl} = \delta_{il}\delta_{jk}$ (with the components of $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}^{-1}$ expressed in the same rectangular coordinate system).

In the derivation, first note that the elastic nominal stress and the overall nominal stress,

$$\mathbf{P}^{\mathrm{e}} = \mathbf{T}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}T}, \quad \mathbf{P} = \mathbf{T} \cdot \mathbf{F}^{T}, \tag{11.14.4}$$

are derived from the elastic strain energy Ψ^{e} as (Lubarda and Benson, 2001)

$$\mathbf{P}^{\mathrm{e}} = \frac{\partial \Psi^{\mathrm{e}}}{\partial \mathbf{F}^{\mathrm{e}}}\,, \quad \mathbf{P} = \frac{\partial \Psi^{\mathrm{e}}}{\partial \mathbf{F}}\,. \tag{11.14.5}$$

The connection between the two tensors is

$$\mathbf{P}^{\mathrm{e}} = \mathbf{F}^{\mathrm{p}} \cdot \mathbf{P}. \tag{11.14.6}$$

The corresponding pseudomoduli tensors are

$$\mathbf{\Lambda}^{\mathrm{e}} = \frac{\partial^2 \Psi^{\mathrm{e}}}{\partial \mathbf{F}^{\mathrm{e}} \otimes \partial \mathbf{F}^{\mathrm{e}}}\,, \quad \mathbf{\Lambda} = \frac{\partial^2 \Psi^{\mathrm{e}}}{\partial \mathbf{F} \otimes \partial \mathbf{F}}\,. \tag{11.14.7}$$

It can be readily verified by partial differentiation that the components of the two pseudomoduli tensors (in the same rectangular coordinate system) are related by

$$\Lambda^{\mathrm{e}}_{ijkl} = F^{\mathrm{p}}_{im}\,\Lambda_{mjnl}\,F^{\mathrm{p}}_{kn}\,. \tag{11.14.8}$$

The pseudomoduli tensor $\mathbf{\Lambda}^{\mathrm{e}}$ appears in the expression

$$\dot{\mathbf{P}}^{\mathrm{e}} = \mathbf{\Lambda}^{\mathrm{e}} \cdot\cdot\, \dot{\mathbf{F}}^{\mathrm{e}}. \tag{11.14.9}$$

By differentiating Eq. (11.14.6), there follows

$$\dot{\mathbf{P}}^{\mathrm{e}} = \mathbf{F}^{\mathrm{p}} \cdot \dot{\mathbf{P}} + \dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{P}. \tag{11.14.10}$$

Substitution of Eqs. (11.14.10) and (11.14.8) into Eq. (11.14.9) gives

$$\dot{\mathbf{P}} = \mathbf{\Lambda} \cdot\cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}}\right) - \mathbf{F}^{\mathrm{p}-1} \cdot \dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{P}. \tag{11.14.11}$$

On the other hand, by differentiating the multiplicative decomposition $\mathbf{F} = \mathbf{F}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}}$, the rate of deformation gradient is

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}} + \mathbf{F}^{\mathrm{e}} \cdot \dot{\mathbf{F}}^{\mathrm{p}}. \tag{11.14.12}$$

Using this, Eq. (11.14.11) can be rewritten as

$$\dot{\mathbf{P}} = \mathbf{\Lambda} \cdot\cdot \left(\dot{\mathbf{F}} - \mathbf{F}^{\mathrm{e}} \cdot \dot{\mathbf{F}}^{\mathrm{p}}\right) - \mathbf{F}^{\mathrm{p}-1} \cdot \dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{P}, \tag{11.14.13}$$

i.e.,

$$\dot{\mathbf{P}} = \mathbf{\Lambda} \cdot\cdot \left[\dot{\mathbf{F}} - \mathbf{F}^{\mathrm{e}} \cdot \dot{\mathbf{F}}^{\mathrm{p}} - \mathbf{\Lambda}^{-1} \cdot\cdot \left(\mathbf{F}^{\mathrm{p}-1} \cdot \dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{P}\right)\right]. \tag{11.14.14}$$

From Eq. (11.14.14) we now identify the plastic part of the rate of deformation gradient as

$$(\dot{\mathbf{F}})^{\mathrm{p}} = \mathbf{F}^{\mathrm{e}} \cdot \dot{\mathbf{F}}^{\mathrm{p}} + \mathbf{\Lambda}^{-1} \cdot\cdot \left(\mathbf{F}^{\mathrm{p}-1} \cdot \dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{P}\right). \tag{11.14.15}$$

The remaining part of $\dot{\mathbf{F}}$ is the elastic part,

$$(\dot{\mathbf{F}})^{\mathrm{e}} = \dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}} - \boldsymbol{\Lambda}^{-1} \cdot\cdot \left(\mathbf{F}^{\mathrm{p}-1} \cdot \dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{P}\right), \tag{11.14.16}$$

complying with Eq. (11.14.2).

Equation (11.14.14) also serves to identify the elastic and plastic parts of the rate of nominal stress. These are

$$(\dot{\mathbf{P}})^{\mathrm{e}} = \boldsymbol{\Lambda} \cdot\cdot\, \dot{\mathbf{F}}, \tag{11.14.17}$$

$$(\dot{\mathbf{P}})^{\mathrm{p}} = -\left[\mathbf{F}^{\mathrm{p}-1} \cdot \dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{P} + \boldsymbol{\Lambda} \cdot\cdot \left(\mathbf{F}^{\mathrm{e}} \cdot \dot{\mathbf{F}}^{\mathrm{p}}\right)\right], \tag{11.14.18}$$

such that

$$\dot{\mathbf{P}} = (\dot{\mathbf{P}})^{\mathrm{e}} + (\dot{\mathbf{P}})^{\mathrm{p}}. \tag{11.14.19}$$

Evidently, by comparing Eqs. (11.14.15) and (11.14.18), there is a relationship between plastic parts of the rate of nominal stress and deformation gradient,

$$(\dot{\mathbf{P}})^{\mathrm{p}} = -\boldsymbol{\Lambda} \cdot\cdot\, (\dot{\mathbf{F}})^{\mathrm{p}}. \tag{11.14.20}$$

11.15. Relationship between $(\dot{\mathbf{P}})^{\mathrm{p}}$ and $(\dot{\mathbf{T}})^{\mathrm{p}}$

To derive the relationship between plastic parts of the rates of nominal and symmetric Piola–Kirchhoff stress,

$$(\dot{\mathbf{P}})^{\mathrm{p}} = \dot{\mathbf{P}} - \boldsymbol{\Lambda} \cdot\cdot\, \dot{\mathbf{F}}, \quad (\dot{\mathbf{T}})^{\mathrm{p}} = \dot{\mathbf{T}} - \boldsymbol{\Lambda}_{(1)} : \dot{\mathbf{E}}, \tag{11.15.1}$$

we first recall the relationships between $\dot{\mathbf{P}}$ and $\dot{\mathbf{T}}$, and $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}_{(1)}$, which were derived in Section 6.4. Following Hill (1984), these can be conveniently cast as

$$\boldsymbol{\Lambda} = \boldsymbol{\mathcal{K}}^T : \boldsymbol{\Lambda}_{(1)} : \boldsymbol{\mathcal{K}} + \boldsymbol{\mathcal{T}}, \quad \dot{\mathbf{P}} = \boldsymbol{\mathcal{K}}^T : \dot{\mathbf{T}} + \boldsymbol{\mathcal{T}} \cdot\cdot\, \dot{\mathbf{F}}. \tag{11.15.2}$$

The tensor $\boldsymbol{\Lambda}_{(1)}$ possesses the reciprocal symmetry $ij \leftrightarrow kl$. The rectangular components of the fourth-order tensors $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{T}}$ are

$$\mathcal{K}_{ijkl} = \frac{1}{2}\left(\delta_{ik} F_{lj} + \delta_{jk} F_{li}\right), \quad \mathcal{T}_{ijkl} = T_{ik}\delta_{jl}. \tag{11.15.3}$$

They obey the symmetry

$$\mathcal{K}_{ijkl} = \mathcal{K}_{jikl}, \quad \mathcal{T}_{ijkl} = \mathcal{T}_{klij}. \tag{11.15.4}$$

The tensor $\boldsymbol{\mathcal{K}}$ is particularly convenient, because in the trace operation with a second-order tensor $\mathbf{A}$ it behaves such that

$$\boldsymbol{\mathcal{K}} \cdot\cdot\, \mathbf{A} = \mathbf{A} \cdot\cdot\, \boldsymbol{\mathcal{K}}^T = \frac{1}{2}\,(\mathbf{F}^T \cdot \mathbf{A} + \mathbf{A}^T \cdot \mathbf{F}), \tag{11.15.5}$$

$$\boldsymbol{\mathcal{K}}^T \cdot\cdot\, \mathbf{A} = \mathbf{A} \cdot\cdot\, \boldsymbol{\mathcal{K}} = \frac{1}{2}\,(\mathbf{A} + \mathbf{A}^T) \cdot \mathbf{F}^T. \tag{11.15.6}$$

In particular,

$$\boldsymbol{\mathcal{K}} \cdot\cdot\, \dot{\mathbf{F}} = \dot{\mathbf{F}} \cdot\cdot\, \boldsymbol{\mathcal{K}}^T = \dot{\mathbf{E}}, \tag{11.15.7}$$

$$\boldsymbol{\mathcal{K}}^T : \mathbf{T} = \mathbf{T} : \boldsymbol{\mathcal{K}} = \mathbf{T} \cdot \mathbf{F}^T = \mathbf{P}. \tag{11.15.8}$$

If $\mathbf{A}$ is symmetric, the trace product $\cdot\cdot$ can be replaced by $:$ product in Eqs. (11.15.5) and (11.15.6).

The relationship between $(\dot{\mathbf{P}})^{\mathrm{p}}$ and $(\dot{\mathbf{T}})^{\mathrm{p}}$ now follows by taking the trace product of the second equation in (11.15.1) with $\boldsymbol{\mathcal{K}}^T$ from the left. Upon using Eq. (11.15.2), this gives

$$(\dot{\mathbf{P}})^{\mathrm{p}} = \boldsymbol{\mathcal{K}}^T : (\dot{\mathbf{T}})^{\mathrm{p}}. \tag{11.15.9}$$

Since

$$(\dot{\mathbf{P}})^{\mathrm{p}} = -\boldsymbol{\Lambda} \cdot\cdot\, (\dot{\mathbf{F}})^{\mathrm{p}}, \quad (\dot{\mathbf{T}})^{\mathrm{p}} = -\boldsymbol{\Lambda}_{(1)} : (\dot{\mathbf{E}})^{\mathrm{p}}, \tag{11.15.10}$$

we, in addition, have

$$(\dot{\mathbf{F}})^{\mathrm{p}} = \boldsymbol{\Lambda}^{-1} \cdot\cdot\, \boldsymbol{\mathcal{K}}^T : \boldsymbol{\Lambda}_{(1)} : (\dot{\mathbf{E}})^{\mathrm{p}}. \tag{11.15.11}$$

Note that

$$\dot{\mathbf{F}} \cdot\cdot\, (\dot{\mathbf{P}})^{\mathrm{p}} = \dot{\mathbf{E}} : (\dot{\mathbf{T}})^{\mathrm{p}}, \tag{11.15.12}$$

which directly follows by taking the trace product of (11.15.9) with $\dot{\mathbf{F}}$ from the left, and by using Eq. (11.15.7).

11.16. Normality Properties

If increments rather than rates are used, we can write Eq. (11.15.12) as

$$\mathrm{d}\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P} = \mathrm{d}\mathbf{E} : \mathrm{d}^{\mathrm{p}}\mathbf{T}. \tag{11.16.1}$$

An analogous expression holds when the increments of $\mathbf{F}$ and $\mathbf{E}$ are used along an unloading elastic branch of the response, i.e.,

$$\delta\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P} = \delta\mathbf{E} : \mathrm{d}^{\mathrm{p}}\mathbf{T}. \tag{11.16.2}$$

If this is positive, we say that the material complies with the normality rule in strain space. Since

$$\mathrm{d}^{\mathrm{p}}\mathbf{P} = -\boldsymbol{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F}, \quad \mathrm{d}^{\mathrm{p}}\mathbf{T} = -\boldsymbol{\Lambda}_{(1)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}, \tag{11.16.3}$$

and

$$\delta\mathbf{P} = \boldsymbol{\Lambda} \cdot\cdot\, \delta\mathbf{F}, \quad \delta\mathbf{T} = \boldsymbol{\Lambda}_{(1)} : \delta\mathbf{E}, \tag{11.16.4}$$

the substitution into Eq. (11.16.2) yields a dual relationship

$$\delta\mathbf{P} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F} = \delta\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}. \tag{11.16.5}$$

When this is negative, the material complies with the normality rule in stress space. We recall from Section 8.5, if the material complies with Ilyushin's postulate of positive net work in an isothermal cycle of strain that involves plastic deformation, the quantity in (11.16.1) must be negative, i.e.,

$$\mathrm{d}\mathbf{F} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{P} = \mathrm{d}\mathbf{E} : \mathrm{d}^{\mathrm{p}}\mathbf{T} < 0. \tag{11.16.6}$$

Equation (11.16.1) does not have a dual relationship, since

$$\mathrm{d}\mathbf{P} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F} \neq \mathrm{d}\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}. \tag{11.16.7}$$

Instead, we can only write

$$\mathrm{d}\mathbf{F} \cdot\cdot\, \boldsymbol{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F} = \mathrm{d}\mathbf{E} : \boldsymbol{\Lambda}_{(1)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}, \tag{11.16.8}$$

or

$$\mathrm{d}\mathbf{P} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F} + \mathrm{d}^{\mathrm{p}}\mathbf{F} \cdot\cdot\, \boldsymbol{\Lambda} \cdot\cdot\, \mathrm{d}^{\mathrm{p}}\mathbf{F} = \mathrm{d}\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E} + \mathrm{d}^{\mathrm{p}}\mathbf{E} : \boldsymbol{\Lambda}_{(1)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}. \tag{11.16.9}$$

If the material is in the hardening range relative to the conjugate measures $\mathbf{E}$ and $\mathbf{T}$, the stress increment $\mathrm{d}\mathbf{T}$, producing plastic deformation $\mathrm{d}^{\mathrm{p}}\mathbf{E}$, is directed outside the yield surface, satisfying $\mathrm{d}\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E} > 0$. If the material is in the softening range, the stress increment producing plastic deformation is directed inside the yield surface, satisfying the reversed inequality. The quantity $\mathrm{d}\mathbf{T} : \mathrm{d}^{\mathrm{p}}\mathbf{E}$, however, is not invariant under the change of strain measure, and the material judged to be in the hardening range relative to one pair of the conjugate stress and strain measures, may be in the softening range relative to another pair.

As an illustration, consider a uniaxial tension of an incompressible rigid-plastic material whose response in the Cauchy stress vs. logarithmic strain

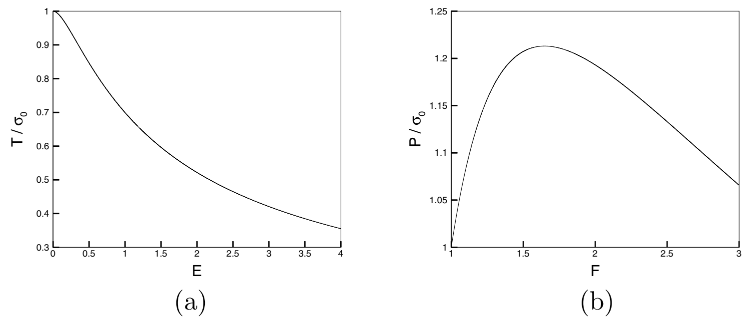

FIGURE 11.3. (a) Piola–Kirchhoff vs. Lagrangian strain, and (b) nominal stress vs. deformation gradient in uniaxial tension of rigid-plastic material with the constant hardening modulus $k = 2\sigma_0$ relative to the Cauchy stress and logarithmic strain measures.

space is described by the linear hardening

$$\sigma = k \, \ln \lambda + \sigma_0. \tag{11.16.10}$$

The constant rate of hardening is k, the initial yield stress is σ_0, and λ is the longitudinal stretch ratio. The corresponding response observed relative to $T = \sigma/\lambda^2$ vs. $E = (\lambda^2 - 1)/2$ measures is

$$T = \frac{1}{1+2E} \left[\sigma_0 + \frac{1}{2} k \, \ln(1+2E) \right]. \tag{11.16.11}$$

A transition from the hardening to softening occurs at

$$E_0 = \frac{1}{2} \left[\exp\left(1 - \frac{2\sigma_0}{k}\right) - 1 \right]. \tag{11.16.12}$$

The response observed relative to $P = \sigma/\lambda$ vs. $F = \lambda$ measures is

$$P = \frac{1}{F} \, (\sigma_0 + k \, \ln F). \tag{11.16.13}$$

A transition from the hardening to softening occurs at

$$F_0 = \exp\left(1 - \frac{\sigma_0}{k}\right). \tag{11.16.14}$$

For example, for $k = 2\sigma_0$ the softening commences at $F_0 = \sqrt{e}$ in the nominal stress vs. deformation gradient, and from the onset of deformation in the Piola–Kirchhoff stress vs. Lagrangian strain measures ($E_0 = 0$). However, a necking of the specimen begins when $\mathrm{d}\sigma/\mathrm{d}(\ln \lambda) = \sigma$, i.e., when

$\lambda = \exp(1 - \sigma_0/k)$. Thus, the necking takes place when

$$\ln \lambda_* = 1 - \frac{\sigma_0}{k}, \quad 2E_* = \exp\left[2\left(1 - \frac{\sigma_0}{k}\right)\right] - 1, \quad F_* = \exp\left(1 - \frac{\sigma_0}{k}\right). \tag{11.16.15}$$

For $k = 2\sigma_0$, this gives

$$\ln \lambda_* = \frac{1}{2} \quad E_* = \frac{1}{2}(e - 1), \quad F_* = \sqrt{e}. \tag{11.16.16}$$

While the onset of softening coincides with the onset of necking when (P, F) measures are used, with $k = 2\sigma_0$ the necking occurs in the softening range relative to (T, E) measures, and in the hardening range relative to $(\sigma, \ln \lambda)$ measures (Fig. 11.3).

11.17. Elastoplastic Deformation of Orthotropic Materials

11.17.1. Principal Axes of Orthotropy

Consider an elastically orthotropic material in its undeformed configuration $\mathcal{B}^0$. Let the unit vectors $\mathbf{a}_i^0$ $(i = 1, 2, 3)$ define the corresponding principal axes of orthotropy (Fig. 11.4). The elastic strain energy function can be most conveniently expressed in the coordinate system with the axes parallel to $\mathbf{a}_i^0$. Denote this representation by

$$\Psi^{\mathrm{e}} = \Psi^{\mathrm{e}}(\mathbf{E}^{\mathrm{e}}), \tag{11.17.1}$$

where $\mathbf{E}^{\mathrm{e}}$ is the Lagrangian strain of purely elastic deformation from $\mathcal{B}^0$. If it is assumed that the material remains orthotropic during elastoplastic deformation, the principal axes of orthotropy in the intermediate configuration $\mathcal{B}^{\mathrm{p}}$ are defined by the unit vectors

$$\hat{\mathbf{a}}_i = \boldsymbol{\mathcal{R}} \cdot \mathbf{a}_i^0, \tag{11.17.2}$$

where $\boldsymbol{\mathcal{R}}$ is an orthogonal rotation tensor. The elastic strain energy relative to the unstressed intermediate configuration, expressed in the original coordinate system with the axes parallel to $\mathbf{a}_i^0$, is (Lubarda, 1991b)

$$\Psi^{\mathrm{e}} = \Psi^{\mathrm{e}}(\boldsymbol{\mathcal{R}}^T \cdot \mathbf{E}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{R}}). \tag{11.17.3}$$

The function Ψ^{e} here is the same function as that used in Eq. (11.17.1) to describe elastic response from the initial undeformed configuration, but its arguments are the components of the rotated strain tensor

$$\hat{\mathbf{E}}^{\mathrm{e}} = \boldsymbol{\mathcal{R}}^T \cdot \mathbf{E}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{R}}, \quad \mathbf{E}^{\mathrm{e}} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{e}T} \cdot \mathbf{F}^{\mathrm{e}} - \mathbf{I}\right). \tag{11.17.4}$$

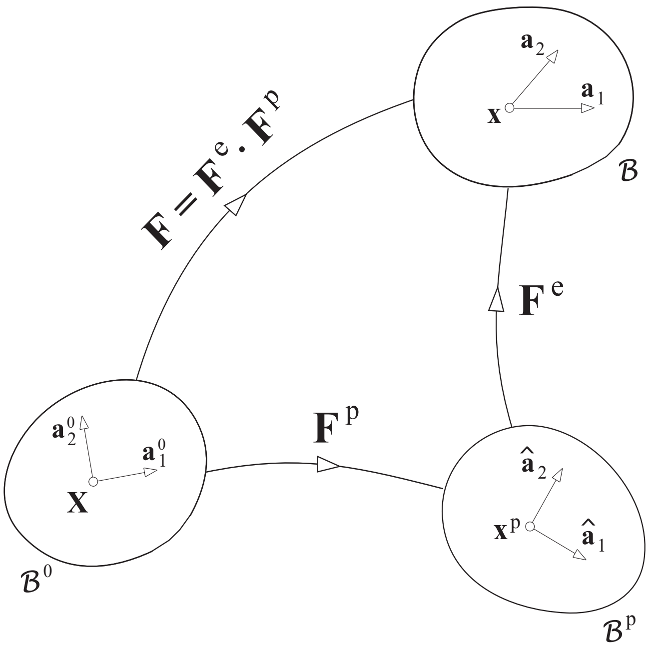

FIGURE 11.4. Multiplicative decomposition of deformation gradient for an orthotropic material. Principal directions of orthotropy $\mathbf{a}_i^0$ in the initial configuration $\mathcal{B}^0$ are rotated to $\hat{\mathbf{a}}_i = \boldsymbol{\mathcal{R}} \cdot \mathbf{a}_i^0$ in the intermediate configuration $\mathcal{B}^{\mathrm{p}}$. They are then convected to the current configuration $\mathcal{B}$ by elastic deformation such that $\mathbf{a}_i = \mathbf{F}^{\mathrm{e}} \cdot \hat{\mathbf{a}}_i$.

The components of the strain tensor $\mathbf{E}^{\mathrm{e}}$, observed in the coordinate system with the axes parallel to $\mathbf{a}_i^0$, are numerically equal to the components of the strain tensor $\hat{\mathbf{E}}^{\mathrm{e}}$, observed in the coordinate system with the axes parallel to $\hat{\mathbf{a}}_i$.

Due to possible discontinuities in displacements and rotations of the material elements at the microscale, caused by plastic deformation, the rotation tensor $\boldsymbol{\mathcal{R}}$ is in general not specified by the overall plastic deformation gradient $\mathbf{F}^{\mathrm{p}}$. In particular, the vectors $\mathbf{a}_i^0$ are not simply convected with the material in the transformation from $\mathcal{B}^0$ to $\mathcal{B}^{\mathrm{p}}$. In contrast, the unit vectors $\hat{\mathbf{a}}_i$ can be considered as embedded in the material during the elastic deformation from $\mathcal{B}^{\mathrm{p}}$ to $\mathcal{B}$. Thus, they become

$$\mathbf{a}_i = \mathbf{F}^{\mathrm{e}} \cdot \hat{\mathbf{a}}_i = \mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\mathcal{R}} \cdot \mathbf{a}_i^0 \tag{11.17.5}$$

in the elastoplastically deformed configuration $\mathcal{B}$. By differentiation, their rate of change is

$$\dot{\mathbf{a}}_i = \left[\dot{\mathbf{F}}^{\rm e} \cdot \mathbf{F}^{{\rm e}-1} + \mathbf{F}^{\rm e} \cdot \left(\dot{\boldsymbol{\mathcal{R}}} \cdot \boldsymbol{\mathcal{R}}^{-1}\right) \cdot \mathbf{F}^{{\rm e}-1}\right] \cdot \mathbf{a}_i. \tag{11.17.6}$$

If $\mathbf{R}^{\rm e}$ is the rotation tensor from the polar decomposition

$$\mathbf{F}^{\rm e} = \mathbf{V}^{\rm e} \cdot \mathbf{R}^{\rm e} = \mathbf{R}^{\rm e} \cdot \mathbf{U}^{\rm e}, \tag{11.17.7}$$

we have

$$\mathbf{a}_i = \mathbf{V}^{\rm e} \cdot \boldsymbol{\mathcal{R}}^{\rm e} \cdot \mathbf{a}_i^0, \quad \boldsymbol{\mathcal{R}}^{\rm e} = \mathbf{R}^{\rm e} \cdot \boldsymbol{\mathcal{R}}. \tag{11.17.8}$$

While both $\boldsymbol{\mathcal{R}}$ and $\mathbf{R}^{\rm e}$ depend on the choice of the intermediate configuration (superposed rotation $\hat{\mathbf{Q}}$), so that

$$\boldsymbol{\mathcal{R}}^* = \hat{\mathbf{Q}} \cdot \boldsymbol{\mathcal{R}}, \quad \mathbf{R}^{{\rm e}*} = \mathbf{R}^{\rm e} \cdot \hat{\mathbf{Q}}^T, \tag{11.17.9}$$

the rotation $\boldsymbol{\mathcal{R}}^{\rm e}$ is a unique quantity, independent of $\hat{\mathbf{Q}}$, i.e.,

$$\boldsymbol{\mathcal{R}}^{{\rm e}*} = \boldsymbol{\mathcal{R}}^{\rm e}. \tag{11.17.10}$$

Apart from rotation of the material elements caused by elastic stretching $\mathbf{V}^{\rm e}$, the directions of the principal axes of orthotropy in the deformed configuration $\mathcal{B}$ are completely specified by the rotation tensor $\boldsymbol{\mathcal{R}}^{\rm e}$.

11.17.2. Partition of the Rate of Deformation

The stress response from $\mathcal{B}^{\rm p}$ to $\mathcal{B}$ is given by

$$\boldsymbol{\tau} = \mathbf{F}^{\rm e} \cdot \frac{\partial \Psi^{\rm e}(\hat{\mathbf{E}}^{\rm e})}{\partial \mathbf{E}^{\rm e}} \cdot \mathbf{F}^{{\rm e}T}, \tag{11.17.11}$$

where $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$ is the Kirchhoff stress. The elastic strain energy function is reckoned per unit unstressed volume, and plastic deformation is assumed to be incompressible. Upon differentiation, we obtain

$$\begin{aligned} \dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{F}}^{\rm e} \cdot \mathbf{F}^{{\rm e}-1}\right) \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}^{\rm e} \cdot \mathbf{F}^{{\rm e}-1}\right)^T &= \boldsymbol{\mathcal{L}}_{(1)} : \left(\dot{\mathbf{F}}^{\rm e} \cdot \mathbf{F}^{{\rm e}-1}\right)_{\rm s} \\ &+ \mathbf{F}^{\rm e} \cdot \left(\frac{\partial^2 \Psi^{\rm e}}{\partial \mathbf{E}^{\rm e} \otimes \partial \boldsymbol{\mathcal{R}}} \cdot \cdot \dot{\boldsymbol{\mathcal{R}}}\right) \cdot \mathbf{F}^{{\rm e}T}. \end{aligned} \tag{11.17.12}$$

The rectangular components of $\boldsymbol{\mathcal{L}}_{(1)}$ are

$$\mathcal{L}^{(1)}_{ijkl} = F^{\rm e}_{iM} F^{\rm e}_{jN} \frac{\partial^2 \Psi^{\rm e}(\hat{\mathbf{E}}^{\rm e})}{\partial E^{\rm e}_{MN}\, \partial E^{\rm e}_{PQ}} F^{\rm e}_{kP} F^{\rm e}_{lQ}. \tag{11.17.13}$$

The last term on the right-hand side of Eq. (11.17.12) can be conveniently rewritten as

$$\mathbf{F}^{\mathrm{e}} \cdot \left(\frac{\partial^2 \Psi^{\mathrm{e}}}{\partial \mathbf{E}^{\mathrm{e}} \otimes \partial \boldsymbol{\mathcal{R}}} \cdot \cdot \dot{\boldsymbol{\mathcal{R}}} \right) \cdot \mathbf{F}^{\mathrm{e}T} = -\boldsymbol{\mathcal{L}}_{(1)} : \mathbf{Z}_{\mathrm{s}} - \boldsymbol{\tau} \cdot \mathbf{Z} - \mathbf{Z}^T \cdot \boldsymbol{\tau}, \qquad (11.17.14)$$

where

$$\mathbf{Z} = \mathbf{F}^{\mathrm{e}-T} \cdot \left(\dot{\boldsymbol{\mathcal{R}}} \cdot \boldsymbol{\mathcal{R}}^{-1} \right) \cdot \mathbf{F}^{\mathrm{e}T}. \qquad (11.17.15)$$

In the transition, the following expressions were utilized

$$\frac{\partial \Psi^{\mathrm{e}}}{\partial \boldsymbol{\mathcal{R}}} = 2\, \boldsymbol{\mathcal{R}}^T \cdot \frac{\partial \Psi^{\mathrm{e}}}{\partial \mathbf{E}^{\mathrm{e}}} \cdot \mathbf{E}^{\mathrm{e}}, \qquad (11.17.16)$$

$$\frac{\partial^2 \Psi^{\mathrm{e}}}{\partial E^{\mathrm{e}}_{IJ}\, \partial \mathcal{R}_{KL}} = \mathcal{R}_{ML} \left(2\, \frac{\partial^2 \Psi^{\mathrm{e}}}{\partial E^{\mathrm{e}}_{IJ}\, \partial E^{\mathrm{e}}_{MN}}\, E^{\mathrm{e}}_{NK} + \frac{\partial \Psi^{\mathrm{e}}}{\partial E^{\mathrm{e}}_{MI}}\, \delta_{JK} + \frac{\partial \Psi^{\mathrm{e}}}{\partial E^{\mathrm{e}}_{MJ}}\, \delta_{IK} \right). \qquad (11.17.17)$$

The right-hand side of Eq. (11.17.14) is also equal to

$$-\boldsymbol{\mathcal{L}}_{(1)} : \mathbf{Z}_{\mathrm{s}} - \boldsymbol{\tau} \cdot \mathbf{Z} - \mathbf{Z}^T \cdot \boldsymbol{\tau} = -\boldsymbol{\mathcal{L}}_{(0)} : \mathbf{Z}_{\mathrm{s}} - \boldsymbol{\tau} \cdot \mathbf{Z}_{\mathrm{a}} + \mathbf{Z}_{\mathrm{a}} \cdot \boldsymbol{\tau}. \qquad (11.17.18)$$

The components of the elastic moduli tensor $\boldsymbol{\mathcal{L}}_{(0)}$ are

$$\mathcal{L}^{(0)}_{ijkl} = \mathcal{L}^{(1)}_{ijkl} + \frac{1}{2}\,(\tau_{ik}\delta_{jl} + \tau_{jk}\delta_{il} + \tau_{il}\delta_{jk} + \tau_{jl}\delta_{ik}). \qquad (11.17.19)$$

Thus, Eq. (11.17.12) becomes

$$\dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} \right)_{\mathrm{a}} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} \right)_{\mathrm{a}} = \boldsymbol{\mathcal{L}}_{(0)} : \left[\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} \right)_{\mathrm{s}} - \mathbf{Z}_{\mathrm{s}} \right] + \mathbf{Z}_{\mathrm{a}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{Z}_{\mathrm{a}}. \qquad (11.17.20)$$

To proceed with the analysis, we recall from Eq. (11.3.4) that

$$\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} \right)_{\mathrm{a}} = \mathbf{W} - \boldsymbol{\omega}^{\mathrm{p}}, \quad \boldsymbol{\omega}^{\mathrm{p}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1} \right) \cdot \mathbf{F}^{\mathrm{e}-1} \right]_{\mathrm{a}}. \qquad (11.17.21)$$

When this is substituted into Eq. (11.17.20), there follows

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_{(0)} : \left[\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} \right)_{\mathrm{s}} - \mathbf{Z}_{\mathrm{s}} \right] - (\boldsymbol{\omega}^{\mathrm{p}} - \mathbf{Z}_{\mathrm{a}}) \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot (\boldsymbol{\omega}^{\mathrm{p}} - \mathbf{Z}_{\mathrm{a}}). \qquad (11.17.22)$$

Consequently, the elastic rate of deformation is given by

$$\mathbf{D}^{\mathrm{e}} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} \right)_{\mathrm{s}} - \mathbf{Z}_{\mathrm{s}} - \boldsymbol{\mathcal{L}}^{-1}_{(0)} : \left[(\boldsymbol{\omega}^{\mathrm{p}} - \mathbf{Z}_{\mathrm{a}}) \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot (\boldsymbol{\omega}^{\mathrm{p}} - \mathbf{Z}_{\mathrm{a}}) \right]. \qquad (11.17.23)$$

The remaining, plastic part of the rate of deformation is

$$\mathbf{D}^{\mathrm{p}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1} \right) \cdot \mathbf{F}^{\mathrm{e}-1} \right]_{\mathrm{s}} + \mathbf{Z}_{\mathrm{s}} + \boldsymbol{\mathcal{L}}^{-1}_{(0)} : \left[(\boldsymbol{\omega}^{\mathrm{p}} - \mathbf{Z}_{\mathrm{a}}) \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot (\boldsymbol{\omega}^{\mathrm{p}} - \mathbf{Z}_{\mathrm{a}}) \right]. \qquad (11.17.24)$$

The spin $(\boldsymbol{\omega}^{\rm p} - \mathbf{Z}_{\rm a})$, appearing in the previous equations, can be expressed from Eqs. (11.17.15) and (11.17.21) as

$$\boldsymbol{\omega}^{\rm p} - \mathbf{Z}_{\rm a} = \left[\mathbf{F}^{\rm e} \cdot \left(\dot{\mathbf{F}}^{\rm p} \cdot \mathbf{F}^{\rm p-1} - \dot{\boldsymbol{\mathcal{R}}} \cdot \boldsymbol{\mathcal{R}}^{-1}\right) \cdot \mathbf{F}^{\rm e-1}\right]_{\rm a}. \tag{11.17.25}$$

11.17.3. Isoclinic Intermediate Configuration

If intermediate configuration is specified by

$$\boldsymbol{\mathcal{R}} = \mathbf{I}, \quad \text{i.e.,} \quad \hat{\mathbf{a}}_i = \mathbf{a}_i^0, \tag{11.17.26}$$

it is referred to as an isoclinic intermediate configuration. The terminology is originally due to Mandel (1973). For an isoclinic intermediate configuration, therefore,

$$\mathbf{a}_i = \mathbf{F}^{\rm e} \cdot \mathbf{a}_i^0 = \mathbf{V}^{\rm e} \cdot \mathbf{R}^{\rm e} \cdot \mathbf{a}_i^0. \tag{11.17.27}$$

If the rotation $\mathbf{R}^{\rm e}$ is determined by the integration from an appropriately constructed constitutive expression for the spin

$$\boldsymbol{\Omega}^{\rm e} = \dot{\mathbf{R}}^{\rm e} \cdot \mathbf{R}^{\rm e-1}, \tag{11.17.28}$$

the stress response and the elastic moduli of an orthotropic material are derived from

$$\boldsymbol{\tau} = 2\,\mathbf{V}^{\rm e} \cdot \frac{\partial \Psi^{\rm e}(\mathbf{R}^{{\rm e}T} \cdot \mathbf{B}^{\rm e} \cdot \mathbf{R}^{\rm e})}{\partial \mathbf{B}^{\rm e}} \cdot \mathbf{V}^{\rm e}, \tag{11.17.29}$$

$$\mathcal{L}^{(1)}_{ijkl} = 4V^{\rm e}_{im}V^{\rm e}_{jn}\,\frac{\partial^2 \Psi^{\rm e}(\mathbf{R}^{{\rm e}T} \cdot \mathbf{B}^{\rm e} \cdot \mathbf{R}^{\rm e})}{\partial B^{\rm e}_{mn}\,\partial B^{\rm e}_{pq}}\,V^{\rm e}_{kp}V^{\rm e}_{lq}, \tag{11.17.30}$$

in terms of $\mathbf{V}^{\rm e}$ and $\mathbf{R}^{\rm e}$.

Since $\dot{\boldsymbol{\mathcal{R}}} = \mathbf{0}$ for an isoclinic intermediate configuration, we have $\mathbf{Z} = \mathbf{0}$ in Eq. (11.17.15). Consequently, from Eqs. (11.17.23) and (11.17.24), the elastic and plastic parts of the rate of deformation become

$$\mathbf{D}^{\rm e} = \left(\dot{\mathbf{F}}^{\rm e} \cdot \mathbf{F}^{\rm e-1}\right)_{\rm s} - \boldsymbol{\mathcal{L}}^{-1}_{(0)} : (\boldsymbol{\omega}^{\rm p} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\rm p}), \tag{11.17.31}$$

$$\mathbf{D}^{\rm p} = \left[\mathbf{F}^{\rm e} \cdot \left(\dot{\mathbf{F}}^{\rm p} \cdot \mathbf{F}^{\rm p-1}\right) \cdot \mathbf{F}^{\rm e-1}\right]_{\rm s} + \boldsymbol{\mathcal{L}}^{-1}_{(0)} : (\boldsymbol{\omega}^{\rm p} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\rm p}). \tag{11.17.32}$$

In particular, if the principal directions of stress remain parallel to $\mathbf{a}_i^0$ during the deformation, the orientation of the principal directions of orthotropy are fixed, and $\mathbf{R}^{\rm e} = \mathbf{I}$.

11.17.4. Orthotropic Yield Criterion

The yield criterion of an orthotropic material can be constructed by using an orthotropic function of the rotated-axes components of the Cauchy stress, i.e.,

$$f(\hat{\boldsymbol{\sigma}}, k) = 0, \quad \hat{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{R}}^{\mathrm{e}T} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\mathcal{R}}^{\mathrm{e}}. \tag{11.17.33}$$

The scalar k specifies the current size of the yield surface. For isotropic hardening, this is a function of an equivalent or generalized plastic strain. Using Hill's (1948) orthotropic criterion, the function f can be expressed as

$$\begin{aligned} f = \big[& f_0(\hat{\sigma}_{22} - \hat{\sigma}_{33})^2 + g_0(\hat{\sigma}_{33} - \hat{\sigma}_{11})^2 + h_0(\hat{\sigma}_{11} - \hat{\sigma}_{22})^2 \\ & + 2l_0\, \hat{\sigma}_{23}^2 + 2m_0\, \hat{\sigma}_{31}^2 + 2n_0\, \hat{\sigma}_{12}^2 \big]^{1/2} - k. \end{aligned} \tag{11.17.34}$$

The plastic part of the rate of deformation is assumed to be normal to the yield surface, and given by

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{H} \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \boldsymbol{\sigma}} \right) : \overset{\circ}{\boldsymbol{\tau}}. \tag{11.17.35}$$

The scalar H is determined from the consistency condition $\dot{f} = 0$. For an isoclinic intermediate configuration,

$$\boldsymbol{\mathcal{R}}^{\mathrm{e}} = \mathbf{R}^{\mathrm{e}}, \quad \hat{\boldsymbol{\sigma}} = \mathbf{R}^{\mathrm{e}T} \cdot \boldsymbol{\sigma} \cdot \mathbf{R}^{\mathrm{e}}. \tag{11.17.36}$$

If the constitutive expression for the spin $\boldsymbol{\Omega}^{\mathrm{e}}$ is available, the rotation $\mathbf{R}^{\mathrm{e}}$ is determined by the integration from Eq. (11.17.28). Equation (11.17.35) then defines the plastic part of the rate of deformation for an orthotropic material.

Additional analysis of the yield criteria and constitutive theory for orthotropic materials is available in Hill (1979,1990,1993), Boehler (1982, 1987), Betten (1988), Ferron, Makkouk, and Morreale (1994), Steinmann, Miehe, and Stein (1996), and Vial-Edwards (1997). For an elastoplastic analysis of the transversely isotropic materials, see Aravas (1992).

11.18. Damage-Elastoplasticity

11.18.1. Damage Variables

If plastic deformation affects the elastic properties, which, for example, can happen due to grain (lattice) rotations in a polycrystalline metal sample and resulting crystallographic texture, additional variables need to be introduced in the constitutive framework to describe these changes. They are referred to as the damage variables. They describe a degradation of the elastic properties and their directional changes produced by plastic deformation. Damage variables may be scalars, vectors, second- or higher-order tensors. Derivation in this section will be restricted to damage variables that are either scalars, second- or fourth-order symmetric tensors, collectively denoted by $\mathbf{d}$.

Damage variables change only during plastic deformation, remaining unaltered by elastic unloading or reverse elastic loading, except for the elastic embedding which convects them with the material (Lubarda, 1994b). Thus, if a damage variable in the configuration $\mathcal{B}$ is $\mathbf{d}$, it becomes $\hat{\mathbf{d}}$ in the intermediate configuration $\mathcal{B}^{\mathrm{p}}$, where $\hat{\mathbf{d}}$ is induced from $\mathbf{d}$ by the elastic deformation $\mathbf{F}^{\mathrm{e}}$. For example, the induced damage variable can be defined by the transformation of a weighted contravariant or covariant type. For the second-order tensor these are

$$\hat{\mathbf{d}} = (\det \mathbf{F}^{\mathrm{e}})^{w}\, \mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{d} \cdot \mathbf{F}^{\mathrm{e}-T}, \quad \hat{\mathbf{d}} = (\det \mathbf{F}^{\mathrm{e}})^{-w}\, \mathbf{F}^{\mathrm{e}T} \cdot \mathbf{d} \cdot \mathbf{F}^{\mathrm{e}}, \qquad (11.18.1)$$

where w is the weight. Transformations of mixed type could also be considered. For the fourth-order tensors the weighted contravariant and covariant transformations are

$$\hat{\mathbf{d}} = (\det \mathbf{F}^{\mathrm{e}})^{w}\, \mathbf{F}^{\mathrm{e}-1}\, \mathbf{F}^{\mathrm{e}-1}\, \mathbf{d}\, \mathbf{F}^{\mathrm{e}-T}\, \mathbf{F}^{\mathrm{e}-T}, \quad \hat{\mathbf{d}} = (\det \mathbf{F}^{\mathrm{e}})^{-w}\, \mathbf{F}^{\mathrm{e}T}\, \mathbf{F}^{\mathrm{e}T}\, \mathbf{d}\, \mathbf{F}^{\mathrm{e}}\, \mathbf{F}^{\mathrm{e}}. \qquad (11.18.2)$$

The products in (11.18.2) are defined such that, for example, the components of the covariant transformation are

$$\hat{d}_{IJKL} = (\det \mathbf{F}^{\mathrm{e}})^{-w}\, F^{\mathrm{e}}_{mI}\, F^{\mathrm{e}}_{nJ}\, d_{mnpq}\, F^{\mathrm{e}}_{pK}\, F^{\mathrm{e}}_{qL}. \qquad (11.18.3)$$

The elastic strain energy per unit unstressed volume in the configuration $\mathcal{B}^{\mathrm{p}}$ is

$$\Psi^{\mathrm{e}} = \Psi^{\mathrm{e}}(\mathbf{E}^{\mathrm{e}}, \hat{\mathbf{d}}). \qquad (11.18.4)$$

The elastic strain energy per unit initial volume in the configuration $\mathcal{B}^0$ is then

$$\Psi = (\det \mathbf{F}^{\mathrm{p}})\, \Psi^{\mathrm{e}} = \Psi(\mathbf{E}^{\mathrm{e}}, \hat{\mathbf{d}}), \qquad (11.18.5)$$

which is equal to Ψ^{e} only when the plastic deformation is incompressible. The function Ψ is an isotropic function of both $\mathbf{E}^{\mathrm{e}}$ and $\hat{\mathbf{d}}$. This means that, under a rigid-body rotation $\hat{\mathbf{Q}}$, superposed to the intermediate configuration,

$$\Psi(\hat{\mathbf{Q}} \cdot \mathbf{E}^{\mathrm{e}} \cdot \hat{\mathbf{Q}}^T, \hat{\mathbf{Q}} \cdot \hat{\mathbf{d}} \cdot \hat{\mathbf{Q}}^T) = \Psi(\mathbf{E}^{\mathrm{e}}, \hat{\mathbf{d}}). \tag{11.18.6}$$

The damage variable in this expression is assumed to be a second-order symmetric tensor. The elastic stress response from $\mathcal{B}^{\mathrm{p}}$ to $\mathcal{B}$ is consequently

$$(\det \mathbf{F}^{\mathrm{e}})\, \boldsymbol{\sigma} = \mathbf{F}^{\mathrm{e}} \cdot \frac{\partial \Psi^{\mathrm{e}}(\mathbf{E}^{\mathrm{e}}, \hat{\mathbf{d}})}{\partial \mathbf{E}^{\mathrm{e}}} \cdot \mathbf{F}^{\mathrm{e}T}, \tag{11.18.7}$$

or

$$\boldsymbol{\tau} = \mathbf{F}^{\mathrm{e}} \cdot \frac{\partial \Psi(\mathbf{E}^{\mathrm{e}}, \hat{\mathbf{d}})}{\partial \mathbf{E}^{\mathrm{e}}} \cdot \mathbf{F}^{\mathrm{e}T}, \tag{11.18.8}$$

where $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$ is the Kirchhoff stress.

11.18.2. Inelastic and Damage Rates of Deformation

Upon differentiation of Eq. (11.18.8), we obtain

$$\begin{aligned}\dot{\boldsymbol{\tau}} - \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right) \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)^T &= \boldsymbol{\mathcal{L}}_{(1)} : \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} \\ &+ \mathbf{F}^{\mathrm{e}} \cdot \left(\frac{\partial^2 \Psi}{\partial \mathbf{E}^{\mathrm{e}} \otimes \partial \hat{\mathbf{d}}} : \dot{\hat{\mathbf{d}}}\right) \cdot \mathbf{F}^{\mathrm{e}T}.\end{aligned} \tag{11.18.9}$$

The rectangular components of $\boldsymbol{\mathcal{L}}_{(1)}$ are

$$\mathcal{L}^{(1)}_{ijkl} = F^{\mathrm{e}}_{iM} F^{\mathrm{e}}_{jN} \frac{\partial^2 \Psi(\mathbf{E}^{\mathrm{e}}, \hat{\mathbf{d}})}{\partial E^{\mathrm{e}}_{MN}\, \partial E^{\mathrm{e}}_{PQ}} F^{\mathrm{e}}_{kP} F^{\mathrm{e}}_{lQ}. \tag{11.18.10}$$

The last term on the right-hand side of Eq. (11.18.9) can be conveniently rewritten as

$$\mathbf{F}^{\mathrm{e}} \cdot \left(\frac{\partial^2 \Psi}{\partial \mathbf{E}^{\mathrm{e}} \otimes \partial \hat{\mathbf{d}}} : \dot{\hat{\mathbf{d}}}\right) \cdot \mathbf{F}^{\mathrm{e}T} = \frac{\partial \boldsymbol{\tau}}{\partial \hat{\mathbf{d}}} : \dot{\hat{\mathbf{d}}}. \tag{11.18.11}$$

Substitution of Eqs. (11.18.11) and (11.17.21) into Eq. (11.18.9) then gives

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_{(0)} : \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} - \boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}} + \frac{\partial \boldsymbol{\tau}}{\partial \hat{\mathbf{d}}} : \dot{\hat{\mathbf{d}}}. \tag{11.18.12}$$

The elastic part of the rate of deformation,

$$\mathbf{D}^{\mathrm{e}} = \boldsymbol{\mathcal{L}}^{-1}_{(0)} : \overset{\circ}{\boldsymbol{\tau}}, \tag{11.18.13}$$

is identified from Eq. (11.18.12) as

$$\mathbf{D}^{\mathrm{e}} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} - \boldsymbol{\mathcal{L}}^{-1}_{(0)} : (\boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}}) + \boldsymbol{\mathcal{L}}^{-1}_{(0)} : \left(\frac{\partial \boldsymbol{\tau}}{\partial \hat{\mathbf{d}}} : \dot{\hat{\mathbf{d}}}\right). \tag{11.18.14}$$

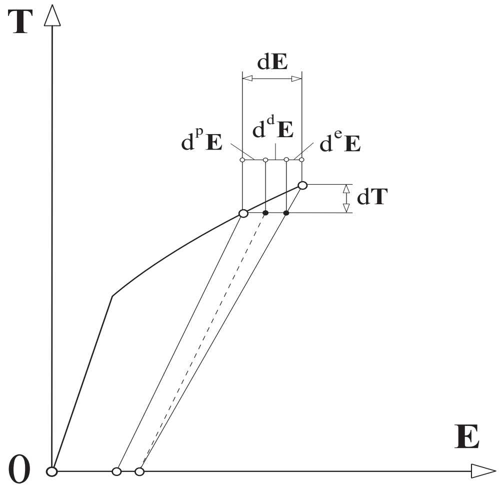

FIGURE 11.5. Geometric interpretation of the partition of the strain increment into its elastic, damage, and plastic parts.

The remaining part of the total rate of deformation is the inelastic part

$$\mathbf{D}^{\mathrm{i}} = \mathbf{D} - \mathbf{D}^{\mathrm{e}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{s}} + \boldsymbol{\mathcal{L}}_{(0)}^{-1} : (\boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}}) - \boldsymbol{\mathcal{L}}_{(0)}^{-1} : \left(\frac{\partial \boldsymbol{\tau}}{\partial \hat{\mathbf{d}}} : \dot{\hat{\mathbf{d}}}\right). \tag{11.18.15}$$

The first two terms on the right-hand side of Eq. (11.18.15) represent the plastic part

$$\mathbf{D}^{\mathrm{p}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{s}} + \boldsymbol{\mathcal{L}}_{(0)}^{-1} : (\boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}}), \tag{11.18.16}$$

while

$$\mathbf{D}^{\mathrm{d}} = -\boldsymbol{\mathcal{L}}_{(0)}^{-1} : \left(\frac{\partial \boldsymbol{\tau}}{\partial \hat{\mathbf{d}}} : \dot{\hat{\mathbf{d}}}\right) \tag{11.18.17}$$

is the damage part of the rate of deformation tensor (Fig. 11.5). These are such that

$$\mathbf{D}^{\mathrm{i}} = \mathbf{D}^{\mathrm{p}} + \mathbf{D}^{\mathrm{d}}, \tag{11.18.18}$$

and

$$\mathbf{D} = \mathbf{D}^{\mathrm{e}} + \mathbf{D}^{\mathrm{i}} = \mathbf{D}^{\mathrm{e}} + \mathbf{D}^{\mathrm{p}} + \mathbf{D}^{\mathrm{d}}. \tag{11.18.19}$$

If the material behavior complies with Ilyushin's postulate, the inelastic part $\mathbf{D}^{\mathrm{i}}$ of the rate of deformation tensor is normal to a locally smooth yield surface in the Cauchy stress space.

11.18.3. Rates of Damage Tensors

For a scalar damage variable, which transforms during the elastic deformation according to

$$\hat{d} = (\det \mathbf{F}^{\mathrm{e}})^{w}\, d, \tag{11.18.20}$$

the rates of d and $\hat{d}$ are related by

$$\dot{\hat{d}} = (\det \mathbf{F}^{\mathrm{e}}) \left[\dot{d} + w\, d \operatorname{tr}\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)\right]. \tag{11.18.21}$$

For a second-order damage tensor $\mathbf{d}$ and a covariant type transformation, we have

$$\hat{\mathbf{d}} = (\det \mathbf{F}^{\mathrm{e}})^{-w}\, \mathbf{F}^{\mathrm{e}T} \cdot \mathbf{d} \cdot \mathbf{F}^{\mathrm{e}}, \quad \dot{\hat{\mathbf{d}}} = (\det \mathbf{F}^{\mathrm{e}})^{-w}\, \mathbf{F}^{\mathrm{e}T} \cdot \overset{\blacktriangle}{\mathbf{d}} \cdot \mathbf{F}^{\mathrm{e}}. \tag{11.18.22}$$

Here,

$$\overset{\blacktriangle}{\mathbf{d}} = \dot{\mathbf{d}} + \mathbf{d} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right) + \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)^{T} \cdot \mathbf{d} - w\, \mathbf{d} \operatorname{tr}\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right) \tag{11.18.23}$$

represents the convected rate associated with a weighted covariant transformation. If the induced tensor $\hat{\mathbf{d}}$ is obtained from $\mathbf{d}$ by a contravariant transformation, then

$$\hat{\mathbf{d}} = (\det \mathbf{F}^{\mathrm{e}})^{w}\, \mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{d} \cdot \mathbf{F}^{\mathrm{e}-T}, \quad \dot{\hat{\mathbf{d}}} = (\det \mathbf{F}^{\mathrm{e}})^{w}\, \mathbf{F}^{\mathrm{e}-1} \cdot \overset{\blacktriangledown}{\mathbf{d}} \cdot \mathbf{F}^{\mathrm{e}-T}. \tag{11.18.24}$$

The convected rate associated with a weighted contravariant transformation is

$$\overset{\blacktriangledown}{\mathbf{d}} = \dot{\mathbf{d}} - \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right) \cdot \mathbf{d} - \mathbf{d} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)^{T} + w\, \mathbf{d} \operatorname{tr}\left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right). \tag{11.18.25}$$

For the fourth-order damage tensor with a covariant transformation, we similarly have

$$\hat{\mathbf{d}} = (\det \mathbf{F}^{\mathrm{e}})^{-w}\, \mathbf{F}^{\mathrm{e}T}\, \mathbf{F}^{\mathrm{e}T} \mathbf{d}\, \mathbf{F}^{\mathrm{e}}\, \mathbf{F}^{\mathrm{e}}, \tag{11.18.26}$$

$$\dot{\hat{\mathbf{d}}} = (\det \mathbf{F}^{\mathrm{e}})^{-w}\, \mathbf{F}^{\mathrm{e}T}\, \mathbf{F}^{\mathrm{e}T}\, \overset{\blacktriangle}{\mathbf{d}}\, \mathbf{F}^{\mathrm{e}}\, \mathbf{F}^{\mathrm{e}}. \tag{11.18.27}$$

The rectangular components of $\overset{\blacktriangle}{\mathbf{d}}$ are

$$\begin{aligned} \overset{\blacktriangle}{d}_{ijkl} = \dot{d}_{ijkl} + L^{\mathrm{e}}_{mi}\, d_{mjkl} + L^{\mathrm{e}}_{mj}\, d_{imkl} + L^{\mathrm{e}}_{mk}\, d_{ijml} + L^{\mathrm{e}}_{ml}\, d_{ijkm} \\ - w\, L^{\mathrm{e}}_{mm}\, d_{ijkl}. \end{aligned} \tag{11.18.28}$$

The notation $\mathbf{L}^{\mathrm{e}} = \dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}$ is used in Eq. (11.18.28). If the induced tensor $\hat{\mathbf{d}}$ is obtained from the fourth-order damage tensor $\mathbf{d}$ by a contravariant transformation, there follows

$$\hat{\mathbf{d}} = (\det \mathbf{F}^{\mathrm{e}})^{w}\, \mathbf{F}^{\mathrm{e}-1}\, \mathbf{F}^{\mathrm{e}-1}\, \mathbf{d}\, \mathbf{F}^{\mathrm{e}-T}\, \mathbf{F}^{\mathrm{e}-T}, \tag{11.18.29}$$

$$\dot{\hat{\mathbf{d}}} = (\det \mathbf{F}^{\mathrm{e}})^{w}\, \mathbf{F}^{\mathrm{e}-1}\, \mathbf{F}^{\mathrm{e}-1}\, \overset{\blacktriangledown}{\mathbf{d}}\, \mathbf{F}^{\mathrm{e}-T}\, \mathbf{F}^{\mathrm{e}-T}, \tag{11.18.30}$$

where

$$\begin{aligned} \overset{\blacktriangledown}{d}_{ijkl} = \dot{d}_{ijkl} - L^{\mathrm{e}}_{im}\, d_{mjkl} - L^{\mathrm{e}}_{jm}\, d_{imkl} - L^{\mathrm{e}}_{km}\, d_{ijml} - L^{\mathrm{e}}_{lm}\, d_{ijkm} \\ + w\, L^{\mathrm{e}}_{mm}\, d_{ijkl}. \end{aligned} \tag{11.18.31}$$

Substituting the expression for $\dot{\hat{\mathbf{d}}}$, corresponding to the tensorial order of the introduced damage variable $\mathbf{d}$ and the transformation rule between $\hat{\mathbf{d}}$ and $\mathbf{d}$, into the expression for the damage part of the rate of deformation, gives

$$\frac{\partial \boldsymbol{\tau}}{\partial \hat{\mathbf{d}}} : \dot{\hat{\mathbf{d}}} = \frac{\partial \boldsymbol{\tau}}{\partial \mathbf{d}} : \overset{\blacktriangle}{\mathbf{d}}, \quad \text{or} \quad \frac{\partial \boldsymbol{\tau}}{\partial \hat{\mathbf{d}}} : \dot{\hat{\mathbf{d}}} = \frac{\partial \boldsymbol{\tau}}{\partial \mathbf{d}} : \overset{\blacktriangledown}{\mathbf{d}}, \tag{11.18.32}$$

and

$$\mathbf{D}^{\mathrm{d}} = -\boldsymbol{\mathcal{L}}^{-1}_{(0)} : \left(\frac{\partial \boldsymbol{\tau}}{\partial \mathbf{d}} : \overset{\blacktriangle}{\mathbf{d}}\right), \quad \text{or} \quad \mathbf{D}^{\mathrm{d}} = -\boldsymbol{\mathcal{L}}^{-1}_{(0)} : \left(\frac{\partial \boldsymbol{\tau}}{\partial \mathbf{d}} : \overset{\blacktriangledown}{\mathbf{d}}\right). \tag{11.18.33}$$

With the specified evolution equation for $\overset{\blacktriangle}{\mathbf{d}}$ or $\overset{\blacktriangledown}{\mathbf{d}}$, this determines the damage part of the rate of deformation.

Further elaboration on the constitutive theory of damage-elastoplasticity can be found in the papers by Simo and Ju (1987), Lehmann (1991), Hansen and Schreyer (1994), Lubarda (1994b), and Lubarda and Krajcinovic (1995). See also the books by Lemaitre and Chaboche (1990), Maugin (1992), Krajcinovic (1996), and Voyiadjis and Kattan (1999).

11.19. Reversed Decomposition $\mathbf{F} = \mathbf{F}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{e}}$

In the wake of Lee's decomposition $\mathbf{F} = \mathbf{F}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}}$, the suggestions were made for an alternative, reversed decomposition $\mathbf{F} = \mathbf{F}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{e}}$ (e.g., Clifton, 1972; Nemat-Nasser, 1979). This decomposition, however, remained far less employed than the original Lee's decomposition. Lubarda (1999) recently demonstrated that the constitutive analysis of elastoplastic behavior can be developed by using the reversed decomposition quite analogously as using Lee's decomposition. The two formulations can be viewed in many respects as dual to each other, both leading to the same final structure of the constitutive equations, although some of the derivation and interpretations are simpler in the case of Lee's decomposition.

The reversed decomposition is introduced as follows. An arbitrary state of elastoplastic deformation, corresponding to the deformation gradient $\mathbf{F}$, is imagined to be reached in two stages. First, it is assumed that all internal mechanisms responsible for plastic deformation are frozen, so that, for example, the critical forces needed to drive dislocations, or the critical resolved shear stresses of the crystalline slip systems, are assigned infinitely large values. The application of the total stress to such material, incapable of plastic deformation, results in the pure elastic deformation $\mathbf{F}_{\mathrm{e}}$. This carries the material from its initial configuration $\mathcal{B}^0$ to the intermediate configuration $\mathcal{B}_{\mathrm{e}}$. Subsequently, the material is plastically unlocked, by defreezing the mechanisms of plastic deformation, which enables the material to flow at the constant stress. The corresponding part of the deformation gradient, associated with the transition from the intermediate $\mathcal{B}_{\mathrm{e}}$ to the final configuration $\mathcal{B}$, is the plastic part of deformation gradient $\mathbf{F}_{\mathrm{p}}$ (Fig. 11.6). Thus, the reversed decomposition

$$\mathbf{F} = \mathbf{F}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{e}}. \tag{11.19.1}$$

The intermediate elastically deformed configuration $\mathcal{B}_{\mathrm{e}}$ is unique, since a superposed rotation to $\mathcal{B}_{\mathrm{e}}$ would rotate the stress state, and the plastic flow from $\mathcal{B}_{\mathrm{e}}$ to $\mathcal{B}$ would not take place at the constant state of stress. In the subsequent analysis it will be assumed that plastic flow is incompressible and that elastic properties of the material are not affected by plastic deformation. Relative to a given orientation of the principal directions of elastic anisotropy, there is in this case a unique

$$\mathbf{F}_{\mathrm{e}} = \mathbf{F}^{\mathrm{e}} \tag{11.19.2}$$

that gives rise to total stress in $\mathcal{B}_{\mathrm{e}}$ and $\mathcal{B}$. This stress is

$$\boldsymbol{\tau} = \mathbf{F}^{\mathrm{e}} \cdot \frac{\partial \Psi^{\mathrm{e}}}{\partial \mathbf{E}^{\mathrm{e}}} \cdot \mathbf{F}^{\mathrm{e}T}. \tag{11.19.3}$$

The elastic strain energy per unit initial volume is Ψ^{e}, while $\mathbf{E}^{\mathrm{e}}$ is the Lagrangian elastic strain relative to its ground state ($\mathcal{B}^{\mathrm{p}}$ in the case of Lee's

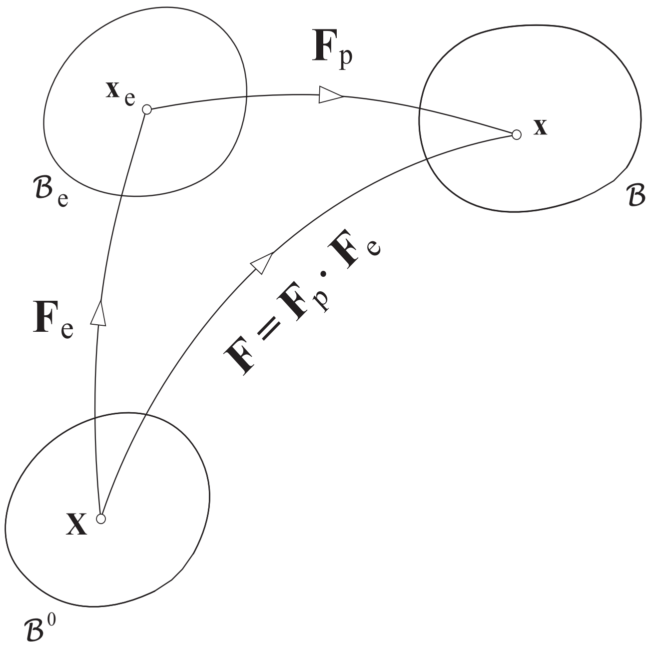

FIGURE 11.6. Schematic representation of the reversed multiplicative decomposition of deformation gradient into its elastic and plastic parts. The intermediate configuration $\mathcal{B}_\mathrm{e}$ is obtained from the initial configuration $\mathcal{B}^0$ by elastic loading to the current stress level, assuming that all inelastic mechanisms of the deformation are momentarily frozen.

decomposition and $\mathcal{B}^0$ in the case of the reversed decomposition, both having the same orientation of the principal axes of anisotropy, relative to the fixed frame of reference). The Kirchhoff stress is $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ designates the Cauchy stress. Therefore, we can write

$$\mathbf{F} = \mathbf{F}^\mathrm{e} \cdot \mathbf{F}^\mathrm{p} = \mathbf{F}_\mathrm{p} \cdot \mathbf{F}^\mathrm{e}. \tag{11.19.4}$$

The same elastic deformation gradient $\mathbf{F}^\mathrm{e}$ appears in both decompositions. The relationship between plastic parts of the deformation gradient is consequently

$$\mathbf{F}^\mathrm{p} = \mathbf{F}^{\mathrm{e}-1} \cdot \mathbf{F}_\mathrm{p} \cdot \mathbf{F}^\mathrm{e}. \tag{11.19.5}$$

If the material is elastically isotropic, an initial rotation $\mathbf{R}^\mathrm{e}$ of $\mathcal{B}^0$ does not affect the stress response, and the relevant part of the total deformation gradient for the constitutive analysis is

$$\mathbf{F} = \mathbf{F}_{\mathrm{p}} \cdot \mathbf{V}^{\mathrm{e}}. \tag{11.19.6}$$

In this case, therefore, we can write

$$\mathbf{F} = \mathbf{V}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}} = \mathbf{F}_{\mathrm{p}} \cdot \mathbf{V}^{\mathrm{e}}. \tag{11.19.7}$$

11.19.1. Elastic Unloading

During elastic loading from $\mathcal{B}^{\mathrm{p}}$ to $\mathcal{B}$, or elastic unloading from $\mathcal{B}$ to $\mathcal{B}^{\mathrm{p}}$, the plastic deformation gradient $\mathbf{F}^{\mathrm{p}}$ of the decomposition $\mathbf{F} = \mathbf{F}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}}$ remains constant. This greatly simplifies the derivation of the corresponding constitutive equations. As shown in Section 11.3, the velocity gradient in $\mathcal{B}$ is

$$\mathbf{L} = \dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} + \mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}, \tag{11.19.8}$$

so that during elastic unloading

$$\dot{\mathbf{F}}^{\mathrm{p}} = \mathbf{0}, \quad \mathbf{L} = \dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}. \tag{11.19.9}$$

In the framework of the reversed decomposition $\mathbf{F} = \mathbf{F}_{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}$, however, the plastic part of deformation gradient $\mathbf{F}_{\mathrm{p}}$ does not remain constant during elastic unloading. In fact, upon complete unloading from an elastoplastic state of deformation to zero stress, the configuration $\mathcal{B}^{\mathrm{p}}$ is reached, and $\mathbf{F}_{\mathrm{p}} = \mathbf{F}^{\mathrm{p}}$ at that instant (Fig. 11.7). Therefore, $\dot{\mathbf{F}}_{\mathrm{p}} \neq \mathbf{0}$ during elastic unloading. This can also be recognized from the general relationship between $\mathbf{F}^{\mathrm{p}}$ and $\mathbf{F}_{\mathrm{p}}$. By differentiating Eq. (11.19.5), we obtain

$$\dot{\mathbf{F}}^{\mathrm{p}} = \mathbf{F}^{\mathrm{e}-1} \cdot \overset{*}{\mathbf{F}}_{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}}, \tag{11.19.10}$$

where

$$\overset{*}{\mathbf{F}}_{\mathrm{p}} = \dot{\mathbf{F}}_{\mathrm{p}} - \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right) \cdot \mathbf{F}_{\mathrm{p}} + \mathbf{F}_{\mathrm{p}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right) \tag{11.19.11}$$

is a convected-type derivative of $\mathbf{F}_{\mathrm{p}}$ relative to elastic deformation. Consequently,

$$\overset{*}{\mathbf{F}}_{\mathrm{p}} = \mathbf{0}, \quad \text{if} \quad \dot{\mathbf{F}}^{\mathrm{p}} = \mathbf{0}, \tag{11.19.12}$$

and in this case

$$\dot{\mathbf{F}}_{\mathrm{p}} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right) \cdot \mathbf{F}_{\mathrm{p}} - \mathbf{F}_{\mathrm{p}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right). \tag{11.19.13}$$

The last expression defines the change of $\mathbf{F}_{\mathrm{p}}$ during elastic unloading.

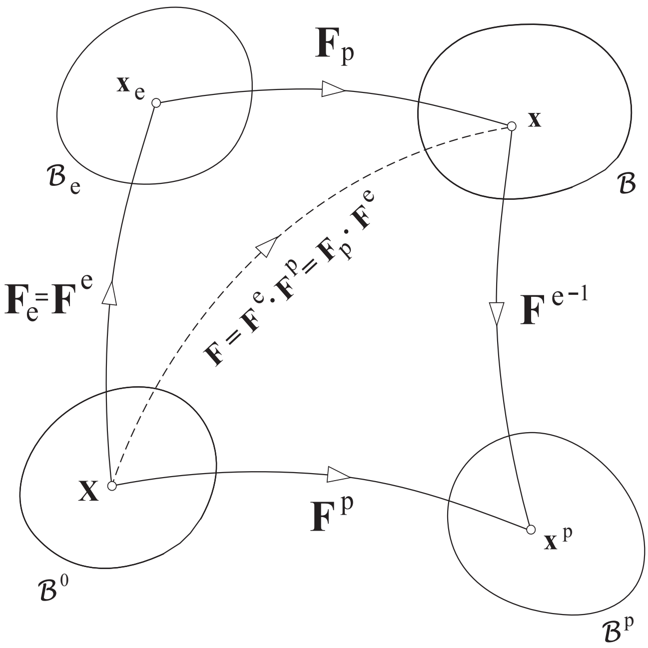

FIGURE 11.7. Plastic part of deformation gradient $\mathbf{F}_{\mathrm{p}}$ does not remain constant during elastic unloading. Upon complete unloading to zero stress, the configuration $\mathcal{B}^{\mathrm{p}}$ is reached, and $\mathbf{F}_{\mathrm{p}} = \mathbf{F}^{\mathrm{p}}$ at that instant.

Furthermore, from Eq. (11.19.10),

$$\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1} = \mathbf{F}^{\mathrm{e}-1} \cdot \left(\overset{*}{\mathbf{F}}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{p}}^{-1} \right) \cdot \mathbf{F}^{\mathrm{e}}, \tag{11.19.14}$$

and the substitution into Eq. (11.19.8) gives

$$\mathbf{L} = \dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} + \overset{*}{\mathbf{F}}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{p}}^{-1}. \tag{11.19.15}$$

11.19.2. Elastic and Plastic Rates of Deformation

If the elastic part of the rate of deformation tensor is defined by a kinetic relation

$$\mathbf{D}^{\mathrm{e}} = \boldsymbol{\mathcal{L}}_{(0)}^{-1} : \overset{\circ}{\boldsymbol{\tau}}, \quad \overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{W}, \tag{11.19.16}$$

it follows that

$$\mathbf{D}^{\mathrm{e}} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1} \right)_{\mathrm{s}} - \boldsymbol{\mathcal{L}}_{(0)}^{-1} : \left(\boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}} \right), \tag{11.19.17}$$

$$\mathbf{D}^{\mathrm{p}} = \left(\overset{*}{\mathbf{F}}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{p}}^{-1}\right)_{\mathrm{s}} + \boldsymbol{\mathcal{L}}_{(0)}^{-1} : (\boldsymbol{\omega}^{\mathrm{p}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \boldsymbol{\omega}^{\mathrm{p}})\,. \tag{11.19.18}$$

The spin $\boldsymbol{\omega}^{\mathrm{p}}$ is

$$\boldsymbol{\omega}^{\mathrm{p}} = \left(\overset{*}{\mathbf{F}}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{p}}^{-1}\right)_{\mathrm{a}} = \left[\mathbf{F}^{\mathrm{e}} \cdot \left(\dot{\mathbf{F}}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{p}-1}\right) \cdot \mathbf{F}^{\mathrm{e}-1}\right]_{\mathrm{a}}. \tag{11.19.19}$$

The elastic part of the rate of deformation can also be expressed as

$$\mathbf{D}^{\mathrm{e}} = \left(\overset{\bullet}{\mathbf{F}}{}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}, \tag{11.19.20}$$

where

$$\overset{\bullet}{\mathbf{F}}{}^{\mathrm{e}} = \dot{\mathbf{F}}^{\mathrm{e}} - \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}} + \mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}}. \tag{11.19.21}$$

The spin $\boldsymbol{\Omega}^{\mathrm{p}}$ is the solution of the matrix equation

$$\mathbf{W} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}} + \left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}}. \tag{11.19.22}$$

This is analogous to the derivation from Section 11.7, based on Lee's decomposition. Therefore,

$$\mathbf{D}^{\mathrm{e}} = \left(\dot{\mathbf{F}}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}} + \left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}, \tag{11.19.23}$$

$$\mathbf{D}^{\mathrm{p}} = \left(\overset{*}{\mathbf{F}}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{p}}^{-1}\right)_{\mathrm{s}} - \left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{s}}. \tag{11.19.24}$$

Summing up Eqs. (11.19.22)–(11.19.24), we obtain an expression for the velocity gradient $\mathbf{L}$. The comparison with Eq. (11.19.15) then gives

$$\mathbf{0} = \left(\overset{*}{\mathbf{F}}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{p}}^{-1}\right)_{\mathrm{a}} - \left(\mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}\right)_{\mathrm{a}}. \tag{11.19.25}$$

Thus, the plastic part of the rate of deformation can be alternatively written as

$$\mathbf{D}^{\mathrm{p}} = \overset{*}{\mathbf{F}}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{p}}^{-1} - \mathbf{F}^{\mathrm{e}} \cdot \boldsymbol{\Omega}^{\mathrm{p}} \cdot \mathbf{F}^{\mathrm{e}-1}. \tag{11.19.26}$$

The result is in accord with Eq. (11.8.2), as can be verified by using the definitions of $\overset{\bullet}{\mathbf{F}}{}^{\mathrm{p}}$ and $\overset{*}{\mathbf{F}}_{\mathrm{p}}$, and the relationship

$$\mathbf{F}_{\mathrm{p}}^{-1} = \mathbf{F}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}-1} \cdot \mathbf{F}^{\mathrm{e}-1}. \tag{11.19.27}$$

The presented derivation demonstrates a duality in the constitutive formulation of large-deformation elastoplasticity based on Lee's decomposition $\mathbf{F} = \mathbf{F}^{\mathrm{e}} \cdot \mathbf{F}^{\mathrm{p}}$ and the reversed decomposition $\mathbf{F} = \mathbf{F}_{\mathrm{p}} \cdot \mathbf{F}_{\mathrm{e}}$, at least for the considered material models. The structure of the kinematic expressions is more involved in the case of the reversed decomposition, partly because during elastic unloading the plastic deformation gradient $\mathbf{F}^{\mathrm{p}}$ of Lee's decomposition remains constant, while $\mathbf{F}_{\mathrm{p}}$ of the reversed decomposition changes, albeit in a definite manner specified by Eq. (11.19.13). It is possible, however, that in some applications the reversed decomposition may have certain advantages. For example, Clifton (1972) found that it is slightly more convenient in the analysis of one-dimensional wave propagation in elastic-viscoplastic solids.

Lee's decomposition has definite advantages in modeling the plasticity with evolving elastic properties. In this case, a set of damage or structural tensors can be attached to the intermediate configuration $\mathcal{B}^{\mathrm{p}}$ to represent its current state of elastic anisotropy. The structural tensors evolve during plastic deformation, depending on the nature of microscopic inelastic processes, as represented by the appropriate evolution equations. The stress response at each instant of deformation is given in terms of the gradient of elastic strain energy with respect to elastic strain, at the current values of the structural tensors. This has been discussed in Section 11.18. In the case of the reversed decomposition, however, the elastic response is defined relative to the initial configuration $\mathcal{B}^0$, which does not contain any information about the evolving elastic properties or subsequently developed elastic anisotropy. Additional remedy has to be introduced to deal with these features of the material response, which is likely to make the reversed decomposition less attractive than the original Lee's decomposition.

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