CHAPTER 10

PLASTIC STABILITY

Hill's theory of uniqueness and plastic stability is presented in this chapter. Exclusion functional and incrementally linear comparison material are first introduced. Eigenmodal deformations and acceleration waves in elastoplastic solids are then discussed. Fundamentals of Rice's localization analysis for various constitutive models are presented. Elastoplastic materials described by associative and nonassociative flow rules, as well as rigid-plastic materials are considered. The effects of yield vertices on localization predictions are examined.

10.1. Elastoplastic Rate-Potentials

The analysis is restricted to isothermal and rate-independent elastoplastic behavior. It was shown in Section 9.2 that the corresponding constitutive structure, for materials with a smooth yield surface, is bilinear and given by

$$\dot{\mathbf{T}}_{(n)} = \boldsymbol{\Lambda}^{\mathrm{ep}}_{(n)} : \dot{\mathbf{E}}_{(n)}. \tag{10.1.1}$$

One branch of the stiffness tensor $\boldsymbol{\Lambda}^{\mathrm{ep}}_{(n)}$ is associated with plastic loading, and the other with elastic unloading or neutral loading, such that

$$\boldsymbol{\Lambda}^{\mathrm{ep}}_{(n)} = \begin{cases} \boldsymbol{\Lambda}^{\mathrm{p}}_{(n)}, & \text{if} \quad \dfrac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \boldsymbol{\Lambda}_{(n)} : \dot{\mathbf{E}}_{(n)} > 0, \\ \\ \boldsymbol{\Lambda}_{(n)}, & \text{if} \quad \dfrac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \boldsymbol{\Lambda}_{(n)} : \dot{\mathbf{E}}_{(n)} \le 0. \end{cases} \tag{10.1.2}$$

The stiffness tensor for plastic loading branch is defined by Eq. (9.2.10), i.e.,

$$\boldsymbol{\Lambda}^{\mathrm{p}}_{(n)} = \boldsymbol{\Lambda}_{(n)} - \frac{1}{h_{(n)}} \left(\boldsymbol{\Lambda}_{(n)} : \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} \right) \otimes \left(\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \boldsymbol{\Lambda}_{(n)} \right). \tag{10.1.3}$$

The elastic stiffness tensor is $\boldsymbol{\Lambda}_{(n)}$. More involved piecewise linear relations, with several or many branches, could be used to represent the behavior at the yield surface vertex (for example, for single crystals of metals deforming by multiple slip). Since $\mathbf{\Lambda}^{\rm ep}_{(n)}$ obeys the reciprocal symmetry, we can introduce the elastoplastic rate-potential function $\chi_{(n)}$, such that

$$\dot{\mathbf{T}}_{(n)} = \frac{\partial \chi_{(n)}}{\partial \dot{\mathbf{E}}_{(n)}}, \quad \chi_{(n)} = \frac{1}{2}\,\mathbf{\Lambda}^{\rm ep}_{(n)} :: \left(\dot{\mathbf{E}}_{(n)} \otimes \dot{\mathbf{E}}_{(n)}\right). \tag{10.1.4}$$

Alternatively, the elastoplastic constitutive structure can be expressed in terms of the rate of nominal stress and the rate of deformation tensor. By conveniently selecting $n = 1$ in Eq. (10.1.1), and by using the relationships

$$\dot{\mathbf{E}}_{(1)} = \frac{1}{2}\left(\mathbf{F}^T \cdot \dot{\mathbf{F}} + \dot{\mathbf{F}}^T \cdot \mathbf{F}\right), \quad \dot{\mathbf{T}}_{(1)} = \left(\dot{\mathbf{P}} - \mathbf{P} \cdot \mathbf{L}^T\right) \cdot \mathbf{F}^{-T}, \tag{10.1.5}$$

from Eqs. (3.8.8) and (3.8.14), it follows that

$$\dot{\mathbf{P}} = \mathbf{\Lambda}^{\rm ep} \cdot\cdot\, \dot{\mathbf{F}}. \tag{10.1.6}$$

The Cartesian components of elastoplastic moduli and pseudomoduli are related by

$$\Lambda^{\rm ep}_{JiLk} = \Lambda^{\rm ep\,(1)}_{JMLN} F_{iM} F_{kN} + T^{(1)}_{JL}\delta_{ik}, \tag{10.1.7}$$

as previously derived in Eq. (6.4.8). Since the pseudomoduli obey reciprocal symmetry ($\Lambda^{\rm ep}_{JiLk} = \Lambda^{\rm ep}_{LkJi}$), we can introduce the rate-potential function χ, such that

$$\dot{\mathbf{P}} = \frac{\partial \chi}{\partial \dot{\mathbf{F}}}, \quad \chi = \frac{1}{2}\,\mathbf{\Lambda}^{\rm ep} \cdot\cdot\cdot\cdot\, (\dot{\mathbf{F}} \otimes \dot{\mathbf{F}}). \tag{10.1.8}$$

The response over entire $\dot{\mathbf{F}}$ space is bilinear, since in the range of elastic unloading or neutral loading $\mathbf{\Lambda}^{\rm ep} = \mathbf{\Lambda}$ (tensor of elastic pseudomoduli), while in the range of plastic loading $\mathbf{\Lambda}^{\rm ep} = \mathbf{\Lambda}^{\rm p}$.

More generally, if inelastic rate response is thoroughly nonlinear (as in the description of actual behavior of polycrystals at yield vertices), we have

$$\dot{\mathbf{P}} = \frac{\partial \chi}{\partial \dot{\mathbf{F}}}, \quad \chi = \frac{1}{2}\,\dot{\mathbf{P}} \cdot\cdot\, \dot{\mathbf{F}}. \tag{10.1.9}$$

In the absence of time-dependent viscous effects, the rate-potential χ is necessarily homogeneous of degree two in $\dot{\mathbf{F}}$.

10.1.1. Current Configuration as Reference

When the current configuration is taken as the reference configuration, Eq. (10.1.8) becomes

$$\underline{\dot{\mathbf{P}}} = \frac{\partial \underline{\chi}}{\partial \mathbf{L}}, \quad \underline{\chi} = \frac{1}{2}\,\underline{\mathbf{\Lambda}}^{\rm ep} \cdot\cdot\cdot\cdot\, (\mathbf{L} \otimes \mathbf{L}), \tag{10.1.10}$$

since

$$\underline{\dot{\mathbf{F}}} = \mathbf{L}. \tag{10.1.11}$$

From Eqs. (6.3.4) and (6.4.16), or directly from Eq. (10.1.7), we have

$$\underline{\Lambda}^{\text{ep}}_{jilk} = \underline{\mathcal{L}}^{\text{ep}\ (1)}_{jilk} + \sigma_{jl}\delta_{ik}, \tag{10.1.12}$$

so that

$$\underline{\chi} = \frac{1}{2}\,\underline{\boldsymbol{\mathcal{L}}}^{\text{ep}}_{(1)} :: (\mathbf{D}\otimes\mathbf{D}) + \frac{1}{2}\,\boldsymbol{\sigma} : \left(\mathbf{L}^T\cdot\mathbf{L}\right). \tag{10.1.13}$$

Alternatively, in view of

$$\underline{\boldsymbol{\mathcal{L}}}^{\text{ep}}_{(0)} = \underline{\boldsymbol{\mathcal{L}}}^{\text{ep}}_{(1)} + 2\underline{\boldsymbol{\mathcal{S}}}, \tag{10.1.14}$$

where $\underline{\boldsymbol{\mathcal{S}}}$ is defined by Eq. (6.3.11), there follows

$$\underline{\chi} = \frac{1}{2}\,\underline{\boldsymbol{\mathcal{L}}}^{\text{ep}}_{(0)} :: (\mathbf{D}\otimes\mathbf{D}) + \frac{1}{2}\,\boldsymbol{\sigma} : \left(\mathbf{L}^T\cdot\mathbf{L} - 2\,\mathbf{D}^2\right). \tag{10.1.15}$$

The rate potentials $\underline{\chi}_{(n)}$ can be introduced such that

$$\underline{\dot{\mathbf{T}}}_{(n)} = \frac{\partial\underline{\chi}_{(n)}}{\partial\mathbf{D}}, \quad \underline{\chi}_{(n)} = \frac{1}{2}\,\underline{\boldsymbol{\mathcal{L}}}^{\text{ep}}_{(n)} :: (\mathbf{D}\otimes\mathbf{D}). \tag{10.1.16}$$

As in Section 7.6, the following relationships hold

$$\underline{\chi}_{(n)} = \underline{\chi}_{(0)} - n\boldsymbol{\sigma} : \mathbf{D}^2 = \underline{\chi}_{(1)} + (1-n)\boldsymbol{\sigma} : \mathbf{D}^2, \tag{10.1.17}$$

and

$$\underline{\chi} = \underline{\chi}_{(n)} + \frac{1}{2}\boldsymbol{\sigma} : \left[\mathbf{L}^T\cdot\mathbf{L} - 2(1-n)\mathbf{D}^2\right]. \tag{10.1.18}$$

In particular,

$$\underline{\overset{\circ}{\boldsymbol{\tau}}} = \underline{\boldsymbol{\mathcal{L}}}^{\text{ep}}_{(0)} : \mathbf{D} = \frac{\partial\underline{\chi}_{(0)}}{\partial\mathbf{D}}, \quad \underline{\chi}_{(0)} = \frac{1}{2}\,\underline{\boldsymbol{\mathcal{L}}}^{\text{ep}}_{(0)} :: (\mathbf{D}\otimes\mathbf{D}). \tag{10.1.19}$$

The tensor $\underline{\boldsymbol{\mathcal{L}}}^{\text{ep}}_{(0)}$ was explicitly given for various constitutive models in Chapter 9. In the range of elastic unloading or neutral loading it is equal to $\underline{\boldsymbol{\mathcal{L}}}_{(0)}$, and in the range of plastic loading it is equal to $\underline{\boldsymbol{\mathcal{L}}}^{\text{p}}_{(0)}$. For example, in the case of isotropic hardening $\underline{\boldsymbol{\mathcal{L}}}^{\text{p}}_{(0)}$ is defined by Eq. (9.4.43), and in the case of linear kinematic hardening by Eq. (9.4.19).

If the response is thoroughly nonlinear,

$$\underline{\dot{\mathbf{P}}} = \frac{\partial\underline{\chi}}{\partial\mathbf{L}}, \quad \underline{\chi} = \frac{1}{2}\,\underline{\dot{\mathbf{P}}}\cdot\cdot\,\mathbf{L}, \tag{10.1.20}$$

where $\underline{\chi}$ is a homogeneous function of degree two in components of the velocity gradient $\mathbf{L}$.

10.2. Reciprocal Relations

For nonlinear incremental response (either thoroughly nonlinear or nonlinear on account of different behavior in loading and unloading), we can write

$$\dot{\mathbf{P}} \cdot\cdot\, \dot{\mathbf{F}} = 2\chi, \tag{10.2.1}$$

where χ is homogeneous of degree two in $\dot{\mathbf{F}}$. Taking the variation of Eq. (10.2.1), associated with an infinitesimal variation $\delta\dot{\mathbf{F}}$, gives

$$\delta\dot{\mathbf{P}} \cdot\cdot\, \dot{\mathbf{F}} + \dot{\mathbf{P}} \cdot\cdot\, \delta\dot{\mathbf{F}} = 2\delta\chi. \tag{10.2.2}$$

Since

$$\dot{\mathbf{P}} \cdot\cdot\, \delta\dot{\mathbf{F}} = \frac{\partial\chi}{\partial\dot{\mathbf{F}}} \cdot\cdot\, \delta\dot{\mathbf{F}} = \delta\chi, \tag{10.2.3}$$

we deduce from Eq. (10.2.2) the reciprocal relation

$$\delta\dot{\mathbf{P}} \cdot\cdot\, \dot{\mathbf{F}} = \dot{\mathbf{P}} \cdot\cdot\, \delta\dot{\mathbf{F}}. \tag{10.2.4}$$

This expression will be used in the derivation of the following reciprocal theorem. Consider a divergence expression

$$\boldsymbol{\nabla}^0 \cdot \left(\dot{\mathbf{P}} \cdot \delta\mathbf{v} - \delta\dot{\mathbf{P}} \cdot \mathbf{v}\right). \tag{10.2.5}$$

Since by Eq. (1.13.13),

$$\boldsymbol{\nabla}^0 \cdot \left(\dot{\mathbf{P}} \cdot \delta\mathbf{v}\right) = \left(\boldsymbol{\nabla}^0 \cdot \dot{\mathbf{P}}\right) \cdot \delta\mathbf{v} + \dot{\mathbf{P}} \cdot\cdot\, \delta\dot{\mathbf{F}}, \tag{10.2.6}$$

and similarly for the second term in (10.2.5), the divergence expression becomes

$$\boldsymbol{\nabla}^0 \cdot \left(\dot{\mathbf{P}} \cdot \delta\mathbf{v} - \delta\dot{\mathbf{P}} \cdot \mathbf{v}\right) = \left(\boldsymbol{\nabla}^0 \cdot \dot{\mathbf{P}}\right) \cdot \delta\mathbf{v} - \left(\boldsymbol{\nabla}^0 \cdot \delta\dot{\mathbf{P}}\right) \cdot \mathbf{v}. \tag{10.2.7}$$

The reciprocal relation (10.2.4) was utilized in the last step. Integrating Eq. (10.2.7) over the reference volume V^0, employing the equations of continuing equilibrium

$$\boldsymbol{\nabla}^0 \cdot \dot{\mathbf{P}} = -\rho^0\, \dot{\mathbf{b}}, \quad \boldsymbol{\nabla}^0 \cdot \delta\dot{\mathbf{P}} = -\rho^0\, \delta\dot{\mathbf{b}}, \tag{10.2.8}$$

and the Gauss theorem, gives

$$\begin{aligned} &\int_{V^0} \rho^0\, \dot{\mathbf{b}} \cdot \delta\mathbf{v}\, \mathrm{d}V^0 + \int_{S^0} \mathbf{n}^0 \cdot \dot{\mathbf{P}} \cdot \delta\mathbf{v}\, \mathrm{d}S^0 \\ &\qquad = \int_{V^0} \rho^0\, \delta\dot{\mathbf{b}} \cdot \mathbf{v}\, \mathrm{d}V^0 + \int_{S^0} \mathbf{n}^0 \cdot \delta\dot{\mathbf{P}} \cdot \mathbf{v}\, \mathrm{d}S^0. \end{aligned} \tag{10.2.9}$$

This is a reciprocal theorem for the considered incrementally nonlinear response (Hill, 1978).

For incrementally linear response, the variations $\delta\mathbf{v}$ and $\delta\dot{\mathbf{P}}$ can be replaced by (finite) differences $\mathbf{v}-\mathbf{v}^*$ and $\dot{\mathbf{P}}-\dot{\mathbf{P}}^*$ of any two (not necessarily nearby) equilibrium fields, and reciprocal relations of Eqs. (10.2.4) and (10.2.9) reduce to

$$\dot{\mathbf{P}}\cdot\cdot\,\dot{\mathbf{F}}^* = \dot{\mathbf{P}}^*\cdot\cdot\,\dot{\mathbf{F}}, \tag{10.2.10}$$

and

$$\begin{aligned}\int_{V^0}\rho^0\,\dot{\mathbf{b}}\cdot\mathbf{v}^*\,\mathrm{d}V^0 + \int_{S^0}\mathbf{n}^0\cdot\dot{\mathbf{P}}\cdot\mathbf{v}^*\,\mathrm{d}S^0 \\ = \int_{V^0}\rho^0\,\dot{\mathbf{b}}^*\cdot\mathbf{v}\,\mathrm{d}V^0 + \int_{S^0}\mathbf{n}^0\cdot\dot{\mathbf{P}}^*\cdot\mathbf{v}\,\mathrm{d}S^0.\end{aligned} \tag{10.2.11}$$

The latter is analogous to Betti's reciprocal theorem of classical elasticity, as discussed for incrementally linear elastic response in Subsection 7.5.1.

10.2.1. Clapeyron's Formula

Suppose that the stress rate field $\dot{\mathbf{P}}$ satisfies the equations of continuing equilibrium,

$$\boldsymbol{\nabla}^0\cdot\dot{\mathbf{P}} + \rho^0\,\dot{\mathbf{b}} = \mathbf{0}. \tag{10.2.12}$$

Then, for any analytically admissible velocity field $\mathbf{v}$, we have

$$\int_{V^0}\dot{\mathbf{P}}\cdot\cdot\,\dot{\mathbf{F}}\,\mathrm{d}V^0 = \int_{V^0}\rho^0\,\dot{\mathbf{b}}\cdot\mathbf{v}\,\mathrm{d}V^0 + \int_{S^0}\mathbf{n}^0\cdot\dot{\mathbf{P}}\cdot\mathbf{v}\,\mathrm{d}S^0, \tag{10.2.13}$$

by the Gauss theorem. For incrementally nonlinear response with $\dot{\mathbf{P}}$ defined by Eq. (10.1.9), χ being homogeneous of degree two in $\dot{\mathbf{F}}$, Eq. (10.2.13) becomes

$$2\int_{V^0}\chi\,\mathrm{d}V^0 = \int_{V^0}\rho^0\,\dot{\mathbf{b}}\cdot\mathbf{v}\,\mathrm{d}V^0 + \int_{S^0}\mathbf{n}^0\cdot\dot{\mathbf{P}}\cdot\mathbf{v}\,\mathrm{d}S^0. \tag{10.2.14}$$

The result is analogous to Clapeyron's formula of linear elasticity, and can be referred to as Clapeyron's formula of incrementally nonlinear response.

10.3. Variational Principle

If the stress rate field $\dot{\mathbf{P}}$ satisfies the equations of continuing equilibrium (10.2.12), then for any analytically admissible (not necessarily infinitesimal) velocity field $\delta\mathbf{v}$, it follows that

$$\int_{V^0}\dot{\mathbf{P}}\cdot\cdot\,\delta\dot{\mathbf{F}}\,\mathrm{d}V^0 = \int_{V^0}\rho^0\,\dot{\mathbf{b}}\cdot\delta\mathbf{v}\,\mathrm{d}V^0 + \int_{S^0}\mathbf{n}^0\cdot\dot{\mathbf{P}}\cdot\delta\mathbf{v}\,\mathrm{d}S^0, \tag{10.3.1}$$

again by the Gauss theorem. Recall that

$$\delta\dot{\mathbf{F}} = \delta\mathbf{v} \otimes \boldsymbol{\nabla}^0. \tag{10.3.2}$$

For incrementally nonlinear response with $\dot{\mathbf{P}}$ defined by Eq. (10.1.9), Eq. (10.3.1) becomes

$$\int_{V^0} \delta\chi \, \mathrm{d}V^0 = \int_{V^0} \rho^0 \, \dot{\mathbf{b}} \cdot \delta\mathbf{v} \, \mathrm{d}V^0 + \int_{S^0} \mathbf{n}^0 \cdot \dot{\mathbf{P}} \cdot \delta\mathbf{v} \, \mathrm{d}S^0. \tag{10.3.3}$$

Assuming that the rate of body forces is independent of the material response (deformation insensitive, dead body loading), Eq. (10.3.3) can be rewritten as

$$\delta\left(\int_{V^0} \chi \, \mathrm{d}V^0 - \int_{V^0} \rho^0 \, \dot{\mathbf{b}} \cdot \mathbf{v} \, \mathrm{d}V^0\right) = \int_{S_t^0} \dot{\mathbf{p}}_n \cdot \delta\mathbf{v} \, \mathrm{d}S_t^0, \tag{10.3.4}$$

provided that $\delta\mathbf{v}$ vanishes on $S_v^0 = S^0 - S_t^0$. If the current configuration is taken as the reference,

$$\delta\left(\int_{V} \underline{\chi} \, \mathrm{d}V - \int_{V} \rho \, \dot{\mathbf{b}} \cdot \mathbf{v} \, \mathrm{d}V\right) = \int_{S_t} \underline{\dot{\mathbf{p}}}_n \cdot \delta\mathbf{v} \, \mathrm{d}S_t, \tag{10.3.5}$$

since

$$\underline{\dot{\mathbf{p}}}_n \, \mathrm{d}S_t = \dot{\mathbf{p}}_n \, \mathrm{d}S_t^0. \tag{10.3.6}$$

The traction rate $\underline{\dot{\mathbf{p}}}_n$ is related to the rate of Cauchy traction $\dot{\mathbf{t}}_n$ by Eq. (3.9.18).

Suppose that the surface data over S_t consists of two parts,

$$\underline{\dot{\mathbf{p}}}_n = \underline{\dot{\mathbf{p}}}_n^{\mathrm{c}} + \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}}, \tag{10.3.7}$$

where $\underline{\dot{\mathbf{p}}}_n^{\mathrm{c}}$ is the controllable part of the incremental loading (independent of material response), and $\underline{\dot{\mathbf{p}}}_n^{\mathrm{s}}$ is the deformation-sensitive part allowing for the deformability of both material and tool (linear homogeneous expression in $\mathbf{v}$ and $\mathbf{L}$), Hill (1978). For instance, in the case of fluid pressure, $\mathbf{t}_n = -p\,\mathbf{n}$, it follows that

$$\dot{\mathbf{t}}_n = -\dot{p}\,\mathbf{n} - p\,\dot{\mathbf{n}}, \tag{10.3.8}$$

where, from Eq. (2.4.18),

$$\dot{\mathbf{n}} = (\mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n})\,\mathbf{n} - \mathbf{n} \cdot \mathbf{L}. \tag{10.3.9}$$

Thus, Eq. (3.9.18) gives

$$\underline{\dot{\mathbf{p}}}_n = -\dot{p}\,\mathbf{n} + p\,(\mathbf{n} \cdot \mathbf{L} - \mathbf{n}\,\mathrm{tr}\,\mathbf{D}). \tag{10.3.10}$$

The first term is deformation insensitive,

$$\underline{\dot{\mathbf{p}}}_n^{\rm c} = -\dot{p}\,\mathbf{n}, \tag{10.3.11}$$

while the remaining part is deformation sensitive,

$$\underline{\dot{\mathbf{p}}}_n^{\rm s} = p\,(\mathbf{n}\cdot\mathbf{L} - \mathbf{n}\,\mathrm{tr}\,\mathbf{D}). \tag{10.3.12}$$

A deformation-sensitive part of the incremental loading is self-adjoint if

$$\int_{S_t} \left(\underline{\dot{\mathbf{p}}}_n^{\rm s}\cdot\mathbf{v}^* - \underline{\dot{\mathbf{p}}}_n^{*\,\rm s}\cdot\mathbf{v}\right)\mathrm{d}S_t = 0, \tag{10.3.13}$$

for any two analytically admissible velocity fields $\mathbf{v}$ and $\mathbf{v}^*$ whose difference vanishes on S_v. Since $\underline{\dot{\mathbf{p}}}_n^{\rm s}$ is linear homogeneous, equivalent definitions are

$$\int_{S_t} \left(\underline{\dot{\mathbf{p}}}_n^{\rm s}\cdot\delta\mathbf{v} - \delta\underline{\dot{\mathbf{p}}}_n^{\rm s}\cdot\mathbf{v}\right)\mathrm{d}S_t = 0, \quad \text{i.e.,} \quad \int_{S_t} \underline{\dot{\mathbf{p}}}_n^{\rm s}\cdot\delta\mathbf{v}\,\mathrm{d}S_t = \frac{1}{2}\,\delta\int_{S_t}\underline{\dot{\mathbf{p}}}_n^{\rm s}\cdot\mathbf{v}\,\mathrm{d}S_t, \tag{10.3.14}$$

where $\delta\mathbf{v}$ is an analytically admissible infinitesimal variation of $\mathbf{v}$ that vanishes on S_v (Hill, *op. cit.*).

A true variational principle can be deduced from Eq. (10.3.5) when the surface data over S_t is self-adjoint in the sense of (10.3.14), since then

$$\delta\underline{\Xi} = 0, \tag{10.3.15}$$

with the variational integral

$$\underline{\Xi} = \int_V \underline{\chi}\,\mathrm{d}V - \int_V \rho\,\dot{\mathbf{b}}\cdot\mathbf{v}\,\mathrm{d}V - \int_{S_t}\left(\underline{\dot{\mathbf{p}}}_n^{\rm c} + \frac{1}{2}\,\underline{\dot{\mathbf{p}}}_n^{\rm s}\right)\cdot\mathbf{v}\,\mathrm{d}S_t. \tag{10.3.16}$$

Among all kinematically admissible velocity fields, the actual velocity field (whether unique or not) of the considered rate boundary-value problem renders stationary the functional $\underline{\Xi}(\mathbf{v})$. In Section 10.5 it will be shown that, under the uniqueness condition formulated in Section 10.4, the variational principle (10.3.15) with (10.3.16) can be strengthened to a minimum principle. Formulation of variational principles in the framework of infinitesimal strain is presented by Hill (1950), Drucker (1958,1960), and Koiter (1960). See also Ponter (1969), Neale (1972), and Sewell (1987).

10.3.1. Homogeneous Data

The incremental data is homogeneous at an instant of deformation process if

$$\dot{\mathbf{b}} = \mathbf{0} \quad \text{in} \quad V, \qquad \mathbf{v} = \mathbf{0} \quad \text{on} \quad S_v, \qquad \underline{\dot{\mathbf{p}}}_n^{\rm c} = \mathbf{0} \quad \text{on} \quad S_t, \tag{10.3.17}$$

at that instant. The corresponding homogeneous boundary value problem is governed by the variational principle

$$\delta\underline{\Xi} = 0, \quad \underline{\Xi} = \int_V \underline{\chi}\,\mathrm{d}V - \frac{1}{2}\int_{S_t} \dot{\underline{\mathbf{p}}}^{\mathrm{s}}_n \cdot \mathbf{v}\,\mathrm{d}S_t. \tag{10.3.18}$$

In addition, the Clapeyron formula (10.2.14) reduces to

$$\int_V \underline{\chi}\,\mathrm{d}V = \frac{1}{2}\int_{S_t} \dot{\underline{\mathbf{p}}}^{\mathrm{s}}_{(n)} \cdot \mathbf{v}\,\mathrm{d}S_t. \tag{10.3.19}$$

A possible nontrivial solution is characterized by both

$$\delta\underline{\Xi} = 0 \quad \text{and} \quad \underline{\Xi} = 0. \tag{10.3.20}$$

For example, if $\underline{\chi}$ is given by Eq. (10.1.15), we have

$$\underline{\Xi} = \frac{1}{2}\int_V \left[\underline{\boldsymbol{\mathcal{L}}}^{\mathrm{ep}}_{(0)} :: (\mathbf{D}\otimes\mathbf{D}) + \boldsymbol{\sigma} : \left(\mathbf{L}^T\cdot\mathbf{L} - 2\,\mathbf{D}^2\right)\right]\mathrm{d}V - \frac{1}{2}\int_{S_t} \dot{\underline{\mathbf{p}}}^{\mathrm{s}}_n \cdot \mathbf{v}\,\mathrm{d}S_t. \tag{10.3.21}$$

Recall that the traction rate $\dot{\underline{\mathbf{p}}}^{\mathrm{s}}_n$ is related to the rate of Cauchy traction by an equation such as (3.9.18). When the geometry of the body is such that an admissible velocity field gives rise to large spins and small strain rates (as in slender beams), the terms proportional to stress within the volume integral in (10.3.21) can be of the same order as the terms proportional to elastoplastic moduli, even when the stress components are small compared to instantaneous moduli.

10.4. Uniqueness of Solution

In this section we consider the uniqueness of solution to incrementally nonlinear boundary-value problem, described by the equations of continuing equilibrium,

$$\boldsymbol{\nabla}\cdot\dot{\underline{\mathbf{P}}} + \rho\,\dot{\mathbf{b}} = \mathbf{0}, \tag{10.4.1}$$

and the boundary conditions

$$\mathbf{v} = \mathbf{v}_0 \quad \text{on} \quad S_v, \qquad \mathbf{n}\cdot\dot{\underline{\mathbf{P}}} = \dot{\underline{\mathbf{p}}}_n \quad \text{on} \quad S_t. \tag{10.4.2}$$

Material response is incrementally nonlinear and governed by Eq. (10.1.9). The incremental body loading is assumed to be deformation-insensitive, while deformation-sensitive part of incremental surface loading is self-adjoint in the spirit of Eq. (10.3.13).

Following Hill (1958, 1961a, 1978), suppose that there are two different solutions of Eqs. (10.4.1) and (10.4.2), $\mathbf{v}$ and $\mathbf{v}^*$. The corresponding velocity gradients are $\mathbf{L}$ and $\mathbf{L}^*$, and the rates of nominal stress are

$$\underline{\dot{\mathbf{P}}} = \frac{\partial \underline{\underline{\chi}}}{\partial \mathbf{L}}, \quad \underline{\dot{\mathbf{P}}}^* = \frac{\partial \underline{\underline{\chi}}}{\partial \mathbf{L}^*}. \tag{10.4.3}$$

Then, since

$$\boldsymbol{\nabla} \cdot \left(\underline{\dot{\mathbf{P}}} - \underline{\dot{\mathbf{P}}}^* \right) = \mathbf{0}, \tag{10.4.4}$$

by the equations of equilibrium, the fields $\left(\underline{\dot{\mathbf{P}}}, \mathbf{L}\right)$ and $\left(\underline{\dot{\mathbf{P}}}^*, \mathbf{L}^*\right)$ necessarily satisfy the condition

$$\int_V \left(\underline{\dot{\mathbf{P}}}^* - \underline{\dot{\mathbf{P}}} \right) \cdot\cdot \left(\mathbf{L}^* - \mathbf{L} \right) \mathrm{d}V = \int_{S_t} \left(\underline{\dot{\mathbf{p}}}_n^{*\,\mathrm{s}} - \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \right) \cdot \left(\mathbf{v}^* - \mathbf{v} \right) \mathrm{d}S_t. \tag{10.4.5}$$

This follows upon application of the Gauss divergence theorem. Consequently, from Eq. (10.4.5) the velocity field $\mathbf{v}$ is unique if

$$\int_V \left(\underline{\dot{\mathbf{P}}}^* - \underline{\dot{\mathbf{P}}} \right) \cdot\cdot \left(\mathbf{L}^* - \mathbf{L} \right) \mathrm{d}V \neq \int_{S_t} \left(\underline{\dot{\mathbf{p}}}_n^{*\,\mathrm{s}} - \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \right) \cdot \left(\mathbf{v}^* - \mathbf{v} \right) \mathrm{d}S_t, \tag{10.4.6}$$

for all kinematically admissible $\mathbf{v}^*$ giving rise to

$$\mathbf{L}^* = \frac{\partial \mathbf{v}^*}{\partial \mathbf{x}}, \quad \underline{\dot{\mathbf{P}}}^* = \frac{\partial \underline{\underline{\chi}}}{\partial \mathbf{L}^*}. \tag{10.4.7}$$

The stress rate $\underline{\dot{\mathbf{P}}}^*$ in (10.4.6) need not be statically admissible, so even if equality sign applies in (10.4.6) for some $\mathbf{v}^*$, the uniqueness is lost only if $\mathbf{v}^*$ gives rise to statically admissible stress-rate field $\underline{\dot{\mathbf{P}}}^*$. Therefore, a sufficient condition for uniqueness is

$$\int_V \left(\underline{\dot{\mathbf{P}}}^* - \underline{\dot{\mathbf{P}}} \right) \cdot\cdot \left(\mathbf{L}^* - \mathbf{L} \right) \mathrm{d}V > \int_{S_t} \left(\underline{\dot{\mathbf{p}}}_n^{*\,\mathrm{s}} - \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \right) \cdot \left(\mathbf{v}^* - \mathbf{v} \right) \mathrm{d}S_t, \tag{10.4.8}$$

i.e.,

$$\int_V \left(\frac{\partial \underline{\underline{\chi}}}{\partial \mathbf{L}^*} - \frac{\partial \underline{\underline{\chi}}}{\partial \mathbf{L}} \right) \cdot\cdot \left(\mathbf{L}^* - \mathbf{L} \right) \mathrm{d}V > \int_{S_t} \left(\underline{\dot{\mathbf{p}}}_n^{*\,\mathrm{s}} - \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \right) \cdot \left(\mathbf{v}^* - \mathbf{v} \right) \mathrm{d}S_t, \tag{10.4.9}$$

for the differences of all distinct kinematically admissible velocity fields $\mathbf{v}$ and $\mathbf{v}^*$.

For a piecewise linear response, the uniqueness condition (10.4.8) becomes

$$\int_V \left(\underline{\boldsymbol{\Lambda}}^{*\,\mathrm{ep}} \cdot\cdot \mathbf{L}^* - \underline{\boldsymbol{\Lambda}}^{\mathrm{ep}} \cdot\cdot \mathbf{L} \right) \cdot\cdot \left(\mathbf{L}^* - \mathbf{L} \right) \mathrm{d}V > \int_{S_t} \left(\underline{\dot{\mathbf{p}}}_n^{*\,\mathrm{s}} - \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \right) \cdot \left(\mathbf{v}^* - \mathbf{v} \right) \mathrm{d}S_t. \tag{10.4.10}$$

The superimposed asterisk to one of the elastoplastic pseudomoduli tensors indicates that different loading branches (elastic or plastic) can correspond to different velocity fields $\mathbf{v}$ and $\mathbf{v}^*$ at each point of the continuum.

The condition (10.4.9), or (10.4.10), does not depend on prescribed $\underline{\dot{\mathbf{p}}}_n^{\mathrm{c}}$, nor does it depend on prescribed velocities on S_v, and is thus likely to be over-sufficient (i.e., not necessary).

The uniqueness condition (10.4.8) can be rewritten in terms of other stress measures. For example, it can be easily shown that

$$\underline{\dot{\mathbf{P}}} \cdot\cdot\, \mathbf{L}^* = \underline{\overset{\circ}{\boldsymbol{\tau}}} : \mathbf{D}^* - \boldsymbol{\sigma} : \left(2\mathbf{D}\cdot\mathbf{D}^* - \mathbf{L}^T \cdot \mathbf{L}^*\right), \tag{10.4.11}$$

so that in (10.4.8) we have

$$\begin{aligned}\left(\underline{\dot{\mathbf{P}}}^* - \underline{\dot{\mathbf{P}}}\right) \cdot\cdot\, (\mathbf{L}^* - \mathbf{L}) = (\underline{\overset{\circ}{\boldsymbol{\tau}}}^* - \underline{\overset{\circ}{\boldsymbol{\tau}}}) : (\mathbf{D}^* - \mathbf{D}) \\ - \boldsymbol{\sigma} : \left[2(\mathbf{D}^* - \mathbf{D})^2 - \left(\mathbf{L}^{*T} - \mathbf{L}^T\right)\cdot(\mathbf{L}^* - \mathbf{L})\right].\end{aligned} \tag{10.4.12}$$

10.4.1. Homogeneous Boundary Value Problem

A homogeneous boundary value problem for incrementally nonlinear material is described by

$$\boldsymbol{\nabla}\cdot\underline{\dot{\boldsymbol{P}}} = \mathbf{0}, \tag{10.4.13}$$

and the boundary conditions

$$\mathbf{w} = \mathbf{0} \quad \text{on} \quad S_v, \qquad \mathbf{n}\cdot\underline{\dot{\boldsymbol{P}}} = \underline{\dot{\boldsymbol{p}}}_n^{\mathrm{s}} \quad \text{on} \quad S_t, \tag{10.4.14}$$

where

$$\boldsymbol{L} = \frac{\partial \mathbf{w}}{\partial \mathbf{x}}, \quad \underline{\dot{\boldsymbol{P}}} = \frac{\partial \underline{\chi}}{\partial \underline{\boldsymbol{L}}}. \tag{10.4.15}$$

This has always a null solution $\mathbf{w} = \mathbf{0}$. If the homogeneous problem also has a nontrivial solution $\mathbf{w} \neq \mathbf{0}$, then from (10.4.5)

$$\int_V \underline{\chi}\,\mathrm{d}V = \frac{1}{2}\int_{S_t} \underline{\dot{\boldsymbol{p}}}_n^{\mathrm{s}} \cdot \mathbf{w}\,\mathrm{d}S_t, \quad 2\,\underline{\chi} = \underline{\dot{\boldsymbol{P}}} \cdot\cdot\, \boldsymbol{L}. \tag{10.4.16}$$

Thus, if the exclusion functional is positive,

$$\underline{\mathcal{F}}(\mathbf{w}) = \int_V \underline{\chi}(\mathbf{w})\,\mathrm{d}V - \frac{1}{2}\int_{S_t} \underline{\dot{\boldsymbol{p}}}_n^{\mathrm{s}}(\mathbf{w})\cdot\mathbf{w}\,\mathrm{d}S_t > 0, \tag{10.4.17}$$

for any kinematically admissible $\mathbf{w}$ giving rise to $\boldsymbol{L} = \partial\mathbf{w}/\partial\mathbf{x}$, the current state of material is incrementally unique (i.e., eigenstates under homogeneous data are excluded). In an eigenstate

$$\underline{\mathcal{F}}(\mathbf{w}) = 0, \tag{10.4.18}$$

for some kinematically admissible $\mathbf{w}$. Such an eigenmode $\mathbf{w}$ makes the exclusion functional stationary within the class of kinematically admissible variations $\delta\mathbf{w}$. Conversely, any kinematically admissible velocity field $\mathbf{w}$ that makes $\underline{\mathcal{F}}$ stationary is an eigenmode. This follows because for homogeneous problem the variational integral of Eq. (10.3.18) is equal to the exclusion functional,

$$\underline{\Xi} = \underline{\mathcal{F}}. \tag{10.4.19}$$

10.4.2. Incrementally Linear Comparison Material

In contrast to incrementally linear response, for incrementally nonlinear and piecewise linear response the difference $\underline{\dot{\mathbf{P}}} - \underline{\dot{\mathbf{P}}}^*$ is not a single-valued function of $\mathbf{v} - \mathbf{v}^*$, but of $\mathbf{v}$ and $\mathbf{v}^*$ individually. This makes direct application of the uniqueness criterion (10.4.8) and (10.4.10) for these materials more difficult. An indirect approach was introduced by Hill (1958, 1959, 1967). It is based on the notion of an incrementally linear comparison material, that is in a sense less stiff than the original material. Denote its rate potential by

$$\underline{\chi}^l = \frac{1}{2}\,\underline{\mathbf{\Lambda}}^l \cdots (\mathbf{L} \otimes \mathbf{L}). \tag{10.4.20}$$

If $\mathbf{v}$ and $\mathbf{v}^*$ are both solutions of the inhomogeneous boundary value problem corresponding to incrementally linear comparison material, then from (10.4.5)

$$\int_V \underline{\mathbf{\Lambda}}^l \cdots [(\mathbf{L}^* - \mathbf{L}) \otimes (\mathbf{L}^* - \mathbf{L})]\, \mathrm{d}V = \int_{S_t} \left(\underline{\dot{\mathbf{p}}}_n^{*\,\mathrm{s}} - \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}}\right) \cdot (\mathbf{v}^* - \mathbf{v})\, \mathrm{d}S_t. \tag{10.4.21}$$

A sufficient condition for uniqueness is therefore

$$\int_V \underline{\mathbf{\Lambda}}^l \cdots [(\mathbf{L}^* - \mathbf{L}) \otimes (\mathbf{L}^* - \mathbf{L})]\, \mathrm{d}V > \int_{S_t} \left(\underline{\dot{\mathbf{p}}}_n^{*\,\mathrm{s}} - \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}}\right) \cdot (\mathbf{v}^* - \mathbf{v})\, \mathrm{d}S_t, \tag{10.4.22}$$

for the difference of all distinct kinematically admissible velocity fields $\mathbf{v}$ and $\mathbf{v}^*$.

Following the development of Section 7.8 for incrementally linear elastic material, consider a homogeneous problem described by (10.4.13) and (10.4.14), where

$$\boldsymbol{L} = \frac{\partial \mathbf{w}}{\partial \mathbf{x}}, \quad \underline{\dot{\boldsymbol{P}}} = \underline{\mathbf{\Lambda}}^l \cdot\cdot\, \boldsymbol{L}. \tag{10.4.23}$$

There is always a null solution $\mathbf{w} = \mathbf{0}$ to this problem. If the homogeneous problem also has a nontrivial solution $\mathbf{w} \neq \mathbf{0}$, then from (10.4.21)

$$\frac{1}{2} \int_V \underline{\boldsymbol{\Lambda}}^l \cdot\cdot\cdot\cdot (\boldsymbol{L} \otimes \boldsymbol{L}) \, \mathrm{d}V = \int_{S_t} \underline{\dot{\boldsymbol{p}}}_n^{\mathrm{s}} \cdot \mathbf{w} \, \mathrm{d}S_t. \tag{10.4.24}$$

The examination of the uniqueness of solution to inhomogeneous problem for incrementally linear comparison material is thus equivalent to examination of the uniqueness of solution to the associated homogeneous problem. Consequently, the uniqueness is assured, i.e., the inequality (10.4.22) is satisfied, if

$$\underline{\mathcal{F}} = \int_V \underline{\chi}^l(\mathbf{w}) \, \mathrm{d}V - \frac{1}{2} \int_{S_t} \underline{\dot{\boldsymbol{p}}}_n^{\mathrm{s}}(\mathbf{w}) \cdot \mathbf{w} \, \mathrm{d}S_t > 0, \ \ \underline{\chi}^l(\mathbf{w}) = \frac{1}{2} \underline{\boldsymbol{\Lambda}}^l \cdot\cdot\cdot\cdot (\boldsymbol{L} \otimes \boldsymbol{L}), \tag{10.4.25}$$

for any kinematically admissible $\mathbf{w}$ giving rise to $\boldsymbol{L} = \partial \mathbf{w} / \partial \mathbf{x}$.

Suppose that, at the given state of deformation, the exclusion condition (10.4.25) is satisfied for incrementally linear material with the rate potential $\underline{\chi}^l$. Then, if

$$\underline{\chi}^l \leq \underline{\chi} \tag{10.4.26}$$

at each point (linear comparison material in this sense being less stiff), the exclusion functional (10.4.17) for incrementally nonlinear material with the rate potential $\underline{\chi}$ is also satisfied, precluding eigenstates under homogeneous data.

More strongly, if (10.4.25) is satisfied and the function $\underline{\chi} - \underline{\chi}^l$ is convex at each point, bifurcation is ruled out for any associated inhomogeneous data (Hill, 1978). Indeed, for convex function $\underline{\chi} - \underline{\chi}^l$, by definition of convexity we can write

$$\underline{\chi}(\mathbf{L}^*) - \underline{\chi}^l(\mathbf{L}^*) - [\underline{\chi}(\mathbf{L}) - \underline{\chi}^l(\mathbf{L})] \geq \frac{\partial(\underline{\chi} - \underline{\chi}^l)}{\partial \mathbf{L}} \cdot\cdot (\mathbf{L}^* - \mathbf{L}), \tag{10.4.27}$$

and likewise

$$\underline{\chi}(\mathbf{L}) - \underline{\chi}^l(\mathbf{L}) - [\underline{\chi}(\mathbf{L}^*) - \underline{\chi}^l(\mathbf{L}^*)] \geq \frac{\partial(\underline{\chi} - \underline{\chi}^l)}{\partial \mathbf{L}^*} \cdot\cdot (\mathbf{L} - \mathbf{L}^*). \tag{10.4.28}$$

The convexity condition (10.4.27) is schematically depicted in (Fig. 10.1). By summing up the above two inequalities, we obtain

$$\left[\frac{\partial(\underline{\chi} - \underline{\chi}^l)}{\partial \mathbf{L}^*} - \frac{\partial(\underline{\chi} - \underline{\chi}^l)}{\partial \mathbf{L}} \right] \cdot\cdot (\mathbf{L}^* - \mathbf{L}) \geq 0. \tag{10.4.29}$$

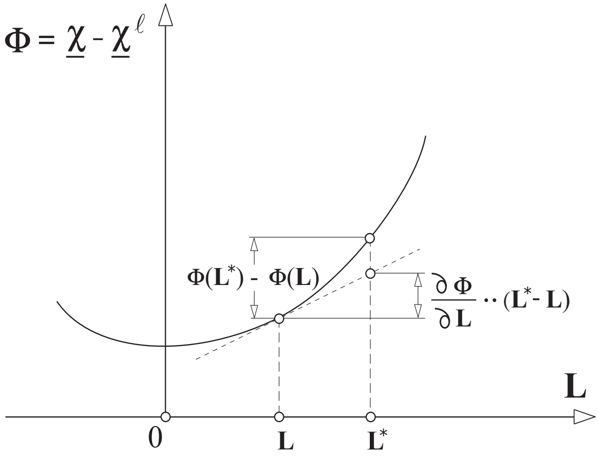

FIGURE 10.1. Schematic illustration of the convexity condition (10.4.27).

In view of Eq. (10.4.3) for the rates of nominal stress, and Eq. (10.4.20) for the rate potential, the inequality (10.4.29) can be recast in the following form

$$\left(\underline{\dot{\mathbf{P}}}^* - \underline{\dot{\mathbf{P}}}\right) \cdot\cdot (\mathbf{L}^* - \mathbf{L}) \geq \left(\underline{\dot{\mathbf{P}}}^* - \underline{\dot{\mathbf{P}}}\right)^l \cdot\cdot (\mathbf{L}^* - \mathbf{L}) \\ = \underline{\mathbf{\Lambda}}^l \cdot\cdot\cdot\cdot [(\mathbf{L}^* - \mathbf{L}) \otimes (\mathbf{L}^* - \mathbf{L})]. \tag{10.4.30}$$

Therefore, if the inequality (10.4.25), implying (10.4.22), is satisfied for incrementally linear comparison material (i.e., if there is no bifurcation for incrementally linear comparison material), the convexity of the function $\underline{\chi} - \underline{\chi}^l$, leading to (10.4.30), assures that the inequality (10.4.8) is also satisfied, ruling out any bifurcation of incrementally nonlinear material at the considered state. On the other hand, if the current configuration is a primary eigenstate for $\underline{\chi}^l$ material, i.e., $\underline{\mathcal{F}} = 0$ in (10.4.25), the bifurcation may still be excluded for $\underline{\chi}$ material, if $\underline{\chi} - \underline{\chi}^l$ is strictly convex in $\boldsymbol{L}$ (strict inequality applies in (10.4.29)).

For an analysis of uniqueness in the case of an incrementally nonlinear material model without a rate potential function, see the paper by Chambon and Caillerie (1999).

10.4.3. Comparison Material for Elastoplastic Response

For elastoplastic response with a piecewise linear relation defined by

$$\underline{\dot{\mathbf{P}}} = \underline{\mathbf{\Lambda}}^{\mathrm{ep}} \cdot\cdot\, \mathbf{L}, \quad \underline{\mathbf{\Lambda}}^{\mathrm{ep}} = \begin{cases} \underline{\mathbf{\Lambda}}^{\mathrm{p}}, & \text{for plastic loading,} \\ \underline{\mathbf{\Lambda}}, & \text{for elastic unloading or neutral loading,} \end{cases} \tag{10.4.31}$$

an incrementally linear comparison material can be taken to be the material whose stiffness is equal to $\underline{\mathbf{\Lambda}}^{\mathrm{p}}$ at plastically stressed points of the continuum. Elsewhere in the continuum, i.e., at elastically stressed points, the comparison material has the stiffness equal to $\underline{\mathbf{\Lambda}}$. The following is a proof of the required condition,

$$(\underline{\mathbf{\Lambda}}^{*\,\mathrm{ep}} \cdot\cdot\, \mathbf{L}^* - \underline{\mathbf{\Lambda}}^{\mathrm{ep}} \cdot\cdot\, \mathbf{L}) \cdot\cdot\, (\mathbf{L}^* - \mathbf{L}) \geq \underline{\mathbf{\Lambda}}^{\mathrm{p}} \cdot\cdot\cdot\cdot\, [(\mathbf{L}^* - \mathbf{L}) \otimes (\mathbf{L}^* - \mathbf{L})] \tag{10.4.32}$$

for the identification of selected incrementally linear comparison material, with the stiffness $\underline{\mathbf{\Lambda}}^{\mathrm{p}}$.

From Eqs. (9.1.1) and (9.1.4), a piecewise linear elastoplastic response is governed by

$$\dot{\mathbf{T}}_{(n)} = \mathbf{\Lambda}^{\mathrm{p}}_{(n)} : \dot{\mathbf{E}}_{(n)} - \left[\dot{\gamma}_{(n)} - \frac{1}{h_{(n)}} \left(\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} \right) \right] \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}}, \tag{10.4.33}$$

where, by Eq. (9.1.13),

$$\mathbf{\Lambda}^{\mathrm{p}}_{(n)} = \mathbf{\Lambda}_{(n)} - \frac{1}{h_{(n)}} \left(\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} \otimes \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} \right). \tag{10.4.34}$$

The loading index is

$$\dot{\gamma}_{(n)} = \frac{1}{h_{(n)}} \left(\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} \right) > 0 \tag{10.4.35}$$

for plastic loading, and $\dot{\gamma}_{(n)} = 0$ for elastic unloading or neutral loading. Consequently,

$$\begin{aligned} \left(\dot{\mathbf{T}}^*_{(n)} - \dot{\mathbf{T}}_{(n)} \right) : \left(\dot{\mathbf{E}}^*_{(n)} - \dot{\mathbf{E}}_{(n)} \right) &= \mathbf{\Lambda}^{\mathrm{p}}_{(n)} :: \left[\left(\dot{\mathbf{E}}^*_{(n)} - \dot{\mathbf{E}}_{(n)} \right) \otimes \left(\dot{\mathbf{E}}^*_{(n)} - \dot{\mathbf{E}}_{(n)} \right) \right] \\ &- \left[\dot{\gamma}^*_{(n)} - \dot{\gamma}_{(n)} - \frac{1}{h_{(n)}} \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \left(\dot{\mathbf{E}}^*_{(n)} - \dot{\mathbf{E}}_{(n)} \right) \right] \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \left(\dot{\mathbf{E}}^*_{(n)} - \dot{\mathbf{E}}_{(n)} \right). \end{aligned} \tag{10.4.36}$$

If both $\dot{\mathbf{E}}_{(n)}$ and $\dot{\mathbf{E}}^*_{(n)}$ correspond to plastic loading from the current state, the terms within square brackets in the second line of Eq. (10.4.36) cancel each other. If one strain rate corresponds to plastic loading and the other to elastic unloading, or if both strain rates correspond to elastic unloading, or if one strain rate corresponds to elastic unloading and the other to neutral loading, the whole expression

$$\left[\dot{\gamma}^*_{(n)} - \dot{\gamma}_{(n)} - \frac{1}{h_{(n)}}\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \left(\dot{\mathbf{E}}^*_{(n)} - \dot{\mathbf{E}}_{(n)}\right)\right] \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \left(\dot{\mathbf{E}}^*_{(n)} - \dot{\mathbf{E}}_{(n)}\right) \tag{10.4.37}$$

is negative. If both strain rates correspond to neutral loading, or one to neutral loading and the other to plastic loading, the above expression vanishes. Thus, from Eq. (10.4.36) it follows that

$$\left(\dot{\mathbf{T}}^*_{(n)} - \dot{\mathbf{T}}_{(n)}\right) : \left(\dot{\mathbf{E}}^*_{(n)} - \dot{\mathbf{E}}_{(n)}\right) \geq \mathbf{\Lambda}^{\mathrm{p}}_{(n)} :: \left[\left(\dot{\mathbf{E}}^*_{(n)} - \dot{\mathbf{E}}_{(n)}\right) \otimes \left(\dot{\mathbf{E}}^*_{(n)} - \dot{\mathbf{E}}_{(n)}\right)\right]. \tag{10.4.38}$$

This means that actual piecewise linear response is more convex than a hypothetical linear response with the stiffness moduli $\mathbf{\Lambda}^{\mathrm{p}}_{(n)}$ over the entire $\dot{\mathbf{E}}_{(n)}$ space.

If the current configuration is taken as the reference, (10.4.38) becomes

$$\left(\underline{\dot{\mathbf{T}}}^*_{(n)} - \underline{\dot{\mathbf{T}}}_{(n)}\right) : (\mathbf{D}^* - \mathbf{D}) \geq \underline{\mathbf{\Lambda}}^{\mathrm{p}}_{(n)} :: \left[(\mathbf{D}^* - \mathbf{D}) \otimes (\mathbf{D}^* - \mathbf{D})\right], \tag{10.4.39}$$

or, since $\underline{\dot{\mathbf{T}}}_{(n)}$ and $\underline{\mathbf{\Lambda}}_{(n)}$ are fully symmetric tensors,

$$\left(\underline{\dot{\mathbf{T}}}^*_{(n)} - \underline{\dot{\mathbf{T}}}_{(n)}\right) \cdot\cdot\, (\mathbf{L}^* - \mathbf{L}) \geq \underline{\mathbf{\Lambda}}^{\mathrm{p}}_{(n)} \cdot\cdot\cdot\cdot \left[(\mathbf{L}^* - \mathbf{L}) \otimes (\mathbf{L}^* - \mathbf{L})\right]. \tag{10.4.40}$$

To express this condition in terms of $\underline{\dot{\mathbf{P}}}$ and $\underline{\mathbf{\Lambda}}^{\mathrm{p}}$, the choice $n = 1$ is conveniently made in Eq. (10.4.40). Since

$$\underline{\dot{\mathbf{T}}}_{(1)} = \underline{\dot{\mathbf{P}}} - \boldsymbol{\sigma} \cdot \mathbf{L}^T, \tag{10.4.41}$$

from the second of Eq. (10.1.5), and recalling the relationship between the components of elastoplastic moduli and pseudomoduli given by Eq. (10.1.7), the substitution into Eq. (10.4.40) gives

$$\left(\underline{\dot{\mathbf{P}}}^* - \underline{\dot{\mathbf{P}}}\right) \cdot\cdot\, (\mathbf{L}^* - \mathbf{L}) \geq \underline{\mathbf{\Lambda}}^{\mathrm{p}} \cdot\cdot\cdot\cdot \left[(\mathbf{L}^* - \mathbf{L}) \otimes (\mathbf{L}^* - \mathbf{L})\right]. \tag{10.4.42}$$

This is precisely the condition (10.4.32).

In conclusion, the bifurcation problem for a piecewise linear elastoplastic material with the stiffness $\underline{\mathbf{\Lambda}}^{\mathrm{ep}}$ is reduced to determining primary eigenstate of incrementally linear comparison material with the stiffness $\underline{\mathbf{\Lambda}}^{\mathrm{p}}$. Among infinitely many deformation modes that are all solutions of given inhomogeneous problem for $\underline{\mathbf{\Lambda}}^{\mathrm{p}}$ material at that state (these being the sums of the increment of the fundamental solution and any multiple of the eigenmode solution), there may be those for which the strain rate at every plastically stressed point is in the plastic loading range of the $\underline{\boldsymbol{\Lambda}}^{\mathrm{ep}}$ material itself. Such deformation modes are then also solutions of the given inhomogeneous rate problem of the $\underline{\boldsymbol{\Lambda}}^{\mathrm{ep}}$ material, which means that a primary bifurcation for this material has been identified (Hill, 1978).

For incrementally linear comparison material bifurcation can occur in an eigenstate for any prescribed traction rates on S_t, and velocities on S_v. In an actual elastoplastic material bifurcation occurs only for those traction rates and prescribed velocities for which there is no elastic unloading in the current plastic region of the body. See also Nguyen (1987,1994), Triantafyllidis (1983), and Petryk (1989).

For solids with corners on their yield surfaces, comparison material is defined as a hypothetical material whose every yield system is active. For example, with a pyramidal vertex formed by k_0 intersecting segments, from Section 9.5 it follows that

$$\boldsymbol{\Lambda}^{\mathrm{p}} = \boldsymbol{\Lambda}_{(n)} - \sum_{i=1}^{k_0}\sum_{j=1}^{k_0} h_{(n)}^{<ij>-1} \left(\boldsymbol{\Lambda}_{(n)} : \frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}} \right) \otimes \left(\frac{\partial f_{(n)}^{<j>}}{\partial \mathbf{T}_{(n)}} : \boldsymbol{\Lambda}_{(n)} \right). \tag{10.4.43}$$

The range of strain rate space in which no elastic unloading occurs on any yield segment is called fully active or total loading range (Sewell, 1972; Hutchinson, 1974). In the context of crystal plasticity, this is further discussed in Chapter 12.

10.5. Minimum Principle

If the uniqueness condition (10.4.8) applies, the variational principle (10.3.15) with (10.3.16) can be strengthened to a minimum principle. Let $\mathbf{v}$ be the actual unique solution of the considered problem, and $\mathbf{v}^*$ any kinematically admissible velocity field. First, it is observed that

$$\begin{aligned} \underline{\Xi}(\mathbf{v}^*) - \underline{\Xi}(\mathbf{v}) = & \int_V (\underline{\chi}^* - \underline{\chi})\,\mathrm{d}V - \int_V \rho\,\dot{\mathbf{b}} \cdot (\mathbf{v}^* - \mathbf{v})\,\mathrm{d}V \\ & - \int_{S_t} \underline{\dot{\mathbf{p}}}_n^{\mathrm{c}} \cdot (\mathbf{v}^* - \mathbf{v})\,\mathrm{d}S_t - \frac{1}{2}\int_{S_t} \left(\underline{\dot{\mathbf{p}}}_n^{*\,\mathrm{s}} + \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \right) \cdot (\mathbf{v}^* - \mathbf{v})\,\mathrm{d}S_t, \end{aligned} \tag{10.5.1}$$

and

$$\frac{1}{2}\left(\underline{\dot{\mathbf{p}}}_n^{*\,\mathrm{s}} + \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \right) = \frac{1}{2}\left(\underline{\dot{\mathbf{p}}}_n^{*\,\mathrm{s}} - \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \right) + \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}}. \tag{10.5.2}$$

Thus,

$$\underline{\Xi}(\mathbf{v}^*) - \underline{\Xi}(\mathbf{v}) = \int_V (\underline{\chi}^* - \underline{\chi})\,\mathrm{d}V - \frac{1}{2}\int_{S_t} \left(\dot{\underline{\mathbf{p}}}_n^{*\,\mathrm{s}} - \dot{\underline{\mathbf{p}}}_n^{\mathrm{s}}\right)\cdot(\mathbf{v}^* - \mathbf{v})\,\mathrm{d}S_t$$
$$- \int_V \rho\,\dot{\mathbf{b}}\cdot(\mathbf{v}^* - \mathbf{v})\,\mathrm{d}V - \int_{S_t} \dot{\underline{\mathbf{p}}}_n\cdot(\mathbf{v}^* - \mathbf{v})\,\mathrm{d}S_t. \tag{10.5.3}$$

The two surface integrals can be expressed by the Gauss theorem as the volume integrals, see (10.2.13), with the result

$$\underline{\Xi}(\mathbf{v}^*) - \underline{\Xi}(\mathbf{v}) = \int_V (\underline{\chi}^* - \underline{\chi})\,\mathrm{d}V - \int_V \dot{\underline{\mathbf{P}}}\cdot\cdot(\mathbf{L}^* - \mathbf{L})\,\mathrm{d}V$$
$$- \frac{1}{2}\int_{S_t} \left(\dot{\underline{\mathbf{p}}}_n^{*\,\mathrm{s}} - \dot{\underline{\mathbf{p}}}_n^{\mathrm{s}}\right)\cdot(\mathbf{v}^* - \mathbf{v})\,\mathrm{d}S_t. \tag{10.5.4}$$

For the minimum principle to hold, it is required to prove that the right hand side of Eq. (10.5.4) is positive. Following Hill (1978), introduce a continuous sequence of kinematically admissible fields

$$\mathbf{v}^+(\alpha) = \mathbf{v} + \alpha(\mathbf{v}^+ - \mathbf{v}), \quad 0 \le \alpha \le 1, \tag{10.5.5}$$

the parameter α being uniform throughout the body. Then, by Eq. (10.5.4),

$$\underline{\Xi}(\mathbf{v}^+) - \underline{\Xi}(\mathbf{v}) = \int_V (\underline{\chi}^+ - \underline{\chi})\,\mathrm{d}V - \int_V \dot{\underline{\mathbf{P}}}\cdot\cdot(\mathbf{L}^+ - \mathbf{L})\,\mathrm{d}V$$
$$- \frac{1}{2}\int_{S_t} \left(\dot{\underline{\mathbf{p}}}_n^{+\,\mathrm{s}} - \dot{\underline{\mathbf{p}}}_n^{\mathrm{s}}\right)\cdot(\mathbf{v}^+ - \mathbf{v})\,\mathrm{d}S_t. \tag{10.5.6}$$

Here,

$$\mathbf{L}^+ = \mathbf{L} + \alpha(\mathbf{L}^* - \mathbf{L}), \tag{10.5.7}$$

and, since $\dot{\underline{\mathbf{p}}}_n^{\mathrm{s}}$ is linear homogeneous in velocity gradient,

$$\dot{\underline{\mathbf{p}}}_n^{+\,\mathrm{s}} = \dot{\underline{\mathbf{p}}}_n^{\mathrm{s}} + \alpha\left(\dot{\underline{\mathbf{p}}}_n^{*\,\mathrm{s}} - \dot{\underline{\mathbf{p}}}_n^{\mathrm{s}}\right). \tag{10.5.8}$$

Consequently,

$$\alpha\,\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\underline{\Xi}(\mathbf{v}^+) - \underline{\Xi}(\mathbf{v})\right] = \int_V \left(\dot{\underline{\mathbf{P}}}^+ - \dot{\underline{\mathbf{P}}}\right)\cdot\cdot(\mathbf{L}^+ - \mathbf{L})\,\mathrm{d}V$$
$$- \int_{S_t} \left(\dot{\underline{\mathbf{p}}}_n^{+\,\mathrm{s}} - \dot{\underline{\mathbf{p}}}_n^{\mathrm{s}}\right)\cdot(\mathbf{v}^+ - \mathbf{v})\,\mathrm{d}S_t > 0, \tag{10.5.9}$$

which is positive by the uniqueness condition (10.4.8), applied to fields $\mathbf{v}$ and $\mathbf{v}^+$. In the derivation it is recalled that $\underline{\chi}$ is a homogeneous function of degree two, so that

$$\alpha\,\frac{\mathrm{d}\chi^+}{\mathrm{d}\alpha} = \frac{\partial\chi^+}{\partial\mathbf{L}^+}\cdot\cdot(\mathbf{L}^+ - \mathbf{L}) = 2\,\chi^+ - \dot{\underline{\mathbf{P}}}^+\cdot\cdot\mathbf{L} = \dot{\underline{\mathbf{P}}}^+\cdot\cdot(\mathbf{L}^+ - \mathbf{L}). \tag{10.5.10}$$

Therefore, in the range $0 < \alpha \leq 1$ the function $\Xi(\mathbf{v}^+) - \Xi(\mathbf{v})$ has a positive gradient,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\Xi(\mathbf{v}^+) - \Xi(\mathbf{v})\right] > 0. \tag{10.5.11}$$

Since $\Xi(\mathbf{v}^+) - \Xi(\mathbf{v})$ is equal to zero for $\alpha = 0$, it follows that

$$\Xi(\mathbf{v}^+) - \Xi(\mathbf{v}) > 0, \quad 0 < \alpha \leq 1, \tag{10.5.12}$$

which is a desired result. Thus,

$$\Xi(\mathbf{v}^*) > \Xi(\mathbf{v}) \tag{10.5.13}$$

for all kinematically admissible velocity fields $\mathbf{v}^*$, which implies a minimum principle.

10.6. Stability of Equilibrium

Consider an equilibrium state of the body whose response is incrementally nonlinear with the rate potential χ. Let the current equilibrium stress field be $\mathbf{P}$, associated with the body force $\mathbf{b}$ within V^0, the traction $\mathbf{p}_n$ over S_t^0, and prescribed displacement on the remaining part of the boundary $S^0 - S_t^0$. Assume that an infinitesimal virtual displacement field $\delta\mathbf{u}$ is imposed on the body ($\delta\mathbf{u} = \mathbf{0}$ on $S^0 - S_t^0$), under dead body force and unchanged controllable part of the surface loading. The work done by applied forces on this virtual displacement is

$$\int_{V^0} \rho^0\, \mathbf{b} \cdot \delta\mathbf{u}\, \mathrm{d}V^0 + \int_{S_t^0} \left(\mathbf{p}_n + \frac{1}{2}\,\delta\mathbf{p}_n^{\mathrm{s}}\right) \cdot \delta\mathbf{u}\, \mathrm{d}S_t^0, \tag{10.6.1}$$

since deformation-sensitive change $\delta\mathbf{p}_n^{\mathrm{s}}$, induced by $\delta\mathbf{u}$, is linear in $\delta\mathbf{u}$. The stress field $\mathbf{P}$ changes to $\mathbf{P} + \delta\mathbf{P}$, where $\delta\mathbf{P}$ is constitutively associated with the displacement increment $\delta\mathbf{u}$ through

$$\delta\mathbf{P} = \frac{\partial\chi}{\partial(\delta\mathbf{F})}\,, \quad \delta\mathbf{F} = \frac{\partial(\delta\mathbf{u})}{\partial\mathbf{X}}\,. \tag{10.6.2}$$

Kinematically admissible neighboring configurations need not be equilibrium configurations, i.e., the stress field $\mathbf{P} + \delta\mathbf{P}$ need not be an equilibrium field. The increment of internal energy associated with virtual change $\delta\mathbf{u}$ is, to second order,

$$\int_{V^0} \left(\mathbf{P} + \frac{1}{2}\,\delta\mathbf{P}\right) \cdot\cdot\, \delta\mathbf{F}\, \mathrm{d}V^0. \tag{10.6.3}$$

According to the energy criterion of stability, the underlying equilibrium configuration is stable if the increase of internal energy due to $\delta\mathbf{u}$ is greater than the work done by already applied forces in the virtual transition. Upon using (10.6.1), (10.6.3) and the formula (3.12.1), the stability condition becomes (Hill, 1958,1978)

$$\int_{V^0} \delta\mathbf{P} \cdot\cdot\, \delta\mathbf{F}\, \mathrm{d}V^0 > \int_{S_t^0} \delta\mathbf{p}_n^{\mathrm{s}} \cdot \delta\mathbf{u}\, \mathrm{d}S_t^0, \tag{10.6.4}$$

or

$$\int_{V^0} \dot{\mathbf{P}} \cdot\cdot\, \dot{\mathbf{F}}\, \mathrm{d}V^0 > \int_{S_t^0} \dot{\mathbf{p}}_n^{\mathrm{s}} \cdot \mathbf{v}\, \mathrm{d}S_t^0, \tag{10.6.5}$$

for all admissible velocity fields $\mathbf{v}$ vanishing on $S_v^0 = S^0 - S_t^0$. Since χ is a homogeneous function of $\dot{\mathbf{F}}$ of degree two, and $\dot{\mathbf{P}} = \partial\chi/\partial\dot{\mathbf{F}}$, (10.6.5) can be rewritten as

$$\int_{V^0} \chi\, \mathrm{d}V^0 > \frac{1}{2}\int_{S_t^0} \dot{\mathbf{p}}_n^{\mathrm{s}} \cdot \mathbf{v}\, \mathrm{d}S_t^0. \tag{10.6.6}$$

If the current configuration is taken as the reference, the stability criterion becomes

$$\int_V \underline{\chi}\, \mathrm{d}V = \frac{1}{2}\int_V \underline{\dot{\mathbf{P}}} \cdot\cdot\, \mathbf{L}\, \mathrm{d}V > \frac{1}{2}\int_{S_t} \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \cdot \mathbf{v}\, \mathrm{d}S_t. \tag{10.6.7}$$

10.7. Relationship between Uniqueness and Stability Criteria

In this section we compare the uniqueness criterion from Section 10.4,

$$\int_V \left(\underline{\dot{\mathbf{P}}}^* - \underline{\dot{\mathbf{P}}}\right) \cdot\cdot\, (\mathbf{L}^* - \mathbf{L})\, \mathrm{d}V > \int_{S_t} \left(\underline{\dot{\mathbf{p}}}_n^{*\,\mathrm{s}} - \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}}\right) \cdot (\mathbf{v}^* - \mathbf{v})\, \mathrm{d}S_t, \tag{10.7.1}$$

with the stability condition

$$\int_V \underline{\dot{\mathbf{P}}} \cdot\cdot\, \mathbf{L}\, \mathrm{d}V > \int_{S_t} \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \cdot \mathbf{v}\, \mathrm{d}S_t. \tag{10.7.2}$$

For kinematically admissible fields $\mathbf{v}$ and $\mathbf{v}^*$ vanishing on $S_v = S - S_t$, the field $\mathbf{v} - \mathbf{v}^*$ also vanishes on S_v and can be used as an admissible field in (10.7.2). The condition (10.7.2) is then equivalent to (10.7.1) only if the response is incrementally linear, so that $\underline{\dot{\mathbf{P}}}^* - \underline{\dot{\mathbf{P}}}$ is a linear function of $\mathbf{v} - \mathbf{v}^*$. For nonlinear and piecewise linear response this is not the case and two conditions are not equivalent.

Suppose the uniqueness condition (10.7.1) is satisfied when S_v is rigidly constrained or absent. Since the field $\mathbf{v}^* = \mathbf{0}$ is an admissible field for this boundary condition, it can be combined in (10.7.1) with any other nonzero admissible field $\mathbf{v}$, reproducing (10.7.2). Thus, when the sufficient condition for uniqueness of the rate boundary value problem at given state is satisfied for rigidly constrained or absent S_v, the underlying equilibrium state is also stable. The converse is not necessarily true for incrementally nonlinear material. The boundary value problem need not have unique solution in a stable state, i.e., (10.7.2) may be satisfied but not (10.7.1). A stable bifurcation could occur, although not under dead loading ($\dot{\mathbf{b}} = \mathbf{0}$ and $\underline{\dot{\mathbf{p}}}_n^{\mathrm{c}} = \mathbf{0}$), since this would imply that

$$\int_V \underline{\dot{\mathbf{P}}} \cdot\cdot\, \mathbf{L}\, \mathrm{d}V = \int_{S_t} \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \cdot \mathbf{v}\, \mathrm{d}S_t, \tag{10.7.3}$$

for the actual velocity field (by the divergence theorem). The loading would have to change such that

$$\begin{aligned} \int_V \rho\, \dot{\mathbf{b}} \cdot \mathbf{v}\, \mathrm{d}V &+ \int_{S_t} \dot{\mathbf{p}}_n^{\mathrm{c}} \cdot \mathbf{v}\, \mathrm{d}S_t \\ &= \int_V \underline{\dot{\mathbf{P}}} \cdot\cdot\, \mathbf{L}\, \mathrm{d}V - \int_{S_t} \underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} \cdot \mathbf{v}\, \mathrm{d}S_t > 0, \end{aligned} \tag{10.7.4}$$

for any actual field at the bifurcation.

Denote by V^{e} the elastically stressed part of the body, and by V^{p} the remaining plastically stressed part (i.e., the part that is at the state of incipient yield), and assume that $\underline{\dot{\mathbf{p}}}_n^{\mathrm{s}} = \mathbf{0}$ on S_t for any kinematically admissible velocity. For rigidly constrained or absent S_v, the uniqueness condition becomes

$$\int_V \underline{\dot{\mathbf{P}}} \cdot\cdot\, \mathbf{L}\, \mathrm{d}V = \int_{V^{\mathrm{e}}} \underline{\dot{\mathbf{P}}} \cdot\cdot\, \mathbf{L}\, \mathrm{d}V^{\mathrm{e}} + \int_{V^{\mathrm{p}}} \underline{\dot{\mathbf{P}}} \cdot\cdot\, \mathbf{L}\, \mathrm{d}V^{\mathrm{p}} > 0. \tag{10.7.5}$$

In the elastic region V^{e} the response is incrementally linear, $\underline{\dot{\mathbf{P}}} = \underline{\boldsymbol{\Lambda}} \cdot\cdot\, \mathbf{L}$. For incrementally linear comparison material in the plastic region V^{p}, we have $\underline{\dot{\mathbf{P}}} = \underline{\boldsymbol{\Lambda}}^{\mathrm{p}} \cdot \mathbf{L}$. Thus, (10.7.5) is replaced with

$$\int_{V^{\mathrm{e}}} \underline{\boldsymbol{\Lambda}} \cdot\cdot\cdot\cdot\, (\mathbf{L} \otimes \mathbf{L})\, \mathrm{d}V^{\mathrm{e}} + \int_{V^{\mathrm{p}}} \underline{\boldsymbol{\Lambda}}^{\mathrm{p}} \cdot\cdot\cdot\cdot\, (\mathbf{L} \otimes \mathbf{L})\, \mathrm{d}V^{\mathrm{p}} > 0, \tag{10.7.6}$$

or, in view of (10.1.10) and (10.1.15),

$$\begin{aligned} \int_{V^{\mathrm{e}}} \underline{\boldsymbol{\mathcal{L}}}_{(0)} :: (\mathbf{D} \otimes \mathbf{D})\, \mathrm{d}V^{\mathrm{e}} &+ \int_{V^{\mathrm{p}}} \underline{\boldsymbol{\mathcal{L}}}_{(0)}^{\mathrm{p}} :: (\mathbf{D} \otimes \mathbf{D}) \mathrm{d}V^{\mathrm{p}} \\ &> \int_V \boldsymbol{\sigma} : \left(2\, \mathbf{D}^2 - \mathbf{L}^T \cdot \mathbf{L}\right) \mathrm{d}V. \end{aligned} \tag{10.7.7}$$

For example, consider pressure-independent isotropic hardening plasticity for which, by Eq. (9.4.26),

$$\underline{\boldsymbol{\mathcal{L}}}_{(0)}^{\mathrm{p}} = \underline{\boldsymbol{\mathcal{L}}}_{(0)} - \frac{2\mu}{1 + h^{\mathrm{p}}/\mu}\, (\mathbb{M} \otimes \mathbb{M}), \tag{10.7.8}$$

where $\mathbb{M}$ is a deviatoric normalized tensor in the direction of the yield surface normal,

$$\mathbb{M} = \frac{\partial f/\partial \boldsymbol{\sigma}}{\|\partial f/\partial \boldsymbol{\sigma}\|}, \quad \left\|\frac{\partial f}{\partial \boldsymbol{\sigma}}\right\| = \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^{1/2}. \tag{10.7.9}$$

The uniqueness condition (10.7.7) then becomes

$$H > \Sigma, \tag{10.7.10}$$

for all kinematically admissible velocity fields, where (Hill, 1958)

$$H = \int_V \underline{\boldsymbol{\mathcal{L}}}_{(0)} :: (\mathbf{D} \otimes \mathbf{D}) \, \mathrm{d}V - \int_{V^{\mathrm{p}}} \frac{2\mu}{1 + h^{\mathrm{p}}/\mu} (\mathbb{M} : \mathbf{D})^2 \, \mathrm{d}V^{\mathrm{p}}, \tag{10.7.11}$$

$$\Sigma = \int_V \boldsymbol{\sigma} : \left(2\mathbf{D}^2 - \mathbf{L}^T \cdot \mathbf{L}\right) \mathrm{d}V. \tag{10.7.12}$$

If the rate of hardening $h^{\mathrm{p}} \to \infty$, we have

$$H^{\infty} = \int_V \underline{\boldsymbol{\mathcal{L}}}_{(0)} :: (\mathbf{D} \otimes \mathbf{D}) \, \mathrm{d}V = \int_V \left[\lambda \, (\mathrm{tr}\, \mathbf{D})^2 + 2\mu \, (\mathbf{D} : \mathbf{D})\right] \mathrm{d}V > 0. \tag{10.7.13}$$

In the ideally plastic limit, $h^{\mathrm{p}} \to 0$ and

$$H^0 = H^{\infty} - 2\mu \int_{V^{\mathrm{p}}} (\mathbb{M} : \mathbf{D})^2 \, \mathrm{d}V^{\mathrm{p}} < H^{\infty}. \tag{10.7.14}$$

For any positive rate of hardening h^{p}, then,

$$H^0 \leq H \leq H^{\infty}. \tag{10.7.15}$$

When h^{p} is the same throughout the volume V^{p}, the uniqueness condition $H > \Sigma$ becomes

$$H^{\infty} - \frac{2\mu}{1 + h^{\mathrm{p}}/\mu} \int_{V^{\mathrm{p}}} (\mathbf{n} : \mathbf{D})^2 \, \mathrm{d}V^{\mathrm{p}} > \Sigma. \tag{10.7.16}$$

Using (10.7.14) to eliminate the integral over V^{p}, this gives

$$H^{\infty} - \Sigma > \frac{1}{1 + h^{\mathrm{p}}/\mu} \left(H^{\infty} - H^0\right), \tag{10.7.17}$$

i.e.,

$$\frac{h^{\mathrm{p}}}{\mu} > \frac{\Sigma - H^0}{H^{\infty} - \Sigma}. \tag{10.7.18}$$

Thus, the solution is certainly unique if, for all kinematically admissible $\mathbf{v}$,

$$\frac{h^{\mathrm{p}}}{\mu} > \beta, \quad \beta = \max_{\mathbf{v}} \left(\frac{\Sigma - H^0}{H^{\infty} - \Sigma}\right). \tag{10.7.19}$$

Consider next stability of the underlying equilibrium configuration. The stability criterion is also given by (10.7.5). This further becomes

$$\int_{V^{\mathrm{e}}} \underline{\mathbf{\Lambda}} \cdot\cdot\cdot\cdot (\mathbf{L} \otimes \mathbf{L})\, \mathrm{d}V^{\mathrm{e}} + \int_{V_u^{\mathrm{p}}} \underline{\mathbf{\Lambda}} \cdot\cdot\cdot\cdot (\mathbf{L} \otimes \mathbf{L})\, \mathrm{d}V_u^{\mathrm{p}} + \int_{V_l^{\mathrm{p}}} \underline{\mathbf{\Lambda}}^{\mathrm{p}} \cdot\cdot\cdot\cdot (\mathbf{L} \otimes \mathbf{L})\, \mathrm{d}V_l^{\mathrm{p}} > 0, \tag{10.7.20}$$

or, in view of (10.1.10) and (10.1.15),

$$\int_{V^{\mathrm{e}}} \underline{\boldsymbol{\mathcal{L}}}_{(0)} :: (\mathbf{D} \otimes \mathbf{D})\, \mathrm{d}V^{\mathrm{e}} + \int_{V_u^{\mathrm{p}}} \underline{\boldsymbol{\mathcal{L}}}_{(0)} :: (\mathbf{D} \otimes \mathbf{D}) \mathrm{d}V_u^{\mathrm{p}} + \int_{V_l^{\mathrm{p}}} \underline{\boldsymbol{\mathcal{L}}}_{(0)}^{\mathrm{p}} :: (\mathbf{D} \otimes \mathbf{D})\, \mathrm{d}V_l^{\mathrm{p}} > \int_V \boldsymbol{\sigma} : \left(2\,\mathbf{D}^2 - \mathbf{L}^T \cdot \mathbf{L}\right) \mathrm{d}V. \tag{10.7.21}$$

Here, V_l^{p} is the part of V^{p} where plastic loading takes place, while V_u^{p} is the part of V^{p} where elastic unloading or neutral loading takes place, for the prescribed $\mathbf{v}$. When Eq. (10.7.8) is incorporated, this becomes

$$H_l > \Sigma, \tag{10.7.22}$$

for all kinematically admissible velocity fields, where

$$H_l = \int_V \underline{\boldsymbol{\mathcal{L}}}_{(0)} :: (\mathbf{D} \otimes \mathbf{D})\, \mathrm{d}V - \int_{V_l^{\mathrm{p}}} \frac{2\mu}{1 + h^{\mathrm{p}}/\mu} (\mathbb{M} : \mathbf{D})^2\, \mathrm{d}V_l^{\mathrm{p}}. \tag{10.7.23}$$

The plastic loading condition in V_l^{p} is

$$\mathbb{M} : \mathbf{D} > 0. \tag{10.7.24}$$

If we define

$$H_l^0 = H^\infty - 2\mu \int_{V_l^{\mathrm{p}}} (\mathbb{M} : \mathbf{D})^2\, \mathrm{d}V_l^{\mathrm{p}}, \tag{10.7.25}$$

the equilibrium is stable when

$$\frac{h^{\mathrm{p}}}{\mu} > \beta_l, \quad \beta_l = \max_{\mathbf{v}} \left(\frac{\Sigma - H_l^0}{H^\infty - \Sigma} \right), \tag{10.7.26}$$

for all kinematically admissible $\mathbf{v}$.

Evidently, since $V_l^{\mathrm{p}} \leq V^{\mathrm{p}}$, we have

$$H^0 \leq H_l^0, \quad \beta_l \leq \beta. \tag{10.7.27}$$

Thus, for certain problems and deformation paths, a state of bifurcation can be reached at an earlier stage than a failure of stability. This could occur at the hardening rate $h^{\mathrm{p}} = \beta\,\mu$, when (10.7.1) fails and uniqueness is no longer certain. If such stable bifurcation occurs, the loading must change with further deformation according to (10.7.4). Assuming that the hardening rate gradually decreases as the deformation proceeds, the stability of equilibrium configuration would be lost at the lower hardening rate $h^{\mathrm{p}} = \beta_l\,\mu$.

10.8. Uniqueness and Stability for Rigid-Plastic Materials

If elastic moduli are assigned infinitely large values, only plastic strain can take place and the model of rigid-plastic behavior is obtained. For example, in the case of isotropic hardening, the rate of deformation is

$$\mathbf{D} = \frac{1}{2h}\,(\mathbb{M} : \overset{\circ}{\boldsymbol{\sigma}})\,\mathbb{M}\,, \tag{10.8.1}$$

provided that $\mathbb{M} : \overset{\circ}{\boldsymbol{\sigma}} > 0$. The hardening modulus is h (with the von Mises yield criterion, $h = h_{\mathrm{t}}$, the tangent modulus in shear test). The response is incompressible and bilinear, since in the hardening range

$$\mathbf{D} = \mathbf{0}, \quad \text{when} \quad \mathbb{M} : \overset{\circ}{\boldsymbol{\sigma}} \leq 0. \tag{10.8.2}$$

Also note that

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} = \overset{\circ}{\boldsymbol{\sigma}}, \tag{10.8.3}$$

since $\operatorname{tr}\mathbf{D} = 0$. By taking the inner product of $\mathbf{D}$ with $\overset{\circ}{\boldsymbol{\sigma}}$ and with itself, it follows that

$$\overset{\circ}{\boldsymbol{\sigma}} : \mathbf{D} = \frac{1}{2h}\,(\mathbb{M} : \overset{\circ}{\boldsymbol{\sigma}})^2, \quad \mathbf{D} : \mathbf{D} = \frac{1}{4h^2}\,(\mathbb{M} : \overset{\circ}{\boldsymbol{\sigma}})^2, \tag{10.8.4}$$

so that

$$\overset{\circ}{\boldsymbol{\sigma}} : \mathbf{D} = 2h\,(\mathbf{D} : \mathbf{D}). \tag{10.8.5}$$

It can be readily shown, when both rates of deformation vanish, or when both rates are different from zero ($\mathbf{D} = \mathbf{0}$ and $\mathbf{D}^* = \mathbf{0}$, or $\mathbf{D} \neq \mathbf{0}$ and $\mathbf{D}^* \neq \mathbf{0}$),

$$(\overset{\circ}{\boldsymbol{\sigma}}{}^* - \overset{\circ}{\boldsymbol{\sigma}}) : (\mathbf{D}^* - \mathbf{D}) = 2h\,(\mathbf{D}^* - \mathbf{D}) : (\mathbf{D}^* : \mathbf{D}). \tag{10.8.6}$$

If one rate of deformation vanishes and the other does not (e.g., $\mathbf{D}^* = \mathbf{0}$ and $\mathbf{D} \neq \mathbf{0}$),

$$\begin{aligned}(\overset{\circ}{\boldsymbol{\sigma}}{}^* - \overset{\circ}{\boldsymbol{\sigma}}) : (\mathbf{D}^* - \mathbf{D}) &= \overset{\circ}{\boldsymbol{\sigma}} : \mathbf{D} - \overset{\circ}{\boldsymbol{\sigma}}{}^* : \mathbf{D} \\ &> 2h\,(\mathbf{D} : \mathbf{D}) = 2h\,(\mathbf{D}^* - \mathbf{D}) : (\mathbf{D}^* : \mathbf{D}),\end{aligned} \tag{10.8.7}$$

since $\overset{\circ}{\boldsymbol{\sigma}}{}^* : \mathbf{D} \leq 0$. Thus, for all pairs $\mathbf{D}$ and $\mathbf{D}^*$, we have

$$(\overset{\circ}{\boldsymbol{\sigma}}{}^* - \overset{\circ}{\boldsymbol{\sigma}}) : (\mathbf{D}^* - \mathbf{D}) \geq 2h\,(\mathbf{D}^* - \mathbf{D}) : (\mathbf{D}^* : \mathbf{D}). \tag{10.8.8}$$

The uniqueness condition (10.4.8) for an elastoplastic material can be written, in view of Eq. (10.4.12), as

$$\int_V \Big\{ (\overset{\circ}{\underline{\boldsymbol{\tau}}}{}^* - \overset{\circ}{\underline{\boldsymbol{\tau}}}) : (\mathbf{D}^* - \mathbf{D}) - \boldsymbol{\sigma} : \big[2\,(\mathbf{D}^* - \mathbf{D})^2 - \big(\mathbf{L}^{*T} - \mathbf{L}^T\big) \cdot (\mathbf{L}^* - \mathbf{L}) \,\big] \Big\}\, \mathrm{d}V > \int_{S_t} \Big(\dot{\underline{\mathbf{p}}}_n^{*\,\mathrm{s}} - \dot{\underline{\mathbf{p}}}_n^{\mathrm{s}} \Big) \cdot (\mathbf{v}^* - \mathbf{v})\, \mathrm{d}S_t. \tag{10.8.9}$$

Having regard to inequality (10.8.8), and $\overset{\circ}{\underline{\boldsymbol{\tau}}} = \overset{\circ}{\boldsymbol{\sigma}}$, a sufficient condition for uniqueness of the boundary value problem for rigid-plastic material is

$$\int_V \Big\{ 2h\,(\mathbf{D}^* - \mathbf{D}) : (\mathbf{D}^* - \mathbf{D}) - \boldsymbol{\sigma} : \big[2\,(\mathbf{D}^* - \mathbf{D})^2 - \big(\mathbf{L}^{*T} - \mathbf{L}^T\big) \cdot (\mathbf{L}^* - \mathbf{L}) \,\big] \Big\}\, \mathrm{d}V > \int_{S_t} \Big(\dot{\underline{\mathbf{p}}}_n^{*\,\mathrm{s}} - \dot{\underline{\mathbf{p}}}_n^{\mathrm{s}} \Big) \cdot (\mathbf{v}^* - \mathbf{v})\, \mathrm{d}S_t. \tag{10.8.10}$$

This also directly follows from the notion of an incrementally linear comparison material that reacts at every plastically stressed point according to plastic loading branch (10.8.1). Although $\boldsymbol{\sigma}$ is undetermined in rigid regions, the integrals in (10.8.10) can be taken over the whole volume, since there is no contribution from rigid regions ($\mathbf{v}$ there being equal to $\mathbf{v}^*$). Furthermore, since for isotropic behavior the principal directions of $\boldsymbol{\sigma}$ and $(\mathbf{D}^* - \mathbf{D})$ coincide, the tensor $\boldsymbol{\sigma} \cdot (\mathbf{D}^* - \mathbf{D})$ is symmetric, and

$$\boldsymbol{\sigma} : \big[2\,(\mathbf{D}^* - \mathbf{D})^2 - \big(\mathbf{L}^{*T} - \mathbf{L}^T\big) \cdot (\mathbf{L}^* - \mathbf{L}) \big] = \boldsymbol{\sigma} : (\mathbf{L}^* - \mathbf{L})^2. \tag{10.8.11}$$

Consequently, the uniqueness is assured if the exclusion functional is positive

$$\underline{\mathcal{F}}(\mathbf{w}) = \int_V \big[2h\,(\boldsymbol{D} : \boldsymbol{D}) - \boldsymbol{\sigma} : \boldsymbol{L}^2 \big]\, \mathrm{d}V - \int_{S_t} \dot{\underline{\boldsymbol{p}}}_n^{\mathrm{s}}(\mathbf{w}) \cdot \mathbf{w}\, \mathrm{d}S_t > 0, \tag{10.8.12}$$

for any incompressible kinematically admissible velocity field $\mathbf{w}$, which gives rise to rate of deformation $\boldsymbol{D}$ (symmetric part of $\boldsymbol{L} = \partial \mathbf{w} / \partial \mathbf{x}$) that is codirectional with $\mathbb{M}$ in the plastic region (though not necessarily in the same sense, since $\mathbf{D}^* - \mathbf{D}$ in (10.8.10) can be in either $\mathbb{M}$ or $-\mathbb{M}$ direction), and equal to zero in the rigid region.

If h is constant throughout plastically stressed region V^{p}, and if $\dot{\underline{\boldsymbol{p}}}_n^{\mathrm{s}} = \mathbf{0}$ on S_t, the uniqueness is certain when

$$2\,h > \max_{\mathbf{w}} \frac{\int_V (\boldsymbol{\sigma} : \boldsymbol{L}^2)\, \mathrm{d}V}{\int_V (\boldsymbol{D} : \boldsymbol{D})\, \mathrm{d}V}\,. \tag{10.8.13}$$

The underlying equilibrium configuration is stable if (10.8.13) holds, but the class of admissible velocity fields is further restricted by the requirement that $\mathbb{M} : \boldsymbol{D}$ is non-negative in the plastic region ($\mathbb{M} : \boldsymbol{D}$ can be either positive, negative or zero in the plastic region for admissible velocity fields in the uniqueness condition, so that this class is wider than the class of admissible velocity fields in the stability condition).

10.8.1. Uniaxial Tension

In the tension test of a specimen with uniform cross-section, the state of stress at an incipient bifurcation is uniform tension $\sigma_{11} = \sigma$, other stress components being equal to zero. An admissible velocity field for the uniqueness condition must be incompressible and give rise to the rate of deformation tensor $\boldsymbol{D}$ parallel to $\boldsymbol{\sigma}'$ (the yield surface normal). This is satisfied when

$$D_{22} = D_{33} = -\frac{1}{2}\, D_{11}, \quad D_{12} = D_{23} = D_{31} = 0. \tag{10.8.14}$$

Thus,

$$\boldsymbol{\sigma} : \boldsymbol{L}^2 = \sigma\,(D_{11}^2 - L_{12}^2 - L_{13}^2), \quad \boldsymbol{D} : \boldsymbol{D} = \frac{3}{2}\, D_{11}^2, \tag{10.8.15}$$

and the condition (10.8.13) gives

$$3\,\frac{h}{\sigma} > \max_{\mathbf{w}} \frac{\int_V \left(D_{11}^2 - L_{12}^2 - L_{13}^2\right) \mathrm{d}V}{\int_V D_{11}^2\, \mathrm{d}V}\,. \tag{10.8.16}$$

The right-hand side is always smaller than one (irrespective of the boundary conditions at the ends and specific representation of admissible functions $\mathbf{w}$), so that fundamental mode of deformation (uniform straining) is certainly unique (and underlying equilibrium configuration stable) for $h > \sigma/3$. With the von Mises yield criterion, $h = h_\mathrm{t}$, and since $h_\mathrm{t} = (1/3)\,\mathrm{d}\sigma/\mathrm{d}e$, where e denotes longitudinal logarithmic strain in uniaxial tension, the deformation mode is unique when the slope of the true stress-strain curve exceeds the current yield stress. As is well-known, at the critical value $\mathrm{d}\sigma/\mathrm{d}e = \sigma$, the applied load attains its maximum value and either further uniform straining or localized necking is possible in principle. Hutchinson and Miles (1974) have demonstrated that in the case of circular cylinder of incompressible elastic-plastic material, an axially symmetric bifurcation of a necking type exists when the true stress reaches a critical value slightly greater than the stress corresponding to the maximum load. The shear free ends of the cylinder with traction-free lateral surface were subject to uniform longitudinal relative displacement. A numerical study of necking in elastoplastic circular cylinders under uniaxial tension with different boundary conditions at the ends was performed by Needleman (1972). Hill and Hutchinson (1975) gave a comprehensive analysis of bifurcation modes from a state of homogeneous in-plane tension of an incompressible rectangular block under plane deformation. The sides of the block were traction-free and the shear-free ends were subject to uniform longitudinal relative displacement. See also Burke and Nix (1979), and Bardet (1991). For the effects of plastic non-normality on bifurcation prediction, see Needleman (1979) and Kleiber (1986). Bufurcation of an incompressible plate under pure bending in plane strain was studied by Triantafyllidis (1980).

10.8.2. Compression of Column

Consider a column of uniform cross-sectional area A, built at one end and loaded at the other by an increasing axial load N. The state of stress is uniaxial compression of amount $\sigma_{11} = -N/A$, except possibly near the ends. For sufficiently long, slender columns possible nonuniformities near the ends can be neglected and the uniqueness condition (10.8.13) gives (Hill, 1957)

$$3\,\frac{A\,h}{N} > \max_{\mathbf{w}} \frac{\int_V \left(L_{12}^2 + L_{13}^2 - D_{11}^2\right) \mathrm{d}V}{\int_V D_{11}^2\,\mathrm{d}V}\,. \tag{10.8.17}$$

The admissible velocity field $\mathbf{w}$ again satisfies the conditions $D_{22} = D_{33} = -D_{11}/2$, and $D_{12} = D_{23} = D_{31} = 0$, i.e.,

$$\begin{gathered}\frac{\partial w_2}{\partial x_2} = \frac{\partial w_3}{\partial x_3} = -\frac{1}{2}\,\frac{\partial w_1}{\partial x_1}\,,\\ \frac{\partial w_1}{\partial x_2} + \frac{\partial w_2}{\partial x_1} = 0, \quad \frac{\partial w_2}{\partial x_3} + \frac{\partial w_3}{\partial x_2} = 0, \quad \frac{\partial w_3}{\partial x_1} + \frac{\partial w_1}{\partial x_3} = 0.\end{gathered} \tag{10.8.18}$$

These have the general solution

$$\begin{aligned}w_1 &= a\,x_1x_2 + b\,x_1x_3 + c\,(2x_1^2 + x_2^2 + x_3^2) + d\,x_1,\\ w_2 &= -\frac{1}{2}\,b\,x_2x_3 - 2\,c\,x_1x_2 - \frac{1}{4}\,a\,(2x_1^2 + x_2^2 - x_3^2) - \frac{1}{2}\,d\,x_2,\\ w_3 &= -\frac{1}{2}\,a\,x_2x_3 - 2\,c\,x_1x_3 - \frac{1}{4}\,b\,(2x_1^2 - x_2^2 + x_3^2) - \frac{1}{2}\,d\,x_3.\end{aligned} \tag{10.8.19}$$

By taking the origin of the coordinate system at the centroid of the fixed end, above functions satisfy the end conditions

$$w_1 = w_2 = w_3 = 0, \quad \frac{\partial w_2}{\partial x_1} = \frac{\partial w_3}{\partial x_1} = 0 \tag{10.8.20}$$

at the origin. Selecting the axes x_2 and x_3 to be the principal centroidal axes of the cross section, the substitution of the expression for w_1 from (10.8.19) into (10.8.17) gives

$$3\,\frac{h}{Nl^2} > \max_{\mathbf{w}} \frac{a^2\left(\frac{1}{3} - \frac{I_3}{Al^2}\right) + b^2\left(\frac{1}{3} - \frac{I_2}{Al^2}\right) + 4c^2\,\frac{I_2+I_3}{Al^2} - \left[\frac{4}{3}\,c^2 + \left(2c + \frac{d}{l}\right)^2\right]}{a^2\,I_3 + b^2\,I_2 + \left[\frac{4}{3}\,c^2 + \left(2c + \frac{d}{l}\right)^2\right] Al^2}\,. \tag{10.8.21}$$

The second moments of the cross sectional area about the x_2 and x_3 axes are I_2 and I_3. The right-hand side in (10.8.21) has a maximum value when the square bracketed term vanishes, which occurs for $c = d = 0$ (for slender columns, $I_2 + I_3 << Al^2$). Thus,

$$3\,\frac{h}{Nl^2} > \frac{1}{3}\max_{\mathbf{w}}\left(\frac{a^2 + b^2}{a^2\,I_3 + b^2\,I_2}\right) - \frac{1}{Al^2}\,. \tag{10.8.22}$$

The term $(Al^2)^{-1}$ can be neglected for slender columns, and

$$\frac{h}{Nl^2} > \frac{1}{9\,I_{\min}}\,. \tag{10.8.23}$$

If $I_3 > I_2$, the maximum occurs for $a = 0$; if $I_2 > I_3$, the maximum occurs for $b = 0$; if $I_2 = I_3$, any ratio a/b can be used. In each case the $\mathbf{w}$ field reduces to pure bending. For example, for circular cross-section of radius R, we obtain

$$h > \frac{4}{9}\,\frac{N}{A}\left(\frac{l}{R}\right)^2. \tag{10.8.24}$$

In the consideration of stability the constants a, b, c, d are not entirely arbitrary in the expressions for admissible functions (10.8.19), but are subject to condition

$$\boldsymbol{\sigma}' : \boldsymbol{D} \geq 0, \quad \text{i.e.,} \quad D_{11} \leq 0. \tag{10.8.25}$$

This gives (Hill, 1957)

$$a\,x_2 + b\,x_3 + 4\,c\,x_1 + d \leq 0, \tag{10.8.26}$$

everywhere in the body. The expression (10.8.21) attains its maximum for $c = 0$, so that

$$3\,\frac{h}{Nl^2} > \max_{\mathbf{w}} \frac{a^2\left(\frac{1}{3} - \frac{I_3}{Al^2}\right) + b^2\left(\frac{1}{3} - \frac{I_2}{Al^2}\right) - \frac{d^2}{l^2}}{a^2 I_3 + b^2 I_2 + Ad^2}\,. \tag{10.8.27}$$

Suppose that the cross-section is a circle of radius R. The value of d which makes the right-hand side of (10.8.27) maximum and fulfills the condition (10.8.26) with $c = 0$ is readily found to be

$$d = -R(a^2 + b^2)^{1/2}. \tag{10.8.28}$$

The condition (10.8.27) consequently becomes

$$\frac{h}{Nl^2} > \frac{4}{45AR^2} - \frac{1}{3Al^2}\,. \tag{10.8.29}$$

Upon neglecting $(Al^2)^{-1}$ term,

$$h > \frac{4}{45}\,\frac{N}{A}\left(\frac{l}{R}\right)^2. \tag{10.8.30}$$

The obtained critical hardening rate for stability of column is 1/5 of that obtained from the condition of uniqueness, which is given by (10.8.24). More general elastoplastic analysis of column failure is presented by Hill and Sewell (1960, 1962). A comprehensive treatment of plastic buckling and post-buckling behavior of columns and other structures is given by Hutchinson (1973, 1974), and Bažant and Cedolin (1991). See also Storåkers (1971, 1977), Sewell (1973), Young (1976), Needleman and Tvergaard (1982), and Nguyen (1994).

10.9. Eigenmodal Deformations

From the analysis in preceding sections it is recognized that there may be particular configurations of the body where nominal tractions are momentarily constant as the body is incrementally deformed in certain ways. The corresponding instantaneous velocity fields are then nontrivial solutions of a homogeneous boundary-value problem. These velocity fields are referred to as eigenmodes. The underlying configurations are the eigenstates. An uniaxial tension specimen of a ductile metal at maximum load is an example of an eigenstate configuration. The presented theory is originally due to Hill (1967).

10.9.1. Eigenstates and Eigenmodes

Consider a solid body whose entire bounding surface is unconstrained ($S_t = S$). The exclusion functional of Eq. (10.4.17) is then

$$\underline{\mathcal{F}}(\mathbf{w}) = \int_V \underline{\chi}(\mathbf{w})\,\mathrm{d}V - \frac{1}{2}\int_S \dot{\underline{\boldsymbol{p}}}^{\mathrm{s}}_n(\mathbf{w})\cdot\mathbf{w}\,\mathrm{d}S. \tag{10.9.1}$$

If equilibrium configuration of an incrementally linear material is stable under all-around dead loads, the strain path cannot bifurcate from that state for any loading rates applied to the state. A sufficient condition for stability and uniqueness is that $\underline{\mathcal{F}}(\mathbf{w}) > 0$ for all admissible velocity fields $\mathbf{w}$. Bifurcation can occur only when a primary eigenstate is reached (first eigenstate reached on a given deformation path), where

$$\underline{\mathcal{F}}(\mathbf{w}) \geq 0, \tag{10.9.2}$$

with the equality sign for some velocity field (eigenmode velocity field).

For a piecewise linear or thoroughly nonlinear material response with the rate potential $\underline{\chi}$, a deformation path could bifurcate under varying load before the primary eigenstate is reached and stability lost. As discussed in Subsection 10.4.2, to prevent bifurcation before an eigenstate is reached, it is sufficient that configuration is stable for incrementally linear comparison material $\underline{\chi}^l$, and that $\underline{\chi} - \underline{\chi}^l$ is a convex function of $\boldsymbol{L}$. The bifurcation may be excluded for $\underline{\chi}$ material even if the configuration is an eigenstate for $\underline{\chi}^l$ material, but $\underline{\chi} - \underline{\chi}^l$ is strictly convex function in that configuration. If the current configuration is a primary eigenstate for $\underline{\chi}^l$ material, and $\underline{\chi} - \underline{\chi}^l$ is merely convex, the configuration may be a primary eigenstate for $\underline{\chi}$ material, provided there is an eigenmode of $\underline{\chi}^l$ material that is also an eigenmode of $\underline{\chi}$ material (giving rise to plastic loading throughout plastically stressed region of $\underline{\chi}$ material).

Suppose that for, either incrementally linear or incrementally nonlinear material, $\underline{\mathcal{F}}(\mathbf{w})$ is positive definite along a loading path from the undeformed state, until a primary eigenstate is reached where $\underline{\mathcal{F}}(\mathbf{w}) \geq 0$ (with equality sign for an eigenmodal field). Since $\underline{\mathcal{F}}(\mathbf{w})$ is non-negative in an eigenstate, vanishing only in an eigenmode, its first variation $\delta\underline{\mathcal{F}}$ must be zero in an eigenmode,

$$\delta\left[\int_V \underline{\chi}(\mathbf{w})\,\mathrm{d}V - \frac{1}{2}\int_S \dot{\underline{\boldsymbol{p}}}^{\mathrm{s}}_n(\mathbf{w})\cdot\mathbf{w}\,\mathrm{d}S\right] = 0. \tag{10.9.3}$$

Thus, in an eigenmode field $\mathbf{w}$,

$$\int_V \underline{\dot{\boldsymbol{P}}} \cdot\cdot\, \delta \boldsymbol{L} \, \mathrm{d}V - \int_S \underline{\dot{\boldsymbol{p}}}_n^{\mathrm{s}} \cdot \delta \mathbf{w} \, \mathrm{d}S = 0, \tag{10.9.4}$$

for all admissible variations $\delta\mathbf{w}$. In addition, the functional itself vanishes in an eigenmode,

$$\int_V \underline{\chi}(\mathbf{w}) \, \mathrm{d}V - \frac{1}{2} \int_S \underline{\dot{\boldsymbol{p}}}_n^{\mathrm{s}}(\mathbf{w}) \cdot \mathbf{w} \, \mathrm{d}S = 0. \tag{10.9.5}$$

Under all-around deformation-insensitive dead loading, the above two conditions reduce to

$$\int_V \underline{\dot{\boldsymbol{P}}} \cdot\cdot\, \delta \boldsymbol{L} \, \mathrm{d}V = 0, \quad \int_V \underline{\chi}(\mathbf{w}) \, \mathrm{d}V = 0. \tag{10.9.6}$$

An eigenmode is in this case a nontrivial solution of homogeneous boundary value problem described by

$$\boldsymbol{\nabla} \cdot \underline{\dot{\boldsymbol{P}}} = \mathbf{0} \quad \text{in} \quad V, \quad \text{and} \quad \mathbf{n} \cdot \underline{\dot{\boldsymbol{P}}} = \mathbf{0} \quad \text{on} \quad S. \tag{10.9.7}$$

10.9.2. Eigenmodal Spin

Suppose that a homogeneous body is uniformly strained from its undeformed configuration to a primary eigenstate configuration. The state of stress and material properties are then uniform at each instant of deformation, and $\underline{\chi}$ is the same function of velocity gradient at every point of the body in the considered configuration. By choosing velocity fields with arbitrary uniform gradient $\boldsymbol{L}$, it follows that $\underline{\mathcal{F}} > 0$ if and only if $\underline{\chi} > 0$ along stable segment of deformation path, and that $\underline{\chi} \geq 0$ in a primary eigenstate. Equality $\underline{\chi} = 0$ applies for an eigenmode velocity field, which also makes $\underline{\chi}$ stationary. Since

$$\delta \underline{\chi} = \underline{\dot{\boldsymbol{P}}} \cdot\cdot\, \delta \boldsymbol{L} = 0 \tag{10.9.8}$$

in an eigenmode for all $\delta\boldsymbol{L}$, we conclude that

$$\underline{\dot{\boldsymbol{P}}} = \frac{\partial \underline{\chi}}{\partial \boldsymbol{L}} = \mathbf{0}. \tag{10.9.9}$$

This means that the nominal stress is stationary in an eigenmode (momentarily constant as the body is incrementally deformed along an eigenmode field).

Since from Section 3.9,

$$\underline{\dot{\boldsymbol{T}}}_{(1)} = \underline{\dot{\boldsymbol{P}}} - \boldsymbol{\sigma} \cdot \boldsymbol{L}^T, \tag{10.9.10}$$

and since local rotational balance requires $\underline{\dot{\boldsymbol{T}}}_{(1)}$ to be symmetric, from (10.9.9) it follows that in an eigenmode

$$\boldsymbol{L} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot \boldsymbol{L}^T, \tag{10.9.11}$$

so that

$$\boldsymbol{\sigma} \cdot \boldsymbol{W} + \boldsymbol{W} \cdot \boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{\sigma}. \tag{10.9.12}$$

This can be solved for $\boldsymbol{W}$ in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{D}$ by using (1.12.12). The solution is an expression for the eigenmodal spin in terms of stress and eigenmodal rate of deformation,

$$\boldsymbol{W} = (\operatorname{tr} \mathbf{S})(\boldsymbol{\sigma} \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{\sigma}) - \mathbf{S} \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{\sigma}) - (\boldsymbol{\sigma} \cdot \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{\sigma}) \cdot \mathbf{S}, \tag{10.9.13}$$

where

$$\mathbf{S} = [(\operatorname{tr} \boldsymbol{\sigma})\, \mathbf{I} - \boldsymbol{\sigma}]^{-1}. \tag{10.9.14}$$

It is assumed that $\mathbf{S}$ exists. When written in terms of components on the principal axes of stress $\boldsymbol{\sigma}$, the required condition for the inverse in Eq. (10.9.14) to exist is

$$\det[(\operatorname{tr} \boldsymbol{\sigma})\, \mathbf{I} - \boldsymbol{\sigma}] = (\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) \neq 0. \tag{10.9.15}$$

The eigenmodal spin components on the principal stress axes are

$$W_{12} = \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}\, D_{12}, \quad W_{23} = \frac{\sigma_2 - \sigma_3}{\sigma_2 + \sigma_3}\, D_{23}, \quad W_{31} = \frac{\sigma_3 - \sigma_1}{\sigma_3 + \sigma_1}\, D_{31}. \tag{10.9.16}$$

Evidently, if the principal axes of $\boldsymbol{D}$ happen to coincide with those of $\boldsymbol{\sigma}$ (as in the case of rigid-plastic von Mises plasticity), the spin of an eigenmode field entirely vanishes. If the stress field has an axis of equilibrium, for example axis 1 in the case when $\sigma_2 + \sigma_3 = 0$, W_{23} is undetermined and D_{23} must vanish. On the other hand, when the stress state is uniaxial, $\sigma_2 = \sigma_3 = 0$, there is no restriction on D_{23} but W_{23} is still undetermined.

It can be readily verified that among all velocity gradients with the fixed strain rates, $\underline{\chi}$ attains its minimum when $\sigma_1 + \sigma_2 > 0$, $\sigma_2 + \sigma_3 > 0$, $\sigma_3 + \sigma_1 > 0$, and when the spin components are determined by (10.9.16).

Indeed, for an elastoplastic material, $\underline{\chi}$ can be written from (10.1.15) as

$$\underline{\chi} = \frac{1}{2}\,\underline{\boldsymbol{\mathcal{L}}}^{\mathrm{p}}_{(0)} :: (\boldsymbol{D}\otimes\boldsymbol{D}) - \frac{1}{2}\,\boldsymbol{\sigma} : \mathbf{D}^2$$

$$+ \frac{1}{2}\left[(\sigma_1+\sigma_2)\,W_{12}^2 + (\sigma_2+\sigma_3)\,W_{23}^2 + (\sigma_3+\sigma_1)\,W_{31}^2\right] \tag{10.9.17}$$

$$- (\sigma_1-\sigma_2)\,D_{12}\,W_{12} - (\sigma_2-\sigma_3)\,D_{23}\,W_{23} - (\sigma_3-\sigma_1)\,D_{31}\,W_{31}.$$

The stationary conditions

$$\frac{\partial\underline{\chi}}{\partial W_{ij}} = 0 \tag{10.9.18}$$

clearly reproduce (10.9.16). The corresponding minimum of $\underline{\chi}$ is

$$\underline{\chi}^0 = \frac{1}{2}\,\underline{\boldsymbol{\mathcal{L}}}^{\mathrm{p}}_{(0)} :: (\boldsymbol{D}\otimes\boldsymbol{D}) - \frac{1}{2}\,\boldsymbol{\sigma} : \mathbf{D}^2$$

$$- \frac{1}{2}\left[\frac{(\sigma_1-\sigma_2)^2}{\sigma_1+\sigma_2}\,D_{12}^2 + \frac{(\sigma_2-\sigma_3)^2}{\sigma_2+\sigma_3}\,D_{23}^2 + \frac{(\sigma_3-\sigma_1)^2}{\sigma_3+\sigma_1}\,D_{31}^2\right]. \tag{10.9.19}$$

For isotropic hardening plasticity, from (9.8.14) we obtain

$$\frac{1}{2}\,\underline{\boldsymbol{\mathcal{L}}}^{\mathrm{p}}_{(0)} :: (\boldsymbol{D}\otimes\boldsymbol{D}) = \frac{1}{2}\,\lambda\,(\mathrm{tr}\,\boldsymbol{D})^2 + \mu\,\boldsymbol{D}:\boldsymbol{D} - \frac{\mu}{1+h^{\mathrm{p}}/\mu}\,(\mathbb{M}:\boldsymbol{D})^2. \tag{10.9.20}$$

Since, for isotropic smooth yield surface, $\mathbb{M}$ has the principal directions parallel to those of stress, $\mathbb{M}_{ij}=0$ for $i\neq j$ on the coordinate axes parallel to the principal stress axes.

If $\boldsymbol{D}$ is the rate of deformation in an eigenmode, then

$$\underline{\chi}^0(\boldsymbol{D}) = 0. \tag{10.9.21}$$

For all other rates of deformation in an eigenstate, $\underline{\chi}^0 > 0$. The uniqueness and stability are assured in any configuration before primary eigenstate is reached if $\underline{\chi}^0$, defined by (10.9.19), is positive definite in that configuration, since then $\underline{\chi}$ is also positive definite in that configuration.

In order that the configuration can qualify as stable by the criterion $\underline{\chi} > 0$ for all $\boldsymbol{L}$, the stress state has to be such that

$$\sigma_1+\sigma_2>0, \quad \sigma_2+\sigma_3>0, \quad \sigma_3+\sigma_1>0, \tag{10.9.22}$$

which means that tension acts on the planes of maximum shear stress. This follows from (10.9.17) by choosing $\boldsymbol{L}$ to be an arbitrary antisymmetric (spin) tensor, so that

$$\underline{\chi} = \frac{1}{2} \left[(\sigma_1 + \sigma_2)\, W_{12}^2 + (\sigma_2 + \sigma_3)\, W_{23}^2 + (\sigma_3 + \sigma_1)\, W_{31}^2 \right]. \tag{10.9.23}$$

Physically, (10.9.22) is imposed, because the opposite inequalities would allow dead loads to do positive work in certain virtual rotations of the body. Note, however, that pure spin cannot by itself be an eigenmode field under triaxial state of stress, since equations of continuing rotational equilibrium (10.9.12) would require that

$$(\sigma_1 + \sigma_2)\, W_{12} = 0, \quad (\sigma_2 + \sigma_3)\, W_{23} = 0, \quad (\sigma_3 + \sigma_1)\, W_{31} = 0. \tag{10.9.24}$$

Thus, unless the stress state has an axis of equilibrium, each spin component must vanish. This is also clear from (10.9.16); if the rate of deformation components are zero in an eigenmode, the eigenmode spin also vanishes. If $\sigma_1 + \sigma_2 = 0$, the spin W_{23} could be nonzero (but would be permissible as an actual mode only if it does not alter the applied tractions, keeping them dead in magnitude and direction, as in the case of uniaxial tension and a spin around the axis of loading).

10.9.3. Eigenmodal Rate of Deformation

The components of rate of deformation D_{ij} of an eigenmode velocity field are nontrivial solutions of the homogeneous system of equations resulting from (10.9.9). Since

$$\underline{\dot{\boldsymbol{P}}} = \underline{\overset{\circ}{\boldsymbol{\tau}}} - \boldsymbol{D} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{W}, \tag{10.9.25}$$

the system of equations is

$$\underline{\boldsymbol{\mathcal{L}}}^{\mathrm{p}}_{(0)} : \boldsymbol{D} - \boldsymbol{D} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{W} = \boldsymbol{0}, \tag{10.9.26}$$

where $\boldsymbol{W}$ is defined in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{D}$ by (10.9.16). Specifically,

$$\boldsymbol{D} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \boldsymbol{W} = \begin{bmatrix} \sigma_1\, D_{11} & \frac{\sigma_1^2+\sigma_2^2}{\sigma_1+\sigma_2}\, D_{12} & \frac{\sigma_1^2+\sigma_3^2}{\sigma_1+\sigma_3}\, D_{13} \\ \frac{\sigma_1^2+\sigma_2^2}{\sigma_1+\sigma_2}\, D_{12} & \sigma_2\, D_{22} & \frac{\sigma_2^2+\sigma_3^2}{\sigma_2+\sigma_3}\, D_{23} \\ \frac{\sigma_1^2+\sigma_3^2}{\sigma_1+\sigma_3}\, D_{13} & \frac{\sigma_2^2+\sigma_3^2}{\sigma_2+\sigma_3}\, D_{23} & \sigma_3\, D_{33} \end{bmatrix}. \tag{10.9.27}$$

For a nontrivial solution of the system of six equations for six unknown components of the rate of deformation to exist, the determinant of the system

(10.9.26) must vanish. This provides a relationship between the instantaneous moduli and applied stress, which characterizes the primary eigenstate.

10.9.4. Uniaxial Tension of Elastic-Plastic Material

If the stress state has an axis of equilibrium, say corresponding to $\sigma_2 + \sigma_3 = 0$, there is only one term proportional to W_{23} that remains in (10.9.17), and for $\sigma_2 \neq \sigma_3$ this term can be made arbitrarily large and negative by appropriately adjusting the sign and magnitude of W_{23}. This means that $\underline{\chi}$ can be negative for some velocity gradients, implying that configuration under stress state with an axis of equilibrium could not qualify as stable. However, if $\sigma_2 = \sigma_3 = 0$, and $\sigma_1 > 0$, $\underline{\chi}$ in (10.9.17) does not depend on W_{23}, having a minimum

$$\underline{\chi}^0 = \frac{1}{2}\,\underline{\boldsymbol{\mathcal{L}}}^{\mathrm{p}}_{(0)} :: (\boldsymbol{D} \otimes \boldsymbol{D}) - \sigma_1 \left(\frac{1}{2}\, D_{11}^2 + D_{12}^2 + D_{13}^2 \right) \tag{10.9.28}$$

in an eigenmode with the spin components

$$W_{12} = D_{12}, \quad W_{31} = -D_{13}. \tag{10.9.29}$$

The configuration under uniaxial tension is thus stable if

$$\begin{aligned} \underline{\chi}^0 &= \frac{1}{2}\lambda\,(D_{11} + D_{22} + D_{33})^2 + \mu(D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2 + 2D_{23}^2 + 2D_{31}^2) \\ &- \frac{2\mu/3}{1 + h^{\mathrm{p}}/\mu} \left(D_{11} - \frac{1}{2} D_{22} - \frac{1}{2} D_{33} \right)^2 - \sigma_1 \left(\frac{1}{2} D_{11}^2 + D_{12}^2 + D_{13}^2 \right) > 0. \end{aligned} \tag{10.9.30}$$

Note that in uniaxial tension

$$\mathbb{M}_{22} = \mathbb{M}_{33} = -\frac{1}{2}\,\mathbb{M}_{11}, \tag{10.9.31}$$

since deviatoric components of uniaxial stress are so related. Thus, $\mathbb{M}_{11} = \sqrt{2/3}$. The function $\underline{\chi}^0$ can be split into two parts. The first part,

$$(2\mu - \sigma_1)(D_{12}^2 + D_{31}^2) + 2\mu D_{23}^2, \tag{10.9.32}$$

is positive for $\sigma_1 < 2\mu$. The function $\underline{\chi}^0$ will be certainly positive if the remaining term is also positive. We then require

$$\begin{aligned} &\frac{1}{2}\lambda\,(D_{11} + D_{22} + D_{33})^2 + \mu(D_{11}^2 + D_{22}^2 + D_{33}^2) \\ &\quad - \frac{2\mu/3}{1 + h^{\mathrm{p}}/\mu} \left(D_{11} - \frac{1}{2} D_{22} - \frac{1}{2} D_{33} \right)^2 - \frac{1}{2}\,\sigma_1\, D_{11}^2 > 0. \end{aligned} \tag{10.9.33}$$

This quadratic form in D_{11}, D_{22}, D_{33} is positive definite if the principal minors of associated matrix are positive definite. The first one is

$$\frac{1}{2}\lambda + \mu - \frac{2\mu/3}{1 + h^{\mathrm{p}}/\mu} - \frac{1}{2}\,\sigma_1 > 0, \tag{10.9.34}$$

which is fulfilled for realistic stress levels. The second one is fulfilled, as well. It remains to examine the determinant

$$\Delta = \begin{vmatrix} \frac{1}{2}\lambda + \mu - \frac{2}{3}\,\alpha\,\mu - \frac{1}{2}\sigma_1 & \frac{1}{2}\lambda + \frac{1}{3}\alpha\,\mu & \frac{1}{2}\lambda + \frac{1}{3}\alpha\,\mu \\ \frac{1}{2}\lambda + \frac{1}{3}\alpha\,\mu & \frac{1}{2}\lambda + \mu - \frac{1}{6}\alpha\,\mu & \frac{1}{2}\lambda - \frac{1}{6}\alpha\,\mu \\ \frac{1}{2}\lambda + \frac{1}{3}\alpha\,\mu & \frac{1}{2}\lambda - \frac{1}{6}\alpha\,\mu & \frac{1}{2}\lambda + \mu - \frac{1}{6}\alpha\,\mu \end{vmatrix}, \tag{10.9.35}$$

where

$$\alpha = \left(1 + \frac{h^{\mathrm{p}}}{\mu}\right)^{-1}. \tag{10.9.36}$$

Upon expansion,

$$\Delta = \frac{1}{2}\,\mu^2\left[(3\lambda + 2\mu)\,(1 - \alpha) - \sigma_1\left(1 + \frac{\lambda}{\mu} - \frac{1}{3}\,\alpha\right)\right], \tag{10.9.37}$$

which is positive when

$$h^{\mathrm{p}} > \frac{\sigma_1/3}{1 - \sigma_1/E}\,. \tag{10.9.38}$$

Here, E stands for the Young's modulus, related to Lamé constants by

$$E = \frac{3\lambda + 2\mu}{1 + \lambda/\mu}\,. \tag{10.9.39}$$

Since physically attainable values of stress are much smaller that the elastic modulus, stability and uniqueness are both practically assured for $\sigma_1 < 3h^{\mathrm{p}}$. The results for triaxial tension of compressible elastic-plastic materials were obtained by Miles (1975). In the next subsection we proceed with a less involved analysis for incompressible materials.

10.9.5. Triaxial Tension of Incompressible Material

For incompressible elastic-plastic material $\underline{\chi}^0$ is the sum of two parts,

$$\begin{aligned} &\mu\left(D_{11}^2 + D_{22}^2 + D_{33}^2\right) - \alpha\,\mu\,(\mathbb{M}_{11}D_{11} + \mathbb{M}_{22}D_{22} + \mathbb{M}_{33}D_{33})^2 \\ &\qquad - \frac{1}{2}\left(\sigma_1\,D_{11}^2 + \sigma_2\,D_2^2 + \sigma_3\,D_{33}^2\right), \end{aligned} \tag{10.9.40}$$

where $D_{33} = -(D_{11} + D_{22})$, and

$$\left(2\,\mu - \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 + \sigma_2}\right) D_{12}^2 + \left(2\,\mu - \frac{\sigma_2^2 + \sigma_3^2}{\sigma_2 + \sigma_3}\right) D_{23}^2 + \left(2\,\mu - \frac{\sigma_3^2 + \sigma_1^2}{\sigma_3 + \sigma_1}\right) D_{31}^2. \tag{10.9.41}$$

The second part is certainly positive for

$$\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 + \sigma_2} < 2\,\mu, \quad \frac{\sigma_2^2 + \sigma_3^2}{\sigma_2 + \sigma_3} < 2\,\mu, \quad \frac{\sigma_3^2 + \sigma_1^2}{\sigma_3 + \sigma_1} < 2\,\mu, \tag{10.9.42}$$

which is expected to be always the case within attainable range of applied stress. For positive definiteness of $\underline{\chi}^0$ it is then sufficient to prove the positive definiteness of (10.9.40) for all volume preserving rate of deformation components. The elements of 2×2 determinant of the corresponding quadratic form are

$$\Delta_{11} = 2\,\mu - \alpha\,\mu\,(\mathbb{M}_{11} - \mathbb{M}_{33})^2 - \frac{1}{2}\,(\sigma_1 + \sigma_3), \tag{10.9.43}$$

$$\Delta_{22} = 2\,\mu - \alpha\,\mu\,(\mathbb{M}_{22} - \mathbb{M}_{33})^2 - \frac{1}{2}\,(\sigma_2 + \sigma_3), \tag{10.9.44}$$

$$\Delta_{12} = \Delta_{21} = \mu - \alpha\,\mu\,(\mathbb{M}_{11} - \mathbb{M}_{33})(\mathbb{M}_{22} - \mathbb{M}_{33}) - \frac{1}{2}\,\sigma_3. \tag{10.9.45}$$

The determinant Δ is accordingly

$$\begin{aligned}\frac{\Delta}{\mu^2} &= 3 - \frac{1}{\mu}\,(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{4\,\mu^2}\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ &- \alpha\left\{3 - \frac{1}{2\mu}\left[(\mathbb{M}_{22} - \mathbb{M}_{33})^2\,\sigma_1 + (\mathbb{M}_{33} - \mathbb{M}_{11})^2\,\sigma_2 + (\mathbb{M}_{11} - \mathbb{M}_{22})^2\,\sigma_3\right]\right\}.\end{aligned} \tag{10.9.46}$$

This is positive when

$$h^{\mathrm{p}} > \frac{\frac{1}{2}\,(\mathbb{M}_{11}^2\sigma_1 + \mathbb{M}_{22}^2\sigma_2 + \mathbb{M}_{33}^2\sigma_3) - \frac{1}{12\mu}\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)}{1 - \frac{1}{3\,\mu}\,(\sigma_1 + \sigma_2 + \sigma_3) + \frac{1}{12\,\mu^2}\,(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)}. \tag{10.9.47}$$

It is recalled that $\mathbb{M}$ is deviatoric and normalized, so that

$$\mathbb{M}_{11} + \mathbb{M}_{22} + \mathbb{M}_{33} = 0, \quad \mathbb{M}_{11}^2 + \mathbb{M}_{22}^2 + \mathbb{M}_{33}^2 = 1. \tag{10.9.48}$$

The critical hardening rate therefore depends on the state of stress, elastic shear modulus μ, and the components of the tensor $\mathbb{M}$ which is normal to the yield surface.

For biaxial tension with $\sigma_3 = 0$, the uniqueness and stability are certain for

$$h^{\mathrm{p}} > \frac{\frac{1}{2}\left(\mathbb{M}_{11}^2\sigma_1 + \mathbb{M}_{22}^2\sigma_2\right) - \frac{1}{12\mu}\,\sigma_1\sigma_2}{1 - \frac{1}{3\,\mu}\,(\sigma_1+\sigma_2) + \frac{1}{12\,\mu^2}\,\sigma_1\sigma_2}\,. \tag{10.9.49}$$

For example, for the von Mises yield criterion,

$$\mathbb{M}_{11} = \frac{2\sigma_1 - \sigma_2}{[6(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2)]^{1/2}}\,, \quad \mathbb{M}_{22} = \frac{2\sigma_2 - \sigma_1}{[6(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2)]^{1/2}}\,, \tag{10.9.50}$$

and the condition (10.9.49) becomes

$$h^{\mathrm{p}} = h_{\mathrm{t}}^{\mathrm{p}} > \frac{4\sigma_1^3 - 3\sigma_1^2\sigma_2 - 3\sigma_1\sigma_2^2 + 4\sigma_2^3}{12\,(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2)}\,, \tag{10.9.51}$$

neglecting terms of the order σ/μ and smaller. For equal biaxial tension $\sigma_1 = \sigma_2 = \sigma$, we have by symmetry

$$\mathbb{M}_{11} = \mathbb{M}_{22} = \frac{1}{\sqrt{6}}\,, \tag{10.9.52}$$

for any isotropic smooth yield surface, and

$$h^{\mathrm{p}} > \frac{\sigma/6}{1 - \sigma/6\,\mu}\,. \tag{10.9.53}$$

For uniaxial tension with $\sigma_2 = \sigma_3 = 0$, $\mathbb{M}_{11} = \sqrt{2/3}$ and the condition (10.9.49) reduces to

$$h^{\mathrm{p}} > \frac{\sigma_1/3}{1 - \sigma_1/3\,\mu}\,. \tag{10.9.54}$$

Since for incompressible elasticity $E = 3\,\mu$, the condition (10.9.54) is in accord with the condition (10.9.38).

10.9.6. Triaxial Tension of Rigid-Plastic Material

For a rigid-plastic material model with isotropic smooth yield surface, the principal directions of the rate of deformation tensor are parallel to those of stress, and eigenmodal spin components are identically equal to zero. The bifurcation and instability are thus both excluded if

$$\underline{\chi} = h\,(\boldsymbol{D} : \boldsymbol{D}) - \frac{1}{2}\,\boldsymbol{\sigma} : \boldsymbol{D}^2 > 0. \tag{10.9.55}$$

Since constitutively admissible $\boldsymbol{D}$ (and thus any eigenmodal rate of deformation) must be codirectional with the stress, the condition (10.9.55) is met when the modulus h satisfies

$$h > \frac{1}{2}\,\boldsymbol{\sigma} : \mathbb{M}^2 = \frac{1}{2}\,(\mathbb{M}_{11}^2\sigma_1 + \mathbb{M}_{22}^2\sigma_2 + \mathbb{M}_{33}^2\sigma_3). \tag{10.9.56}$$

The tensor $\mathbb{M}$ is normal to the smooth yield surface $f = 0$, having principal directions parallel to those of stress. Equivalently, we can write

$$h > \frac{1}{2}\,(\sigma + \boldsymbol{\sigma}' : \mathbb{M}^2), \quad \sigma = \frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\sigma}\,. \tag{10.9.57}$$

Expressed in terms of the principal stress components, and with the von Mises yield condition, this gives for biaxial tension

$$h > \frac{4\sigma_1^3 - 3\sigma_1^2\sigma_2 - 3\sigma_1\sigma_2^2 + 4\sigma_2^3}{12(\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2)}\,, \tag{10.9.58}$$

as originally derived by Swift (1952)[1], and for triaxial tension

$$h > \frac{1}{2}\left[\sigma + \frac{(\sigma_1-\sigma)^3 + (\sigma_2-\sigma)^3 + (\sigma_3-\sigma)^3}{(\sigma_1-\sigma)^2 + (\sigma_2-\sigma)^2 + (\sigma_3-\sigma)^2}\right], \tag{10.9.59}$$

as derived by Hill (1967). For equal biaxial tension $\sigma_1 = \sigma_2 = \sigma$, $h > \sigma/6$, while for uniaxial tension with $\sigma_2 = \sigma_3$, $h > \sigma_1/3$, for any isotropic smooth yield surface, in accord with the results from previous subsections.

10.10. Acceleration Waves in Elastoplastic Solids

During wave propagation in a medium, certain field variables can be discontinuous across the wave front. If displacement discontinuity is precluded by assumption that the failure does not occur, the strongest possible discontinuity is in the velocity of the particle. This is called a shock wave. If the velocity is continuous, but acceleration is discontinuous across the wave front, the wave is called an acceleration wave. Weaker waves are characterized by discontinuities in higher time derivatives of the velocity field (e.g., Janssen, Datta, and Jahsman, 1972; Clifton, 1974; Ting, 1976).

Consider a portion of the deforming body momentarily bounded in part by the surface S, embedded in the material and deforming with it, and in part by the surface Σ which propagates relative to the material. If the enclosed volume at the considered instant is V, then, for any continuous differentiable field $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathbf{T}\,\mathrm{d}V = \int_V \frac{\partial}{\partial t}(\rho\,\mathbf{T})\,\mathrm{d}V + \int_S \rho\,\mathbf{T}\,\mathbf{v}\cdot\mathrm{d}\mathbf{S} + \int_\Sigma \rho\,\mathbf{T}\,c\,\mathrm{d}\Sigma. \tag{10.10.1}$$

The particle velocity is $\mathbf{v}$, and c is the propagation speed of the surface Σ in the direction of its outward normal, both relative to a fixed observer.

[1]Published as the first paper in the first volume of the *Journal of the Mechanics and Physics of Solids*.

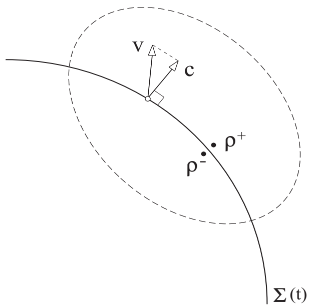

FIGURE 10.2. A surface of discontinuity $\Sigma(t)$ propagates relative to material with the speed c in the direction of its outward normal. The mass densities ahead and behind Σ are ρ^+ and ρ^-.

The above formula, which can be viewed as a modified Reynolds transport theorem of Eq. (3.2.6), will be used to derive the jump conditions across the wave front.

10.10.1. Jump Conditions for Shock Waves

Suppose that a mass density is discontinuous across Σ. Then, take

$$\mathbf{T} = 1, \tag{10.10.2}$$

and apply Eq. (10.10.1) to a thin slice of material immediately ahead and behind Σ. Summing up the resulting expressions, and implementing the conservation of mass condition, gives in the limit

$$c[\![\rho]\!] - [\![\rho\mathbf{v}]\!] \cdot \mathbf{n} = 0, \tag{10.10.3}$$

where $\mathbf{n}$ is the unit normal to Σ in the direction of propagation of Σ (Thomas, 1961). The brackets $[\![\ ]\!]$ designate the jump of the enclosed quantity across the surface Σ, e.g.,

$$[\![\,\rho\,]\!] = \rho^+ - \rho^-. \tag{10.10.4}$$

The superposed plus indicates the value at the point just ahead of Σ, and minus just behind the Σ (Fig. 10.2).

By taking

$$\mathbf{T} = \mathbf{v} \tag{10.10.5}$$

in Eq. (10.10.1), and by implementing Eq. (3.3.1), we similarly obtain

$$\mathbf{n}\cdot[\![\boldsymbol{\sigma}]\!]+\rho^{-}\left(c-\mathbf{v}^{-}\cdot\mathbf{n}\right)[\![\mathbf{v}]\!]=\mathbf{0}, \tag{10.10.6}$$

which relates the discontinuities in stress and velocity across the surface Σ. For further analysis of shock waves in elastic-plastic solids, see Wilkins (1964), Germain and Lee (1973), Ting (1976), and Drugan and Shen (1987).

10.10.2. Jump Conditions for Acceleration Waves

In an acceleration wave, the velocity and stress fields are continuous across Σ, but the acceleration $\dot{\mathbf{v}}=\mathrm{d}\mathbf{v}/\mathrm{d}t$ is not. To derive the corresponding jump condition across Σ, substitute

$$\mathbf{T}=\dot{\mathbf{v}} \tag{10.10.7}$$

in Eq. (10.10.1). In view of equations of motion and the relationship between the true and nominal tractions, we first have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\dot{\mathbf{v}}\,\mathrm{d}V &= \frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{t}_n\,\mathrm{d}S+\int_V \rho\,\dot{\mathbf{b}}\,\mathrm{d}V\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{S^0}\mathbf{p}_n\,\mathrm{d}S^0+\int_V\rho\,\dot{\mathbf{b}}\,\mathrm{d}V=\int_{S^0}\dot{\mathbf{P}}^T\cdot\mathbf{n}^0\,\mathrm{d}S^0+\int_V\rho\,\dot{\mathbf{b}}\,\mathrm{d}V.\end{aligned} \tag{10.10.8}$$

Further, the Nanson's relation (2.2.17) and Eq. (3.9.17) give

$$\int_{S^0}\dot{\mathbf{P}}^T\cdot\mathbf{n}^0\,\mathrm{d}S^0=\int_S\underline{\dot{\mathbf{P}}}^T\cdot\mathbf{n}\,\mathrm{d}S, \tag{10.10.9}$$

so that Eq. (10.10.1) becomes

$$\begin{aligned}&\int_S\mathbf{n}\cdot\underline{\dot{\mathbf{P}}}\,\mathrm{d}S+\int_V\rho\,\dot{\mathbf{b}}\,\mathrm{d}V\\ &\qquad=\int_V\frac{\partial}{\partial t}(\rho\,\dot{\mathbf{v}})\,\mathrm{d}V+\int_S\rho\,\dot{\mathbf{v}}\,\mathbf{v}\cdot\mathrm{d}\mathbf{S}+\int_\Sigma\rho\,\dot{\mathbf{v}}\,c\,\mathrm{d}\Sigma.\end{aligned} \tag{10.10.10}$$

Applying this to a thin slice of material just ahead and behind of Σ, the volume integrals vanish in the limit, and the summation yields

$$\mathbf{n}\cdot[\![\underline{\dot{\mathbf{P}}}]\!]+\rho\,c_r\,[\![\dot{\mathbf{v}}]\!]=\mathbf{0}. \tag{10.10.11}$$

Here,

$$c_r=c-\mathbf{v}\cdot\mathbf{n} \tag{10.10.12}$$

is the speed of Σ relative to the material. Equation (10.10.11) relates the jumps in the acceleration and stress rate across the surface Σ.

A characteristic segment of the wave is defined as the discontinuity in the gradient of the particle velocity across the wave front,

$$\boldsymbol{\eta} = \left[\!\left[\frac{\partial \mathbf{v}}{\partial n} \right]\!\right]. \tag{10.10.13}$$

The geometric and kinematic conditions of compatibility for the velocity field (Thomas, 1961; Hill, 1961 b) give

$$[\![\, \mathbf{L} \,]\!] = \left[\!\left[\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right]\!\right] = \boldsymbol{\eta} \otimes \mathbf{n}, \quad \left[\!\left[\frac{\partial \mathbf{v}}{\partial t} \right]\!\right] = -c\, \boldsymbol{\eta}, \tag{10.10.14}$$

provided that $\mathbf{v}$ is continuous across Σ. Since

$$\dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{L} \cdot \mathbf{v}, \tag{10.10.15}$$

a discontinuity in the acceleration is related to discontinuity in the velocity gradient by

$$[\![\, \dot{\mathbf{v}} \,]\!] = -c_r\, \boldsymbol{\eta}. \tag{10.10.16}$$

10.10.3. Propagation Condition

Substitution of Eq. (10.10.16) into Eq. (10.10.11) gives

$$\mathbf{n} \cdot [\![\, \underline{\dot{\mathbf{P}}} \,]\!] = \rho\, c_r^2\, \boldsymbol{\eta}. \tag{10.10.17}$$

Suppose that on both sides of Σ the plastic loading takes place. Since the stress and pseudomoduli are continuous across Σ in an acceleration wave, we have

$$[\![\, \underline{\dot{\mathbf{P}}} \,]\!] = [\![\, \underline{\boldsymbol{\Lambda}}^{\mathrm{p}} \cdot\cdot\, \mathbf{L} \,]\!] = \underline{\boldsymbol{\Lambda}}^{\mathrm{p}} \cdot\cdot\, [\![\, \mathbf{L} \,]\!] = \underline{\boldsymbol{\Lambda}}^{\mathrm{p}} \cdot\cdot\, (\boldsymbol{\eta} \otimes \mathbf{n}). \tag{10.10.18}$$

Combining Eqs. (10.10.17) and (10.10.18), therefore,

$$\mathbf{n} \cdot \underline{\boldsymbol{\Lambda}}^{\mathrm{p}} : (\mathbf{n} \otimes \boldsymbol{\eta}) = \rho\, c_r^2\, \boldsymbol{\eta}, \tag{10.10.19}$$

i.e.,

$$\underline{\mathbf{A}}^{\mathrm{p}} \cdot \boldsymbol{\eta} = \rho\, c_r^2\, \boldsymbol{\eta}. \tag{10.10.20}$$

The rectangular components of the real matrix $\underline{\mathbf{A}}^{\mathrm{p}}$ are

$$\underline{A}^{\mathrm{p}}_{ij} = \underline{\Lambda}^{\mathrm{p}}_{kilj} n_k n_l. \tag{10.10.21}$$

They depend on the current state of stress and material properties (embedded in $\underline{\boldsymbol{\Lambda}}^{\mathrm{p}}$), and the direction of propagation $\mathbf{n}$. In view of reciprocal symmetry ($\underline{\Lambda}^{\mathrm{p}}_{kilj} = \underline{\Lambda}^{\mathrm{p}}_{ljki}$), it follows that, in addition to be real, $\underline{\mathbf{A}}^{\mathrm{p}}$ is also symmetric ($\underline{A}^{\mathrm{p}}_{ij} = \underline{A}^{\mathrm{p}}_{ji}$). Thus, the eigenvalues $\rho\, c_r^2$ in Eq. (10.10.20) are all real. There is a wave propagating in the direction $\mathbf{n}$, carrying a discontinuity

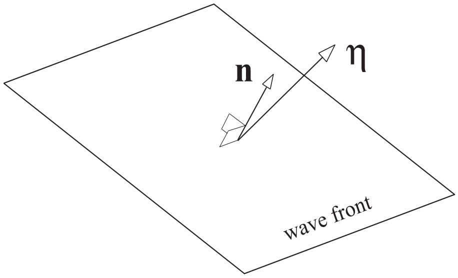

FIGURE 10.3. Plane wave propagating in the direction $\mathbf{n}$. The vector $\boldsymbol{\eta}$ is the polarization of the wave, which defines direction of the particle velocity.

$\boldsymbol{\eta}$, if the corresponding c_r^2 is positive. This is assured in the states where $\underline{\mathbf{A}}^{\mathrm{p}}$ is positive definite, since

$$\boldsymbol{\eta} \cdot \underline{\mathbf{A}}^{\mathrm{p}} \cdot \boldsymbol{\eta} = \rho\, c_r^2\, (\boldsymbol{\eta} \cdot \boldsymbol{\eta}). \tag{10.10.22}$$

The condition for nontrivial $\boldsymbol{\eta}$ to exist in the eigenvalue problem (10.10.20) is

$$\det(\underline{\mathbf{A}}^{\mathrm{p}} - \rho\, c_r^2\, \mathbf{I}) = 0. \tag{10.10.23}$$

A wave that carries a discontinuity in the velocity gradient also carries a discontinuity in the stress gradient. This is (Hill, 1961 b)

$$\left[\!\left[\, \frac{\partial \sigma_{ij}}{\partial x_m} \,\right]\!\right] = \frac{1}{c_r} \left(\sigma_{ij}\delta_{kl} - \sigma_{jk}\delta_{il} - \underline{\Lambda}^{\mathrm{p}}_{ijkl} \right) n_k n_m\, \eta_l . \tag{10.10.24}$$

In view of the relationship between the moduli in Eq. (10.1.15), we also have

$$\left(\underline{\mathcal{L}}^{\mathrm{p}\,(1)}_{kilj}\, n_k n_l \right) \eta_j = (\rho\, c_r^2 - \sigma_n)\eta_i , \tag{10.10.25}$$

where $\sigma_n = \sigma_{ij} n_i n_j$ is the normal stress over Σ.

Propagation of Plane Waves

There is an analogy between governing equations for acceleration waves and plane waves. Indeed, consider the rate equations of motion,

$$\boldsymbol{\nabla} \cdot \underline{\dot{\mathbf{P}}} = \rho\, \frac{\mathrm{d}^2 \mathbf{v}}{\mathrm{d}t^2}\,, \quad \underline{\dot{\mathbf{P}}} = \underline{\boldsymbol{\Lambda}} \cdot\cdot\, \mathbf{L}, \tag{10.10.26}$$

whose solutions are sought in the form of a plane wave propagating with a speed c in the direction $\mathbf{n}$,

$$\mathbf{v} = \boldsymbol{\eta} f(\mathbf{n} \cdot \mathbf{x} - ct). \tag{10.10.27}$$

The vector $\boldsymbol{\eta}$ is the polarization of the wave (Fig. 10.3). On substituting (10.10.27) into (10.10.26), we obtain the propagation condition

$$\underline{\mathbf{A}}^{\mathrm{p}} \cdot \boldsymbol{\eta} = \rho\, c^2\, \boldsymbol{\eta}, \quad \underline{A}^{\mathrm{p}}_{ij} = \underline{\Lambda}^{\mathrm{p}}_{kilj} n_k n_l. \tag{10.10.28}$$

The second-order tensor $\underline{\mathbf{A}}^{\mathrm{p}}$ is referred to as the acoustic tensor. Thus, ρc^2 is an eigenvalue and $\boldsymbol{\eta}$ is an eigenvector of $\underline{\mathbf{A}}^{\mathrm{p}}$. Since $\underline{\mathbf{A}}^{\mathrm{p}}$ is real and symmetric, c^2 must be real. If $c^2 > 0$ (assured by positive definiteness of $\underline{\mathbf{A}}^{\mathrm{p}}$), there is a stability with respect to propagation of small disturbances, superposed to finitely deformed current state. Equation (10.10.28) then admits three linearly independent plane progressive waves for each direction of propagation $\mathbf{n}$. Small amplitude plane waves can propagate along a given direction in three distinct, mutually orthogonal modes. These modes are generally neither longitudinal nor transverse (i.e., $\boldsymbol{\eta}$ is neither parallel nor normal to $\mathbf{n}$). For $c^2 = 0$, there is a transition from stability to instability. The latter is associated with $c^2 < 0$, and a divergent growth of initial disturbance. These fundamental results were established by Hadamar (1903) in the context of elastic stability, and for inelasticity by Thomas (1961), Hill (1962), and Mandel (1966).

If $\underline{\mathbf{\Lambda}}^{\mathrm{p}}$ does not possess reciprocal symmetry (nonassociative plasticity), $\underline{\mathbf{A}}^{\mathrm{p}}$ is not symmetric, and it may happen that at some states of deformation and material parameters two eigenvalues in Eq. (10.10.28) are complex conjugates (one is always real), which means that a flutter type instability may occur (Rice, 1977; Bigoni, 1995). Uniqueness and stability criteria for elastoplastic materials with nonassociative flow rules were studied by Maier (1970), Raniecki (1979), Needleman (1979), Raniecki and Bruhns (1981), Bruhns (1984), Bigoni and Hueckel (1991), Ottosen and Runesson (1991), Bigoni and Zaccaria (1992,1993), Neilsen and Schreyer (1993), and others.

10.10.4. Stationary Discontinuity

When the matrix $\underline{\mathbf{A}}^{\mathrm{p}}$ has a zero eigenvalue ($c_r = 0$), there is a discontinuity surface that does not travel relative to the material (stationary discontinuity, in Hadamar's terminology). This happens if and only if a discontinuity surface normal $\mathbf{n}$ satisfies

$$\det\left(\underline{\mathbf{A}}^{\mathrm{p}}\right) = \det\left(\underline{\Lambda}^{\mathrm{p}}_{ijkl} n_i n_k\right) = 0. \tag{10.10.29}$$

The corresponding eigenvector $\boldsymbol{\eta}$ is a nontrivial solution of the homogeneous system of equations

$$\underline{\mathbf{A}}^{\mathrm{p}} \cdot \boldsymbol{\eta} = \mathbf{0}. \tag{10.10.30}$$

Equation (10.10.24) does not determine discontinuity in the stress gradient across stationary discontinuity (since $c_r = 0$), but it does impose a condition on the current moduli and stress components there. This is

$$(\boldsymbol{\eta} \cdot \mathbf{n}\, \delta_{ik} - \eta_i n_k)\, \sigma_{kj} = \underline{\Lambda}^{\mathrm{p}}_{ijkl}\, n_k \eta_l, \tag{10.10.31}$$

if discontinuity in the velocity gradient actually occurs. Since material particles remain on the surface of stationary discontinuity, there is no jump in acceleration or nominal traction rate across Σ, so that

$$[\![\, \dot{\mathbf{v}} \,]\!] = \mathbf{0}, \quad \mathbf{n} \cdot [\![\, \dot{\underline{\mathbf{P}}} \,]\!] = \mathbf{0}. \tag{10.10.32}$$

Note that

$$\mathbf{n} \cdot [\![\, \dot{\underline{\mathbf{P}}} \,]\!] = \underline{\mathbf{A}}^{\mathrm{p}} \cdot \boldsymbol{\eta}. \tag{10.10.33}$$

Furthermore, since

$$\dot{\underline{\mathbf{P}}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\, \mathrm{tr}\, \mathbf{D} - \mathbf{L} \cdot \boldsymbol{\sigma}, \tag{10.10.34}$$

it follows that

$$\mathbf{n} \cdot [\![\, \dot{\underline{\mathbf{P}}} \,]\!] = \mathbf{n} \cdot [\![\, \dot{\boldsymbol{\sigma}} \,]\!]. \tag{10.10.35}$$

In proof, let

$$\boldsymbol{\eta} = \mathbf{b} + g\mathbf{n}, \tag{10.10.36}$$

where $\mathbf{b} \cdot \mathbf{n} = 0$ and g is a scalar function. Then, since $[\![\, \mathbf{L} \,]\!] = \boldsymbol{\eta} \otimes \mathbf{n}$, we have

$$\mathrm{tr}\, [\![\, \mathbf{D} \,]\!] = g, \quad \mathbf{n} \cdot [\![\, \mathbf{L} \,]\!] = g\mathbf{n}, \tag{10.10.37}$$

thus the result.

10.11. Analysis of Plastic Flow Localization

Consider an equilibrium configuration of uniformly strained homogeneous body. Suppose that increments of deformation (velocity) are prescribed on the boundary of the body, giving rise to uniform velocity gradient $\mathbf{L}^0$ throughout the body. The question is if there could be another statically and constitutively admissible velocity gradient field, associated with the same velocity boundary conditions. All-around displacement conditions are imposed to rule out geometric instabilities, such as buckling or necking, which could precede localization. We wish to examine if the bifurcation field can be characterized by localization of deformation within a planar band with normal $\mathbf{n}$, such that

$$\mathbf{L} = \mathbf{L}^0 + \boldsymbol{\eta} \otimes \mathbf{n}, \quad \text{i.e.,} \quad [\![\, \mathbf{L} \,]\!] = \boldsymbol{\eta} \otimes \mathbf{n}, \tag{10.11.1}$$

across the band. As discussed in the previous subsection, this could happen in the band whose normal $\mathbf{n}$ satisfies the condition (10.10.29), assuring that there is a nontrivial solution for $\boldsymbol{\eta}$ in equations

$$\underline{\Lambda}^{\mathrm{p}}_{ijkl} n_i n_k \eta_l = 0. \tag{10.11.2}$$

Here,

$$\underline{\Lambda}^{\mathrm{p}}_{ijkl} = \underline{\mathcal{L}}^{\mathrm{p}\,(1)}_{ijkl} + \sigma_{ik}\delta_{jl} = \underline{\mathcal{L}}^{\mathrm{p}\,(0)}_{ijkl} + \mathcal{R}_{ijkl}, \tag{10.11.3}$$

and

$$\mathcal{R}_{ijkl} = \frac{1}{2}\left(\sigma_{ik}\delta_{jl} - \sigma_{jk}\delta_{il} - \sigma_{il}\delta_{jk} - \sigma_{jl}\delta_{ik}\right). \tag{10.11.4}$$

It is noted that Eq. (10.11.2) can also be deduced through an eigenmodal analysis of the type used in Section 7.9.

10.11.1. Elastic-Plastic Materials

Following Rice (1977), suppose that elastoplastic response is described by a nonassociative flow rule, with the instantaneous elastoplastic stiffness

$$\underline{\boldsymbol{\mathcal{L}}}^{\mathrm{p}}_{(0)} = \underline{\boldsymbol{\mathcal{L}}}_{(0)} - \frac{1}{\hat{\mathbb{Q}} : \mathbb{P} + H}\, \hat{\mathbb{P}} \otimes \hat{\mathbb{Q}}, \tag{10.11.5}$$

where

$$\mathbb{P} = \frac{\partial \pi}{\partial \boldsymbol{\sigma}}, \quad \mathbb{Q} = \frac{\partial f}{\partial \boldsymbol{\sigma}}. \tag{10.11.6}$$

The potential function and the yield function are denoted by π and f, and

$$\hat{\mathbb{Q}} = \underline{\boldsymbol{\mathcal{L}}}_{(0)} : \mathbb{Q}, \quad \hat{\mathbb{P}} = \underline{\boldsymbol{\mathcal{L}}}_{(0)} : \mathbb{P}, \quad \hat{\mathbb{Q}} : \mathbb{P} = \mathbb{Q} : \underline{\boldsymbol{\mathcal{L}}}_{(0)} : \mathbb{P}. \tag{10.11.7}$$

Equation (10.11.5) can be derived from the general expression (9.8.9), with the current state used as the reference, and with elastic and plastic parts of the rate of deformation defined with respect to stress rate $\overset{\circ}{\underline{\boldsymbol{\tau}}}$. Note that $\mathbb{P}$ and $\mathbb{Q}$ are not normalized. In particular, with isotropic elastic stiffness,

$$\underline{\boldsymbol{\mathcal{L}}}_{(0)} = \lambda\, \mathbf{I} \otimes \mathbf{I} + 2\mu\, \boldsymbol{I}, \tag{10.11.8}$$

we have

$$\hat{\mathbb{Q}} \otimes \hat{\mathbb{P}} = (\lambda \operatorname{tr} \mathbb{Q}\; \mathbf{I} + 2\mu\, \mathbb{Q}) \otimes (\lambda \operatorname{tr} \mathbb{P}\; \mathbf{I} + 2\mu\, \mathbb{P}), \tag{10.11.9}$$

and

$$\hat{\mathbb{Q}} : \mathbb{P} = \lambda\,(\operatorname{tr}\mathbb{Q})(\operatorname{tr}\mathbb{P}) + 2\mu\,\mathbb{Q} : \mathbb{P}. \tag{10.11.10}$$

A nontrivial solution for $\boldsymbol{\eta}$ is sought in equations

$$\left(\underline{\mathcal{L}}^{(0)}_{ijkl} - \frac{1}{\hat{\mathbb{Q}} : \mathbb{P} + H}\,\hat{\mathbb{P}}_{ij}\hat{\mathbb{Q}}_{kl} + \mathcal{R}_{ijkl}\right) n_i n_k \eta_l = 0. \tag{10.11.11}$$

They can be rewritten in direct notation as

$$\mathbf{C}\cdot\boldsymbol{\eta} - \frac{1}{\hat{\mathbb{Q}} : \mathbb{P} + H}\,\hat{\mathbb{P}}\cdot\mathbf{n}\,(\mathbf{n}\cdot\hat{\mathbb{Q}}\cdot\boldsymbol{\eta}) + \mathbf{R}\cdot\boldsymbol{\eta} = \mathbf{0}. \tag{10.11.12}$$

The second-order tensors $\mathbf{C}$ and $\mathbf{R}$ are introduced by

$$C_{jl} = \underline{\mathcal{L}}^{(0)}_{ijkl} n_i n_k, \quad R_{jl} = \mathcal{R}_{ijkl} n_i n_k. \tag{10.11.13}$$

In view of the representation for $\underline{\boldsymbol{\mathcal{L}}}_{(0)}$, the tensor $\mathbf{C}$ and its inverse are explicitly given by

$$\mathbf{C} = \mu\left(\mathbf{I} + \frac{1}{1-2\nu}\,\mathbf{n}\otimes\mathbf{n}\right), \quad \mathbf{C}^{-1} = \frac{1}{\mu}\left[\mathbf{I} - \frac{1}{2(1-\nu)}\,\mathbf{n}\otimes\mathbf{n}\right], \tag{10.11.14}$$

where ν is the Poisson ratio. Multiplying (10.11.12) by $\mathbf{C}^{-1}$ gives

$$(\mathbf{I}+\mathbf{B})\cdot\boldsymbol{\eta} = \frac{1}{\hat{\mathbb{Q}} : \mathbb{P} + H}\,\mathbf{C}^{-1}\cdot\hat{\mathbb{P}}\cdot\mathbf{n}\,(\mathbf{n}\cdot\hat{\mathbb{Q}}\cdot\boldsymbol{\eta}), \tag{10.11.15}$$

i.e.,

$$\boldsymbol{\eta} = \frac{1}{\hat{\mathbb{Q}} : \mathbb{P} + H}\,(\mathbf{I}+\mathbf{B})^{-1}\cdot\mathbf{C}^{-1}\cdot\hat{\mathbb{P}}\cdot\mathbf{n}\,(\mathbf{n}\cdot\hat{\mathbb{Q}}\cdot\boldsymbol{\eta}), \tag{10.11.16}$$

where

$$\mathbf{B} = \mathbf{C}^{-1}\cdot\mathbf{R}. \tag{10.11.17}$$

Since the components of the matrix $\mathbf{R}$ are of the order of stress, which is ordinarily much smaller than the elastic modulus, the components of matrix $\mathbf{B}$ are small comparing to one. Thus the inverse matrix $(\mathbf{I}+\mathbf{B})^{-1}$ can be determined accurately by retaining few leading terms in the expansion

$$(\mathbf{I}+\mathbf{B})^{-1} = \mathbf{I} - \mathbf{B} + \mathbf{B}\cdot\mathbf{B} - \cdots. \tag{10.11.18}$$

Equation (10.11.16) enables an easy identification of the critical hardening rate for the localization. Upon multiplication by $\mathbf{n}\cdot\hat{\mathbb{Q}}$ and the cancellation of $\mathbf{n}\cdot\hat{\mathbb{Q}}\cdot\boldsymbol{\eta}$, there follows

$$H = \mathbf{n}\cdot\hat{\mathbb{Q}}\cdot(\mathbf{I}+\mathbf{B})^{-1}\cdot\mathbf{C}^{-1}\cdot\hat{\mathbb{P}}\cdot\mathbf{n} - \hat{\mathbb{Q}} : \mathbb{P}. \tag{10.11.19}$$

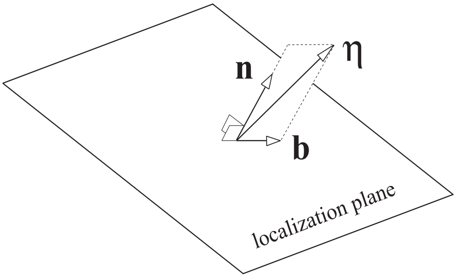

FIGURE 10.4. Localization plane (stationary discontinuity) with normal **n**. The localization vector $\boldsymbol{\eta}$ defines velocity discontinuity across the plane. The component **b** in the plane of localization corresponds to shear band localization.

Furthermore, Eq. (10.11.16) by inspection gives the characteristic segment (localization vector)

$$\boldsymbol{\eta} \propto (\mathbf{I} + \mathbf{B})^{-1} \cdot \mathbf{C}^{-1} \cdot \hat{\mathbb{P}} \cdot \mathbf{n}, \tag{10.11.20}$$

to within a scalar multiple.

If the **B** components are neglected (which is equivalent to approximating $\overset{\circ}{\underline{\boldsymbol{\tau}}}$ with $\dot{\boldsymbol{\sigma}}$ in the elastoplastic constitutive structure), Eq. (10.11.19) becomes

$$\begin{aligned}\frac{H}{\mu} &= 4\,\mathbf{n} \cdot \mathbb{Q} \cdot \mathbb{P} \cdot \mathbf{n} - \frac{2}{1-\nu}\,\mathbb{Q}_n\,\mathbb{P}_n - 2\,\mathbb{Q} : \mathbb{P} \\ &+ \frac{2\nu}{1-\nu}\left[\,(\operatorname{tr}\mathbb{Q})\mathbb{P}_n + (\operatorname{tr}\mathbb{P})\mathbb{Q}_n - (\operatorname{tr}\mathbb{Q})(\operatorname{tr}\mathbb{P})\right],\end{aligned} \tag{10.11.21}$$

where

$$\mathbb{Q}_n = \mathbf{n} \cdot \mathbb{Q} \cdot \mathbf{n}, \quad \mathbb{P}_n = \mathbf{n} \cdot \mathbb{P} \cdot \mathbf{n}. \tag{10.11.22}$$

The localization vector is

$$\boldsymbol{\eta} \propto \mathbb{P} \cdot \mathbf{n} - \frac{1}{2(1-\nu)}\,(\mathbb{P}_n - 2\nu\operatorname{tr}\mathbb{P})\,\mathbf{n}. \tag{10.11.23}$$

Observe that

$$\mathbf{n} \cdot \boldsymbol{\eta} \propto \frac{1}{2(1-\nu)}\left[(1-2\nu)\mathbb{P}_n + 2\nu\operatorname{tr}\mathbb{P}\right], \tag{10.11.24}$$

so that the component of $\boldsymbol{\eta}$ in the plane of localization (Fig. 10.4) is

$$\mathbf{b} = \boldsymbol{\eta} - (\mathbf{n} \cdot \boldsymbol{\eta})\mathbf{n} \propto \mathbb{P} \cdot \mathbf{n} - \mathbb{P}_n\,\mathbf{n}. \tag{10.11.25}$$

If

$$\mu\,\mathbb{P}_n + \lambda\operatorname{tr}\mathbb{P} = 0, \tag{10.11.26}$$

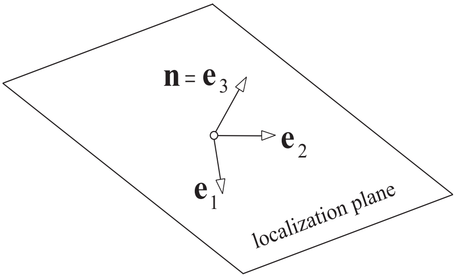

FIGURE 10.5. Localization plane with normal $\mathbf{n}$ in the coordinate direction $\mathbf{e}_3$. The other two coordinate directions $\mathbf{e}_1$ and $\mathbf{e}_2$ are in the plane of localization.

the shear band localization occurs ($\boldsymbol{\eta} = \mathbf{b}$).

Particularly simple representation of the expression for the critical hardening rate is obtained when the coordinate system is used with one axis in the direction $\mathbf{n}$ ($n_i = \delta_{i3}$). This is (Rice, 1977)

$$\frac{H}{\mu} = -2\,\mathbb{Q}_{\alpha\beta}\mathbb{P}_{\alpha\beta} - \frac{2\nu}{1-\nu}\,\mathbb{Q}_{\alpha\alpha}\,\mathbb{P}_{\beta\beta}, \tag{10.11.27}$$

where $\alpha, \beta = 1, 2$ denote the components on orthogonal axes in the plane of localization (Fig. 10.5). In the case of associative plasticity ($\mathbb{Q} = \mathbb{P}$), Eq. (10.11.27) shows that H at localization cannot be positive (i.e., softening is required for localization), at least when $\mathbf{B}$ terms are neglected, as assumed in (10.11.27).

A study of bifurcation in the form of shear bands from the nonhomogeneous stress state in the necked region of a tensile specimen is given by Iwakuma and Nemat-Nasser (1982). See also Ortiz, Leroy, and Needleman (1987), and Ramakrishnan and Atluri (1994). For the effects of elastic anisotropy on strain localization, the paper by Rizzi and Loret (1997) can be consulted.

10.11.2. Localization in Pressure-Sensitive Materials

For pressure-sensitive dilatant materials considered in Subsection 9.8.1, the yield and potential functions are such that

$$\mathbb{Q} = \frac{\boldsymbol{\sigma}'}{2J_2^{1/2}} + \frac{1}{3}\,\mu^*\,\mathbf{I}, \quad \mathbb{P} = \frac{\boldsymbol{\sigma}'}{2J_2^{1/2}} + \frac{1}{3}\,\beta\,\mathbf{I}, \tag{10.11.28}$$

where μ^* is the frictional parameter, and β the dilatancy factor. Thus,

$$\mathbb{Q}_n = \frac{\sigma'_n}{2J_2^{1/2}} + \frac{1}{3}\,\mu^*, \quad \mathbb{P}_n = \frac{\sigma'_n}{2J_2^{1/2}} + \frac{1}{3}\,\beta, \tag{10.11.29}$$

$$\mathbf{n}\cdot\mathbb{Q}\cdot\mathbb{P}\cdot\mathbf{n} = \frac{\mathbf{n}\cdot\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}'\cdot\mathbf{n}}{4J_2} + \frac{\sigma'_n}{6J_2^{1/2}}\,(\beta+\mu^*) + \frac{1}{9}\,\beta\mu^*, \tag{10.11.30}$$

$$\mathbb{Q}:\mathbb{P}_n = \frac{1}{2} + \frac{1}{3}\,\beta\mu^*, \quad \operatorname{tr}\mathbb{Q} = \mu^*, \quad \operatorname{tr}\mathbb{P} = \beta. \tag{10.11.31}$$

The deviatoric normal stress in the localization plane is $\sigma'_n = \mathbf{n}\cdot\boldsymbol{\sigma}'\cdot\mathbf{n}$. Substitution into Eq. (10.11.21), therefore, gives

$$\begin{aligned}\frac{H}{\mu} = \frac{\mathbf{n}\cdot\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}'\cdot\mathbf{n}}{J_2} &- \frac{1}{2(1-\nu)}\,\frac{\sigma_n'^{\,2}}{J_2} + \frac{1+\nu}{3(1-\nu)}\,\frac{\sigma'_n}{J_2^{1/2}}\,(\beta+\mu^*)\\ &- \frac{4(1+\nu)}{9(1-\nu)}\,\beta\mu^* - 1.\end{aligned} \tag{10.11.32}$$

If localization occurs, it will take place in the plane whose normal $\mathbf{n}$ maximizes the hardening rate H in Eq. (10.11.32) (H being a nonincreasing function of the amount of deformation imposed on the material). The problem was originally formulated and solved by Rudnicki and Rice (1975). To find the localization plane and the corresponding critical hardening rate, it is convenient to choose the coordinate axes parallel to principal stress axes. With respect to these axes,

$$\sigma'_n = (2\sigma'_2+\sigma'_3)n_2^2 + (2\sigma'_3+\sigma'_2)n_3^2 - (\sigma'_2+\sigma'_3), \tag{10.11.33}$$

and

$$\mathbf{n}\cdot\boldsymbol{\sigma}'\cdot\boldsymbol{\sigma}'\cdot\mathbf{n} = (\sigma'_2+\sigma'_3)^2 - \sigma'_3(2\sigma'_2+\sigma'_3)n_2^2 - \sigma'_2(2\sigma'_3+\sigma'_2)n_3^2, \tag{10.11.34}$$

since

$$n_1^2+n_2^2+n_3^2 = 1, \quad \sigma'_1+\sigma'_2+\sigma'_3 = 0. \tag{10.11.35}$$

Consequently, Eq. (10.11.32) becomes

$$\begin{aligned}\frac{H}{\mu} = \frac{1}{J_2}\big\{&\sigma'_2\sigma'_3 - \sigma'_3(2\sigma'_2+\sigma'_3)n_2^2 - \sigma'_2(2\sigma'_3+\sigma'_2)n_3^2\\ &- \frac{1}{2(1-\nu)}\left[(2\sigma'_2+\sigma'_3)n_2^2 + (2\sigma'_3+\sigma'_2)n_3^2 - (\sigma'_2+\sigma'_3)\right]^2\\ &+ \frac{1+\nu}{3(1-\nu)}\,J_2^{1/2}\,(\beta+\mu^*)\left[(2\sigma'_2+\sigma'_3)n_2^2 + (2\sigma'_3+\sigma'_2)n_3^2 - (\sigma'_2+\sigma'_3)\right]\\ &- \frac{4(1+\nu)}{9(1-\nu)}\,J_2\beta\mu^*\big\}.\end{aligned} \tag{10.11.36}$$

The stationary conditions

$$\frac{\partial H}{\partial n_2} = 0, \quad \frac{\partial H}{\partial n_3} = 0 \tag{10.11.37}$$

then yield

$$\begin{aligned}(2\sigma_2' + \sigma_3')n_2\,[\sigma_2' + \nu\sigma_3' + \frac{1+\nu}{3}\,J_2^{1/2}\,(\beta+\mu^*) \\ - (2\sigma_2' + \sigma_3')n_2^2 - (2\sigma_3' + \sigma_2')n_3^2] = 0,\end{aligned} \tag{10.11.38}$$

$$\begin{aligned}(2\sigma_3' + \sigma_2')n_3\,[\sigma_3' + \nu\sigma_2' + \frac{1+\nu}{3}\,J_2^{1/2}\,(\beta+\mu^*) \\ - (2\sigma_2' + \sigma_3')n_2^2 - (2\sigma_3' + \sigma_2')n_3^2] = 0.\end{aligned} \tag{10.11.39}$$

Note that

$$2\sigma_2' + \sigma_3' = \sigma_2 - \sigma_1 \le 0, \quad 2\sigma_3' + \sigma_2' = \sigma_3 - \sigma_1 \le 0. \tag{10.11.40}$$

If all principal stresses are distinct, there are three possibilities to satisfy Eqs. (10.11.38) and (10.11.39). These are

$$\begin{gathered} n_2 = 0, \quad n_3 \ne 0, \\ n_2 \ne 0, \quad n_3 = 0, \\ n_2 = n_3 = 0. \end{gathered} \tag{10.11.41}$$

If $n_2 = 0$, Eq. (10.11.39) gives

$$(2\sigma_3' + \sigma_2')n_3^2 - (\sigma_3' + \nu\sigma_2') = \frac{1+\nu}{3}\,J_2^{1/2}(\beta+\mu^*), \tag{10.11.42}$$

i.e.,

$$n_3^2 = \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3} - (1+\nu)\,\frac{J_2^{1/2}}{\sigma_1 - \sigma_3}\left(\frac{\sigma_2'}{J_2^{1/2}} + \frac{\beta+\mu^*}{3}\right). \tag{10.11.43}$$

The value of n_3^2 must be between zero and one, $0 \le n_3^2 \le 1$. For positive β and μ^*, this is assured if

$$\beta + \mu^* \le \sqrt{3}. \tag{10.11.44}$$

In proof, one can use the connections

$$\frac{\sigma_1'}{J_2^{1/2}} = \left(1 - \frac{3}{4}\,\frac{\sigma_2'^{\,2}}{J_2}\right)^{1/2} - \frac{1}{2}\,\frac{\sigma_2'}{J_2^{1/2}}, \quad \frac{\sigma_3'}{J_2^{1/2}} = -\left(1 - \frac{3}{4}\,\frac{\sigma_2'^{\,2}}{J_2}\right)^{1/2} - \frac{1}{2}\,\frac{\sigma_2'}{J_2^{1/2}}, \tag{10.11.45}$$

which follow, for example, by solving

$$\sigma_2'^{\,2} + \sigma_3'^{\,2} + \sigma_2'\sigma_3' = J_2 \tag{10.11.46}$$

as a quadratic equation for σ_3' in terms of σ_2' and J_2. It is observed that

$$-\frac{1}{\sqrt{3}} \le \frac{\sigma_2'}{J_2^{1/2}} \le \frac{1}{\sqrt{3}}\,. \tag{10.11.47}$$

The lower bound is associated with axially-symmetric tension ($\sigma_1 > \sigma_2 = \sigma_3$), and the upper bound with axially-symmetric compression ($\sigma_1 = \sigma_2 > \sigma_3$). Substituting $n_2 = 0$ and Eq. (10.11.42) into Eq. (10.11.36) gives the critical hardening rate associated with the choice $n_2 = 0$,

$$\frac{H_{(2)}}{\mu} = -\frac{\sigma_2'^{\,2}}{J_2} + \frac{1-\nu}{2}\left(\frac{1+\nu}{1-\nu}\,\frac{\beta+\mu^*}{3} - \frac{\sigma_2'}{J_2^{1/2}}\right)^2 - \frac{4(1+\nu)}{9(1-\nu)}\,\beta\mu^*. \tag{10.11.48}$$

This can be rearranged as

$$\frac{H_{(2)}}{\mu} = \frac{1+\nu}{9(1-\nu)}\,(\beta-\mu^*)^2 - \frac{1+\nu}{2}\left(\frac{\sigma_2'}{J_2^{1/2}} + \frac{\beta+\mu^*}{3}\right)^2, \tag{10.11.49}$$

which was originally derived by Rudnicki and Rice (1975). See also Perrin and Leblond (1993).

The second solution of Eqs. (10.11.38) and (10.11.39) is associated with $n_3 = 0$. In this case

$$n_2^2 = -\frac{\sigma_2-\sigma_3}{\sigma_1-\sigma_2} - (1+\nu)\,\frac{J_2^{1/2}}{\sigma_1-\sigma_2}\left(\frac{\sigma_3'}{J_2^{1/2}} + \frac{\beta+\mu^*}{3}\right), \tag{10.11.50}$$

which must meet the condition $0 \le n_2^2 \le 1$. The critical hardening rate is consequently

$$\frac{H_{(3)}}{\mu} = \frac{1+\nu}{9(1-\nu)}\,(\beta-\mu^*)^2 - \frac{1+\nu}{2}\left(\frac{\sigma_3'}{J_2^{1/2}} + \frac{\beta+\mu^*}{3}\right)^2. \tag{10.11.51}$$

The remaining solution of Eqs. (10.11.38) and (10.11.39) is associated with $n_2 = n_3 = 0$. The corresponding critical hardening rate $H_{(2,3)}$ can be calculated from Eq. (10.11.36).

Among the three values $H_{(2)}$, $H_{(3)}$ and $H_{(2,3)}$, the truly critical hardening rate is the largest of them. For realistic values of material properties β and μ^*, $H_{(2,3)}$ is always smaller than $H_{(2)}$ and $H_{(3)}$. This is expected on physical grounds because there is no shear stress in the localization plane associated with $H_{(2,3)}$ (localization plane being the principal stress plane), which greatly diminishes a tendency toward localization. We thus examine the inequality $H_{(2)} > H_{(3)}$. From (10.11.49) and (10.11.51), this is satisfied

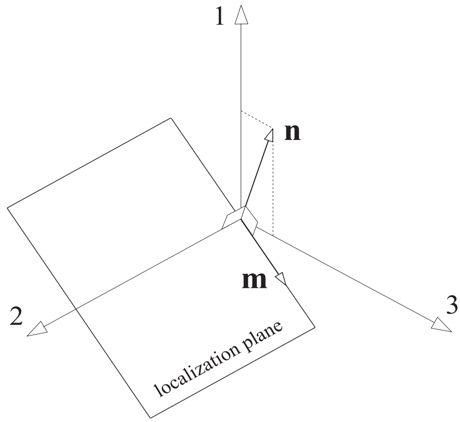

FIGURE 10.6. The localization plane according to considered pressure sensitive material model has its normal **n** perpendicular to the intermediate principal stress σ_2, so that in the coordinate system of principal stresses $\mathbf{n} = \{n_1, 0, (1 - n_1^2)^{1/2}\}$.

if

$$\left(\frac{\sigma_2'}{J_2^{1/2}} + \frac{\beta + \mu^*}{3}\right)^2 < \left(\frac{\sigma_3'}{J_2^{1/2}} + \frac{\beta + \mu^*}{3}\right)^2, \tag{10.11.52}$$

i.e.,

$$\frac{\sigma_1'}{J_2^{1/2}} > \frac{2}{3}\,(\beta + \mu^*). \tag{10.11.53}$$

The result can be expressed by using the first of expressions (10.11.45) as

$$\left(1 - \frac{3}{4}\,\frac{\sigma_2'^{\,2}}{J_2}\right)^{1/2} - \frac{1}{2}\,\frac{\sigma_2'}{J_2^{1/2}} > \frac{2}{3}\,(\beta + \mu^*). \tag{10.11.54}$$

In view of (10.11.47), a conservative bound assuring that $H_{(2)} > H_{(3)}$ is

$$\beta + \mu^* < \frac{\sqrt{3}}{2}\,, \tag{10.11.55}$$

whereas the condition

$$\beta + \mu^* > \sqrt{3} \tag{10.11.56}$$

assures that $H_{(3)} > H_{(2)}$. For the range of β and μ^* values used in constitutive modeling of fissured rocks, the latter case appears to be exceptional (Rudnicki and Rice, *op. cit.*). Thus, the localization will most likely occur in the plane whose normal is perpendicular to σ_2 direction ($n_2 = 0$), and the critical hardening rate is defined by Eq. (10.11.49); see Fig. 10.6.

It remains to examine a possibility for localization in the plane whose normal is perpendicular to σ_1 direction ($n_1 = 0$). The corresponding critical hardening rate would be

$$\frac{H_{(1)}}{\mu} = \frac{1+\nu}{9(1-\nu)}\,(\beta-\mu^*)^2 - \frac{1+\nu}{2}\left(\frac{\sigma_1'}{J_2^{1/2}} + \frac{\beta+\mu^*}{3}\right)^2. \tag{10.11.57}$$

This is greater than $H_{(2)}$ if

$$\frac{\sigma_3'}{J_2^{1/2}} > \frac{2}{3}\,(\beta+\mu^*). \tag{10.11.58}$$

However, from the second of expressions (10.11.45) it can be observed that $\sigma_3'/J_2^{1/2}$ is always negative in the range defined by (10.11.47). For frictional materials showing positive dilatancy, $\beta+\mu^* > 0$, the condition (10.11.58) is, therefore, never met. It could, however, be of interest in the study of loose granular materials which compact during shear, and thus exhibit negative dilatancy.

The expression for the critical hardening rate (10.11.49) reveals that localization in considered pressure-dependent dilatant materials is possible with positive hardening rate, depending on the value of $\sigma_3'/J_2^{1/2}$. The most critical (prompt to localization) is the state of stress

$$\frac{\sigma_2'}{J_2^{1/2}} = -\frac{\beta+\mu^*}{3}\,, \tag{10.11.59}$$

for which the critical hardening rate is

$$\frac{H_{(2)}}{\mu} = \frac{1+\nu}{9(1-\nu)}\,(\beta-\mu^*)^2. \tag{10.11.60}$$

The localization occurs in the plane whose normal is defined by

$$n_1^2 = \frac{\sigma_1-\sigma_2}{\sigma_1-\sigma_3}\,,\quad n_2 = 0,\quad n_3^2 = \frac{\sigma_2-\sigma_3}{\sigma_1-\sigma_3}\,. \tag{10.11.61}$$

Returning to Eqs. (10.11.38) and (10.11.39), if $\sigma_1 = \sigma_2 > \sigma_3$, n_2 remains unspecified by Eq. (10.11.38), which is satisfied by $2\sigma_2' + \sigma_3' = 0$, while Eq. (10.11.39) determines n_3. The critical hardening rate is still defined by Eq. (10.11.49), with $\sigma_2' = (\sigma_2-\sigma_3)/3$.

The presented analysis in this subsection is based on the expression (10.11.21), which does not account for **B** terms, of the order of stress divided by elastic modulus. Inclusion of these terms and examination of their effects on the critical hardening rate and localization is given in the paper by

Rudnicki and Rice (1975). Further analysis of stability in the absence of plastic normality is available in Rice and Rudnicki (1980), Chau and Rudnicki (1990), and Li and Drucker (1994). Shear band formation in concrete was studied by Ortiz (1987). The book by Bažant and Cedolin (1991) provides additional references.

10.11.3. Rigid-Plastic Materials

For rigid-plastic materials the stress rate cannot be expressed in terms of the rate of deformation, so that localization condition cannot be put in the form (10.11.2). Instead, we impose conditions

$$[\![\,\mathbf{L}\,]\!] = \boldsymbol{\eta} \otimes \mathbf{n}, \quad \mathbf{n} \cdot [\![\,\dot{\boldsymbol{\sigma}}\,]\!] = \mathbf{0} \tag{10.11.62}$$

directly, following the procedure by Rice (1977). The constitutive structure for nonassociative rigid-plastic response is

$$\mathbf{D} = \frac{1}{H}\,\mathbb{P} \otimes \mathbb{Q} : \overset{\circ}{\boldsymbol{\sigma}}, \tag{10.11.63}$$

so that

$$[\![\,\mathbf{D}\,]\!] = \frac{1}{H}\,\mathbb{P} \otimes \mathbb{Q} : [\![\,\overset{\circ}{\boldsymbol{\sigma}}\,]\!], \quad [\![\,\overset{\circ}{\boldsymbol{\sigma}}\,]\!] = [\![\,\dot{\boldsymbol{\sigma}}\,]\!] - [\![\,\mathbf{W}\,]\!] \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot [\![\,\mathbf{W}\,]\!]. \tag{10.11.64}$$

Consequently,

$$\begin{aligned} \frac{1}{2}\,(\boldsymbol{\eta} \otimes \mathbf{n} + \mathbf{n} \otimes \boldsymbol{\eta}) = \frac{1}{H}\,\mathbb{P} \otimes \mathbb{Q} : \Big[[\![\,\dot{\boldsymbol{\sigma}}\,]\!] - \frac{1}{2}\,(\boldsymbol{\eta} \otimes \mathbf{n} - \mathbf{n} \otimes \boldsymbol{\eta}) \cdot \boldsymbol{\sigma} \\ + \frac{1}{2}\,\boldsymbol{\sigma} \cdot (\boldsymbol{\eta} \otimes \mathbf{n} - \mathbf{n} \otimes \boldsymbol{\eta}) \Big]. \end{aligned} \tag{10.11.65}$$

This is evidently satisfied if $\mathbb{P}$ has the representation

$$\mathbb{P} = \frac{1}{2}\,(\boldsymbol{\nu} \otimes \mathbf{n} + \mathbf{n} \otimes \boldsymbol{\nu}), \tag{10.11.66}$$

for some vector $\boldsymbol{\nu}$, and if the localization vector is codirectional with $\boldsymbol{\nu}$,

$$\boldsymbol{\eta} = k\,\boldsymbol{\nu}. \tag{10.11.67}$$

Therefore, the localization can occur on the plane with normal $\mathbf{n}$ only if the state of stress is such that $\mathbb{P}$ has a special, rather restrictive representation given by (10.11.66). If the coordinate axes are selected with one axis parallel to $\mathbf{n}$ ($n_i = \delta_{i3}$), we have

$$\mathbb{P}_{\alpha\beta} = 0, \quad \alpha, \beta = 1, 2. \tag{10.11.68}$$

The intermediate principal value of such tensor is equal to zero ($\mathbb{P}_2 = 0$), so that $\mathbb{P}$ is a biaxial tensor with a spectral representation

$$\mathbb{P} = \mathbb{P}_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbb{P}_3 \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{10.11.69}$$

where $\mathbb{P}_1 \geq 0$, $\mathbb{P}_3 \leq 0$, and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the principal directions of $\mathbb{P}$. It follows that

$$\mathbf{n} = \frac{1}{\sqrt{\mathbb{P}_1 - \mathbb{P}_3}} \left(\sqrt{\mathbb{P}_1}\, \mathbf{e}_1 + \sqrt{-\mathbb{P}_3}\, \mathbf{e}_3 \right), \tag{10.11.70}$$

$$\boldsymbol{\nu} = \sqrt{\mathbb{P}_1 - \mathbb{P}_3} \left(\sqrt{\mathbb{P}_1}\, \mathbf{e}_1 - \sqrt{-\mathbb{P}_3}\, \mathbf{e}_3 \right). \tag{10.11.71}$$

For example, it can be readily verified that this complies with

$$\mathbb{P} = \mathbb{P}_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbb{P}_3 \mathbf{e}_3 \otimes \mathbf{e}_3 = \frac{1}{2} (\boldsymbol{\nu} \otimes \mathbf{n} + \mathbf{n} \otimes \boldsymbol{\nu}). \tag{10.11.72}$$

If neither $\mathbb{P}_1$ nor $\mathbb{P}_3$ vanishes, there are two possible localization planes, one with normal $\mathbf{n}$ defined by (10.11.70) and localization vector proportional to (10.11.71), and the other with

$$\mathbf{n} = \frac{1}{\sqrt{\mathbb{P}_1 - \mathbb{P}_3}} \left(\sqrt{\mathbb{P}_1}\, \mathbf{e}_1 - \sqrt{-\mathbb{P}_3}\, \mathbf{e}_3 \right), \tag{10.11.73}$$

$$\boldsymbol{\nu} = \sqrt{\mathbb{P}_1 - \mathbb{P}_3} \left(\sqrt{\mathbb{P}_1}\, \mathbf{e}_1 + \sqrt{-\mathbb{P}_3}\, \mathbf{e}_3 \right), \tag{10.11.74}$$

since $\boldsymbol{\eta}$ and $\mathbf{n}$ appear symmetrically in (10.11.66). If either $\mathbb{P}_1$ or $\mathbb{P}_3$ vanishes, there is one possible plane of localization. For instance, if $\mathbb{P}_3 = 0$, the localization plane has the normal $\mathbf{n} = \mathbf{e}_1$, and the corresponding $\boldsymbol{\nu} = \mathbb{P}_1 \mathbf{n}$. Observe that, in general,

$$\mathbf{n} \cdot \mathbb{P} \cdot \mathbf{n} = \mathbf{n} \cdot \boldsymbol{\nu} = \mathbb{P}_1 + \mathbb{P}_3, \quad \boldsymbol{\nu} \cdot \boldsymbol{\nu} = (\mathbb{P}_1 - \mathbb{P}_3)^2, \tag{10.11.75}$$

$$\boldsymbol{\nu} \cdot \mathbb{P} \cdot \boldsymbol{\nu} = (\boldsymbol{\nu} \cdot \boldsymbol{\nu}) \mathbf{n} \cdot \mathbb{P} \cdot \mathbf{n} = (\boldsymbol{\nu} \cdot \boldsymbol{\nu})(\mathbf{n} \cdot \boldsymbol{\nu}). \tag{10.11.76}$$

The component of the localization vector in the plane of localization is

$$\mathbf{b} = \boldsymbol{\eta} - (\mathbf{n} \cdot \boldsymbol{\eta})\mathbf{n} = 2k \sqrt{\frac{-\mathbb{P}_1 \mathbb{P}_3}{\mathbb{P}_1 - \mathbb{P}_3}} \left(\sqrt{-\mathbb{P}_3}\, \mathbf{e}_1 - \sqrt{\mathbb{P}_1}\, \mathbf{e}_3 \right). \tag{10.11.77}$$

In the case of incompressible plastic flow, $\operatorname{tr} \mathbb{P} = \mathbb{P}_1 + \mathbb{P}_3 = 0$, and

$$\mathbf{n} \cdot \boldsymbol{\nu} = 0, \tag{10.11.78}$$

so that bifurcation vector $\boldsymbol{\eta}$ is in the plane of localization. The plane of localization is in this case the plane of maximum shear stress, since from Eq. (10.11.73),

$$\mathbf{n} = \frac{1}{\sqrt{2}}\,(\mathbf{e}_1 - \mathbf{e}_3). \tag{10.11.79}$$

Returning to Eq. (10.11.65), the substitution of (10.11.66) and (10.11.67) yields

$$k\left\{H + \frac{1}{2}\,\mathbb{Q} : [(\boldsymbol{\nu}\otimes\mathbf{n} - \mathbf{n}\otimes\boldsymbol{\nu})\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot(\boldsymbol{\nu}\otimes\mathbf{n} - \mathbf{n}\otimes\boldsymbol{\nu})]\right\} = \mathbb{Q} : [\![\,\dot{\boldsymbol{\sigma}}\,]\!]. \tag{10.11.80}$$

We impose now the remaining discontinuity condition $\mathbf{n}\cdot[\![\,\dot{\boldsymbol{\sigma}}\,]\!] = \mathbf{0}$. With the orthogonal axes $1,2$ in the plane of localization, and the axis 3 in the direction of $\mathbf{n}$, it follows that

$$[\![\,\dot{\sigma}_{3j}\,]\!] = 0, \quad (j = 1,2,3) \tag{10.11.81}$$

and

$$\mathbb{Q} : [\![\,\dot{\boldsymbol{\sigma}}\,]\!] = \mathbb{Q}_{\alpha\beta}[\![\,\dot{\sigma}_{\alpha\beta}\,]\!], \quad \alpha,\beta = 1,2. \tag{10.11.82}$$

The condition (10.11.80) is accordingly

$$k\left\{H + \frac{1}{2}\,\mathbb{Q} : [(\boldsymbol{\nu}\otimes\mathbf{n} - \mathbf{n}\otimes\boldsymbol{\nu})\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}\cdot(\boldsymbol{\nu}\otimes\mathbf{n} - \mathbf{n}\otimes\boldsymbol{\nu})]\right\} = \mathbb{Q}_{\alpha\beta}[\![\,\dot{\sigma}_{\alpha\beta}\,]\!]. \tag{10.11.83}$$

Suppose that plastic normality is obeyed, so that $\mathbb{P} = \mathbb{Q}$ (associative plasticity). The right-hand side of (10.11.83) is then equal to zero, because $\mathbb{P}_{\alpha\beta} = 0$ by Eq. (10.11.68). Thus, if localization occurs ($k \neq 0$), the bracketed term on the left-hand side of (10.11.83) must vanish. This gives the critical hardening rate

$$H = \frac{1}{2}\,[\boldsymbol{\nu}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\nu} - (\boldsymbol{\nu}\cdot\boldsymbol{\nu})\,\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}]\,. \tag{10.11.84}$$

If the principal directions of stress tensor $\boldsymbol{\sigma}$ are parallel to those of $\mathbf{D}$ and thus $\mathbb{P}$, its spectral decomposition is

$$\boldsymbol{\sigma} = \sigma_1\mathbf{e}_1\otimes\mathbf{e}_1 + \sigma_2\mathbf{e}_2\otimes\mathbf{e}_2 + \sigma_3\mathbf{e}_3\otimes\mathbf{e}_3. \tag{10.11.85}$$

In view of (10.11.70) and (10.11.71), then,

$$\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n} = \frac{1}{\mathbb{P}_1 - \mathbb{P}_3}\,(\mathbb{P}_1\sigma_1 - \mathbb{P}_3\sigma_3), \quad \boldsymbol{\nu}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{\nu} = (\mathbb{P}_1 - \mathbb{P}_3)\,(\mathbb{P}_1\sigma_1 - \mathbb{P}_3\sigma_3). \tag{10.11.86}$$

Since

$$\boldsymbol{\nu} \cdot \boldsymbol{\nu} = (\mathbb{P}_1 - \mathbb{P}_3)^2, \tag{10.11.87}$$

Equation (10.11.86) shows that

$$\boldsymbol{\nu} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = (\boldsymbol{\nu} \cdot \boldsymbol{\nu})\, \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}, \tag{10.11.88}$$

and from Eq. (10.11.84) the critical hardening rate is

$$H = 0. \tag{10.11.89}$$

If principal directions of $\boldsymbol{\sigma}$ are not parallel to those of $\mathbf{D}$ (as in the case of anisotropic hardening rigid-plastic response), the critical hardening rate is not necessarily equal to zero. Furthermore, in the case of nonassociative plastic response (plastic non-normality) it is possible that some of the components $\mathbb{Q}_{\alpha\beta}$ are nonzero. In that case, since the components $[\![\dot{\sigma}_{\alpha\beta}]\!]$ are unrestricted, the condition (10.11.83) permits $k \neq 0$, and thus localization for any value of the hardening rate H. Rice (1977) indicates that the inclusion of elastic effects mitigates this strong tendency for localization in the absence of normality, but the tendency remains.

Since $\mathbb{P}$ and $\mathbf{D}$ are coaxial tensors by (10.11.63), from Eq. (10.11.68) it follows that

$$D_{\alpha\beta} = 0, \quad (\alpha, \beta = 1, 2) \tag{10.11.90}$$

in the plane of localization. Therefore, if the deformation field is such that a nondeforming plane does not exist, the localization cannot occur within the considered constitutive and localization framework. For example, it has been long known that rigid-plastic model with a smooth yield surface predicts an unlimited ductility in thin sheets under positive in-plane principal stretch rates (e.g., with von Mises yield condition and associative flow rule, $2\sigma_2 > \sigma_1$ for positive stretch rate D_2, contrary to the requirement $2\sigma_2 = \sigma_1$ for the existence of nondeforming plane of localization). Since localization actually occurs in these experiments, constitutive models simulating the yield-vertex have been employed to explain the experimental observations (Stören and Rice, 1975). Alternatively, imperfection studies were used in which, rather than being perfectly homogeneous, the sheet was assumed to contain an imperfection in the form of a long thin slice of material with slightly different properties from the material outside (Marciniak and Kuczynski, 1967;

Anand and Spitzig, 1980). Detailed summary and results for various material models can be found in the papers by Needleman (1976), and Needleman and Tvergaard (1983,1992). See also Petryk and Thermann (1996). We discuss below the yield vertex effects on localization in rigid-plastic, and incompressible elastic-plastic materials.

10.11.4. Yield Vertex Effects on Localization

A constitutive model simulating formation and effects of the vertex at the loading point of the yield surface was presented in Subsections 9.8.2 and 9.11.2. In the case of rigid-plasticity with pressure-independent associative flow rule, the rate of deformation is defined by

$$\mathbf{D} = \frac{1}{h}(\mathbb{M} \otimes \mathbb{M}) : \overset{\circ}{\boldsymbol{\sigma}} + \frac{1}{h_1}\left[\overset{\circ}{\boldsymbol{\sigma}}{}' - (\mathbb{M} \otimes \mathbb{M}) : \overset{\circ}{\boldsymbol{\sigma}}\right]. \tag{10.11.91}$$

The normalized tensor

$$\mathbb{M} = \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}}{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}} : \dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^{1/2}}, \tag{10.11.92}$$

is a deviatoric second-order tensor, f being a pressure-independent yield function. For example,

$$\mathbb{M} = \frac{\boldsymbol{\sigma}'}{(2J_2)^{1/2}}, \quad \text{if} \quad f = J_2^{1/2} = \left(\frac{1}{2}\,\boldsymbol{\sigma}' : \boldsymbol{\sigma}'\right)^{1/2}. \tag{10.11.93}$$

The hardening modulus of the vertex response

$$h_1 > h \tag{10.11.94}$$

governs the response to part of the stress increment directed tangentially to what is taken to be a smooth yield surface through the considered stress point. Since

$$\mathbb{M} : \mathbf{D} = \frac{1}{h}\,(\mathbb{M} : \overset{\circ}{\boldsymbol{\sigma}}), \tag{10.11.95}$$

the inverse constitutive expression is

$$\overset{\circ}{\boldsymbol{\sigma}}{}' = h_1\mathbf{D} - (h_1 - h)(\mathbb{M} \otimes \mathbb{M}) : \mathbf{D}, \tag{10.11.96}$$

i.e.,

$$\overset{\circ}{\boldsymbol{\sigma}} = \dot{\sigma}\,\mathbf{I} + h_1\mathbf{D} - (h_1 - h)(\mathbb{M} \otimes \mathbb{M}) : \mathbf{D}. \tag{10.11.97}$$

Here,

$$\sigma = \frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\sigma}, \quad \mathrm{tr}\,\mathbf{D} = \mathrm{tr}\,\mathbb{M} = 0. \tag{10.11.98}$$

The jump condition $\mathbf{n} \cdot [\![\dot{\boldsymbol{\sigma}}]\!] = \mathbf{0}$ is consequently

$$\begin{aligned} &[\![\dot{\sigma}]\!] \, \mathbf{n} + \mathbf{n} \cdot [\![\mathbf{W}]\!] \cdot \boldsymbol{\sigma} - \mathbf{n} \cdot \boldsymbol{\sigma} \cdot [\![\mathbf{W}]\!] + h_1 \mathbf{n} \cdot [\![\mathbf{D}]\!] \\ &\quad - (h_1 - h)(\mathbf{n} \cdot \mathbb{M})(\mathbb{M} : [\![\mathbf{D}]\!]) = \mathbf{0}. \end{aligned} \tag{10.11.99}$$

Since $[\![\mathbf{L}]\!] = \boldsymbol{\eta} \otimes \mathbf{n}$, and $\operatorname{tr} \mathbf{L} = 0$ for incompressible material, $\boldsymbol{\eta}$ must be perpendicular to $\mathbf{n}$. Hence,

$$\boldsymbol{\eta} = g \, \mathbf{m}, \quad \mathbf{m} \cdot \mathbf{n} = 0, \tag{10.11.100}$$

where g is a scalar function (bifurcation amplitude), and $\mathbf{m}$ is a unit vector in the plane of localization. Therefore,

$$[\![\mathbf{L}]\!] = g(\mathbf{m} \otimes \mathbf{n}), \tag{10.11.101}$$

and (10.11.99) becomes

$$[\![\dot{\sigma}]\!] \, \mathbf{n} - \frac{1}{2} \, g \big[\mathbf{m} \cdot \boldsymbol{\sigma} + \sigma_{mn} \, \mathbf{n} - (h_1 + \sigma_n) \, \mathbf{m} + 2(h_1 - h) \mathbb{M}_{mn} (\mathbf{n} \cdot \mathbb{M}) \big] = \mathbf{0}, \tag{10.11.102}$$

where

$$\sigma_{mn} = \mathbf{m} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}, \quad \sigma_n = \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n}, \quad \mathbb{M}_{mn} = \mathbf{m} \cdot \mathbb{M} \cdot \mathbf{n}. \tag{10.11.103}$$

Performing a scalar product of Eq. (10.11.102) with unit vectors $\mathbf{n}$, $\mathbf{m}$ and $\mathbf{p} = \mathbf{m} \times \mathbf{n}$ ($\mathbf{m}$ and $\mathbf{p}$ thus both being in the plane of localization), yields

$$g \big[\sigma_{mn} + (h_1 - h) \mathbb{M}_{mn} \, \mathbb{M}_n \big] = [\![\dot{\sigma}]\!], \tag{10.11.104}$$

$$\sigma_m - \sigma_n - h_1 + 2(h_1 - h) \mathbb{M}_{mn}^2 = 0, \tag{10.11.105}$$

$$\sigma_{mp} + 2(h_1 - h) \mathbb{M}_{mn} \, \mathbb{M}_{np} = 0, \tag{10.11.106}$$

with no summation over repeated index n.

If h_1 is considered to be a constant vertex hardening modulus, localization will occur in the plane for which h is maximum. By taking a variation of (10.11.105) corresponding to $\delta \mathbf{n} \propto \mathbf{p}$ (so that $\mathbf{m}$ remains perpendicular to $\mathbf{n} + \delta \mathbf{n}$, i.e., $\delta \mathbf{m} = \mathbf{0}$), and by setting $\delta h = 0$, it follows that

$$\sigma_{np} - 2(h_1 - h) \mathbb{M}_{mn} \, \mathbb{M}_{mp} = 0, \tag{10.11.107}$$

with no sum on m. Equations (10.11.106) and (10.11.107) are both satisfied if the axis $\mathbf{p}$ is along one of the principal stress axes, provided that $\mathbb{M}$ and $\boldsymbol{\sigma}$ are coaxial tensors (isotropic hardening), for then

$$\sigma_{mp} = \sigma_{np} = 0, \quad \mathbb{M}_{mp} = \mathbb{M}_{np} = 0. \tag{10.11.108}$$

In the case of von Mises yield condition, $\mathbb{M}$ is given by Eq. (10.11.93), and Eqs. (10.11.106) and (10.11.107) are satisfied only if

$$\sigma_{mp} = \sigma_{np} = 0, \tag{10.11.109}$$

so that the axis $\mathbf{p}$ must be codirectional with one of the principal stress axes (Rice, 1977). In the case of a plasticity model without a vertex, we have found in the previous subsection that the axis of intermediate principal stress is in the plane of localization. Since the vertex model reduces to a nonvertex model in the limit $h_1 \to \infty$, we conclude that $\mathbf{p} = \mathbf{e}_2$, and therefore

$$\mathbf{n} = n_1\,\mathbf{e}_1 + (1 - n_1^2)^{1/2}\,\mathbf{e}_3, \quad \mathbf{m} = -(1 - n_1^2)^{1/2}\,\mathbf{e}_1 + n_1\,\mathbf{e}_3. \tag{10.11.110}$$

Consequently,

$$\sigma_m - \sigma_n = (\sigma_1 - \sigma_3)(1 - 2n_1^2), \quad \mathbb{M}_{mn}^2 = (\mathbb{M}_1 - \mathbb{M}_3)^2 n_1^2 (1 - n_1^2), \tag{10.11.111}$$

so that Eq. (10.11.105) becomes

$$2(h_1 - h)(\mathbb{M}_1 - \mathbb{M}_3)^2 n_1^2 (1 - n_1^2) + (\sigma_1 - \sigma_3)(1 - 2n_1^2) - h_1 = 0. \tag{10.11.112}$$

Performing the variation corresponding to δn_1 and setting $\delta h = 0$ gives

$$n_1^2 = \frac{1}{2}\left[1 - \frac{\sigma_1 - \sigma_3}{(h_1 - h)(\mathbb{M}_1 - \mathbb{M}_3)^2}\right]. \tag{10.11.113}$$

For this to be acceptable, $0 \le n_1^2 \le 1$. The condition $n_1^2 \le 1$ is satisfied for $h_1 > h$, while $n_1^2 \ge 0$ gives

$$h_1 - h \ge \frac{\sigma_1 - \sigma_3}{(\mathbb{M}_1 - \mathbb{M}_3)^2}\,. \tag{10.11.114}$$

If vertex effects are neglected ($h_1 \to \infty$), Eq. (10.11.113) reproduces the result $n_1^2 = 1/2$ from the previous subsection. Substituting (10.11.113) back into (10.11.112) gives a quadratic equation for the critical hardening rate h,

$$(h_1 - h)^2 - \frac{2h_1}{(\mathbb{M}_1 - \mathbb{M}_3)^2}(h_1 - h) + \frac{(\sigma_1 - \sigma_3)^2}{(\mathbb{M}_1 - \mathbb{M}_3)^4} = 0. \tag{10.11.115}$$

With the von Mises yield condition, we have

$$\mathbb{M}_1 - \mathbb{M}_3 = \frac{\sigma_1' - \sigma_3'}{(2J_2)^{1/2}}\,, \tag{10.11.116}$$

and since, by Eqs. (10.11.45),

$$\frac{\sigma_1' - \sigma_3'}{J_2^{1/2}} = 2\left(1 - \frac{3}{4}\,\frac{{\sigma_2'}^2}{J_2}\right)^{1/2}, \tag{10.11.117}$$

we obtain

$$n_1^2 = \frac{1}{2}\left[1 - \frac{1}{h_1 - h}\,\frac{J_2^{1/2}}{\left(1 - \frac{3}{4}\,\frac{\sigma_2'^{\,2}}{J_2}\right)^{1/2}}\right], \tag{10.11.118}$$

and

$$\left(1 - \frac{3}{4}\,\frac{\sigma_2'^{\,2}}{J_2}\right)(h_1 - h)^2 - h_1(h_1 - h) + J_2 = 0. \tag{10.11.119}$$

Alternatively, Eq. (10.11.119) can be written as (Rice, 1977)

$$\frac{3}{4}\,\frac{\sigma_2'^{\,2}}{J_2}\,(h_1 - h)^2 + h(h_1 - h) - J_2 = 0. \tag{10.11.120}$$

In order that $n_1^2 \geq 0$, from Eq. (10.11.118) it follows that the hardening rate at localization must satisfy the condition

$$\frac{h}{h_1} \leq 1 - 2\,\frac{J_2}{h_1^2}\,. \tag{10.11.121}$$

Under this condition, the critical hardening rate is, from Eq. (10.11.119),

$$\frac{h}{h_1} = 1 - \frac{1 \pm \sqrt{1 - 4(1-u)J_2/h_1^2}}{2(1-u)}\,, \tag{10.11.122}$$

where

$$u = \frac{3\sigma_2'^{\,2}}{4J_2}\,. \tag{10.11.123}$$

Plus sign should be used if localization occurs at negative h, and minus sign if it occurs at positive h, provided that h meets the condition (10.11.121). If the ratio J_2/h_1^2 is sufficiently small, the condition (10.11.122) gives

$$\frac{h}{h_1} = -\frac{u}{1-u} + \frac{J_2}{h_1^2} + \cdots\,. \tag{10.11.124}$$

In this case, unless plane stain conditions prevail ($u \to 0$), strain softening is required for localization ($h < 0$).

An analysis of localization for elastic-plastic materials with yield vertex effects is more involved, but for an incompressible elastic-plastic material the results can be easily deduced from the rigid-plastic analysis. Addition of elastic part of the rate of deformation ($\mathbf{D}^{\mathrm{e}} = \overset{\circ}{\boldsymbol{\sigma}}{}'/2\mu$) to plastic part gives

$$\mathbf{D} = \left(\frac{1}{h} - \frac{1}{h_1}\right)(\mathbb{M} \otimes \mathbb{M}) : \overset{\circ}{\boldsymbol{\sigma}} + \left(\frac{1}{h_1} + \frac{1}{2\mu}\right)\overset{\circ}{\boldsymbol{\sigma}}{}'. \tag{10.11.125}$$

Evidently, the corresponding localization results can be directly obtained from previously derived results for rigid-plastic material, if the replacements

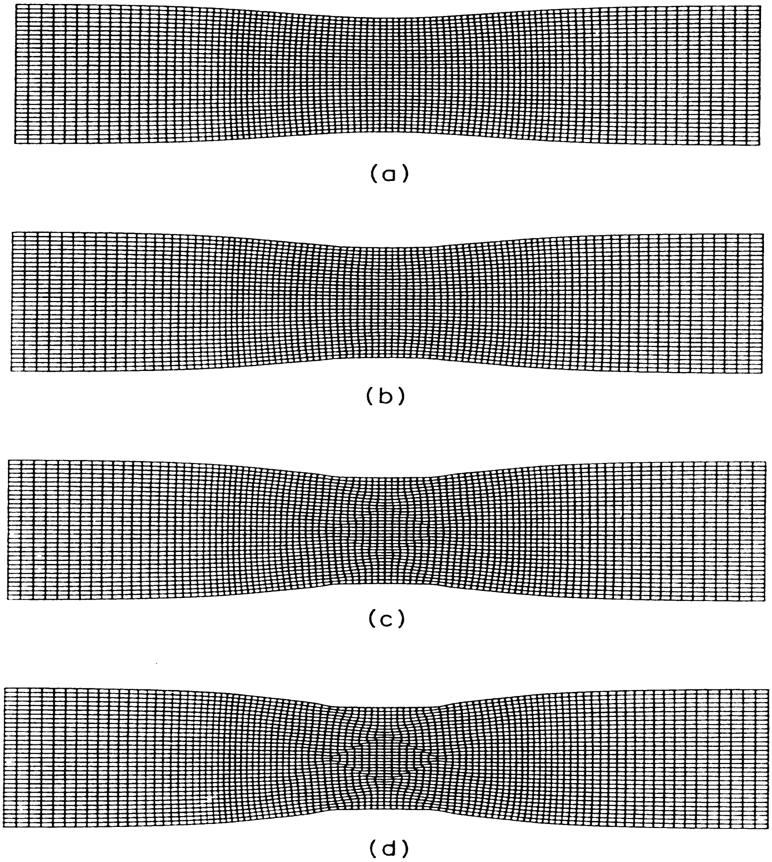

FIGURE 10.7. The neck development obtained by finite element calculations and J_2 corner theory. An initial thickness inhomogeneity grows into the necking mode. At high local strain levels the bands of intense shear deformation develop in the necked region (from Tvergaard, Needleman, and Lo, 1981; with permission from Elsevier Science).

are made

$$\frac{1}{h} \to \frac{1}{h} + \frac{1}{2\mu}, \quad \frac{1}{h_1} \to \frac{1}{h_1} + \frac{1}{2\mu}. \tag{10.11.126}$$

Numerical evaluations reveal that the critical h for localization at states other than plane strain is considerably less negative than the critical h predicted by an analysis without the yield vertex effects (Rudnicki and Rice, 1975; Rice, 1977).

There has been a number of localization studies based on the more involved corner theories of plasticity. The phenomenological J_2 corner theory of Christoffersen and Hutchinson (1979) has been frequently utilized (e.g., Hutchinson and Tvergaard, 1981; Tvergaard, Needleman, and Lo, 1981). Details of the localization predictions can be found in the original papers and reviews (Tvergaard, 1992; Needleman and Tvergaard, 1992). For example, Fig. 10.7 from Tvergaard, Needleman, and Lo (1981) shows the neck development obtained by finite element calculations and J_2 corner theory. An initial imperfection in the form of a long wave-length thickness inhomogeneity grows into the necking mode. Subsequently, at sufficiently high local strain levels, the bands of intense shear deformation develop in the necked region. The localization in rate-dependent solids and under dynamic loading conditions was studied by Anand, Kim, and Shawki (1987), Needleman (1988,1989), Batra and Kim (1990), Xu and Needleman (1992), and others.

References

Anand, L., Kim, K. H., and Shawki, T. G. (1987), Onset of shear localization in viscoplastic solids, *J. Mech. Phys. Solids*, Vol. 35, pp. 407–429.

Anand, L. and Spitzig, W. A. (1980), Initiation of localized shear bands in plane strain, *J. Mech. Phys. Solids*, Vol. 28, pp. 113–128.

Bardet, J. P. (1991), Analytical solutions for the plane-strain bifurcation of compressible solids, *J. Appl. Mech.*, Vol. 58, pp. 651–657.

Batra, R. C. and Kim, C. H. (1990), Effect of viscoplastic flow rules on the initiation and growth of shear bands at high strain rates, *J. Mech. Phys. Solids*, Vol. 38, pp. 859–874.

Bažant, Z. P. and Cedolin, L. (1991), *Stability of Structures: Elastic, Inelastic, Fracture, and Damage Theories*, Oxford University Press, New York.

Burke, M. A. and Nix, W. D. (1979), A numerical study of necking in the plane tension test, *Int. J. Solids Struct.*, Vol. 15, pp. 379–393.

Bigoni, D. (1995), On flutter instability in elastoplastic constitutive models, *Int. J. Solids Struct.*, Vol. 32, pp. 3167–3189.

Bigoni, D. and Hueckel, T. (1991), Uniqueness and localization – associative and nonassociative elastoplasticity, *Int. J. Solids Struct.*, Vol. 28, pp. 197–213.

Bigoni, D. and Zaccaria, D. (1992), Loss of strong ellipticity in nonassociative elastoplasticity, *J. Mech. Phys. Solids*, Vol. 40, pp. 1313–1331.

Bigoni, D. and Zaccaria, D. (1993), On strain localization analysis of elastoplastic materials at finite strains, *Int. J. Plasticity*, Vol. 9, pp. 21–33.

Bruhns, O. T. (1984), Bifurcation problems in plasticity, in *The Constitutive Law in Thermoplasticity*, ed. Th. Lehmann, pp. 465–540, Springer-Verlag, Wien.

Chambon, R. and Caillerie, D. (1996), Existence and uniqueness theorems for boundary value problems involving incrementally nonlinear models, *Int. J. Solids Struct.*, Vol. 36, pp. 5089–5099.

Chau, K.-T. and Rudnicki, J. W. (1990), Bifurcations of compressible pressure-sensitive materials in plane strain tension and compression, *J. Mech. Phys. Solids*, Vol. 38, pp. 875–898.

Clifton, R. J. (1974), Plastic waves: Theory and experiment, in *Mechanics Today*, Vol. 1, ed. S. Nemat-Nasser, pp. 102–167, Pergamon Press, New York.

Christoffersen, J. and Hutchinson, J. W. (1979), A class of phenomenological corner theories of plasticity, *J. Mech. Phys. Solids*, Vol. 27, pp. 465–487.

Drucker, D. C. (1958), Variational principles in mathematical theory of plasticity, *Proc. Symp. Appl. Math.*, Vol. 8, pp. 7–22, McGraw-Hill, New York.

Drucker, D. C. (1960), Plasticity, in *Structural Mechanics – Proc. 1st Symp. Naval Struct. Mechanics*, eds. J. N. Goodier and N. J. Hoff, pp. 407–455, Pergamon Press, New York.

Drugan, W. J. and Shen, Y. (1987), Restrictions on dynamically propagating surfaces of strong discontinuity in elastic-plastic solids, *J. Mech. Phys. Solids*, Vol. 35, pp. 771–787.

Germain, P. and Lee, E. H. (1973), On shock waves in elastic-plastic solids, *J. Mech. Phys. Solids*, Vol. 21, pp. 359–382.

Hadamar, J. (1903), *Leçons sur la Propagation des Ondes et les Équations de L'Hydrodinamique*, Hermann, Paris.

Hill, R. (1950), *The Mathematical Theory of Plasticity*, Oxford University Press, London.

Hill, R. (1957), On the problem of uniqueness in the theory of a rigid-plastic solid – III, *J. Mech. Phys. Solids*, Vol. 5, pp. 153–161.

Hill, R. (1958), A general theory of uniqueness and stability in elastic-plastic solids, *J. Mech. Phys. Solids*, Vol. 6, pp. 236–249.

Hill, R. (1959), Some basic principles in the mechanics of solids without natural time, *J. Mech. Phys. Solids*, Vol. 7, pp. 209–225.

Hill, R. (1961 a), Bifurcation and uniqueness in non-linear mechanics of continua, in *Problems of Continuum Mechanics – N. I. Muskhelishvili Anniversary Volume*, pp. 155-164, SIAM, Philadelphia.

Hill, R. (1961 b), Discontinuity relations in mechanics of solids, in *Progress in Solid Mechanics*, eds. I. N. Sneddon and R. Hill, Vol. 2, pp. 244–276, North-Holland, Amsterdam.

Hill, R. (1962), Acceleration waves in solids, *J. Mech. Phys. Solids*, Vol. 10, pp. 1–16.

Hill, R. (1967), Eigenmodal deformations in elastic/plastic continua, *J. Mech. Phys. Solids*, Vol. 15, pp. 371–385.

Hill, R. (1978), Aspects of invariance in solid mechanics, *Adv. Appl. Mech.*, Vol. 18, pp. 1–75.

Hill, R. and Hutchinson, J. W. (1975), Bifurcation phenomena in the plane tension test, *J. Mech. Phys. Solids*, Vol. 23, pp. 239–264.

Hill, R. and Sewell, M. J. (1960), A general theory of inelastic column failure – I and II, *J. Mech. Phys. Solids*, Vol. 8, pp. 105–118.

Hill, R. and Sewell, M. J. (1962), A general theory of inelastic column failure – III, *J. Mech. Phys. Solids*, Vol. 10, pp. 285–300.

Hutchinson, J. W. (1973), Post-bifurcation behavior in the plastic range, *J. Mech. Phys. Solids*, Vol. 21, pp. 163–190.

Hutchinson, J. W. (1974), Plastic buckling, *Adv. Appl. Mech.*, Vol. 14, pp. 67–144.

Hutchinson, J. W. and Miles, J. P. (1974), Bifurcation analysis of the onset of necking in an elastic-plastic cylinder under uniaxial tension, *J. Mech. Phys. Solids*, Vol. 22, pp. 61–71.

Hutchinson, J. W. and Tvergaard, V. (1981), Shear band formation in plane strain, *Int. J. Solids Struct.*, Vol. 17, pp. 451–470.

Iwakuma, T. and Nemat-Nasser, S. (1982), An analytical estimate of shear band initiation in a necked bar, *Int. J. Solids Struct.*, Vol. 18, pp. 69–83.

Janssen, D. M., Datta, S. K., and Jahsman, W. E. (1972), Propagation of weak waves in elastic-plastic solids, *J. Mech. Phys. Solids*, Vol. 20, pp. 1–18.

Kleiber, M. (1986), On plastic localization and failure in plane strain and round void containing tensile bars, *Int. J. Plasticity*, Vol. 2, pp. 205–221.

Koiter, W. (1960), General theorems for elastic-plastic solids, in *Progress in Solid Mechanics*, eds. I. N. Sneddon and R. Hill, Vol. 1, pp. 165–221, North-Holland, Amsterdam.

Li, M. and Drucker, D. C. (1994), Instability and bifurcation of a nonassociated extended Mises model in the hardening regime, *J. Mech. Phys. Solids*, Vol. 42, pp. 1883–1904.

Maier, G. (1970), A minimum principle for incremental elastoplasticity with non-associated flow laws, *J. Mech. Phys. Solids*, Vol. 18, pp. 319–330.

Mandel, J. (1966), Conditions de stabilité et postulat de Drucker, in *Rheology and Soil Mechanics*, eds. J. Kravtchenko and P. M. Sirieys, pp. 58–68, Springer-Verlag, Berlin.

Marciniak, K. and Kuczynski, K. (1967), Limit strains in the process of stretch forming sheet metal, *Int. J. Mech. Sci.*, Vol. 9, pp. 609–620.

Miles, J. P. (1975), The initiation of necking in rectangular elastic-plastic specimens under uniaxial and biaxial tension, *J. Mech. Phys. Solids*, Vol. 23, pp. 197–213.

Neale, K. W. (1972), A general variational theorem for the rate problem in elasto-plasticity, *Int. J. Solids Struct.*, Vol. 8, pp. 865–876.

Needleman, A. (1972), A numerical study of necking in cylindrical bars, *J. Mech. Phys. Solids*, Vol. 20, pp. 111–127.

Needleman, A. (1976), Necking of pressurized spherical membranes, *J. Mech. Phys. Solids*, Vol. 24, pp. 339–359.

Needleman, A. (1979), Non-normality and bifurcation in plane strain tension and compression, *J. Mech. Phys. Solids*, Vol. 27, pp. 231–254.

Needleman, A. (1988), Material rate dependence and mesh sensitivity in localization problems, *Comput. Meth. Appl. Mech. Engrg.*, Vol. 67, pp. 69–85.

Needleman, A. (1989), Dynamic shear band development in plane strain, *J. Appl. Mech.*, Vol. 56, pp. 1–9.

Needleman, A. and Tvergaard, V. (1982), Aspects of plastic postbuckling behavior, in *Mechanics of Solids – The Rodney Hill 60th Anniversary Volume*, eds. H. G. Hopkins and M. J. Sewell, pp. 453–498, Pergamon Press, Oxford.

Needleman, A. and Tvergaard, V. (1983), Finite element analysis of localization in plasticity, in *Finite Elements Special Problems in Solid Mechanics, Vol. 5*, eds. J. T. Oden and G. F. Carey, pp. 94–157, Prentice-Hall, Englewood Cliffs, New Jersey.

Needleman, A. and Tvergaard, V. (1992), Analyses of plastic flow localization in metals, *Appl. Mech. Rev.*, Vol. 47, No. 3, Part 2, pp. S3–S18.

Neilsen, M. K. and Schreyer, H. L. (1993), Bifurcation in elastic-plastic materials, *Int. J. Solids Struct.*, Vol. 30, pp. 521–544.

Nguyen, Q. S. (1987), Bifurcation and post-bifurcation analysis in plasticity and brittle fracture, *J. Mech. Phys. Solids*, Vol. 35, pp. 303–324.

Nguyen, Q. S. (1994), Bifurcation and stability in dissipative media (plasticity, friction, fracture), *Appl. Mech. Rev.*, Vol. 47, No. 1, Part 1, pp. 1–31.

Ortiz, M. (1987), An analytical study of the localized failure modes of concrete, *Mech. Mater.*, Vol. 6, pp. 159–174.

Ortiz, M., Leroy, Y., and Needleman, A. (1987), A finite element method for localized failure analysis, *Comp. Meth. Appl. Mech. Engin.*, Vol. 61, pp. 189–214.

Ottosen, N. S. and Runesson, K. (1991), Discontinuous bifurcations in a nonassociated Mohr material, *Mech. Mater.*, Vol. 12, pp. 255–265.

Perrin, G. and Leblond, J. B. (1993), Rudnicki and Rice's analysis of strain localization revisited, *J. Appl. Mech.*, Vol. 60, pp. 842–846.

Petryk, H. (1989), On constitutive inequalities and bifurcation in elastic-plastic solids with a yield-surface vertex, *J. Mech. Phys. Solids*, Vol. 37, pp. 265–291.

Petryk, H. and Thermann, K. (1996), Post-critical plastic deformation of biaxially stretched sheets, *Int. J. Solids Struct.*, Vol. 33, pp. 689–705.

Ponter, A. R. S. (1969), Energy theorems and deformation bounds for constitutive relations associated with creep and plastic deformation of metals, *J. Mech. Phys. Solids*, Vol. 17, pp. 493–509.

Ramakrishnan, N. and Atluri, S. N. (1994), On shear band formation: I. Constitutive relationship for a dual yield model, *Int. J. Plasticity*, Vol. 10, pp. 499–520.

Raniecki, B. (1979), Uniqueness criteria in solids with non-associated plastic flow laws at finite deformations, *Bull. Acad. Polon. Sci. Techn.*, Vol. 27, pp. 391–399.

Raniecki, B. and Bruhns, O. T. (1981), Bounds to bifurcation stresses in solids with non-associated plastic flow law at finite strains, *J. Mech. Phys. Solids*, Vol. 29, pp. 153–172.

Rice, J. R. (1977), The localization of plastic deformation, in *Theoretical and Applied Mechanics – Proc. 14th Int. Congr. Theor. Appl. Mech.*, ed. W. T. Koiter, pp. 207–220, North-Holland, Amsterdam.

Rice, J. R. and Rudnicki, J. W. (1980), A note on some features of the theory of localization of deformation, *Int. J. Solids Struct.*, Vol. 16, pp. 597–605.

Rizzi, E. and Loret, B. (1997), Qualitative analysis of strain localization. Part I: Transversely isotropic elasticity and isotropic plasticity, *Int. J. Plasticity*, Vol. 13, pp. 461–499.

Rudnicki, J. W. and Rice, J. R. (1975), Conditions for the localization of deformation in pressure-sensitive dilatant materials, *J. Mech. Phys. Solids*, Vol. 23, pp. 371–394.

Sewell, M. J. (1972), A survey of plastic buckling, in *Stability*, ed. H. Leipholz, pp. 85–197, University of Waterloo Press, Ontario.

Sewell, M. J. (1973), A yield-surface corner lowers the buckling stress of an elastic-plastic plate under compression, *J. Mech. Phys. Solids*, Vol. 21, pp. 19–45.

Sewell, M. J. (1987), *Maximum and Minimum Principles – A Unified Approach with Applications*, Cambridge University Press, Cambridge.

Storåkers, B. (1971), Bifurcation and instability modes in thick-walled rigid-plastic cylinders under pressure, *J. Mech. Phys. Solids*, Vol. 19, pp. 339–351.

Storåkers, B. (1977), On uniqueness and stability under configuration-dependent loading of solids with or without a natural time, *J. Mech. Phys. Solids*, Vol. 25, pp. 269–287.

Stören, S. and Rice, J. R. (1975), Localized necking in thin sheets, *J. Mech. Phys. Solids*, Vol. 23, pp. 421–441.

Swift, H. W. (1952), Plastic instability under plane stress, *J. Mech. Phys. Solids*, Vol. 1, pp. 1–18.

Thomas, T. Y. (1961), *Plastic Flow and Fracture in Solids*, Academic Press, New York.

Ting, T. C. T. (1976), Shock waves and weak discontinuities in anisotropic elastic-plastic media, in *Propagation of Shock Waves in Solids*, ed. E. Varley, pp. 41–64, ASME, New York.

Triantafyllidis, N. (1980), Bifurcation phenomena in pure bending, *J. Mech. Phys. Solids*, Vol. 28, pp. 221–245.

Triantafyllidis, N. (1983), On the bifurcation and postbifurcation analysis of elastic-plastic solids under general prebifurcation conditions, *J. Mech. Phys. Solids*, Vol. 31, pp. 499–510.

Tvergaard, V. (1992), On the computational prediction of plastic strain localization, in *Mechanical Behavior of Materials – VI*, eds. M. Jono and T. Inoue, pp. 189–196, Pergamon Press, Oxford.

Tvergaard, V., Needleman, A., and Lo, K. K. (1981), Flow localization in the plane strain tensile test, *J. Mech. Phys. Solids*, Vol. 29, pp. 115–142.

Wilkins, M. L. (1964), Calculation of plastic flow, in *Methods in Computational Physics – Advances in Research and Applications*, eds. B. Alder, S. Fernbach, and M. Rotenberg, pp. 211–263, Academic Press, New York.

Xu, X.-P. and Needleman, A. (1992), The influence of nucleation criterion on shear localization in rate-sensitive porous plastic solids, *Int. J. Plasticity*, Vol. 8, pp. 315–330.

Young, N. J. B. (1976), Bifurcation phenomena in the plane compression test, *J. Mech. Phys. Solids*, Vol. 24, pp. 77–91.