CHAPTER 9

PHENOMENOLOGICAL PLASTICITY

This chapter contains a detailed analysis of phenomenological constitutive equations for large deformation elastoplasticity. First eight sections are devoted to rate-independent models of isothermal elastoplastic behavior. Formulations in stress and strain space are both given. Different hardening models of metal plasticity are discussed, including isotropic, kinematic, combined and multisurface hardening models. Constitutive equations accounting for the yield vertices are also included. Pressure-dependent and nonassociative flow rules are then analyzed, with an application to rock mechanics. Constitutive theories of thermoplasticity, rate-dependent plasticity and viscoplasticity are considered in Sections 9.9 and 9.10. The final section of the chapter deals with the deformation theory of plasticity.

9.1. Formulation in Strain Space

In the rate-independent elastoplastic theory with the yield surface in strain space, the stress rate is decomposed into elastic and plastic parts, such that

$$\dot{\mathbf{T}}_{(n)} = \dot{\mathbf{T}}^{\mathrm{e}}_{(n)} + \dot{\mathbf{T}}^{\mathrm{p}}_{(n)} = \mathbf{\Lambda}_{(n)} : \dot{\mathbf{E}}_{(n)} - \dot{\gamma}_{(n)} \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} . \tag{9.1.1}$$

The function $g_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right)$ is the yield function, and

$$\dot{\gamma}_{(n)} > 0 \tag{9.1.2}$$

is the loading index, both corresponding to selected strain measure and reference state. The yield surface is defined by

$$g_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right) = 0. \tag{9.1.3}$$

Assuming an incrementally linear response and a continuity of the response between loading and unloading, defined by Eq. (8.2.8), the loading index can be written as

$$\dot{\gamma}_{(n)} = \frac{1}{h_{(n)}} \left(\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} \right), \quad \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} > 0. \tag{9.1.4}$$

The parameter

$$h_{(n)} > 0 \tag{9.1.5}$$

is a scalar function of the plastic state on the yield surface, to be determined from the consistency condition and a given representation of the yield function. If the strain rate is such that

$$\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} \leq 0, \tag{9.1.6}$$

only elastic deformation takes place, and

$$\dot{\gamma}_{(n)} = 0. \tag{9.1.7}$$

An alternative derivation of (9.1.4) is based on the consistency condition for continuing plastic deformation. This can be expressed as

$$\mathrm{d}g_{(n)} = \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \mathrm{d}\mathbf{E}_{(n)} + \mathrm{d}^{\mathrm{p}} g_{(n)} = 0, \tag{9.1.8}$$

where

$$\mathrm{d}^{\mathrm{p}} g_{(n)} = g_{(n)} \left(\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H} \right) - g_{(n)} \left(\mathbf{E}_{(n)}, \mathcal{H} \right) \tag{9.1.9}$$

is the plastic part of the increment of $\mathrm{d}^{\mathrm{p}} g_{(n)}$, due to change of the internal structure. Writing

$$\mathrm{d}^{\mathrm{p}} g_{(n)} = -h_{(n)} \, \mathrm{d}\gamma_{(n)}, \tag{9.1.10}$$

Equation (9.1.8) yields Eq. (9.1.4).

When Eq. (9.1.4) is substituted into Eq. (9.1.1), the constitutive equation for elastoplastic loading becomes (Hill, 1967a, 1978)

$$\dot{\mathbf{T}}_{(n)} = \left[\mathbf{\Lambda}_{(n)} - \frac{1}{h_{(n)}} \left(\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} \otimes \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} \right) \right] : \dot{\mathbf{E}}_{(n)}. \tag{9.1.11}$$

The fourth-order tensor within the square brackets is the elastoplastic stiffness tensor, associated with the considered stress and strain measures and the reference state. Within the employed framework based on the Green-elasticity and normality rule, the elastoplastic stiffness tensor obeys the reciprocal or self-adjoint symmetry (with respect to first and second pair of indices), in addition to symmetries in the first two and last two indices (minor symmetry), associated with the symmetry of the stress and strain tensors. The formulation of elastoplasticity theory based on the yield surface in strain space was also studied by Naghdi and Trapp (1975), Casey and Naghdi (1981,1983), Yoder and Iwan (1981), Klisinski, Mróz, and Runesson (1992), and Negahban (1995). A review by Naghdi (1990) contains additional related references.

It is of interest to invert the constitutive structure (9.1.11), and express the strain rate in terms of the stress rate. By taking a trace product of Eq. (9.1.11) with $\mathbf{M}_{(n)} = \boldsymbol{\Lambda}_{(n)}^{-1}$, there follows

$$\mathbf{M}_{(n)} : \dot{\mathbf{T}}_{(n)} = \dot{\mathbf{E}}_{(n)} - \frac{1}{h_{(n)}} \mathbf{M}_{(n)} : \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} \left(\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} \right). \tag{9.1.12}$$

A trace product of (9.1.12) with $\partial g_{(n)}/\partial \mathbf{E}_{(n)}$ gives

$$\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} = \frac{h_{(n)}}{H_{(n)}} \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \mathbf{M}_{(n)} : \dot{\mathbf{T}}_{(n)}, \tag{9.1.13}$$

where

$$H_{(n)} = h_{(n)} - \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \mathbf{M}_{(n)} : \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}}. \tag{9.1.14}$$

For plastic loading the quantity in Eq. (9.1.13) must be positive. The substitution of Eq. (9.1.13) into Eq. (9.1.12) yields a desired inverted form

$$\dot{\mathbf{E}}_{(n)} = \left[\mathbf{M}_{(n)} + \frac{1}{H_{(n)}} \left(\mathbf{M}_{(n)} : \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} \right) \otimes \left(\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \mathbf{M}_{(n)} \right) \right] : \dot{\mathbf{T}}_{(n)}. \tag{9.1.15}$$

If current state is taken as the reference state, Eq. (9.1.11) becomes

$$\underline{\dot{\mathbf{T}}}_{(n)} = \left[\underline{\boldsymbol{\Lambda}}_{(n)} - \frac{1}{\underline{h}_{(n)}} \left(\frac{\partial \underline{g}_{(n)}}{\partial \underline{\mathbf{E}}_{(n)}} \otimes \frac{\partial \underline{g}_{(n)}}{\partial \underline{\mathbf{E}}_{(n)}} \right) \right] : \mathbf{D}. \tag{9.1.16}$$

Incorporating Eq. (6.3.6) for $\underline{\dot{\mathbf{T}}}_{(n)}$, and Eq. (6.3.13) for $\underline{\boldsymbol{\Lambda}}_{(n)} = \underline{\boldsymbol{\mathcal{L}}}_{(n)}$, gives

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} = \left[\underline{\boldsymbol{\mathcal{L}}}_{(0)} - \frac{1}{\underline{h}_{(n)}} \left(\frac{\partial \underline{g}_{(n)}}{\partial \underline{\mathbf{E}}_{(n)}} \otimes \frac{\partial \underline{g}_{(n)}}{\partial \underline{\mathbf{E}}_{(n)}} \right) \right] : \mathbf{D}. \tag{9.1.17}$$

It is noted that

$$\underline{\dot{\mathbf{T}}}^{\mathrm{p}}_{(n)} = -\frac{1}{\underline{h}_{(n)}} \left(\frac{\partial \underline{g}_{(n)}}{\partial \underline{\mathbf{E}}_{(n)}} \otimes \frac{\partial \underline{g}_{(n)}}{\partial \underline{\mathbf{E}}_{(n)}} \right) : \mathbf{D} = \overset{\circ}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}}, \tag{9.1.18}$$

for all n. In particular, the gradient $\partial \underline{g}_{(n)}/\partial \underline{\mathbf{E}}_{(n)}$ at the yield point is in the same direction for all n.

9.1.1. Translation and Expansion of the Yield Surface

Let the yield surface in strain space be defined by

$$g_{(n)}\left(\mathbf{E}_{(n)} - \mathbf{E}^{\mathrm{p}}_{(n)},\, k_{(n)}\right) = 0, \tag{9.1.19}$$

where $g_{(n)}$ is an isotropic function of its tensor argument, and $\mathbf{E}^{\mathrm{p}}_{(n)}$ represents the center of the current yield surface (Fig. 9.1). This yield surface translates and expands in strain space, although it physically corresponds to isotropic hardening in stress space. The current center of the yield surface is determined by integration from an appropriate evolution equation, along a given deformation path. For instance, the evolution of $\mathbf{E}^{\mathrm{p}}_{(n)}$ can be described by

$$\dot{\mathbf{E}}^{\mathrm{p}}_{(n)} = -\mathbf{M}_{(n)} : \dot{\mathbf{T}}^{\mathrm{p}}_{(n)} = \dot{\gamma}_{(n)}\, \mathbf{M}_{(n)} : \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}}\,. \tag{9.1.20}$$

The scalar function

$$k_{(n)} = k_{(n)}\left(\varphi_{(n)}\right) \tag{9.1.21}$$

in Eq. (9.1.19) specifies the size of the current yield surface. The parameter $\varphi_{(n)}$ accounts for the history of plastic deformation, and can be taken as

$$\varphi_{(n)} = -\int_0^t \left(\frac{1}{2}\,\dot{\mathbf{T}}^{\mathrm{p}}_{(n)} : \dot{\mathbf{T}}^{\mathrm{p}}_{(n)}\right)^{1/2} \mathrm{d}t. \tag{9.1.22}$$

The consistency condition for continuing plastic deformation is

$$\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \left(\dot{\mathbf{E}}_{(n)} - \dot{\mathbf{E}}^{\mathrm{p}}_{(n)}\right) + \frac{\partial g_{(n)}}{\partial k_{(n)}}\,\frac{\mathrm{d}k_{(n)}}{\mathrm{d}\varphi_{(n)}}\,\dot{\varphi}_{(n)} = 0. \tag{9.1.23}$$

Substitution of Eqs. (9.1.20) and (9.1.22) into Eq. (9.1.23) gives the loading index as in Eq. (9.1.4), with

$$h_{(n)} = \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \mathbf{M}_{(n)} : \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} + \frac{\partial g_{(n)}}{\partial k_{(n)}}\,\frac{\mathrm{d}k_{(n)}}{\mathrm{d}\varphi_{(n)}}\left(\frac{1}{2}\,\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}}\right)^{1/2}. \tag{9.1.24}$$

Suppose that current state on the yield surface is taken as the reference state for the strain measure, so that $\underline{\mathbf{E}}_{(n)} = \mathbf{0}$. Then,

$$-\underline{\mathbf{E}}^{\mathrm{p}}_{(n)} = \boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(-n)}, \tag{9.1.25}$$

where $\boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(-n)}$ is a spatial measure of elastic strain at the current yield state, relative to the state at the center of the yield surface. To recognize this,

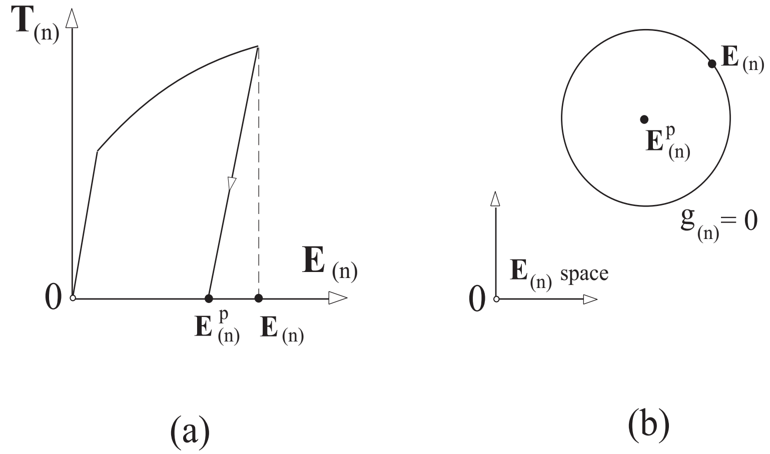

FIGURE 9.1. (a) Uniaxial stress-strain curve. (b) Yield surface in strain space corresponding to isotropic hardening. The center of the yield surface is at the plastic state of strain $\mathbf{E}^{\mathrm{p}}_{(n)}$, corresponding to zero state of stress.

denote by $\mathbf{F}^{\mathrm{e}} = \mathbf{V}^{\mathrm{e}}\cdot\mathbf{R}^{\mathrm{e}}$ the deformation gradient from the state at the center of the yield surface to the current state on the yield surface. It follows that

$$\underline{\mathbf{F}}^{\mathrm{p}} = (\mathbf{F}^{\mathrm{e}})^{-1}, \quad \underline{\mathbf{U}}^{\mathrm{p}} = (\mathbf{V}^{\mathrm{e}})^{-1}, \tag{9.1.26}$$

and

$$\underline{\mathbf{E}}^{\mathrm{p}}_{(n)} = \frac{1}{2n}\left[(\underline{\mathbf{U}}^{\mathrm{p}})^{2n} - \mathbf{I}\right] = -\boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(-n)}. \tag{9.1.27}$$

Thus, Eqs. (9.1.4) and (9.1.24) give

$$\dot{\underline{\gamma}}_{(n)} = \frac{1}{\underline{h}_{(n)}}\left(\frac{\partial \underline{g}_{(n)}}{\partial \boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(-n)}} : \mathbf{D}\right), \tag{9.1.28}$$

where

$$\underline{h}_{(n)} = \frac{\partial \underline{g}_{(n)}}{\partial \boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(-n)}} : \underline{\mathbf{M}}_{(n)} : \frac{\partial \underline{g}_{(n)}}{\partial \boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(-n)}} + \frac{\partial g_{(n)}}{\partial k_{(n)}}\,\frac{\mathrm{d}\underline{k}_{(n)}}{\mathrm{d}\underline{\varphi}_{(n)}}\left(\frac{1}{2}\,\frac{\partial \underline{g}_{(n)}}{\partial \boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(-n)}} : \frac{\partial \underline{g}_{(n)}}{\partial \boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(-n)}}\right)^{1/2}. \tag{9.1.29}$$

It is recalled from Eq. (9.1.18) that all stress rates $\dot{\underline{\mathbf{T}}}^{\mathrm{p}}_{(n)}$ are equal to each other, and thus all the history parameters $\underline{\varphi}_{(n)}$ are also equal (independent of n); see Eq. (9.1.22). These general expressions are next specialized by assuming that the elastic component of strain is infinitesimally small.

If elastic deformation within the yield surface is infinitesimal, all strain measures $\boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(-n)}$ reduce to infinitesimal elastic strain $\boldsymbol{\varepsilon}^{\mathrm{e}}$, whose deviatoric part is related to Cauchy stress by

$$\boldsymbol{\varepsilon}^{\mathrm{e}\,\prime} = \frac{1}{2\mu}\,\boldsymbol{\sigma}^{\prime}. \tag{9.1.30}$$

For example, let the yield surface be specified by

$$g = 4\mu^2\left[\frac{1}{2}\,\boldsymbol{\varepsilon}^{\mathrm{e}\,\prime} : \boldsymbol{\varepsilon}^{\mathrm{e}\,\prime} - k^2(\varphi)\right] = 0\,. \tag{9.1.31}$$

The factor $4\mu^2$ is introduced for the sake of comparison with the corresponding yield surface in stress space, considered later in Subsection 9.2.1. From Eqs. (9.1.18) and (9.1.31), we have

$$\frac{\partial g}{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}} = 4\mu^2\boldsymbol{\varepsilon}^{\mathrm{e}\,\prime}, \quad \overset{\circ}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}} = -4\mu^2\dot{\gamma}\,\boldsymbol{\varepsilon}^{\mathrm{e}\,\prime}, \tag{9.1.32}$$

while Eqs. (9.1.22), (9.1.24), and (9.1.29) give

$$\dot{\varphi} = -4\mu^2 k\,\dot{\gamma}, \quad \dot{\gamma} = \frac{4\mu^2}{h}\,(\boldsymbol{\varepsilon}^{\mathrm{e}\,\prime} : \mathbf{D})\,, \quad h = 16\mu^3 k^2\left(1 - 2\mu\,\frac{\mathrm{d}k}{\mathrm{d}\varphi}\right). \tag{9.1.33}$$

Consequently,

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} = \left(\underline{\boldsymbol{\mathcal{L}}}_{(0)} - \frac{2\mu}{1 - 2\mu\,\mathrm{d}k/\mathrm{d}\varphi}\,\frac{\boldsymbol{\varepsilon}^{\mathrm{e}\,\prime}\otimes\boldsymbol{\varepsilon}^{\mathrm{e}\,\prime}}{\boldsymbol{\varepsilon}^{\mathrm{e}\,\prime} : \boldsymbol{\varepsilon}^{\mathrm{e}\,\prime}}\right) : \mathbf{D}. \tag{9.1.34}$$

The elastic stiffness or moduli tensor is taken as

$$\underline{\boldsymbol{\mathcal{L}}}_{(0)} = \lambda\,\mathbf{I}\otimes\mathbf{I} + 2\mu\,\boldsymbol{I}. \tag{9.1.35}$$

A similar approach to derive elastoplastic constitutive equations with the yield surface in strain space was used, within infinitesimal strain context, by Yoder and Iwan (1981).

It is convenient to express the elastic stiffness tensor (9.1.35) in an alternative form as

$$\underline{\boldsymbol{\mathcal{L}}}_{(0)} = 2\mu\,\boldsymbol{J} + 3\kappa\,\boldsymbol{K}, \tag{9.1.36}$$

where

$$\kappa = \lambda + \frac{2}{3}\,\mu \tag{9.1.37}$$

is the elastic bulk modulus. The base tensors $\boldsymbol{J}$ and $\boldsymbol{K}$ sum to give the fourth-order unit tensor, $\boldsymbol{J} + \boldsymbol{K} = \boldsymbol{I}$. The rectangular components of $\boldsymbol{I}$ and

$\boldsymbol{K}$ are

$$I_{ijkl} = \frac{1}{2}\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right), \quad K_{ijkl} = \frac{1}{3}\,\delta_{ij}\,\delta_{kl}\,. \tag{9.1.38}$$

These are convenient base tensors, because $\boldsymbol{J} : \boldsymbol{J} = \boldsymbol{J}$ and $\boldsymbol{K} : \boldsymbol{K} = \boldsymbol{K}$, as well as $\boldsymbol{J} : \boldsymbol{K} = \boldsymbol{K} : \boldsymbol{J} = \mathbf{0}$ (Hill, 1965; Walpole, 1981). In the trace operation with any second-order tensor $\mathbf{A}$, the tensor $\boldsymbol{J}$ extracts its deviatoric part, while the tensor $\boldsymbol{K}$ extracts its spherical part ($\boldsymbol{J} : \mathbf{A} = \mathbf{A}'$ and $\boldsymbol{K} : \mathbf{A} = \mathbf{A} - \mathbf{A}'$). It is then easily verified that the inverse of (9.1.36) is simply

$$\underline{\boldsymbol{\mathcal{L}}}_{(0)}^{-1} = \frac{1}{2\mu}\,\boldsymbol{J} + \frac{1}{3\kappa}\,\boldsymbol{K}. \tag{9.1.39}$$

9.2. Formulation in Stress Space

In the rate-independent elastoplastic theory with the yield surface in stress space, the strain rate is decomposed as the sum of elastic and plastic parts, such that

$$\dot{\mathbf{E}}_{(n)} = \dot{\mathbf{E}}^{\mathrm{e}}_{(n)} + \dot{\mathbf{E}}^{\mathrm{p}}_{(n)} = \mathbf{M}_{(n)} : \dot{\mathbf{T}}_{(n)} + \dot{\gamma}_{(n)}\,\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}\,. \tag{9.2.1}$$

The function $f_{(n)}\left(\mathbf{T}_{(n)}, \mathcal{H}\right)$ is the yield function, and $\dot{\gamma}_{(n)} > 0$ is the loading index, both corresponding to selected measure and reference state. The yield surface is

$$f_{(n)}\left(\mathbf{T}_{(n)}, \mathcal{H}\right) = 0. \tag{9.2.2}$$

Assuming an incrementally linear response and a continuity of the response, the loading index can be expressed as

$$\dot{\gamma}_{(n)} = \frac{1}{H_{(n)}}\left(\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)}\right). \tag{9.2.3}$$

The scalar function $H_{(n)}$ is determined from the consistency condition and a given representation of the yield function. Substitution of Eq. (9.2.3) into Eq. (9.2.1) gives

$$\dot{\mathbf{E}}_{(n)} = \left[\mathbf{M}_{(n)} + \frac{1}{H_{(n)}}\left(\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} \otimes \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}\right)\right] : \dot{\mathbf{T}}_{(n)}. \tag{9.2.4}$$

The fourth-order tensor within the square brackets is the elastoplastic compliance tensor associated with the considered stress and strain measures and the reference state.

The relationship between $h_{(n)}$ in Eq. (9.1.11) and $H_{(n)}$ in Eq. (9.2.4) can be obtained by equating Eqs. (9.1.4) and (9.2.3), i.e.,

$$\frac{1}{h_{(n)}}\left(\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)}\right) = \frac{1}{H_{(n)}}\left(\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)}\right). \tag{9.2.5}$$

Substituting Eq. (9.2.4) for $\dot{\mathbf{E}}_{(n)}$ and by using the relationship between the yield surface normals in stress and strain space,

$$\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} = \mathbf{M}_{(n)} : \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}}, \tag{9.2.6}$$

there follows

$$H_{(n)} = h_{(n)} - \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}, \tag{9.2.7}$$

in agreement with Eq. (9.1.14). Consequently, Eq. (9.2.4) is equivalent to Eq. (9.1.15).

The scalar parameter $H_{(n)}$ can be positive, negative or equal to zero. Three types of response are thus possible within this constitutive framework. They are

$$\begin{aligned}
H_{(n)} > 0, &\quad \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)} > 0 \quad \text{hardening},\\
H_{(n)} < 0, &\quad \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)} < 0 \quad \text{softening},\\
H_{(n)} = 0, &\quad \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)} = 0 \quad \text{ideally plastic}.
\end{aligned} \tag{9.2.8}$$

Starting from the current yield surface in stress space, the stress point moves outward in the case of hardening, inward in the case of softening, and tangentially to the yield surface in the case of ideally plastic response. In the case of softening, $\dot{\mathbf{E}}_{(n)}$ is not uniquely determined by the prescribed stress rate $\dot{\mathbf{T}}_{(n)}$, since either Eq. (9.2.4) applies, or the elastic unloading expression

$$\dot{\mathbf{E}}_{(n)} = \mathbf{M}_{(n)} : \mathbf{T}_{(n)}. \tag{9.2.9}$$

In the case of ideally plastic response, the plastic part of the strain rate is indeterminate to the extent of an arbitrary positive multiple, since $\dot{\gamma}_{(n)}$ in Eq. (9.2.3) is indeterminate.

Inverted form of Eq. (9.2.4) can be obtained along similar lines as used to invert Eq. (9.1.11). The result is

$$\dot{\mathbf{T}}_{(n)} = \left[\mathbf{\Lambda}_{(n)} - \frac{1}{h_{(n)}}\left(\mathbf{\Lambda}_{(n)} : \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}\right) \otimes \left(\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \mathbf{\Lambda}_{(n)}\right)\right] : \dot{\mathbf{E}}_{(n)}, \tag{9.2.10}$$

where

$$h_{(n)} = H_{(n)} + \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \boldsymbol{\Lambda}_{(n)} : \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}\,, \tag{9.2.11}$$

which is in agreement with Eq. (9.1.14). If current state is taken as the reference state, Eq. (9.2.4) becomes

$$\mathbf{D} = \left[\underline{\mathbf{M}}_{(n)} + \frac{1}{\underline{H}_{(n)}} \left(\frac{\partial \underline{f}_{(n)}}{\partial \underline{\mathbf{T}}_{(n)}} \otimes \frac{\partial \underline{f}_{(n)}}{\partial \underline{\mathbf{T}}_{(n)}} \right) \right] : \underline{\dot{\mathbf{T}}}_{(n)}. \tag{9.2.12}$$

9.2.1. Yield Surface in Cauchy Stress Space

It is most convenient to apply Eq. (9.2.12) for $n = 0$. In the near neighborhood of the current stress state on the yield surface, the conjugate stress to logarithmic strain (relative to the state on the yield surface) is, from Eq. (3.6.17),

$$\mathbf{T}_{(0)} = \mathbf{R} \cdot \boldsymbol{\tau} \cdot \mathbf{R}^T + \mathcal{O}(\boldsymbol{\tau} \cdot \mathbf{E}_{(n)}^2), \tag{9.2.13}$$

where $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$ is the Kirchhoff stress. On the other hand, for $n \neq 0$,

$$\mathbf{T}_{(n)} = \mathbf{R} \cdot \boldsymbol{\tau} \cdot \mathbf{R}^T + \mathcal{O}(\boldsymbol{\tau} \cdot \mathbf{E}_{(n)}), \tag{9.2.14}$$

by Eq. (3.6.16). In the last two equations, the deformation gradient $\mathbf{F}$ and the rotation $\mathbf{R}$ are measured from the current, deformed configuration as the reference. Thus,

$$f_{(0)}(\mathbf{T}_{(0)}, \mathcal{H}) \approx f(\boldsymbol{\sigma}, \mathcal{H}) \tag{9.2.15}$$

in the near neighborhood of the current yield state, where

$$f(\boldsymbol{\sigma}, \mathcal{H}) = 0 \tag{9.2.16}$$

represents the yield surface in the Cauchy stress space. Equation (9.2.12) consequently becomes

$$\mathbf{D} = \left[\underline{\boldsymbol{\mathcal{M}}}_{(0)} + \frac{1}{H} \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \boldsymbol{\sigma}} \right) \right] : \underline{\overset{\circ}{\boldsymbol{\tau}}}. \tag{9.2.17}$$

The tensor

$$\underline{\mathbf{M}}_{(0)} = \underline{\boldsymbol{\mathcal{M}}}_{(0)} = \underline{\boldsymbol{\mathcal{L}}}_{(0)}^{-1} \tag{9.2.18}$$

is the corresponding instantaneous compliance tensor, and H is an appropriate scalar function of the deformation history.

The elastic and plastic parts of the rate of deformation tensor $\mathbf{D}$, corresponding to $\overset{\circ}{\underline{\boldsymbol{\tau}}}$, are

$$\mathbf{D}^{\mathrm{e}}_{(0)} = \underline{\boldsymbol{\mathcal{M}}}_{(0)} : \overset{\circ}{\underline{\boldsymbol{\tau}}}, \quad \mathbf{D}^{\mathrm{p}}_{(0)} = \frac{1}{H}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \boldsymbol{\sigma}}\right) : \overset{\circ}{\underline{\boldsymbol{\tau}}}. \tag{9.2.19}$$

If elastic component of strain is neglected, a model of rigid-plasticity is obtained. The rate of deformation is due to plastic deformation only, so that

$$\mathbf{D} = \frac{1}{H}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \boldsymbol{\sigma}}\right) : \overset{\circ}{\underline{\boldsymbol{\tau}}}. \tag{9.2.20}$$

9.3. Nonuniqueness of the Rate of Deformation Partition

Within the considered framework of conjugate stress and strain tensors, there are infinitely many partitions of the rate of deformation tensor, one associated with each n. Thus, we can write (Lubarda, 1994)

$$\mathbf{D} = \mathbf{D}^{\mathrm{e}}_{(0)} + \mathbf{D}^{\mathrm{p}}_{(0)} = \mathbf{D}^{\mathrm{e}}_{(n)} + \mathbf{D}^{\mathrm{p}}_{(n)}. \tag{9.3.1}$$

The elastic parts of $\mathbf{D}$ are defined by

$$\mathbf{D}^{\mathrm{e}}_{(0)} = \underline{\boldsymbol{\mathcal{M}}}_{(0)} : \overset{\circ}{\underline{\boldsymbol{\tau}}}, \quad \mathbf{D}^{\mathrm{e}}_{(n)} = \underline{\boldsymbol{\mathcal{M}}}_{(n)} : \underline{\dot{\mathbf{T}}}_{(n)}, \tag{9.3.2}$$

where

$$\underline{\dot{\mathbf{T}}}_{(n)} = \overset{\circ}{\underline{\boldsymbol{\tau}}} - 2n\,\underline{\boldsymbol{\mathcal{S}}} : \mathbf{D}, \quad \underline{\boldsymbol{\mathcal{L}}}_{(n)} = \underline{\boldsymbol{\mathcal{L}}}_{(0)} - 2n\,\underline{\boldsymbol{\mathcal{S}}}. \tag{9.3.3}$$

The fourth-order tensor $\underline{\boldsymbol{\mathcal{S}}}$ is defined in Eq. (6.3.11) as

$$\underline{\mathcal{S}}_{ijkl} = \frac{1}{4}\left(\sigma_{ik}\delta_{jl} + \sigma_{jk}\delta_{il} + \sigma_{il}\delta_{jk} + \sigma_{jl}\delta_{ik}\right). \tag{9.3.4}$$

Since, from Eq. (8.8.22),

$$\underline{\dot{\mathbf{T}}}^{\mathrm{p}}_{(0)} = \underline{\dot{\mathbf{T}}}^{\mathrm{p}}_{(n)}, \tag{9.3.5}$$

and since

$$\underline{\dot{\mathbf{T}}}^{\mathrm{p}}_{(n)} = -\underline{\boldsymbol{\mathcal{L}}}_{(n)} : \mathbf{D}^{\mathrm{p}}_{(n)} = -\left(\underline{\boldsymbol{\mathcal{L}}}_{(0)} - 2n\,\underline{\boldsymbol{\mathcal{S}}}\right) : \mathbf{D}^{\mathrm{p}}_{(n)}, \tag{9.3.6}$$

the following relationships hold

$$\mathbf{D}^{\mathrm{p}}_{(0)} = \mathbf{D}^{\mathrm{p}}_{(n)} - 2n\,\underline{\boldsymbol{\mathcal{M}}}_{(0)} : \underline{\boldsymbol{\mathcal{S}}} : \mathbf{D}^{\mathrm{p}}_{(n)}, \tag{9.3.7}$$

$$\mathbf{D}^{\mathrm{e}}_{(0)} = \mathbf{D}^{\mathrm{e}}_{(n)} + 2n\,\underline{\boldsymbol{\mathcal{M}}}_{(0)} : \underline{\mathbf{S}} : \mathbf{D}^{\mathrm{p}}_{(n)}. \tag{9.3.8}$$

Alternatively, these can be expressed as

$$\mathbf{D}^{\mathrm{p}}_{(n)} = \mathbf{D}^{\mathrm{p}}_{(0)} + 2n\,\underline{\boldsymbol{\mathcal{M}}}_{(n)} : \underline{\boldsymbol{\mathcal{S}}} : \mathbf{D}^{\mathrm{p}}_{(0)}, \tag{9.3.9}$$

$$\mathbf{D}^{\mathrm{e}}_{(n)} = \mathbf{D}^{\mathrm{e}}_{(0)} - 2n\,\underline{\boldsymbol{\mathcal{M}}}_{(n)} : \underline{\boldsymbol{S}} : \mathbf{D}^{\mathrm{p}}_{(0)}. \tag{9.3.10}$$

The relative difference between the components of elastic (and plastic) rate of deformation tensors for various n are thus of the order of Cauchy stress over elastic modulus. In the sequel, the elastic and plastic parts of the rate of deformation tensor corresponding to $\overset{\circ}{\underline{\boldsymbol{\tau}}}$ will be designated simply by $\mathbf{D}^{\mathrm{e}}$ and $\mathbf{D}^{\mathrm{p}}$, i.e.,

$$\mathbf{D}^{\mathrm{e}}_{(0)} = \mathbf{D}^{\mathrm{e}}, \quad \mathbf{D}^{\mathrm{p}}_{(0)} = \mathbf{D}^{\mathrm{p}}. \tag{9.3.11}$$

9.4. Hardening Models in Stress Space

9.4.1. Isotropic Hardening

The experimental determination of the yield surface is commonly done with respect to Cauchy stress. Suppose that this is given by

$$f(\sigma, K) = 0, \tag{9.4.1}$$

where f is an isotropic function of $\boldsymbol{\sigma}$, and

$$K = K(\vartheta) \tag{9.4.2}$$

is a scalar function which defines the size of the yield surface. The hardening model in which the yield surface expands during plastic deformation, preserving its shape, is known as the isotropic hardening model. Since f is taken to be an isotropic function of stress, the material is assumed to be isotropic. The history parameter ϑ is the effective (generalized) plastic strain, defined by

$$\vartheta = \int_0^t \left(2\,\mathbf{D}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}}\right)^{1/2} \mathrm{d}t. \tag{9.4.3}$$

In view of the isotropy of the function f, we may write

$$f(\boldsymbol{\sigma}, K) = f\left(\mathbf{R}^T \cdot \boldsymbol{\sigma} \cdot \mathbf{R}, K\right) \approx f(\mathbf{T}_{(0)}, K). \tag{9.4.4}$$

The approximation holds in the near neighborhood of the current state, relative to which $\mathbf{R}$ and $\mathbf{T}_{(0)}$ are measured. The consistency condition for continuing plastic deformation,

$$\dot{f} = 0, \tag{9.4.5}$$

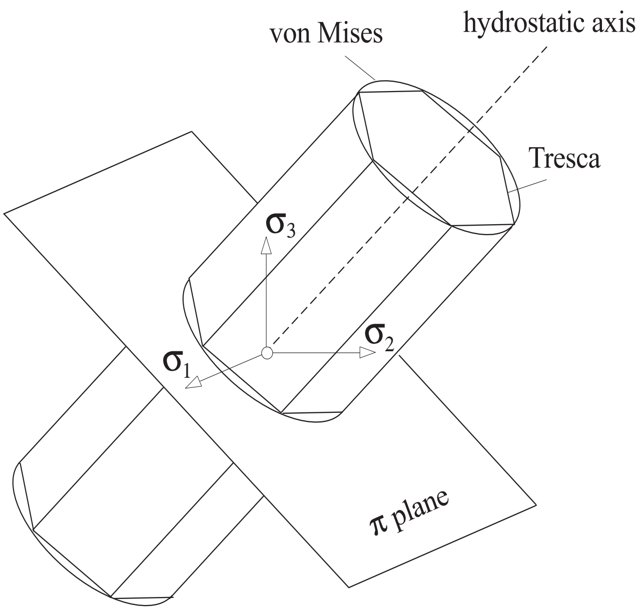

FIGURE 9.2. Von Mises and Tresca yield surfaces in principal stress space. The yield cylinder and the yield prism have their axis parallel to the hydrostatic axis, which is perpendicular to the π plane ($\sigma_1 + \sigma_2 + \sigma_3 = 0$).

gives

$$\frac{\partial f}{\partial \underline{\mathbf{T}}_{(0)}} : \dot{\underline{\mathbf{T}}}_{(0)} = \frac{\partial f}{\partial \boldsymbol{\sigma}} : \overset{\circ}{\underline{\boldsymbol{\tau}}} = -\frac{\partial f}{\partial K} \frac{\mathrm{d}K}{\mathrm{d}\vartheta} \dot{\vartheta}. \tag{9.4.6}$$

Upon substitution of Eqs. (9.4.3), the loading index becomes

$$\dot{\gamma} = \frac{1}{H} \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \overset{\circ}{\underline{\boldsymbol{\tau}}} \right), \quad H = -\frac{\partial f}{\partial K} \frac{\mathrm{d}K}{\mathrm{d}\vartheta} \left(2 \frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}} \right)^{1/2}. \tag{9.4.7}$$

J_2 Flow Theory of Plasticity

For nonporous metals the onset of plastic deformation and plastic yielding is unaffected by a moderate superimposed pressure. The yield condition for such materials can consequently be written as an isotropic function of the deviatoric part of Cauchy stress, i.e., as a function of its second and third invariant,

$$f(J_2, J_3, K) = 0. \tag{9.4.8}$$

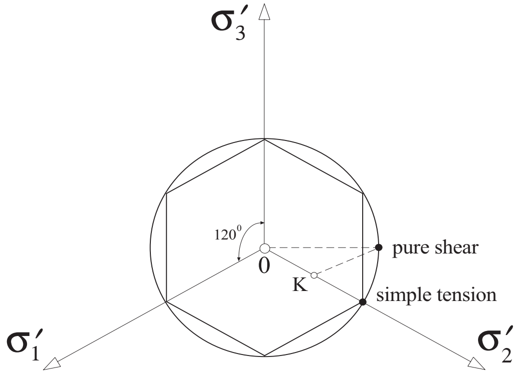

FIGURE 9.3. The trace of the von Mises and Tresca yield surfaces in the π plane. The states of simple tension and pure shear are indicated.

The classical examples are the Tresca maximum shear stress criterion and the von Mises yield criterion (Fig. 9.2). In the latter case,

$$f = J_2 - K^2(\vartheta) = 0, \quad J_2 = \frac{1}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'. \tag{9.4.9}$$

The corresponding plasticity theory is known as the J_2 flow theory of plasticity. If the yield stress in uniaxial tension is σ_Y, and in shear loading τ_Y, we have (Fig. 9.3)

$$K = \frac{1}{\sqrt{3}}\,\sigma_Y = \tau_Y. \tag{9.4.10}$$

For the J_2 plasticity,

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \boldsymbol{\sigma}', \quad \mathbf{D}^{\mathrm{p}} = \dot{\gamma}\,\boldsymbol{\sigma}', \tag{9.4.11}$$

and

$$\dot{\vartheta} = 2K\,\dot{\gamma}, \quad H = 4K^2 h_{\mathrm{t}}^{\mathrm{p}}, \quad \dot{\gamma} = \frac{1}{4K^2 h_{\mathrm{t}}^{\mathrm{p}}}\left(\boldsymbol{\sigma}' : \underline{\overset{\circ}{\boldsymbol{\tau}}}\right). \tag{9.4.12}$$

The plastic tangent modulus in shear test is

$$h_{\mathrm{t}}^{\mathrm{p}} = \frac{\mathrm{d}K}{\mathrm{d}\vartheta}\,. \tag{9.4.13}$$

Equation (9.4.11) implies that plastic deformation is isochoric

$$\mathrm{tr}\,\mathbf{D}^{\mathrm{p}} = 0. \tag{9.4.14}$$

The total rate of deformation is

$$\mathbf{D} = \left[\underline{\boldsymbol{\mathcal{M}}}_{(0)} + \frac{1}{4K^2 h_{\mathrm{t}}^{\mathrm{p}}} \left(\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}' \right) \right] : \overset{\circ}{\underline{\boldsymbol{\tau}}}. \tag{9.4.15}$$

For infinitesimal elasticity the elastic compliance tensor can be taken as

$$\underline{\boldsymbol{\mathcal{M}}}_{(0)} = \frac{1}{2\mu} \left(\boldsymbol{I} - \frac{\lambda}{2\mu + 3\lambda} \mathbf{I} \otimes \mathbf{I} \right) = \frac{1}{2\mu} \boldsymbol{J} + \frac{1}{3\kappa} \boldsymbol{K}. \tag{9.4.16}$$

By using Eq. (9.4.9) to express K in terms of stress, Eq. (9.4.15) is rewritten as

$$\mathbf{D} = \left(\underline{\boldsymbol{\mathcal{M}}}_{(0)} + \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}} \frac{\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}'}{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} \right) : \overset{\circ}{\underline{\boldsymbol{\tau}}}. \tag{9.4.17}$$

The plastic loading condition in the hardening range is

$$\boldsymbol{\sigma}' : \overset{\circ}{\underline{\boldsymbol{\tau}}} > 0. \tag{9.4.18}$$

The inverse equation is

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} = \left(\underline{\boldsymbol{\mathcal{L}}}_{(0)} - \frac{2\mu}{1 + h_{\mathrm{t}}^{\mathrm{p}}/\mu} \frac{\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}'}{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} \right) : \mathbf{D}, \tag{9.4.19}$$

which applies for

$$\boldsymbol{\sigma}' : \mathbf{D} > 0. \tag{9.4.20}$$

Note that

$$\underline{\boldsymbol{\mathcal{L}}}_{(0)} : \boldsymbol{\sigma}' = 2\mu \, \boldsymbol{\sigma}'. \tag{9.4.21}$$

In retrospect, the plastic rate of deformation can be expressed either in terms of stress rate or total rate of deformation as

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}} \frac{\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}'}{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} : \overset{\circ}{\underline{\boldsymbol{\tau}}} = \frac{1}{1 + h_{\mathrm{t}}^{\mathrm{p}}/\mu} \frac{\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}'}{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} : \mathbf{D}. \tag{9.4.22}$$

An often utilized expression for $K = K(\vartheta)$ corresponds to nonlinear hardening that saturates to linear hardening at large ϑ (Fig. 9.4), i.e.,

$$K = K_0 + h_1 \vartheta + (K_1 - K_0) \left[1 - \exp\left(-\frac{h_0 - h_1}{K_1 - K_0} \, \vartheta \right) \right]. \tag{9.4.23}$$

The corresponding plastic tangent modulus is

$$h_{\mathrm{t}}^{\mathrm{p}} = h_1 + (h_0 - h_1) \exp\left(-\frac{h_0 - h_1}{K_1 - K_0} \, \vartheta \right). \tag{9.4.24}$$

In the case of linear hardening, $K = K_0 + h_{\mathrm{t}}^{\mathrm{p}} \vartheta$, where $h_{\mathrm{t}}^{\mathrm{p}}$ is a constant. For ideal (perfect) plasticity, $h_{\mathrm{t}}^{\mathrm{p}} = 0$ can be substituted in the expression on the far right-hand side of Eq. (9.4.22), since $\boldsymbol{\sigma}' : \boldsymbol{\sigma}' = 2K_0^2$, where K_0 is the constant radius of the yield surface.

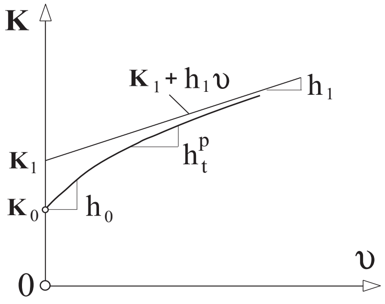

FIGURE 9.4. Nonlinear hardening that saturates to linear hardening with the rate h_1 at large ϑ, according to Eq. (9.4.17). The initial yield stress is K_0 and the initial hardening rate is h_0. The plastic tangent modulus at an arbitrary ϑ is $h_{\mathrm{t}}^{\mathrm{p}}$.

Constitutive structures (9.4.17) and (9.4.19) have been used in analytical and numerical treatments of various plastic deformation problems (e.g., Hutchinson, 1973; McMeeking and Rice, 1975; Neale, 1981; Needleman, 1982). More generally, when f is defined by Eq. (9.4.8), we can write

$$\mathbf{D} = \left(\underline{\boldsymbol{\mathcal{M}}}_{(0)} + \frac{1}{2h^{\mathrm{p}}}\, \mathbb{M} \otimes \mathbb{M} \right) : \overset{\circ}{\underline{\boldsymbol{\tau}}}, \tag{9.4.25}$$

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} = \left(\underline{\boldsymbol{\mathcal{L}}}_{(0)} - \frac{2\mu}{1 + h^{\mathrm{p}}/\mu}\, \mathbb{M} \otimes \mathbb{M} \right) : \mathbf{D}. \tag{9.4.26}$$

The normalized tensor $\mathbb{M}$ is in the direction of outward normal to the yield surface, and h^{p} is the hardening parameter. They are defined by

$$\mathbb{M} = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}}}{\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}} \right)^{1/2}}\,, \quad h^{\mathrm{p}} = -\, \frac{\frac{\partial f}{\partial K}\, h_{\mathrm{t}}^{\mathrm{p}}}{\left(2\, \frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}} \right)^{1/2}}\,. \tag{9.4.27}$$

If f is given by Eq. (9.4.9), then

$$h^{\mathrm{p}} = h_{\mathrm{t}}^{\mathrm{p}} = \frac{\mathrm{d}K}{\mathrm{d}\vartheta}\,. \tag{9.4.28}$$

Derived equations are in accord with the constitutive structure (9.1.34), obtained within formulation based on the yield surface in strain space. This can be easily verified by observing that

$$K = 2\mu k, \quad \frac{\mathrm{d}K}{\mathrm{d}\vartheta} = 2\mu\, \frac{\mathrm{d}k}{\mathrm{d}\varphi}\, \frac{\dot{\varphi}}{\dot{\vartheta}}\,, \quad \frac{\dot{\varphi}}{\dot{\vartheta}} = -\mu, \tag{9.4.29}$$

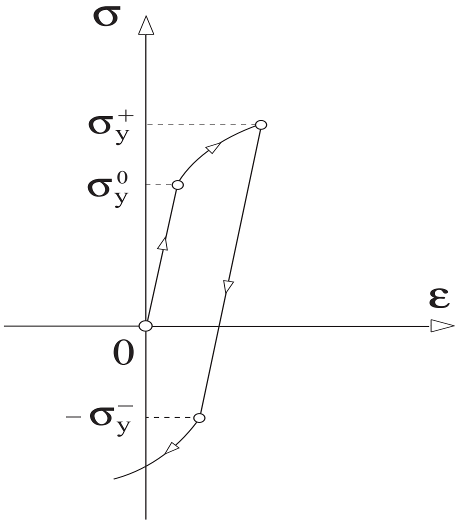

FIGURE 9.5. Illustration of the Bauschinger effect ($|\sigma_y^-| < \sigma_y^+$) in uniaxial tension. The Cauchy stress is σ and the logarithmic strain is ϵ.

and

$$H - h = -4\mu K^2. \tag{9.4.30}$$

The formulation of the constitutive equations for isotropic hardening plasticity within the framework of infinitesimal strain is presented in standard texts or review papers, such as Hill (1950), Drucker (1960), and Naghdi (1960). Derivation of classical Prandtl–Reuss equations for elastic-ideally plastic, and Levy–Mises equations for rigid-ideally plastic material models is also there given. The effects of the third invariant of the stress deviator on plastic deformation are discussed by Novozhilov (1952), Ohashi, Tokuda, and Yamashita (1975), and Gupta and Meyers (1992, 1994). The book by Zyczkowski (1981) contains a comprehensive list of references to various other topics of classical plasticity.

9.4.2. Kinematic Hardening

To account for the Bauschinger effect (Fig. 9.5) and anisotropic hardening, and thus provide better description of material response under cyclic loading, a simple model of kinematic hardening was introduced by Melan (1938) and

Prager (1955,1956). According to this model, the initial yield surface does not change its size and shape during plastic deformation, but translates in the stress space according to some prescribed rule. If the yield condition is pressure-independent, it is assumed that

$$f\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha},\, K_0) = 0, \quad K_0 = \text{const.}, \tag{9.4.31}$$

where $\boldsymbol{\alpha}$ represents the current center of the yield locus in the deviatoric plane $\operatorname{tr}\boldsymbol{\sigma} = 0$ (back stress), and f is an isotropic function of the stress difference $\boldsymbol{\sigma}' - \boldsymbol{\alpha}$. The back stress in the plane $\operatorname{tr}\boldsymbol{\sigma} = \text{const.}$ would be $\boldsymbol{\alpha} + (\operatorname{tr}\boldsymbol{\sigma}/3)\mathbf{I}$. The size of the yield locus is specified by the constant K_0. By an analysis similar to that used in the previous subsection, the consistency condition for continuing plastic deformation can be written as

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} : \left(\underline{\overset{\circ}{\boldsymbol{\tau}}} - \underline{\overset{\circ}{\boldsymbol{\alpha}}}\right) = 0, \tag{9.4.32}$$

where $\partial f/\partial \boldsymbol{\sigma} = \partial f/\partial \boldsymbol{\sigma}'$. Suppose that the yield surface instantaneously translates so that the evolution of back stress is governed by

$$\underline{\overset{\circ}{\boldsymbol{\alpha}}} = c(\boldsymbol{\alpha}, \vartheta)\, \mathbf{D}^{\mathrm{p}} + \mathbf{C}(\boldsymbol{\alpha}, \vartheta)\, (\mathbf{D}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}})^{1/2}\,, \tag{9.4.33}$$

where c and $\mathbf{C}$ are the appropriate scalar and tensor functions of $\boldsymbol{\alpha}$ and ϑ. This representation is in accord with assumed time-independence of plastic deformation, which requires Eq. (9.4.33) to be homogeneous function of degree one in the components of plastic rate of deformation. Since the plastic rate of deformation is

$$\mathbf{D}^{\mathrm{p}} = \dot{\gamma}\, \frac{\partial f}{\partial \boldsymbol{\sigma}}\,, \tag{9.4.34}$$

the substitution of Eq. (9.4.33) into Eq. (9.4.32) gives the loading index

$$\dot{\gamma} = \frac{1}{H}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \underline{\overset{\circ}{\boldsymbol{\tau}}}\right), \quad H = c\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}}\right) + \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^{1/2} \left(\mathbf{C} : \frac{\partial f}{\partial \boldsymbol{\sigma}}\right). \tag{9.4.35}$$

If the yield condition is specified by

$$f = \frac{1}{2}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) - K_0^2 = 0, \tag{9.4.36}$$

then

$$\frac{\partial f}{\partial \boldsymbol{\sigma}} = \boldsymbol{\sigma}' - \boldsymbol{\alpha}, \quad \mathbf{D}^{\mathrm{p}} = \dot{\gamma}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha}), \tag{9.4.37}$$

and

$$\dot{\gamma} = \frac{1}{H}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : \underline{\overset{\circ}{\boldsymbol{\tau}}}, \quad H = 2K_0\left[cK_0 + \frac{1}{\sqrt{2}}\,\mathbf{C} : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})\right]. \tag{9.4.38}$$

Consequently,

$$\mathbf{D} = \left[\underline{\boldsymbol{\mathcal{M}}}_{(0)} + \frac{1}{H} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \right] : \overset{\circ}{\underline{\boldsymbol{\tau}}}. \tag{9.4.39}$$

Linear and Nonlinear Kinematic Hardening

When $\mathbf{C} = \mathbf{0}$ and c is taken to be a constant, the model with evolution equation (9.4.33) reduces to Prager's linear kinematic hardening (Fig. 9.6). The plastic tangent modulus $h_{\mathrm{t}}^{\mathrm{p}}$ from the shear test is constant, and related to c by

$$c = 2h_{\mathrm{t}}^{\mathrm{p}}. \tag{9.4.40}$$

In this case, Eq. (9.4.39) becomes

$$\mathbf{D} = \left[\underline{\boldsymbol{\mathcal{M}}}_{(0)} + \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}} \frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\sigma}' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})} \right] : \overset{\circ}{\underline{\boldsymbol{\tau}}}, \tag{9.4.41}$$

with plastic loading condition in the hardening range

$$(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : \overset{\circ}{\underline{\boldsymbol{\tau}}} > 0. \tag{9.4.42}$$

The inverse equation is

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} = \left[\underline{\boldsymbol{\mathcal{L}}}_{(0)} - \frac{2\mu}{1 + h_{\mathrm{t}}^{\mathrm{p}}/\mu} \frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\sigma}' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})} \right] : \mathbf{D}, \tag{9.4.43}$$

provided that

$$(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : \mathbf{D} > 0. \tag{9.4.44}$$

In retrospect, the evolution equation for the back stress

$$\overset{\circ}{\underline{\boldsymbol{\alpha}}} = 2h_{\mathrm{t}}^{\mathrm{p}}\, \mathbf{D}^{\mathrm{p}} \tag{9.4.45}$$

can be expressed in terms of the stress rate or the rate of deformation as

$$\overset{\circ}{\underline{\boldsymbol{\alpha}}} = \frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\sigma}' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})} : \overset{\circ}{\underline{\boldsymbol{\tau}}}, \quad (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : \overset{\circ}{\underline{\boldsymbol{\tau}}} > 0, \tag{9.4.46}$$

$$\overset{\circ}{\underline{\boldsymbol{\alpha}}} = \frac{2h_{\mathrm{t}}^{\mathrm{p}}}{1 + h_{\mathrm{t}}^{\mathrm{p}}/\mu} \frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\sigma}' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})} : \mathbf{D}, \quad (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : \mathbf{D} > 0. \tag{9.4.47}$$

A nonlinear kinematic hardening model of Armstrong and Frederick (1966) is obtained if $\mathbf{C}$ in Eq. (9.4.33) is taken to be proportional to $\boldsymbol{\alpha}$,

$$\mathbf{C} = -c_0\, \boldsymbol{\alpha}, \tag{9.4.48}$$

where c_0 is a constant material parameter. In this case

$$\overset{\circ}{\underline{\boldsymbol{\alpha}}} = 2h\, \mathbf{D}^{\mathrm{p}} - c_0\, \boldsymbol{\alpha}\, (\mathbf{D}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}})^{1/2}, \tag{9.4.49}$$

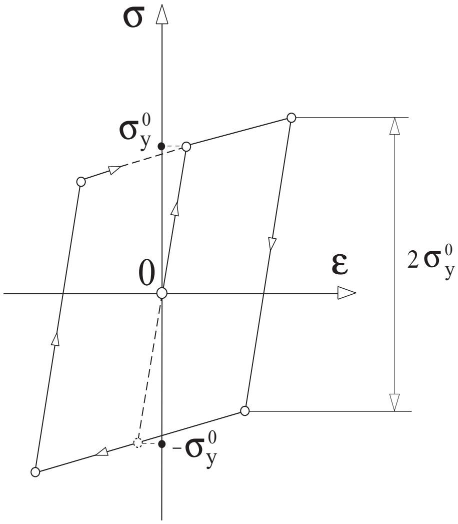

FIGURE 9.6. One-dimensional stress-strain response according to linear kinematic hardening model.

with h as another material parameter. The added nonlinear term in Eq. (9.4.49), referred to as a recall term, gives rise to hardening moduli for reversed plastic loading that are in better agreement with experimental data. It follows that

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2h(1-m)} \frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\sigma}' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})} : \overset{\circ}{\underline{\boldsymbol{\tau}}}, \tag{9.4.50}$$

where

$$m = \frac{c_0}{2\,h} \frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : \boldsymbol{\alpha}}{[(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})]^{1/2}}\,. \tag{9.4.51}$$

In modeling cyclic plasticity it may be convenient to additively decompose the back stress $\boldsymbol{\alpha}$ into two or more constituents, and construct separate evolution equation for each of these. For details, see Moosbrugger and McDowell (1989), Ohno and Wang (1993), and Jiang and Kurath (1996).

Ziegler (1959) used an evolution equation for back stress in the form

$$\overset{\circ}{\underline{\boldsymbol{\alpha}}} = \dot{\beta}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha}). \tag{9.4.52}$$

The proportionality factor $\dot{\beta}$ can be determined from the consistency condition in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ (Fig. 9.7). Detailed analysis is available in the book by Chakrabarty (1987). Duszek and Perzyna (1991) suggested an

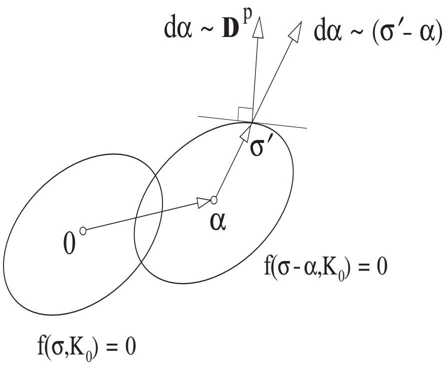

FIGURE 9.7. Translation of the yield surface according to kinematic hardening model. The center of the yield surface is the back stress $\boldsymbol{\alpha}$. Its evolution is governed by $\mathrm{d}\boldsymbol{\alpha} \sim \mathbf{D}^{\mathrm{p}}$ according to Prager's model, and by $\mathrm{d}\boldsymbol{\alpha} \sim (\boldsymbol{\sigma}' - \boldsymbol{\alpha})$ according to Ziegler's model.

evolution equation that is a linear combination of the Prager and Ziegler hardening rules. See also Ishlinsky (1954), Backhaus (1968, 1972), Eisenberg and Phillips (1968), and Lehmann (1972).

9.4.3. Combined Isotropic–Kinematic Hardening

In this hardening model the yield surface expands and translates during plastic deformation (Fig. 9.8), so that

$$f\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha},\, K_\alpha) = 0, \quad K_\alpha = K_\alpha(\vartheta). \tag{9.4.53}$$

The scalar function $K_\alpha(\vartheta)$, with ϑ defined by Eq. (9.4.3), specifies expansion of the yield surface, while (9.4.33) specifies its translation. The resulting constitutive equation for the plastic part of rate of deformation is

$$\mathbf{D}^{\mathrm{p}} = \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\sigma}}\,, \quad \dot{\gamma} = \frac{1}{H}\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \overset{\circ}{\underline{\boldsymbol{\tau}}}\right), \tag{9.4.54}$$

with

$$H = c\left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}}\right) + \left(\frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}}\right)^{1/2}\left(\mathbf{C} : \frac{\partial f}{\partial \boldsymbol{\sigma}} - \sqrt{2}\, h_\alpha^{\mathrm{p}}\,\frac{\partial f}{\partial K_\alpha}\right). \tag{9.4.55}$$

The rate of the yield surface expansion is

$$h_\alpha^{\mathrm{p}} = \frac{\mathrm{d}K_\alpha}{\mathrm{d}\vartheta}\,. \tag{9.4.56}$$

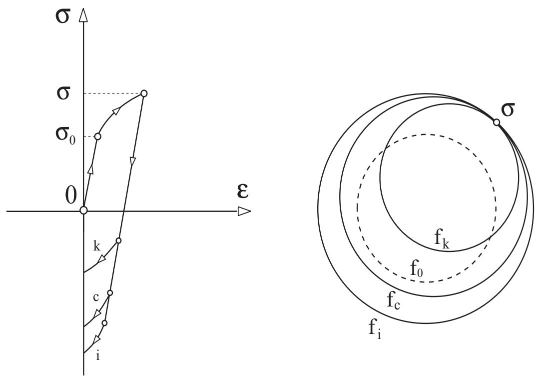

FIGURE 9.8. Geometric illustration of isotropic, kinematic and combined hardening. The initial yield surface (f_0) expands in the case of isotropic, translates in the case of kinematic (f_k), and expands and translates in the case of combined or mixed hardening (f_c).

If the yield surface is

$$\frac{1}{2}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) = K_\alpha^2(\vartheta), \tag{9.4.57}$$

where $\boldsymbol{\alpha}$ represents its current center, and $K_\alpha(\vartheta)$ its current radius, and if the evolution equation for back stress $\boldsymbol{\alpha}$ is given by Eq. (9.4.49), we obtain

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2h_\alpha^{\mathrm{p}} + 2h(1-m)}\,\frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\sigma}' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})} : \overset{\circ}{\underline{\boldsymbol{\tau}}}. \tag{9.4.58}$$

The parameter m is again specified by Eq. (9.4.51). This clearly encompasses the previously considered purely isotropic and kinematic hardening models. For purely kinematic hardening $h_\alpha^{\mathrm{p}} = 0$, and for purely isotropic hardening $h_\alpha^{\mathrm{p}} = h_{\mathrm{t}}^{\mathrm{p}}$ (plastic tangent modulus in simple shear).

For example, when $m = c_0 = 0$, and when the hardening is linear with the yield stress $K = K_0 + h_{\mathrm{t}}^{\mathrm{p}}\vartheta$, where $h_{\mathrm{t}}^{\mathrm{p}} = \text{const.}$, we can write

$$h = (1-r)h_{\mathrm{t}}^{\mathrm{p}}, \quad K_\alpha = K_0 + r h_{\mathrm{t}}^{\mathrm{p}}\vartheta, \tag{9.4.59}$$

and $h_\alpha^{\mathrm{p}} = r h_{\mathrm{t}}^{\mathrm{p}}$. The parameter $0 \le r \le 1$ defines the amount of combined hardening. The value $r = 1$ corresponds to purely isotropic, and $r = 0$ to purely kinematic hardening. Equation (9.4.59) can be extended to the case of nonlinear hardening $K = K(\vartheta)$ by defining

$$h = (1-r)\frac{\mathrm{d}K}{\mathrm{d}\vartheta}\,, \quad K_\alpha = K_0 + r(K - K_0). \tag{9.4.60}$$

Moreton, Moffat, and Parkinson (1981) observed large translations together with moderately small isotropic expansion and distortion of the yield surface in experiments with pressure vessel steels. Detailed description of the measured yield loci can be found in Naghdi, Essenburg, and Koff (1958), Bertsch and Findley (1962), Hecker (1976), Phillips and Lee (1979), Shiratori, Ikegami, and Yoshida (1979), Phillips and Das (1985), Stout, Martin, Helling, and Canova (1985), Wu, Lu, and Pan (1995), and Barlat *et al.* (1997).

9.4.4. Mróz Multisurface Model

More involved hardening models were suggested to better treat nonlinearities in stress-strain loops, cyclic hardening or softening, cyclic creep and stress relaxation. In order to describe nonlinear hardening and provide gradual transition from elastic to plastic deformation, Mróz (1967, 1976) introduced a multiyield surface model in which there is a field of hardening moduli, one for each yield surface. Initially the yield surfaces are assumed to be concentric (Fig. 9.9). When the stress point reaches the innermost surface $f_{<1>} = 0$, the plastic deformation develops according to linear hardening model with the plastic tangent modulus $h^{\mathrm{p}}_{\mathrm{t}\,<1>}$, until the activated yield surface reaches the next surface $f_{<2>} = 0$. Subsequent plastic deformation develops according to linear hardening model with the plastic tangent modulus $h^{\mathrm{p}}_{\mathrm{t}\,<2>}$, until the next surface is reached, etc. Suppose that pressure-independent yield surfaces are defined by

$$f_{<i>} = \frac{1}{2}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>}) - K^2_{<i>} = 0, \quad i = 1, 2, \cdots N. \tag{9.4.61}$$

The centers of the individual surfaces are $\boldsymbol{\alpha}_{<i>}$, and their sizes are specified by the constants $K_{<i>}$ (determined by fitting the nonlinear stress-strain curve in pure shear test). For simplicity, only translation of the yield surfaces is considered. To ascertain that two surfaces in contact have coincident outward normals, the active yield surface

$$f_{<i>} = 0 \tag{9.4.62}$$

translates in the direction of the stress difference $\boldsymbol{\sigma}'_{<i+1>} - \boldsymbol{\sigma}'$, where $\boldsymbol{\sigma}'$ is the current stress state on the yield surface $f_{<i>} = 0$, and $\boldsymbol{\sigma}'_{<i+1>}$ is the

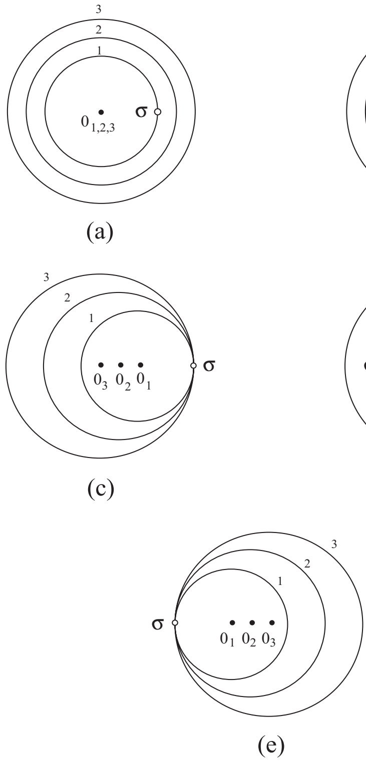

FIGURE 9.9. Illustration of the Mróz multisurface hardening model with the help of three initially concentric surfaces. Sequential translation of the surfaces are indicated corresponding to uniaxial monotonic loading in (b) and (c), and reversed loading in (d) and (e).

stress state on the subsequent yield surface

$$f_{<i+1>} = 0. \tag{9.4.63}$$

This stress state is defined by the requirement that the yield surface normals at $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}'_{<i+1>}$ are parallel (Fig. 9.10). Thus, the evolution law for back stress is

$$\overset{\circ}{\underline{\boldsymbol{\alpha}}}_{<i>} = \dot{\beta}_{<i>} \left(\boldsymbol{\sigma}'_{<i+1>} - \boldsymbol{\sigma}' \right), \tag{9.4.64}$$

where

$$\frac{1}{K_{<i+1>}} \left(\boldsymbol{\sigma}'_{<i+1>} - \boldsymbol{\alpha}_{<i+1>} \right) = \frac{1}{K_{<i>}} \left(\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>} \right). \tag{9.4.65}$$

Inserting Eq. (9.4.65) into Eq. (9.4.64),

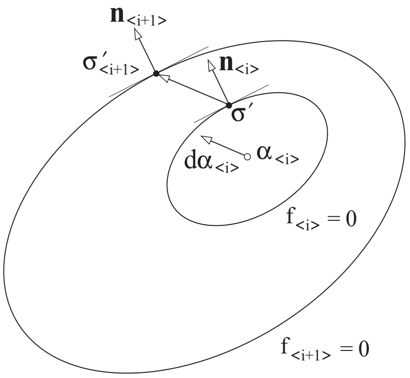

FIGURE 9.10. Translation of the surface $f_{<i>} = 0$ in Mróz's model is specified by $\mathrm{d}\boldsymbol{\alpha}_{<i>} \sim (\boldsymbol{\sigma}'_{<i+1>} - \boldsymbol{\sigma}')$, where $\boldsymbol{\sigma}'_{<i+1>}$ is the stress state on the surface $f_{<i+1>} = 0$, with the normal $\mathbf{n}_{<i+1>}$ parallel to $\mathbf{n}_{<i>}$ at the state of stress $\boldsymbol{\sigma}'$ on the yield surface $f_{<i>} = 0$.

$$\underline{\overset{\circ}{\boldsymbol{\alpha}}}_{<i>} = \dot{\beta}_{<i>} \left[\frac{K_{<i+1>}}{K_{<i>}} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>}) - (\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i+1>}) \right]. \qquad (9.4.66)$$

The consistency condition

$$\dot{f}_{<i>} = 0 \qquad (9.4.67)$$

gives

$$(\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>}) : \underline{\overset{\circ}{\boldsymbol{\tau}}} = (\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>}) : \underline{\overset{\circ}{\boldsymbol{\alpha}}}_{i}. \qquad (9.4.68)$$

Combined with Eq. (9.4.66), this defines

$$\dot{\beta}_{<i>} = \frac{1}{2B_{<i>}} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>}) : \underline{\overset{\circ}{\boldsymbol{\tau}}}, \qquad (9.4.69)$$

where

$$B_{<i>} = K_{<i>} K_{<i+1>} - \frac{1}{2} (\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i+1>}). \qquad (9.4.70)$$

The plastic part of the rate of deformation tensor, during the loading between the active yield surface $f_{<i>} = 0$ and the nearby surface $f_{<i+1>} = 0$, is defined by the linear kinematic hardening law with the plastic tangent modulus $h^{\mathrm{p}}_{\mathrm{t}\,<i>}$. This gives, from Eq. (9.4.41),

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2h^{\mathrm{p}}_{\mathrm{t}\,<i>}} \left[\frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>}) \otimes (\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha}_{<i>})} \right] : \underline{\overset{\circ}{\boldsymbol{\tau}}}. \qquad (9.4.71)$$

Further details, including the incorporation of isotropic component of hardening and determination of material parameters, can be found in cited Mróz's

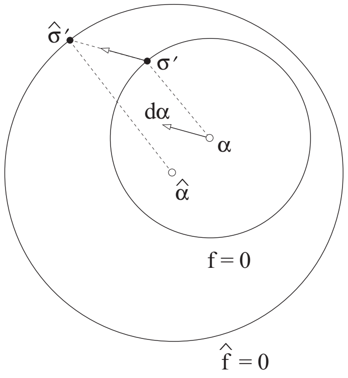

FIGURE 9.11. Schematic representation of the loading and bounding surface in the two-surface hardening model. The loading surface translates toward the bounding surface in the direction $\hat{\boldsymbol{\sigma}}' - \boldsymbol{\sigma}'$.

papers. See also Iwan (1967), Desai and Siriwardane (1984), Khan and Huang (1995), and Jiang and Sehitoglu (1996a, b).

9.4.5. Two-Surface Model

Dafalias and Popov (1975, 1976), and Krieg (1975) suggested the hardening model which uses the yield (loading) surface and the limit (bounding) surface (Fig. 9.11). A smooth transition from elastic to plastic regions on loading is assured by introducing a continuous variation of the plastic tangent modulus between the two surfaces, i.e.,

$$h_{\rm t}^{\rm p} = h_{\rm t}^{\rm p}(\eta, \vartheta). \tag{9.4.72}$$

The scalar

$$\eta = \left[(\hat{\boldsymbol{\sigma}}' - \boldsymbol{\sigma}') : (\hat{\boldsymbol{\sigma}}' - \boldsymbol{\sigma}')\right]^{1/2} \tag{9.4.73}$$

is a measure of the distance between the current stress state $\boldsymbol{\sigma}'$ on the loading surface, and the corresponding, appropriately defined state of stress $\hat{\boldsymbol{\sigma}}'$ on the bounding surface. Only deviatoric parts of stress are used for pressure-independent plasticity. Suppose that the loading surface can translate and expand, such that

$$f = \frac{1}{2}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) - K_\alpha^2(\vartheta) = 0, \quad \vartheta = \int_0^t \left(2\,\mathbf{D}^{\rm p} : \mathbf{D}^{\rm p}\right)^{1/2} {\rm d}t. \tag{9.4.74}$$

The bounding surface is assumed to only translate, i.e.,

$$\hat{f} = \frac{1}{2}\,(\hat{\boldsymbol{\sigma}}\,' - \hat{\boldsymbol{\alpha}}) : (\hat{\boldsymbol{\sigma}}\,' - \hat{\boldsymbol{\alpha}}) - \hat{K}^2 = 0, \quad \hat{K} = \text{const.} \tag{9.4.75}$$

Translation of the loading surface is defined as in the Mróz's model, and it is in the direction of the stress difference $\hat{\boldsymbol{\sigma}}\,' - \boldsymbol{\sigma}\,'$. The current stress state on the loading surface $f = 0$ is $\boldsymbol{\sigma}\,'$, while $\hat{\boldsymbol{\sigma}}\,'$ is the stress state on the bounding surface $\hat{f} = 0$, where the surface normal is parallel to the loading surface normal at $\boldsymbol{\sigma}\,'$. Thus, the evolution law for back stress $\boldsymbol{\alpha}$ is

$$\overset{\circ}{\underline{\boldsymbol{\alpha}}} = \dot{\beta}\,(\hat{\boldsymbol{\sigma}}\,' - \boldsymbol{\sigma}\,') = \dot{\beta}\left[\frac{\hat{K}}{K_\alpha}\,(\boldsymbol{\sigma}\,' - \boldsymbol{\alpha}) - (\boldsymbol{\sigma}\,' - \hat{\boldsymbol{\alpha}})\right]. \tag{9.4.76}$$

The translation of the bounding surface is governed by a linear kinematic hardening rule

$$\overset{\circ}{\underline{\hat{\boldsymbol{\alpha}}}} = 2\hat{h}_{\mathrm{t}}^{\mathrm{p}}\,\mathbf{D}^{\mathrm{p}}, \tag{9.4.77}$$

where $\hat{h}_{\mathrm{t}}^{\mathrm{p}}$ is the corresponding, constant plastic tangent modulus. The plastic part of the rate of deformation tensor is taken to be

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\left[\frac{(\boldsymbol{\sigma}\,' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\sigma}\,' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}\,' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}\,' - \boldsymbol{\alpha})}\right] : \overset{\circ}{\underline{\boldsymbol{\tau}}}. \tag{9.4.78}$$

The consistency condition for the loading surface gives

$$(\boldsymbol{\sigma}\,' - \boldsymbol{\alpha}) : (\overset{\circ}{\underline{\boldsymbol{\tau}}} - \overset{\circ}{\underline{\boldsymbol{\alpha}}}) - 2\,h_\alpha^{\mathrm{p}}\,K_\alpha\,(2\,\mathbf{D}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}})^{1/2} = 0, \tag{9.4.79}$$

with $h_\alpha^{\mathrm{p}} = \mathrm{d}K_\alpha/\mathrm{d}\vartheta$. In view of Eq. (9.4.76) and (9.4.78), Eq. (9.4.79) defines

$$\dot{\beta} = \frac{1}{2B}\left(1 - \frac{h_\alpha^{\mathrm{p}}}{h_{\mathrm{t}}^{\mathrm{p}}}\right)(\boldsymbol{\sigma}\,' - \boldsymbol{\alpha}) : \overset{\circ}{\underline{\boldsymbol{\tau}}}, \tag{9.4.80}$$

where

$$B = K_\alpha\hat{K} - \frac{1}{2}\,(\boldsymbol{\sigma}\,' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}\,' - \hat{\boldsymbol{\alpha}}). \tag{9.4.81}$$

Finally, the consistency condition for the bounding surface,

$$(\hat{\boldsymbol{\sigma}}\,' - \hat{\boldsymbol{\alpha}}) : (\overset{\circ}{\underline{\hat{\boldsymbol{\tau}}}} - \overset{\circ}{\underline{\hat{\boldsymbol{\alpha}}}}) = 0, \tag{9.4.82}$$

specifies the stress rate $\overset{\circ}{\underline{\hat{\boldsymbol{\tau}}}}$ on the bounding surface that corresponds to a prescribed stress rate $\overset{\circ}{\underline{\boldsymbol{\tau}}}$ on the loading surface. Upon substitution of Eqs. (9.4.77) and (9.4.78) into Eq. (9.4.82), there follows

$$(\hat{\boldsymbol{\sigma}}\,' - \hat{\boldsymbol{\alpha}}) : \overset{\circ}{\underline{\hat{\boldsymbol{\tau}}}} = \frac{\hat{K}}{K_\alpha}\,\frac{\hat{h}_{\mathrm{t}}^{\mathrm{p}}}{h_{\mathrm{t}}^{\mathrm{p}}}\,(\boldsymbol{\sigma}\,' - \boldsymbol{\alpha}) : \overset{\circ}{\underline{\boldsymbol{\tau}}}. \tag{9.4.83}$$

The Mróz's assumption (9.4.65) was utilized, so that

$$\hat{\boldsymbol{\sigma}}' - \hat{\boldsymbol{\alpha}} = \frac{\hat{K}}{K_\alpha}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha}). \tag{9.4.84}$$

Further analysis, including the incorporation of isotropic component of hardening for the bounding surface, and the specification of material parameters, can be found in the cited papers. See also McDowell (1985, 1987), Chaboche (1986), Hashiguchi (1981, 1988), and Ellyin (1989) for the generalization of the model and discussion of its performance. There has also been a study of cyclic hardening and softening using continuously evolving parameters and only one yield surface, presented by Haupt and Kamlah (1995), and Ristinmaa (1995). The papers by Caulk and Naghdi (1978), Drucker and Palgen (1981), and Naghdi and Nikkel (1986) address the modeling of saturation hardening under cyclic loading, and related problems.

9.5. Yield Surface with Vertex in Strain Space

Suppose that the yield surface in strain space (Fig. 9.12) has a pyramidal vertex, formed by k_0 intersecting segments (hyperplanes) such that, near the vertex,

$$\prod_{i=1}^{k_0} g_{(n)}^{<i>}\left(\mathbf{E}_{(n)}, \mathcal{H}\right) = 0, \quad k_0 \geq 2. \tag{9.5.1}$$

If the material obeys Ilyushin's postulate, from (8.5.10) it follows that $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ lies within the cone of limiting inward normals to active segments of the yield vertex, i.e.,

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = -\sum_{i=1}^{k} \mathrm{d}\gamma_{(n)}^{<i>}\,\frac{\partial g_{(n)}^{<i>}}{\partial \mathbf{E}_{(n)}}\,, \quad \mathrm{d}\gamma_{(n)}^{<i>} > 0. \tag{9.5.2}$$

Thus,

$$\mathrm{d}\mathbf{T}_{(n)} = \boldsymbol{\Lambda}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} - \sum_{i=1}^{k} \mathrm{d}\gamma_{(n)}^{<i>}\,\frac{\partial g_{(n)}^{<i>}}{\partial \mathbf{E}_{(n)}}\,, \tag{9.5.3}$$

where k is the number of active vertex segments ($\mathrm{d}\gamma_{(n)}^{<i>} = 0$ for $k < i \leq k_0$). If the strain rate is in a fully active range, so that plastic loading takes place with respect to all vertex segments, we have $k = k_0$. The scalars $\mathrm{d}\gamma_{(n)}^{<i>}$ depend on the current values of $\mathbf{E}_{(n)}$, $\mathcal{H}$, and their increments. The

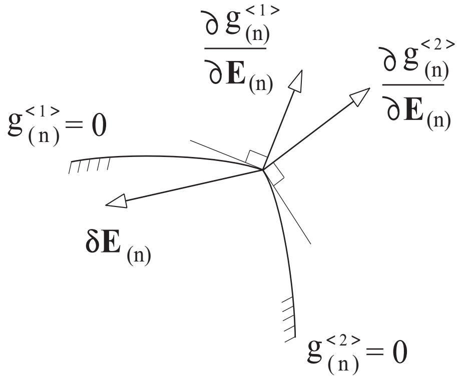

FIGURE 9.12. Yield surface vertex in strain space. Elastic strain increment $\delta\mathbf{E}_{(n)}$ is directed along or inside the vertex segments.

consistency condition for each active vertex segment is

$$\frac{\partial g_{(n)}^{<i>}}{\partial \mathbf{E}_{(n)}} : \mathrm{d}\mathbf{E}_{(n)} + \mathrm{d}^{\mathrm{p}} g_{(n)}^{<i>} = 0, \quad \mathrm{d}\gamma_{(n)}^{<i>} > 0, \tag{9.5.4}$$

where

$$\mathrm{d}^{\mathrm{p}} g_{(n)}^{<i>} = g_{(n)}^{<i>}\left(\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}\right) - g_{(n)}^{<i>}\left(\mathbf{E}_{(n)}, \mathcal{H}\right). \tag{9.5.5}$$

In the case when the vertex segment is not active,

$$\frac{\partial g_{(n)}^{<i>}}{\partial \mathbf{E}_{(n)}} : \mathrm{d}\mathbf{E}_{(n)} + \mathrm{d}^{\mathrm{p}} g_{(n)}^{<i>} \leq 0, \quad \mathrm{d}\gamma_{(n)}^{<i>} = 0. \tag{9.5.6}$$

It is assumed that the vertex segment can harden even if it is inactive, due to cross or latent hardening produced by the ongoing plastic deformation from the neighboring active vertex segments. Equality sign in (9.5.6) applies if the yield state remains on the intersection of active and inactive vertex segments.

Suppose that

$$\mathrm{d}^{\mathrm{p}} g_{(n)}^{<i>} = -\sum_{j=1}^{k} h_{(n)}^{<ij>}\, \mathrm{d}\gamma_{(n)}^{<j>} < 0, \tag{9.5.7}$$

where $h_{(n)}^{<ij>}$ are plastic moduli, in general nonsymmetric and dependent on the current plastic state. The quantity in (9.5.7) is negative because of (9.5.4), and because the scalar product of the increment of elastoplastic strain and the outer normal to any active yield segment at the vertex is positive,

$$\frac{\partial g_{(n)}^{<i>}}{\partial \mathbf{E}_{(n)}} : \mathrm{d}\mathbf{E}_{(n)} > 0, \quad i = 1, 2, \ldots, k. \tag{9.5.8}$$

Substitution into (9.5.4) and (9.5.6) gives

$$\frac{\partial g_{(n)}^{<i>}}{\partial \mathbf{E}_{(n)}} : \mathrm{d}\mathbf{E}_{(n)} = \sum_{j=1}^{k} h_{(n)}^{<ij>}\, \mathrm{d}\gamma_{(n)}^{<j>}, \quad \mathrm{d}\gamma_{(n)}^{<i>} > 0, \tag{9.5.9}$$

$$\frac{\partial g_{(n)}^{<i>}}{\partial \mathbf{E}_{(n)}} : \mathrm{d}\mathbf{E}_{(n)} \leq \sum_{j=1}^{k} h_{(n)}^{<ij>}\, \mathrm{d}\gamma_{(n)}^{<j>}, \quad \mathrm{d}\gamma_{(n)}^{<i>} = 0. \tag{9.5.10}$$

If the matrix of plastic moduli $h_{(n)}^{<ij>}$ is positive definite (thus, nonsingular), (9.5.9) gives a unique set of values

$$\mathrm{d}\gamma_{(n)}^{<i>} = \sum_{j=1}^{k} h_{(n)}^{<ij>-1}\, \frac{\partial g_{(n)}^{<j>}}{\partial \mathbf{E}_{(n)}} : \mathrm{d}\mathbf{E}_{(n)}, \tag{9.5.11}$$

for a prescribed strain increment $\mathrm{d}\mathbf{E}_{(n)}$. Elements of the matrix inverse to plastic moduli matrix $h_{(n)}^{<ij>}$ are denoted by $h_{(n)}^{<ij>-1}$. The substitution of Eq. (9.5.11) into Eq. (9.5.3) then gives

$$\dot{\mathbf{T}}_{(n)} = \left(\mathbf{\Lambda}_{(n)} - \sum_{i=1}^{k}\sum_{j=1}^{k} h_{(n)}^{<ij>-1}\, \frac{\partial g_{(n)}^{<i>}}{\partial \mathbf{E}_{(n)}} \otimes \frac{\partial g_{(n)}^{<j>}}{\partial \mathbf{E}_{(n)}} \right) : \dot{\mathbf{E}}_{(n)}. \tag{9.5.12}$$

This extends the constitutive structure (9.1.11) with a smooth yield surface in strain space to the case when the yield surface has a vertex.

The trace product of (9.5.2) with $\mathrm{d}\mathbf{E}_{(n)}$ yields, upon substitution of (9.5.9),

$$\dot{\mathbf{T}}_{(n)}^{\mathrm{p}} : \dot{\mathbf{E}}_{(n)} = -\sum_{i=1}^{k}\sum_{j=1}^{k} h_{(n)}^{<ij>}\, \dot{\gamma}_{(n)}^{<i>} \dot{\gamma}_{(n)}^{<j>}. \tag{9.5.13}$$

For positive definite matrix of plastic moduli this is clearly negative, in accord with (8.5.10) and Ilyushin's postulate. On the other hand, elastic increments from the yield state at the vertex are directed inside the yield surface and, thus, satisfy a set of k_0 inequalities

$$\frac{\partial g_{(n)}^{<i>}}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} \leq 0, \quad i = 1, 2, \ldots, k_0. \tag{9.5.14}$$

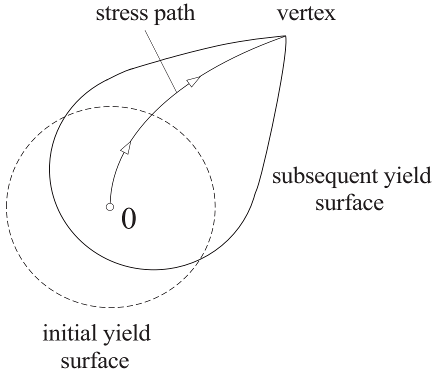

FIGURE 9.13. Development of the vertex at the loading point of the subsequent yield surface.

9.6. Yield Surface with Vertex in Stress Space

Physical theories of plasticity (Batdorf and Budiansky, 1949, 1954; Sanders, 1954; Hill, 1966, 1967b; Hutchinson, 1970) imply formation of the corner or vertex at the loading point of the yield surface (Fig. 9.13). Although sharp corners are seldom seen experimentally, the yield surfaces with relatively high curvature at the loading point are often observed (Hecker, 1972, 1976; Naghdi, 1990). Suppose then that the yield surface in stress space has a pyramidal vertex (Fig. 9.14), formed by k_0 intersecting segments such that, near the vertex,

$$\prod_{i=1}^{k_0} f_{(n)}^{<i>}\left(\mathbf{T}_{(n)}, \mathcal{H}\right) = 0, \quad k_0 \geq 2. \tag{9.6.1}$$

If the material obeys Ilyushin's postulate, from the analysis in Subsection 8.5.1 it follows that $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ lies within the cone of limiting outward normals to active segments of the yield vertex, so that

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = \sum_{i=1}^{k} \mathrm{d}\gamma_{(n)}^{<i>} \frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}}, \quad \mathrm{d}\gamma_{(n)}^{<i>} > 0, \tag{9.6.2}$$

and

$$\mathrm{d}\mathbf{E}_{(n)} = \mathbf{M}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} + \sum_{i=1}^{k} \mathrm{d}\gamma_{(n)}^{<i>} \frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}}. \tag{9.6.3}$$

It is assumed that plastic loading is taking place through k active vertex segments. If the stress rate is in a fully active range, so that plastic loading takes place with respect to all vertex segments, $k = k_0$. (Specification

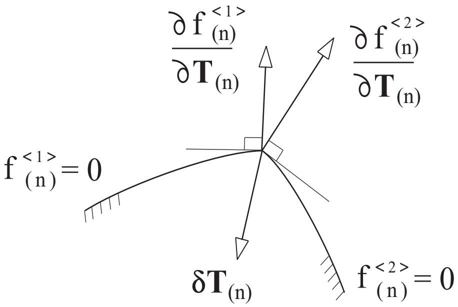

FIGURE 9.14. Yield surface vertex in stress space. Elastic stress increment $\delta\mathbf{T}_{(n)}$ is directed along or inside the vertex segments.

of fully active range and dissection of the stress rate space into pyramidal regions of partially active range is discussed, in the context of crystal plasticity, in Section 12.13). The scalars $\mathrm{d}\gamma_{(n)}^{<i>}$ depend on the current values of $\mathbf{T}_{(n)}$, $\mathcal{H}$, and their increments. The consistency condition for each active vertex segment is

$$\frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}} : \mathrm{d}\mathbf{T}_{(n)} + \mathrm{d}^{\mathrm{p}} f_{(n)}^{<i>} = 0, \quad \mathrm{d}\gamma_{(n)}^{<i>} > 0, \tag{9.6.4}$$

where

$$\mathrm{d}^{\mathrm{p}} f_{(n)}^{<i>} = f_{(n)}^{<i>}\left(\mathbf{T}_{(n)}, \mathcal{H} + d\mathcal{H}\right) - f_{(n)}^{<i>}\left(\mathbf{T}_{(n)}, \mathcal{H}\right). \tag{9.6.5}$$

If the vertex segment is not active,

$$\frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}} : \mathrm{d}\mathbf{T}_{(n)} + \mathrm{d}^{\mathrm{p}} f_{(n)}^{<i>} \leq 0, \quad \mathrm{d}\gamma_{(n)}^{<i>} = 0. \tag{9.6.6}$$

Consistent with the analysis of the yield vertex in strain space, it is assumed that the vertex segment can harden even if it is inactive, due to cross or latent hardening produced by ongoing plastic deformation associated with the neighboring active vertex segments. Equality sign in (9.6.6) applies if the yield state remains on the intersection of active and inactive vertex segments.

Suppose that

$$\mathrm{d}^{\mathrm{p}} f_{(n)}^{<i>} = -\sum_{j=1}^{k} H_{(n)}^{<ij>}\, \mathrm{d}\gamma_{(n)}^{<j>} < 0, \tag{9.6.7}$$

where $H_{(n)}^{<ij>}$ are plastic moduli, in general nonsymmetric and dependent on the current plastic state. Substitution into (9.6.4) and (9.6.6) gives

$$\frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}} : \mathrm{d}\mathbf{T}_{(n)} = \sum_{j=1}^{k} H_{(n)}^{<ij>}\, \mathrm{d}\gamma_{(n)}^{<j>}, \quad \mathrm{d}\gamma_{(n)}^{<i>} > 0, \tag{9.6.8}$$

$$\frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}} : \mathrm{d}\mathbf{T}_{(n)} \leq \sum_{j=1}^{k} H_{(n)}^{<ij>}\, \mathrm{d}\gamma_{(n)}^{<j>}, \quad \mathrm{d}\gamma_{(n)}^{<i>} = 0. \tag{9.6.9}$$

The relationship between the moduli $H_{(n)}^{<ij>}$ and $h_{(n)}^{<ij>}$ can be derived by recalling that

$$f_{(n)}^{<i>}\left(\mathbf{T}_{(n)}, \mathcal{H}\right) = f_{(n)}^{<i>}\left[\mathbf{T}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right), \mathcal{H}\right] = g_{(n)}^{<i>}\left(\mathbf{E}_{(n)}, \mathcal{H}\right). \tag{9.6.10}$$

Thus,

$$\begin{aligned}
\mathrm{d}^{\mathrm{p}} g_{(n)}^{<i>} &= g_{(n)}^{<i>}\left(\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}\right) - g_{(n)}^{<i>}\left(\mathbf{E}_{(n)}, \mathcal{H}\right) \\
&= f_{(n)}^{<i>}\left[\mathbf{T}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}\right), \mathcal{H} + d\mathcal{H}\right] \\
&- f_{(n)}^{<i>}\left[\mathbf{T}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right), \mathcal{H}\right] \\
&= f_{(n)}^{<i>}\left[\mathbf{T}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right) + \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}, \mathcal{H} + d\mathcal{H}\right] \\
&- f_{(n)}^{<i>}\left[\mathbf{T}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right), \mathcal{H}\right],
\end{aligned} \tag{9.6.11}$$

which gives

$$\mathrm{d}^{\mathrm{p}} g_{(n)}^{<i>} = \mathrm{d}^{\mathrm{p}} f_{(n)}^{<i>} + \frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}} : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}. \tag{9.6.12}$$

Upon substitution of (9.5.2), (9.5.7), and (9.6.7) into Eq. (9.6.12), there follows

$$H_{(n)}^{<ij>} = h_{(n)}^{<ij>} - \frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}} : \frac{\partial g_{(n)}^{<j>}}{\partial \mathbf{E}_{(n)}}. \tag{9.6.13}$$

Since

$$\frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}} = \mathbf{M}_{(n)} : \frac{\partial g_{(n)}^{<j>}}{\partial \mathbf{E}_{(n)}}, \tag{9.6.14}$$

the differences of plastic moduli $H_{(n)}^{<ij>} - h_{(n)}^{<ij>}$ form a symmetric matrix, provided that the elastic moduli tensor $\mathbf{M}_{(n)}$ obeys the reciprocal symmetry.

If the matrix of plastic moduli $H_{(n)}^{<ij>}$ is nonsingular, inversion of (9.6.8) gives

$$\mathrm{d}\gamma_{(n)}^{<i>} = \sum_{j=1}^{k} H_{(n)}^{<ij>-1} \frac{\partial f_{(n)}^{<j>}}{\partial \mathbf{T}_{(n)}} : \mathrm{d}\mathbf{T}_{(n)}, \tag{9.6.15}$$

for a prescribed stress increment $d\mathbf{T}_{(n)}$. Elements of the matrix inverse to plastic moduli matrix $H_{(n)}^{<ij>}$ are denoted by $H_{(n)}^{<ij>-1}$. The substitution of Eq. (9.6.15) into Eq. (9.6.3) gives

$$\dot{\mathbf{E}}_{(n)} = \left(\mathbf{M}_{(n)} + \sum_{i=1}^{k} \sum_{j=1}^{k} H_{(n)}^{<ij>-1} \, \frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}} \otimes \frac{\partial f_{(n)}^{<j>}}{\partial \mathbf{T}_{(n)}} \right) : \dot{\mathbf{T}}_{(n)}. \qquad (9.6.16)$$

This extends the constitutive structure (9.2.4) with a smooth yield surface in stress space to the case when the yield surface has a vertex.

Upon substitution of (9.6.8), the trace product of (9.6.2) with $d\mathbf{T}_{(n)}$ yields

$$\dot{\mathbf{T}}_{(n)} : \dot{\mathbf{E}}_{(n)}^{\mathrm{p}} = \sum_{i=1}^{k} \sum_{j=1}^{k} H_{(n)}^{<ij>} \, \dot{\gamma}_{(n)}^{<i>} \dot{\gamma}_{(n)}^{<j>}. \qquad (9.6.17)$$

In the hardening range the plastic moduli $H_{(n)}^{<ij>}$ form a positive definite matrix, so that the quantity in (9.6.17) is positive. In this case, for a prescribed rate of stress $\dot{\mathbf{T}}_{(n)}$, the plastic response is unique and given by (9.6.16). In the softening range the quantity in (9.6.17) is negative. For a prescribed rate of stress, either plastic response given by (9.6.16) applies, or elastic response $\dot{\mathbf{E}}_{(n)} = \mathbf{M}_{(n)} : \dot{\mathbf{T}}_{(n)}$ takes place. In the case of ideal plasticity (vanishing self and latent hardening rates), $\dot{\gamma}_{(n)}^{<i>}$ in Eq. (9.6.2) are indeterminate by the constitutive analysis.

Elastic increments from the yield state at the vertex are always directed inside the yield surface and thus satisfy a set of k_0 inequalities

$$\frac{\partial f_{(n)}^{<i>}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)} \leq 0, \quad i = 1, 2, \ldots, k_0, \qquad (9.6.18)$$

which are dual to (9.5.14).

The papers by Koiter (1953), Mandel (1965), Sewell (1974), Hill (1978), and Ottosen and Ristinmaa (1996) offer further analysis of the plasticity theory with yield corners or vertices.

9.7. Pressure-Dependent Plasticity

For porous metals, concrete and geomaterials like soils and rocks, plastic deformation has its origin in pressure dependent microscopic processes. The corresponding yield condition depends on both deviatoric and hydrostatic

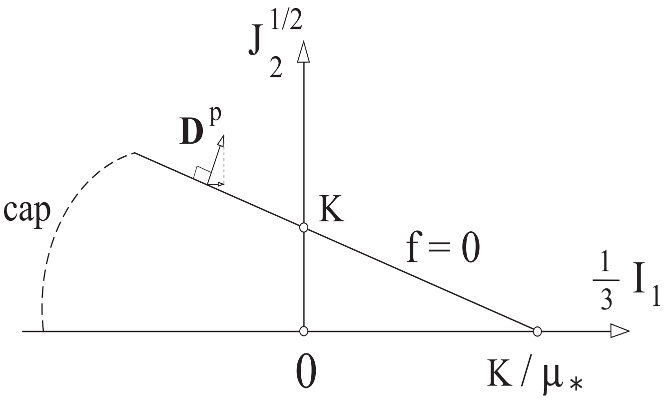

FIGURE 9.15. The Drucker–Prager yield condition shown in the coordinates of stress invariants I_1 and J_2. The yield stress in pure shear is K, and the frictional parameter is μ_*. The horizontal projection of the plastic rate of deformation indicates plastic dilatation according to normality and associative flow rule. At high pressure a cap is used to close the cone.

parts of the stress tensor. Constitutive modeling of such materials is the concern of this section.

9.7.1. Drucker–Prager Condition for Geomaterials

Drucker and Prager (1952) suggested that the yielding in soils occurs when the shear stress on octahedral planes overcomes cohesive and frictional resistance to sliding on these planes, i.e., when

$$\tau_{\mathrm{oct}} = \tau_{\mathrm{frict}} + \sqrt{\frac{2}{3}}\,K, \tag{9.7.1}$$

where

$$\tau_{\mathrm{oct}} = \left(\frac{2}{3}\,J_2\right)^{1/2}, \quad \tau_{\mathrm{frict}} = -\mu^*\sigma_{\mathrm{oct}} = -\frac{1}{3}\,\mu^* I_1. \tag{9.7.2}$$

The coefficient of internal friction (material parameter) is μ^*. The first invariant of the Cauchy stress tensor is I_1, and J_2 is the second invariant of deviatoric part of the Cauchy stress,

$$I_1 = \mathrm{tr}\,\boldsymbol{\sigma}, \quad J_2 = \frac{1}{2}\,\boldsymbol{\sigma}' : \boldsymbol{\sigma}'. \tag{9.7.3}$$

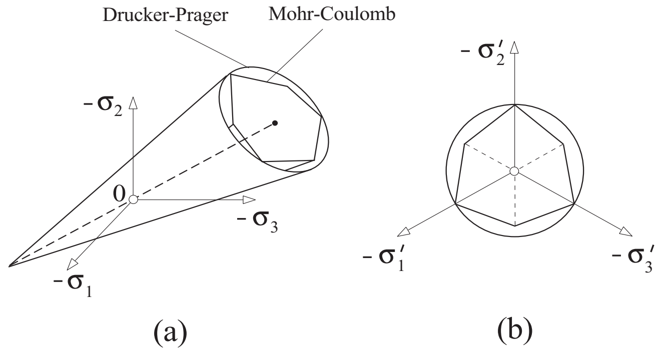

FIGURE 9.16. The Drucker–Prager cone and the Mohr–Coulomb pyramid matched along the compressive meridian, shown in (a) principal stress space, and (b) deviatoric plane.

The yield condition is consequently

$$f = J_2^{1/2} + \frac{1}{3}\,\mu_* I_1 - K = 0, \tag{9.7.4}$$

where the parameter

$$\mu_* = \sqrt{\frac{3}{2}}\,\mu^* \tag{9.7.5}$$

is conveniently introduced (Fig. 9.15). This geometrically represents a cone in the principal stress space with its axis parallel to the hydrostatic axis (Fig. 9.16). The radius of the circle in the deviatoric (π) plane is $\sqrt{2}\,K$, where K is the yield stress in simple shear. The angle of the cone is $\tan^{-1}(\sqrt{2}\,\mu_*/3)$. The yield stresses in uniaxial tension and compression are, according to Eq. (9.7.4),

$$Y^+ = \frac{\sqrt{3}\,K}{1+\mu_*/\sqrt{3}}\,, \quad Y^- = \frac{\sqrt{3}\,K}{1-\mu_*/\sqrt{3}}\,. \tag{9.7.6}$$

For the yield condition to be physically meaningful, the restriction must hold

$$\mu_* < \sqrt{3}. \tag{9.7.7}$$

If the compressive states of stress are considered positive (as commonly done in geomechanics, e.g., Jaeger and Cook, 1976; Salençon, 1977), a minus sign appears in front of the second term of f in Eq. (9.7.4). For the effects of the third stress invariant on plastic deformation of pressure sensitive materials, see Bardet (1990) and the references therein. The second and third deviatoric stress invariants define the Lode angle θ by (e.g., Chen and Han, 1988)

$$\cos(3\theta) = \left(\frac{27J_3^2}{4J_2^3}\right)^{1/2}. \tag{9.7.8}$$

When the Drucker–Prager cone is applied to porous rocks, it overestimates the yield stress at higher pressures, and inadequately predicts inelastic volume changes. To circumvent the former, DiMaggio and Sandler (1971) introduced an ellipsoidal cap to close the cone at certain level of pressure. Other shapes of the cap were also used. Details can be found in Chen and Han (1988), and Lubarda, Mastilovic, and Knap (1996).

Constitutive analysis of inelastic response of concrete has been studied extensively. Representative references include Ortiz and Popov (1982), Ortiz (1985), Pietruszczak, Jiang, and Mirza (1988), Faruque and Chang (1990), Voyiadjis and Abu-Lebdeh (1994), Lubarda, Krajcinovic, and Mastilovic (1994), and Lade and Kim (1995). Pressure-dependent response of granular materials was modeled by Mehrabadi and Cowin (1981), Christoffersen, Mehrabadi, and Nemat-Nasser (1981), Dorris and Nemat-Nasser (1982), Anand (1983), Chandler (1985), Harris (1992), and others.

9.7.2. Gurson Yield Condition for Porous Metals

Based on a rigid-perfectly plastic analysis of spherically symmetric deformation around a spherical cavity, Gurson (1977) suggested a yield condition for porous metals in the form

$$f = J_2 + \frac{2}{3}\,\upsilon Y_0^2 \cosh\left(\frac{I_1}{2Y_0}\right) - (1+\upsilon^2)\frac{Y_0^2}{3} = 0, \tag{9.7.9}$$

where υ is the porosity (void/volume fraction), and $Y_0 = \text{const.}$ is the tensile yield stress of the matrix material (Fig. 9.17). Generalization to include hardening matrix material is also possible. The change in porosity during plastic deformation is given by the evolution equation

$$\dot{\upsilon} = (1-\upsilon)\,\mathrm{tr}\,\mathbf{D}^{\mathrm{p}}. \tag{9.7.10}$$

Other evolution equations, which take into account nucleation and growth of voids, have been considered (e.g., Tvergaard and Needleman, 1984). To improve its predictions and agreement with experimental data, Tvergaard (1982) introduced two additional material parameters in the structure of the Gurson yield criterion. Mear and Hutchinson (1985) incorporated the effects of anisotropic (kinematic) hardening by replacing J_2 in Eq. (9.7.9)

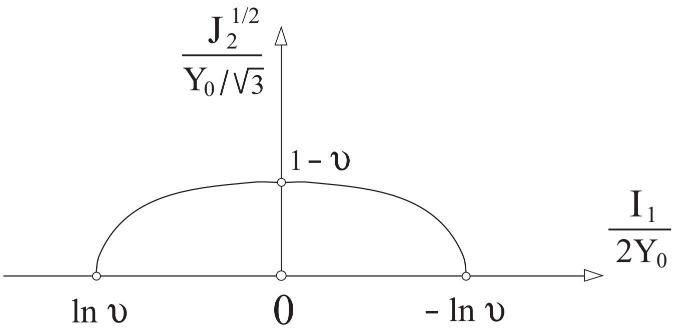

FIGURE 9.17. Gurson yield condition for porous metals with the void/volume fraction v. The tensile yield stress of the matrix material is Y_0.

with $(1/2)(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ defines the intersection of the current axis of the yield surface, parallel to hydrostatic axis, with the deviatoric plane. Yield functions and flow rules for porous pressure-dependent polymeric materials were analyzed by Lee and Oung (2000).

9.7.3. Constitutive Equations

The pressure-dependent yield conditions considered in two previous subsections are of the type

$$f(J_2, I_1, \mathcal{H}) = 0, \tag{9.7.11}$$

where $\mathcal{H}$ designates the appropriate history parameters. If it is assumed that the considered materials obey Ilyushin's postulate, the plastic part of the rate of deformation tensor is normal to the yield surface, and

$$\mathbf{D}^{\mathrm{p}} = \dot{\gamma}\,\frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad \frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{\partial f}{\partial J_2}\,\boldsymbol{\sigma}' + \frac{\partial f}{\partial I_1}\,\mathbf{I}. \tag{9.7.12}$$

The loading index can be expressed as

$$\dot{\gamma} = \frac{1}{H}\left(\frac{\partial f}{\partial J_2}\,\boldsymbol{\sigma}' + \frac{\partial f}{\partial I_1}\,\mathbf{I}\right) : \overset{\circ}{\underline{\boldsymbol{\tau}}}, \tag{9.7.13}$$

where H is an appropriate hardening modulus. The plastic part of the rate of deformation, corresponding to $\overset{\circ}{\underline{\boldsymbol{\tau}}}$, is again denoted by $\mathbf{D}^{\mathrm{p}}$. Substitution of Eq. (9.7.13) into Eq. (9.7.12), therefore, gives

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{H}\left[\left(\frac{\partial f}{\partial J_2}\,\boldsymbol{\sigma}' + \frac{\partial f}{\partial I_1}\,\mathbf{I}\right) \otimes \left(\frac{\partial f}{\partial J_2}\,\boldsymbol{\sigma}' + \frac{\partial f}{\partial I_1}\,\mathbf{I}\right)\right] : \overset{\circ}{\underline{\boldsymbol{\tau}}}. \tag{9.7.14}$$

The volumetric part of the plastic rate of deformation is

$$\operatorname{tr} \mathbf{D}^{\mathrm{p}} = \frac{3}{H} \frac{\partial f}{\partial I_1} \left(\frac{\partial f}{\partial J_2} \boldsymbol{\sigma}' + \frac{\partial f}{\partial I_1} \mathbf{I} \right) : \overset{\circ}{\underline{\boldsymbol{\tau}}}. \tag{9.7.15}$$

Geomaterials

For the Drucker–Prager yield condition,

$$\frac{\partial f}{\partial J_2} = \frac{1}{2} J_2^{-1/2}, \quad \frac{\partial f}{\partial I_1} = \frac{1}{3} \mu_*, \tag{9.7.16}$$

and

$$H = h_{\mathrm{t}}^{\mathrm{p}} = \frac{\mathrm{d}K}{\mathrm{d}\vartheta}, \quad \vartheta = \int_0^t (2\, \mathbf{D}^{\mathrm{p}\prime} : \mathbf{D}^{\mathrm{p}\prime})^{1/2} \, \mathrm{d}t. \tag{9.7.17}$$

The relationship $K = K(\vartheta)$ between the shear yield stress K, under given superimposed pressure, and the generalized shear plastic strain ϑ is assumed to be known. Note that $\dot{\vartheta} = \dot{\gamma}$.

Alternatively, the hardening modulus can be expressed as

$$H = \frac{1}{3} \left(1 - \frac{\mu_*}{\sqrt{3}} \right)^2 \frac{\mathrm{d}Y^-}{\mathrm{d}\vartheta}, \tag{9.7.18}$$

where Y^- is the yield stress in uniaxial compression. The generalized plastic strain is in this case defined by

$$\vartheta = \frac{1 - \mu_*/\sqrt{3}}{(1 + 2\mu_*^2/3)^{1/2}} \int_0^t \left(\frac{2}{3} \mathbf{D}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}} \right)^{1/2} \mathrm{d}t, \tag{9.7.19}$$

which coincides with the longitudinal strain in uniaxial compression test. The relationship between ϑ and γ is

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\gamma} = \frac{1}{\sqrt{3}} \left(1 - \frac{\mu_*}{\sqrt{3}} \right). \tag{9.7.20}$$

Porous Metals

For the Gurson yield condition we have

$$\frac{\partial f}{\partial J_2} = 1, \quad \frac{\partial f}{\partial I_1} = \frac{1}{3} \upsilon \, Y_0 \sinh\left(\frac{I_1}{2Y_0} \right), \tag{9.7.21}$$

and

$$H = \frac{2}{3} \upsilon(1 - \upsilon) Y_0^3 \sinh\left(\frac{I_1}{2Y_0} \right) \left[\upsilon - \cosh\left(\frac{I_1}{2Y_0} \right) \right]. \tag{9.7.22}$$

From Eqs. (9.7.10) and (9.7.12) it follows that the porosity evolves according to

$$\dot{\upsilon} = \dot{\gamma}\, \upsilon(1 - \upsilon) Y_0 \sinh\left(\frac{I_1}{2Y_0} \right). \tag{9.7.23}$$

Further analysis of inelastic deformation of porous materials can be found in Lee (1988), Cocks (1989), Qiu and Weng (1993), and Sun (1995).

9.8. Nonassociative Plasticity

Constitutive equations in which plastic part of the rate of strain is normal to a locally smooth yield surface $f_{(n)} = 0$ in the conjugate stress space,

$$\dot{\mathbf{E}}^{\mathrm{p}}_{(n)} = \dot{\gamma}_{(n)} \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}, \tag{9.8.1}$$

are referred to as the associative flow rules. As discussed in Section 8.5, a sufficient condition for this constitutive structure is that the material obeys Ilyushin's postulate. However, many pressure-dependent dilatant materials, with internal frictional effects, are not well described by associative flow rules. For example, associative flow rules largely overestimate inelastic volume changes in geomaterials like rocks and soils (Rudnicki and Rice, 1975; Rice, 1977), and in certain high-strength steels exhibiting the strength-differential effect by which the yield strength is higher in compression than in tension (Spitzig, Sober, and Richmond, 1975; Casey and Sullivan, 1985; Lee, 1988). For such materials, plastic part of the rate of strain is taken to be normal to the plastic potential surface

$$\pi_{(n)} = 0, \tag{9.8.2}$$

which is distinct from the yield surface

$$f_{(n)} = 0. \tag{9.8.3}$$

The resulting constitutive structure,

$$\dot{\mathbf{E}}^{\mathrm{p}}_{(n)} = \dot{\gamma}_{(n)} \frac{\partial \pi_{(n)}}{\partial \mathbf{T}_{(n)}}, \tag{9.8.4}$$

is known as the nonassociative flow rule (e.g., Mróz, 1963; Nemat-Nasser, 1983; Runesson and Mróz, 1989).

The consistency condition $\dot{f}_{(n)} = 0$ gives

$$\dot{\gamma}_{(n)} = \frac{1}{H_{(n)}} \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)}, \tag{9.8.5}$$

where $H_{(n)}$ is an appropriate hardening modulus. Thus,

$$\dot{\mathbf{E}}^{\mathrm{p}}_{(n)} = \frac{1}{H_{(n)}} \left(\frac{\partial \pi_{(n)}}{\partial \mathbf{T}_{(n)}} \otimes \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} \right) : \dot{\mathbf{T}}_{(n)}. \tag{9.8.6}$$

The overall constitutive structure is

$$\dot{\mathbf{E}}_{(n)} = \left[\mathbf{M}_{(n)} + \frac{1}{H_{(n)}}\left(\frac{\partial \pi_{(n)}}{\partial \mathbf{T}_{(n)}} \otimes \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}\right)\right] : \dot{\mathbf{T}}_{(n)}. \tag{9.8.7}$$

Since

$$\pi_{(n)} \neq f_{(n)}, \tag{9.8.8}$$

the elastoplastic compliance tensor in Eq. (9.8.7) does not possess a reciprocal symmetry. In an inverted form, the constitutive equation (9.8.7) becomes

$$\dot{\mathbf{T}}_{(n)} = \left[\mathbf{\Lambda}_{(n)} - \frac{1}{h_{(n)}}\left(\mathbf{\Lambda}_{(n)} : \frac{\partial \pi_{(n)}}{\partial \mathbf{T}_{(n)}}\right) \otimes \left(\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \mathbf{\Lambda}_{(n)}\right)\right] : \dot{\mathbf{E}}_{(n)}, \tag{9.8.9}$$

where

$$h_{(n)} = H_{(n)} + \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \mathbf{\Lambda}_{(n)} : \frac{\partial \pi_{(n)}}{\partial \mathbf{T}_{(n)}}\,. \tag{9.8.10}$$

9.8.1. Plastic Potential for Geomaterials

To better describe inelastic behavior of geomaterials whose yield is governed by the Drucker–Prager yield condition of Eq. (9.7.4), a nonassociative flow rule can be used with the plastic potential (Fig. 9.18)

$$\pi = J_2^{1/2} + \frac{1}{3}\,\beta\, I_1 - K = 0. \tag{9.8.11}$$

The material parameter β is in general different from the friction parameter μ_* of Eq. (9.7.4). Thus,

$$\mathbf{D}^{\mathrm{p}} = \dot{\gamma}\,\frac{\partial \pi}{\partial \boldsymbol{\sigma}} = \dot{\gamma}\left(\frac{1}{2}\,J_2^{-1/2}\boldsymbol{\sigma}' + \frac{1}{3}\,\beta\,\mathbf{I}\right). \tag{9.8.12}$$

The loading index $\dot{\gamma}$ is determined from the consistency condition. Assuming known the relationship

$$K = K(\vartheta) \tag{9.8.13}$$

between the shear yield stress and the generalized shear plastic strain ϑ, defined by Eq. (9.7.17), the condition $\dot{f} = 0$ gives

$$\dot{\gamma} = \frac{1}{H}\left(\frac{1}{2}\,J_2^{-1/2}\boldsymbol{\sigma}' + \frac{1}{3}\,\mu_*\,\mathbf{I}\right) : \overset{\circ}{\boldsymbol{\tau}}, \quad H = h_{\mathrm{t}}^{\mathrm{p}} = \frac{\mathrm{d}K}{\mathrm{d}\vartheta}\,. \tag{9.8.14}$$

Alternatively, assuming known the relationship

$$Y^- = Y^-(\vartheta) \tag{9.8.15}$$

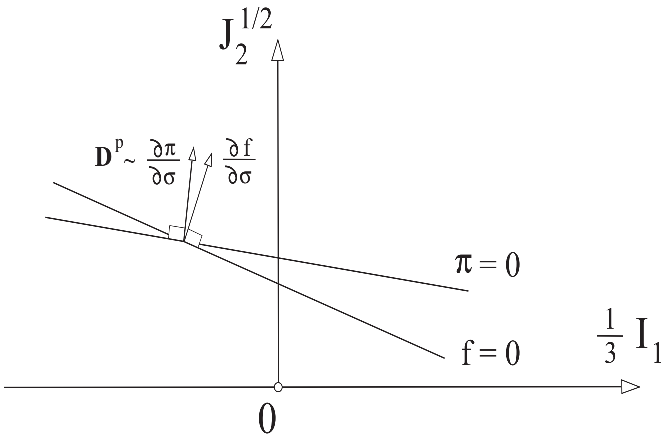

FIGURE 9.18. Illustration of a nonassociative flow rule. The plastic rate of deformation $\mathbf{D}^{\mathrm{p}}$ is normal to the flow potential $\pi = 0$, which is distinct from the yield surface $f = 0$.

between the yield stress in uniaxial compression and the generalized plastic strain

$$\vartheta = \frac{1 - \beta/\sqrt{3}}{(1 + 2\beta^2/3)^{1/2}} \int_0^t \left(\frac{2}{3}\, \mathbf{D}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}} \right)^{1/2} \mathrm{d}t, \tag{9.8.16}$$

the hardening modulus is

$$H = \frac{1}{3} \left(1 - \frac{\mu_*}{\sqrt{3}} \right) \left(1 - \frac{\beta}{\sqrt{3}} \right) \frac{\mathrm{d}Y^-}{\mathrm{d}\vartheta}\,. \tag{9.8.17}$$

The substitution of Eq. (9.8.14) into Eq. (9.8.12) gives

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{H} \left[\left(\frac{1}{2}\, J_2^{-1/2} \boldsymbol{\sigma}' + \frac{1}{3}\, \beta\, \mathbf{I} \right) \otimes \left(\frac{1}{2}\, J_2^{-1/2} \boldsymbol{\sigma}' + \frac{1}{3}\, \mu_*\, \mathbf{I} \right) \right] : \underline{\overset{\circ}{\boldsymbol{\tau}}}. \tag{9.8.18}$$

The deviatoric and spherical parts are

$$\mathbf{D}^{\mathrm{p}\prime} = \frac{1}{2H}\, \frac{\boldsymbol{\sigma}'}{J_2^{1/2}} \left(\frac{\boldsymbol{\sigma}' : \underline{\overset{\circ}{\boldsymbol{\tau}}}}{2\, J_2^{1/2}} + \frac{1}{3}\, \mu_*\, \mathrm{tr}\, \underline{\overset{\circ}{\boldsymbol{\tau}}} \right), \tag{9.8.19}$$

$$\mathrm{tr}\, \mathbf{D}^{\mathrm{p}} = \frac{\beta}{H} \left(\frac{\boldsymbol{\sigma}' : \underline{\overset{\circ}{\boldsymbol{\tau}}}}{2\, J_2^{1/2}} + \frac{1}{3}\, \mu_*\, \mathrm{tr}\, \underline{\overset{\circ}{\boldsymbol{\tau}}} \right). \tag{9.8.20}$$

To physically interpret the parameter β, we observe from Eq. (9.8.12) that

$$\left(2\, \mathbf{D}^{\mathrm{p}\prime} : \mathbf{D}^{\mathrm{p}\prime} \right)^{1/2} = \dot{\gamma}, \quad \mathrm{tr}\, \mathbf{D}^{\mathrm{p}} = \beta\, \dot{\gamma}, \tag{9.8.21}$$

i.e.,

$$\beta = \frac{\operatorname{tr} \mathbf{D}^{\mathrm{p}}}{\left(2\,\mathbf{D}^{\mathrm{p}\prime} : \mathbf{D}^{\mathrm{p}\prime}\right)^{1/2}} \,. \tag{9.8.22}$$

Thus, β is the ratio of the volumetric and shear part of the plastic rate of deformation, which is often called the dilatancy factor (Rudnicki and Rice, 1975). Representative values of the friction coefficient and the dilatancy factor for fissured rocks, listed by Rudnicki and Rice (*op. cit.*), indicate that

$$\mu_* = 0.3 \div 1, \quad \beta = 0.1 \div 0.5. \tag{9.8.23}$$

The frictional parameter and inelastic dilatancy of the material actually change with the progression of inelastic deformation, but are here treated as constants. For a more elaborate analysis, which accounts for their variation, the paper by Nemat-Nasser and Shokooh (1980) can be consulted. Note also that

$$\dot{\gamma} = \frac{\boldsymbol{\sigma} : \mathbf{D}^{\mathrm{p}\prime}}{J_2^{1/2}} \,. \tag{9.8.24}$$

The deviatoric and spherical parts of the total rate of deformation are, respectively,

$$\mathbf{D}' = \frac{\overset{\circ}{\underline{\boldsymbol{\tau}}}{}'}{2\mu} + \frac{1}{2H}\,\frac{\boldsymbol{\sigma}'}{J_2^{1/2}} \left(\frac{\boldsymbol{\sigma}' : \overset{\circ}{\underline{\boldsymbol{\tau}}}}{2\,J_2^{1/2}} + \frac{1}{3}\,\mu_* \operatorname{tr} \overset{\circ}{\underline{\boldsymbol{\tau}}} \right), \tag{9.8.25}$$

$$\operatorname{tr} \mathbf{D} = \frac{1}{3\kappa} \operatorname{tr} \overset{\circ}{\underline{\boldsymbol{\tau}}} + \frac{\beta}{H} \left(\frac{\boldsymbol{\sigma}' : \overset{\circ}{\underline{\boldsymbol{\tau}}}}{2\,J_2^{1/2}} + \frac{1}{3}\,\mu_* \operatorname{tr} \overset{\circ}{\underline{\boldsymbol{\tau}}} \right). \tag{9.8.26}$$

These can be inverted to give the deviatoric and spherical parts of the stress rate as

$$\overset{\circ}{\underline{\boldsymbol{\tau}}}{}' = 2\mu \left[\mathbf{D}' - \frac{1}{c}\,\frac{\boldsymbol{\sigma}'}{J_2^{1/2}} \left(\frac{\boldsymbol{\sigma}' : \mathbf{D}}{2\,J_2^{1/2}} + \mu_*\,\frac{\kappa}{2\mu} \operatorname{tr} \mathbf{D} \right) \right], \tag{9.8.27}$$

$$\operatorname{tr} \overset{\circ}{\underline{\boldsymbol{\tau}}} = \frac{3\kappa}{c} \left[\left(1 + \frac{H}{\mu} \right) \operatorname{tr} \mathbf{D} - \beta\,\frac{\boldsymbol{\sigma}' : \mathbf{D}}{J_2^{1/2}} \right], \tag{9.8.28}$$

where

$$c = 1 + \frac{H}{\mu} + \mu_* \beta\,\frac{\kappa}{\mu} \,. \tag{9.8.29}$$

If the friction coefficient μ_* is equal to zero, Eqs. (9.8.27) and (9.8.28) reduce to

$$\overset{\circ}{\underline{\boldsymbol{\tau}}}{}' = 2\mu \left[\mathbf{D}' - \frac{1}{1 + H/\mu}\,\frac{(\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}') : \mathbf{D}}{2\,J_2} \right], \tag{9.8.30}$$

$$\operatorname{tr} \overset{\circ}{\underline{\boldsymbol{\tau}}} = 3\kappa \left(\operatorname{tr} \mathbf{D} - \frac{\beta}{1 + H/\mu} \, \frac{\boldsymbol{\sigma}' : \mathbf{D}}{J_2^{1/2}} \right). \tag{9.8.31}$$

With a vanishing dilatancy factor ($\beta = 0$), Eqs. (9.8.30) and (9.8.31) coincide with the constitutive equations of isotropic hardening pressure-independent metal plasticity (Subsection 9.4.1). Other nonassociative models for geological materials are discussed by Desai and Hasmini (1989).

Constitutive Inequalities

Returning to Eq. (9.8.18), a trace product with $\overset{\circ}{\underline{\boldsymbol{\tau}}}$ gives

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} : \mathbf{D}^{\mathrm{p}} = \frac{1}{H} \left[\left(\frac{1}{2} \, J_2^{-1/2} \boldsymbol{\sigma}' + \frac{1}{3} \, \beta \, \mathbf{I} \right) : \overset{\circ}{\underline{\boldsymbol{\tau}}} \right] \left[\left(\frac{1}{2} \, J_2^{-1/2} \boldsymbol{\sigma}' + \frac{1}{3} \, \mu_* \, \mathbf{I} \right) : \overset{\circ}{\underline{\boldsymbol{\tau}}} \right]. \tag{9.8.32}$$

In the hardening range ($H > 0$), from Eq. (9.8.14) it follows that

$$\left(\frac{1}{2} \, J_2^{-1/2} \boldsymbol{\sigma}' + \frac{1}{3} \, \mu_* \, \mathbf{I} \right) : \overset{\circ}{\underline{\boldsymbol{\tau}}} > 0, \tag{9.8.33}$$

since $\dot{\gamma} > 0$. Thus, from Eq. (9.8.32) the sign of $\overset{\circ}{\underline{\boldsymbol{\tau}}} : \mathbf{D}^{\mathrm{p}}$ is determined by the sign of

$$\left(\frac{1}{2} \, J_2^{-1/2} \boldsymbol{\sigma}' + \frac{1}{3} \, \beta \, \mathbf{I} \right) : \overset{\circ}{\underline{\boldsymbol{\tau}}}. \tag{9.8.34}$$

Depending on the state of stress and the type of incipient loading, this can be either positive or negative. Therefore, in the framework of nonassociative plasticity, the quantity $\overset{\circ}{\underline{\boldsymbol{\tau}}} : \mathbf{D}^{\mathrm{p}}$ can be negative even in the hardening range. This is in contrast to associative plasticity, where $\overset{\circ}{\underline{\boldsymbol{\tau}}} : \mathbf{D}^{\mathrm{p}}$ is always positive in the hardening range, by Eq. (8.8.8). Similarly, $\overset{\circ}{\underline{\boldsymbol{\tau}}} : \mathbf{D}^{\mathrm{p}}$ can be positive in the softening range. Illustrative examples can be found in the article by Lubarda, Mastilovic, and Knap (1996).

The fact that $\overset{\circ}{\underline{\boldsymbol{\tau}}} : \mathbf{D}^{\mathrm{p}}$ can be negative in the hardening range does not necessarily imply that material becomes unstable. Whether an instability actually occurs at a given state of stress and material constitution is answered by a bifurcation-type analysis, such as used by Rudnicki and Rice (*op. cit.*). For example, they found that for certain states of stress, localization is possible even in the hardening range, for materials described by a

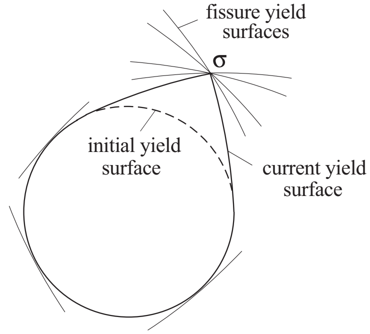

FIGURE 9.19. Macroscopic yield surface formed as an envelope of individual fissure yield surfaces. The yield vertex forms at the loading point due to sliding on favorably oriented fissure surfaces.

nonassociative flow rule. This is never the case for materials with an associative flow rule. Plastic instability and bifurcation analysis are considered in Chapter 10.

9.8.2. Yield Vertex Model for Fissured Rocks

In a brittle rock, modeled to contain a collection of randomly oriented fissures, inelastic deformation results from frictional sliding on the fissure surfaces. Inelastic dilatancy under overall compressive loads is a consequence of opening the fissures at asperities and local tensile fractures at some angle at the edges of fissures. Individual yield surface may be associated with each fissure. Expressed in terms of the resolved shear stress in the plane of fissure with normal $\mathbf{n}$, this is

$$\mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{m} + \mu_* \, \mathbf{n} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} = \text{const.}, \qquad (9.8.35)$$

where μ_* is the friction coefficient between the surfaces of the fissure, and $\mathbf{m}$ is the sliding direction (direction of the maximum shear stress in the plane of fissure). The macroscopic yield surface is the envelope of individual yield surfaces (Fig. 9.19) for fissures of all orientations (Rudnicki and Rice, 1975). This is similar to slip model of metal plasticity (Batdorf and Budiansky, 1949, 1954; Sanders, 1954; Hill, 1967b).

Continued stressing in the same direction will cause continuing sliding on favorably oriented (already activated) fissures, and will initiate sliding for a progressively greater number of orientations. After certain amount of inelastic deformation, the macroscopic yield envelope develops a vertex at the loading point. The stress increment normal to the original stress direction will initiate or continue sliding of fissure surfaces for some fissure orientations. In isotropic hardening idealization with a smooth yield surface, however, a stress increment tangential to the yield surface will cause only elastic deformation, overestimating the stiffness of the response. In order to take into account the effect of the yield vertex in an approximate way, Rudnicki and Rice (*op. cit.*) introduced a second plastic modulus H_1, which governs the response to part of the stress increment directed tangentially to what is taken to be the smooth yield surface through the same stress point (Fig. 9.20). Since no vertex formation is associated with hydrostatic stress increments, tangential stress increments are taken to be deviatoric, and Eq. (9.8.19) is replaced with

$$\mathbf{D}^{\mathrm{p}\prime} = \frac{1}{2H}\,\frac{\boldsymbol{\sigma}'}{J_2^{1/2}}\left(\frac{\boldsymbol{\sigma}' : \overset{\circ}{\underline{\boldsymbol{\tau}}}}{2\,J_2^{1/2}} + \frac{1}{3}\,\mu_*\,\mathrm{tr}\,\overset{\circ}{\underline{\boldsymbol{\tau}}}\right) + \frac{1}{2H_1}\left(\overset{\circ}{\underline{\boldsymbol{\tau}}}{}' - \frac{\boldsymbol{\sigma}' : \overset{\circ}{\underline{\boldsymbol{\tau}}}}{2\,J_2}\,\boldsymbol{\sigma}'\right). \quad (9.8.36)$$

The dilation induced by the small tangential stress increment is assumed to be negligible, so that Eq. (9.8.20) still applies for $\mathrm{tr}\,\mathbf{D}^{\mathrm{p}}$. The constitutive structure in Eq. (9.8.36) is intended to model the response at a yield surface vertex for small deviations from proportional ("straight ahead") loading $\overset{\circ}{\underline{\boldsymbol{\tau}}} \sim \boldsymbol{\sigma}'$.

The expressions for the rate of stress in terms of the rate of deformation are obtained by inversion of the expression for the rate of deformation corresponding to Eqs. (9.8.20) and (9.8.36). The results are

$$\overset{\circ}{\underline{\boldsymbol{\tau}}}{}' = 2\mu\left[\frac{1}{b}\,\mathbf{D}' - \frac{a}{bc}\,\frac{(\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}') : \mathbf{D}}{2\,J_2} - \frac{1}{c}\,\mu_*\,\frac{\kappa}{2\mu}\,\frac{\boldsymbol{\sigma}'}{J_2^{1/2}}\,\mathrm{tr}\,\mathbf{D}\right], \quad (9.8.37)$$

$$\mathrm{tr}\,\overset{\circ}{\underline{\boldsymbol{\tau}}} = \frac{3\kappa}{c}\left[\left(1 + \frac{H}{\mu}\right)\mathrm{tr}\,\mathbf{D} - \beta\,\frac{\boldsymbol{\sigma}' : \mathbf{D}}{J_2^{1/2}}\right]. \quad (9.8.38)$$

The parameters a and b are given by

$$a = 1 - \frac{H}{H_1} - \mu_*\beta\,\frac{\kappa}{H_1}\,, \quad b = 1 + \frac{\mu}{H_1}\,, \quad (9.8.39)$$

and c is defined by Eq. (9.8.29).

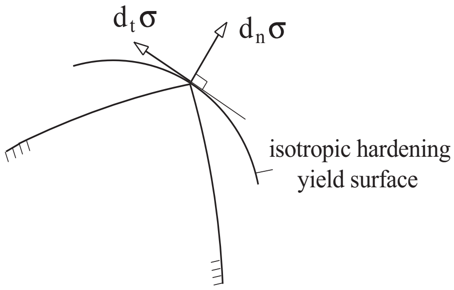

FIGURE 9.20. A stress increment from a yield vertex decomposed in the normal and tangential direction relative to an isotropic hardening smooth yield surface passing through the vertex. The tangential component $\mathrm{d_t}\sigma$ does not cause plastic flow for smooth yield idealization, but it does for the yield vertex.

Another model in which the plastic rate of deformation depends on the component of stress rate tangential to smooth yield surface was proposed by Hashiguchi (1993).

9.9. Thermoplasticity

Nonisothermal plasticity is considered in this section assuming that the temperature is not too high, so that creep deformation can be neglected. The analysis may also be adequate for certain applications under high stresses of short duration, where the temperature increase is more pronounced but the viscous (creep) strains have no time to develop (Prager, 1958; Kachanov, 1971). Thus, infinitesimal changes of stress and temperature applied to the material at a given state produce a unique infinitesimal change of strain, independently of the speed with which these changes are made. Rate-dependent plasticity will be considered in Section 9.10.

The formulation of thermoplastic analysis under described conditions can proceed by introducing a nonisothermal yield condition in either stress or strain space. For example, the yield function in stress space is defined by

$$f_{(n)}\left(\mathbf{T}_{(n)}, \theta, \mathcal{H}\right) = 0, \tag{9.9.1}$$

where θ is the temperature, and $\mathcal{H}$ is the pattern of internal rearrangements. The response within the yield surface is thermoelastic. If the Gibbs energy per unit reference volume, relative to selected stress and strain measures, is

$$\Phi_{(n)} = \Phi_{(n)}\left(\mathbf{T}_{(n)}, \theta, \mathcal{H}\right), \tag{9.9.2}$$

the strain is

$$\mathbf{E}_{(n)} = \frac{\partial \Phi_{(n)}}{\partial \mathbf{T}_{(n)}}. \tag{9.9.3}$$

Consider the stress state $\mathbf{T}_{(n)}$ on the current yield surface. The rates of stress and temperature associated with thermoplastic loading satisfy the consistency condition $\dot{f}_{(n)} = 0$, which gives

$$\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)} + \frac{\partial f_{(n)}}{\partial \theta} : \dot{\theta} - H_{(n)}\,\dot{\gamma}_{(n)} = 0. \tag{9.9.4}$$

The hardening parameter is

$$H_{(n)} = H_{(n)}\left(\mathbf{T}_{(n)}, \theta, \mathcal{H}\right), \tag{9.9.5}$$

and the loading index

$$\dot{\gamma}_{(n)} > 0. \tag{9.9.6}$$

Three types of thermoplastic response are possible,

$$\begin{aligned}
&H_{(n)} > 0, \quad \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)} + \frac{\partial f_{(n)}}{\partial \theta} : \dot{\theta} > 0 \quad \text{thermoplastic hardening},\\
&H_{(n)} < 0, \quad \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)} + \frac{\partial f_{(n)}}{\partial \theta} : \dot{\theta} < 0 \quad \text{thermoplastic softening},\\
&H_{(n)} = 0, \quad \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)} + \frac{\partial f_{(n)}}{\partial \theta} : \dot{\theta} = 0 \quad \text{ideally thermoplastic}.
\end{aligned} \tag{9.9.7}$$

This parallels the isothermal classification of Eq. (9.2.8).

Since rate-independence is assumed, the constitutive relationship of thermoplasticity must be homogeneous of degree one in the rates of stress, strain and temperature. For thermoplastic part of the rate of strain this is satisfied by the normality structure

$$\dot{\mathbf{E}}^{\mathrm{p}}_{(n)} = \dot{\gamma}_{(n)}\,\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}. \tag{9.9.8}$$

In view of Eq. (9.9.4), this becomes

$$\dot{\mathbf{E}}^{\mathrm{p}}_{(n)} = \frac{1}{H_{(n)}}\left(\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)} + \frac{\partial f_{(n)}}{\partial \theta} : \dot{\theta}\right)\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}. \tag{9.9.9}$$

The strain rate is the sum of thermoelastic and thermoplastic parts,

$$\dot{\mathbf{E}}_{(n)} = \dot{\mathbf{E}}^{\mathrm{e}}_{(n)} + \dot{\mathbf{E}}^{\mathrm{p}}_{(n)}. \tag{9.9.10}$$

The thermoelastic part is governed by

$$\dot{\mathbf{E}}^{\mathrm{e}}_{(n)} = \frac{\partial^2 \Phi_{(n)}}{\partial \mathbf{T}_{(n)} \otimes \partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)} + \frac{\partial^2 \Phi_{(n)}}{\partial \mathbf{T}_{(n)} \partial \theta}\, \dot{\theta}. \tag{9.9.11}$$

For example, if the Gibbs energy is taken to be

$$\begin{aligned}\Phi_{(n)} = \frac{1}{4\mu_{(n)}} \left(\operatorname{tr} \mathbf{T}^2_{(n)} - \frac{\lambda_{(n)}}{3\lambda_{(n)} + 2\mu_{(n)}} \operatorname{tr}^2 \mathbf{T}_{(n)} \right) \\ + \alpha_{(n)}(\theta) \operatorname{tr} \mathbf{T}_{(n)} + \beta_{(n)}(\theta, \mathcal{H}),\end{aligned} \tag{9.9.12}$$

we obtain

$$\dot{\mathbf{E}}^{\mathrm{e}}_{(n)} = \frac{1}{2\mu_{(n)}} \left(\boldsymbol{I} - \frac{\lambda_{(n)}}{2\mu_{(n)} + 3\lambda_{(n)}} \mathbf{I} \otimes \mathbf{I} \right) : \dot{\mathbf{T}}_{(n)} + \alpha'_{(n)}(\theta)\, \dot{\theta}\, \mathbf{I}. \tag{9.9.13}$$

The Lamé type elastic constants corresponding to selected stress and strain measures are $\lambda_{(n)}$ and $\mu_{(n)}$. The scalar function $\alpha_{(n)}$ is an appropriate function of the temperature. Its temperature gradient is $\alpha'_{(n)} = \mathrm{d}\alpha_{(n)}/\mathrm{d}\theta$.

9.9.1. Isotropic and Kinematic Hardening

Suppose that a nonisothermal yield condition in the Cauchy stress space is a temperature-dependent von Mises condition

$$f = \frac{1}{2} \boldsymbol{\sigma}' : \boldsymbol{\sigma}' - [\varphi(\theta)\, K(\vartheta)]^2 = 0. \tag{9.9.14}$$

The thermoplastic part of the deformation rate is then

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2\varphi h^{\mathrm{p}}_{\mathrm{t}}} \left(\frac{\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}'}{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} : \overset{\circ}{\underline{\boldsymbol{\tau}}} - \boldsymbol{\sigma}' \frac{\varphi'}{\varphi}\, \dot{\theta} \right), \tag{9.9.15}$$

where

$$h^{\mathrm{p}}_{\mathrm{t}} = \frac{\mathrm{d}K}{\mathrm{d}\vartheta}, \quad \varphi' = \frac{\mathrm{d}\varphi}{\mathrm{d}\theta}. \tag{9.9.16}$$

Combining Eqs. (9.9.13) and (9.9.15), the total rate of deformation becomes

$$\begin{aligned}\mathbf{D} = \left[\frac{1}{2\mu} \left(\boldsymbol{I} - \frac{\lambda}{2\mu + 3\lambda} \mathbf{I} \otimes \mathbf{I} \right) + \frac{1}{2\varphi h^{\mathrm{p}}_{\mathrm{t}}} \frac{\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}'}{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} \right] : \overset{\circ}{\underline{\boldsymbol{\tau}}} \\ + \left[\alpha'(\theta)\, \mathbf{I} - \frac{\varphi'}{2\varphi^2 h^{\mathrm{p}}_{\mathrm{t}}} \boldsymbol{\sigma}' \right] \dot{\theta}.\end{aligned} \tag{9.9.17}$$

The inverse constitutive equation for the stress rate is

$$\begin{aligned}\overset{\circ}{\underline{\boldsymbol{\tau}}} = \left(\lambda\, \mathbf{I} \otimes \mathbf{I} + 2\mu\, \boldsymbol{I} - \frac{2\mu}{1 + \varphi h^{\mathrm{p}}_{\mathrm{t}}/\mu} \frac{\boldsymbol{\sigma}' \otimes \boldsymbol{\sigma}'}{\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} \right) : \mathbf{D} \\ - \left[(3\lambda + 2\mu)\, \alpha'\, \mathbf{I} - \frac{1}{1 + \varphi h^{\mathrm{p}}_{\mathrm{t}}/\mu} \frac{\varphi'}{\varphi} \boldsymbol{\sigma}' \right] \dot{\theta}.\end{aligned} \tag{9.9.18}$$

This can be viewed as a generalization of an infinitesimal strain formulation for a rigid-thermoplastic material, given by Prager (1958). See also Boley and Weiner (1960), Drucker (1960), Lee and Wierzbicki (1967), Lee (1969), Lubarda (1986,1989), and Naghdi (1960,1990).

In the case of thermoplasticity with linear kinematic hardening ($c = 2h_{\mathrm{t}}^{\mathrm{p}}$), and the temperature-dependent yield surface

$$f = \frac{1}{2}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) - [\varphi(\theta)K]^2 = 0, \quad K = \text{const.}, \tag{9.9.19}$$

the thermoplastic rate of deformation is

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\left[\frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\sigma}' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})} : \overset{\circ}{\underline{\boldsymbol{\tau}}} - \frac{\varphi'}{\varphi}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha})\,\dot{\theta}\right]. \tag{9.9.20}$$

Thermoelastic portion of the rate of deformation is as in Eq. (9.9.17), so that inversion of the expression for the total rate of deformation gives

$$\begin{aligned}\overset{\circ}{\underline{\boldsymbol{\tau}}} = &\left[\lambda\,\mathbf{I} \otimes \mathbf{I} + 2\mu\,\boldsymbol{I} - \frac{2\mu}{1 + h_{\mathrm{t}}^{\mathrm{p}}/\mu}\,\frac{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) \otimes (\boldsymbol{\sigma}' - \boldsymbol{\alpha})}{(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})}\right] : \mathbf{D} \\ &- \left[(3\lambda + 2\mu)\,\alpha'\,\mathbf{I} - \frac{1}{1 + h_{\mathrm{t}}^{\mathrm{p}}/\mu}\,\frac{\varphi'}{\varphi}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha})\right]\dot{\theta}.\end{aligned} \tag{9.9.21}$$

Additional analysis of the rate-type constitutive equations of thermoplasticity was presented by Green and Naghdi (1965), De Boer (1977), Lehmann (1985), Zdebel and Lehmann (1987), Wang and Ohno (1991), McDowell (1992), Lucchesi and Šilhavý (1993), and Casey (1998). Experimental investigations of nonisothermal yield surfaces were reported by Phillips (1974, 1982), and others.

9.10. Rate-Dependent Plasticity

There are two types of constitutive equations used in modeling the rate-dependent plastic response of metals and alloys. In one approach, there is no yield surface in the model and plastic deformation commences from the onset of loading, although it may be exceedingly small below certain levels of applied stress. This type of modeling is particularly advocated by researchers in materials science, who view inelastic deformation process as inherently time-dependent. For example, this view is supported by the dislocation dynamics study of crystallographic slip in metals, as reported by Johnston and Gilman (1959). Since there is no separation of time-independent and creep effects, the modeling is often referred to as a unified creep–plasticity theory (Hart, 1970; Bodner and Partom, 1975; Miller, 1976,1987; Krieg, 1977; Estrin and Mecking, 1986). The second approach uses the notion of the static yield surface and dynamic loading surface, and is referred to as a viscoplastic modeling.

In his analysis of rate-dependent behavior of metals, Rice (1970,1971) showed that the plastic rate of strain can be derived from a scalar flow potential $\Omega_{(n)}$, as its gradient

$$\dot{\mathbf{E}}^{\mathrm{p}}_{(n)} = \frac{\partial \Omega_{(n)}\left(\mathbf{T}_{(n)}, \theta, \mathcal{H}\right)}{\partial \mathbf{T}_{(n)}}\,, \tag{9.10.1}$$

provided that the rate of shearing on any given slip system within a crystalline grain depends on local stresses only through the resolved shear stress. The history of deformation is represented by the pattern of internal rearrangements $\mathcal{H}$, and the absolute temperature is θ (Section 4.5). Geometrically, the plastic part of the strain rate is normal to surfaces of constant flow potential in stress space (see also Section 8.4). There is no yield surface in the model, and plastic deformation commences from the onset of loading. Time-independent behavior can be recovered, under certain idealizations – neglecting creep and rate effects, as an appropriate limit. In this limit, at each instant of deformation there is a range of stress space over which the flow potential is constant. The current yield surface is then a boundary of this range, a singular clustering of all surfaces of constant flow potential.

9.10.1. Power-Law and Johnson–Cook Models

The power-law representation of the flow potential in the Cauchy stress space is

$$\Omega = \frac{2\dot{\gamma}^0}{m+1}\left(\frac{J_2^{1/2}}{K}\right)^m J_2^{1/2}, \quad J_2 = \frac{1}{2}\,\boldsymbol{\sigma}' : \boldsymbol{\sigma}', \tag{9.10.2}$$

where $K = K(\theta, \mathcal{H})$ is the reference shear stress, $\dot{\gamma}^0$ is the reference shear strain rate to be selected for each material, and m is the material parameter (of the order of 100 for metals at room temperature and strain rates below $10^4\ \mathrm{s}^{-1}$; Nemat-Nasser, 1992). The corresponding plastic part of the rate of deformation is

$$\mathbf{D}^{\mathrm{p}} = \dot{\gamma}^0\left(\frac{J_2^{1/2}}{K}\right)^m \frac{\boldsymbol{\sigma}'}{J_2^{1/2}}\,. \tag{9.10.3}$$

The equivalent plastic strain ϑ, defined by Eq. (9.4.3), is usually used as the only history parameter, and the reference shear stress depends on ϑ and θ according to

$$K = K^0 \left(1 + \frac{\vartheta}{\vartheta^{\,0}}\right)^{\alpha} \exp\left(-\beta\, \frac{\theta - \theta_0}{\theta_{\rm m} - \theta_0}\right). \tag{9.10.4}$$

Here, K^0 and ϑ^0 are the normalizing stress and strain, θ_0 and $\theta_{\rm m}$ are the room and melting temperatures, and α and β are the material parameters. The total rate of deformation is

$$\mathbf{D} = \underline{\mathbf{M}} : \overset{\circ}{\underline{\boldsymbol{\tau}}} + \dot{\gamma}^0 \left(\frac{J_2^{1/2}}{K}\right)^m \frac{\boldsymbol{\sigma}\,'}{J_2^{1/2}}\,. \tag{9.10.5}$$

The instantaneous elastic compliance tensor $\underline{\mathbf{M}}$ is defined, for infinitesimal elasticity, by Eq. (9.4.16). From the onset of loading the deformation rate consists of elastic and plastic constituents, although for large m the plastic contribution may be small if J_2 is less than K. The inverted form of (9.10.5), expressing $\overset{\circ}{\underline{\boldsymbol{\tau}}}$ in terms of $\mathbf{D}$, is

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} = \underline{\boldsymbol{\Lambda}} : \mathbf{D} - 2\mu\dot{\gamma}^0 \left(\frac{J_2^{1/2}}{K}\right)^m \frac{\boldsymbol{\sigma}\,'}{J_2^{1/2}}\,, \tag{9.10.6}$$

where $\underline{\boldsymbol{\Lambda}} = \underline{\mathbf{M}}^{-1}$. The elastic shear modulus is μ.

Another representation of the flow potential, constructed according to Johnson–Cook (1983) model, is

$$\Omega = \frac{2\dot{\gamma}^0}{a}\, K \exp\left[a\left(\frac{J_2^{1/2}}{K} - 1\right)\right]. \tag{9.10.7}$$

The reference shear stress is

$$K = K^0 \left[1 + b\left(\frac{\vartheta}{\vartheta^{\,0}}\right)^c\right]\left[1 - \left(\frac{\theta - \theta_0}{\theta_{\rm m} - \theta_0}\right)^d\right], \tag{9.10.8}$$

where a, b, c, d are the material parameters. The corresponding plastic part of the rate of deformation becomes

$$\mathbf{D}^{\rm p} = \dot{\gamma}^0 \,\exp\left[a\left(\frac{J_2^{1/2}}{K} - 1\right)\right] \frac{\boldsymbol{\sigma}\,'}{J_2^{1/2}}\,. \tag{9.10.9}$$

Similar expressions can be obtained for other models and the choices of the flow potential (e.g., Zerilli and Armstrong, 1987; see also a section on the physically based constitutive equations in the review by Meyers, 1999). Since there is no yield surface and loading/unloading criteria, some authors refer to these constitutive models as nonlinearly viscoelastic models (e.g.,

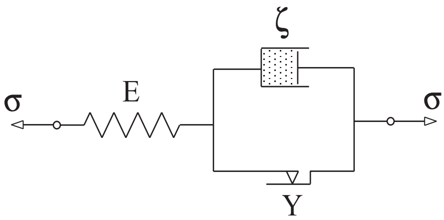

FIGURE 9.21. One-dimensional rheological model of elastic-viscoplastic response. The elastic modulus is E, the viscosity coefficient is ζ, and the yield stress of plastic element is Y.

Bardenhagen, Stout, and Gray, 1997). By selecting an appropriate large value of the parameter m, however, these rate-dependent models are able to reproduce almost rate-independent behavior. The function x^m in that sense can be considered to be a regularizing function (x stands for $J_2^{1/2}/K$). Other examples of regularizing functions are $\tanh(x/m)$, and $[\exp(x)-1]^m$.

9.10.2. Viscoplasticity Models

For high strain rate applications in dynamic plasticity (e.g., Cristescu, 1967; Cristescu and Suliciu, 1982; Clifton, 1983, 1985) viscoplastic models are often used. One dimensional rheological model of viscoplastic response is shown in Fig. 9.21. There are two surfaces in viscoplastic modeling, a static yield surface and a dynamic loading surface. Consider a simple model of J_2 viscoplasticity. The flow potential can be taken as

$$\Omega = \frac{1}{\zeta}\langle\, J_2^{1/2} - K_{\rm s}(\vartheta)\,\rangle^2, \tag{9.10.10}$$

where ζ is the viscosity coefficient, and $K_{\rm s}(\vartheta)$ represents the shear stress – plastic strain relationship from the (quasi) static shear test. The Macauley brackets are used, such that

$$\langle\psi\rangle = \begin{cases} \psi, & \text{if } \psi \geq 0, \\ 0, & \text{if } \psi < 0, \end{cases} \tag{9.10.11}$$

i.e., $\langle\psi\rangle = (\psi + |\psi|)/2$. The positive difference

$$J_2^{1/2} - K_{\rm s}(\vartheta) \tag{9.10.12}$$

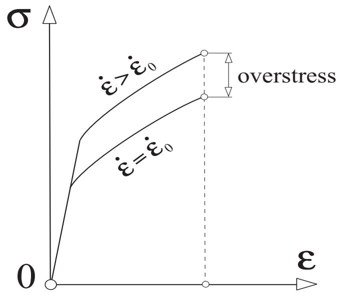

FIGURE 9.22. Stress-strain curves for quasi-static and dynamic loading conditions. The overstress measure is the difference between dynamic and static stress at a given amount of strain.

between the measures of the current dynamic stress and corresponding static stress (at a given level of equivalent plastic strain ϑ) is known as the overstress measure (Sokolovskii, 1948; Malvern, 1951). This is illustrated for uniaxial loading in Fig. 9.22. The plastic part of the rate of deformation is

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{\zeta}\left[J_2^{1/2} - K_{\mathrm{s}}(\vartheta)\right]\frac{\boldsymbol{\sigma}'}{J_2^{1/2}}, \quad J_2^{1/2} - K_{\mathrm{s}}(\vartheta) > 0. \tag{9.10.13}$$

In the case of uniaxial loading σ with static yield stress $\sigma_Y > 0$, above gives

$$D^{\mathrm{p}} = \sqrt{\frac{2}{3}}\,\frac{1}{\zeta}\,(\sigma - \mathcal{P}\sigma), \quad \mathcal{P}\sigma = \sigma_Y\,\mathrm{sign}(\sigma). \tag{9.10.14}$$

This encompasses both tensile and compressive loading. When the operator $\mathcal{P}$ is applied to axial stress, it maps a tensile stress $\sigma > 0$ to σ_Y, and a compressive stress $\sigma < 0$ to $-\sigma_Y$ (Duvaut and Lions, 1976; Simo and Hughes, 1998).

The inverted form of Eq. (9.10.13) is

$$\boldsymbol{\sigma}' = \zeta\,\mathbf{D}^{\mathrm{p}} + 2K_{\mathrm{s}}(\vartheta)\,\frac{\mathbf{D}^{\mathrm{p}}}{(2\mathbf{D}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}})^{1/2}}, \tag{9.10.15}$$

which shows that the rate-dependence in the model comes from the first term on the right-hand side only. In quasi-static tests, viscosity ζ is taken to be equal to zero, and Eq. (9.10.15) reduces to time-independent, von Mises isotropic hardening plasticity. In this case, the flow potential Ω is constant within the elastic range bounded by the yield surface $J_2^{1/2} = K_{\mathrm{s}}(\vartheta)$. The total rate of deformation is obtained by adding to (9.10.13) the elastic part of rate of deformation, such that

$$\mathbf{D} = \underline{\mathbf{M}} : \overset{\circ}{\boldsymbol{\tau}} + \frac{1}{\zeta}\left[J_2^{1/2} - K_{\mathrm{s}}(\vartheta)\right]\frac{\boldsymbol{\sigma}'}{J_2^{1/2}}\,. \tag{9.10.16}$$

The inverted form of (9.10.16), in the case of infinitesimal elastic strain, is

$$\overset{\circ}{\boldsymbol{\tau}} = \underline{\boldsymbol{\Lambda}} : \mathbf{D} - \frac{2\mu}{\zeta}\left[J_2^{1/2} - K_{\mathrm{s}}(\vartheta)\right]\frac{\boldsymbol{\sigma}'}{J_2^{1/2}}\,, \tag{9.10.17}$$

where μ is the elastic shear modulus.

Perzyna Model

More general representation for Ω is obtained by using the Perzyna (1963, 1966) viscoplastic model. For example, by taking

$$\Omega = \frac{C}{m+1}\langle\, f(\boldsymbol{\sigma}) - K_{\mathrm{s}}(\vartheta)\,\rangle^{m+1}, \tag{9.10.18}$$

we obtain

$$\mathbf{D}^{\mathrm{p}} = C\,[f(\boldsymbol{\sigma}) - K_{\mathrm{s}}(\vartheta)]^m\,\frac{\partial f}{\partial \boldsymbol{\sigma}}\,, \quad f(\boldsymbol{\sigma}) - K_{\mathrm{s}}(\vartheta) > 0. \tag{9.10.19}$$

If

$$f = J_2^{1/2}, \quad C = \frac{2}{\zeta}, \quad K_{\mathrm{s}}(\vartheta) = K^0 = \mathrm{const.}, \tag{9.10.20}$$

Equation (9.10.19) gives

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{\zeta}\left(J_2^{1/2} - K^0\right)^m\frac{\boldsymbol{\sigma}'}{J_2^{1/2}}\,. \tag{9.10.21}$$

This is is a generalization of the nonlinear Bingham model (e.g., Shames and Cozzarelli, 1992). In the case when

$$K_{\mathrm{s}}(\vartheta) = 0, \quad f = J_2^{1/2}, \quad C = \frac{2\dot{\gamma}^0}{K^m}\,, \tag{9.10.22}$$

Equation (9.10.19) reproduces the power-law J_2 creep of Eq. (9.10.3). See also Eisenberg and Yen (1981), and Bammann and Krieg (1987). The rate-dependent inelastic deformation of porous materials was studied by Duva and Hutchinson (1984), Haghi and Anand (1992), and Leblond, Perrin, and Suquet (1994).

Viscoplasticity with Isotropic–Kinematic Hardening

Other generalizations of Eq. (9.10.13) are possible. For example, suppose that the static yield condition is of a combined, isotropic–kinematic hardening type. The center of the yield surface is the back stress $\boldsymbol{\alpha}$ and the current radius of the yield surface is $K_\alpha(\vartheta)$. The dynamic loading condition is then

$$\hat{f} = \frac{1}{2}\,(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha}) - \hat{K}^2 = 0, \tag{9.10.23}$$

where $\hat{K}$ is the current radius of the loading surface. Consequently, the plastic rate of deformation becomes

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{\zeta}\,\langle\, \|\boldsymbol{\sigma}' - \boldsymbol{\alpha}\| - \sqrt{2}\,K_\alpha \rangle\, \frac{\boldsymbol{\sigma}' - \boldsymbol{\alpha}}{\|\boldsymbol{\sigma}' - \boldsymbol{\alpha}\|}\,. \tag{9.10.24}$$

For convenience, we introduced the norm

$$\|\boldsymbol{\sigma}' - \boldsymbol{\alpha}\| = [(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) : (\boldsymbol{\sigma}' - \boldsymbol{\alpha})]^{1/2} = \sqrt{2}\,\hat{K}. \tag{9.10.25}$$

An accompanying evolution equation for the back stress $\boldsymbol{\alpha}$ is usually of the type given by Eq. (9.4.49). The viscosity parameter ζ can be a function of the introduced state variables. The potential function Ω, associated with Eq. (9.10.24), is

$$\Omega = \frac{1}{2\zeta}\,\langle\, \|\boldsymbol{\sigma}' - \boldsymbol{\alpha}\| - \sqrt{2}\,K_\alpha \rangle^2, \tag{9.10.26}$$

such that $\mathbf{D}^{\mathrm{p}} = \partial\Omega/\partial\boldsymbol{\sigma}$. Since

$$\mathbf{D}^{\mathrm{p}} = \|\mathbf{D}^{\mathrm{p}}\|\, \frac{\boldsymbol{\sigma}' - \boldsymbol{\alpha}}{\|\boldsymbol{\sigma}' - \boldsymbol{\alpha}\|}\,, \tag{9.10.27}$$

comparison with (9.10.24) identifies

$$\|\mathbf{D}^{\mathrm{p}}\| = \frac{1}{\zeta}\,\langle\, \|\boldsymbol{\sigma}' - \boldsymbol{\alpha}\| - \sqrt{2}\,K_\alpha \rangle. \tag{9.10.28}$$

Thus, the connection

$$\Omega = \frac{\zeta}{2}\,\|\mathbf{D}^{\mathrm{p}}\|^2\,. \tag{9.10.29}$$

The deviatoric symmetric tensor

$$\mathbf{d}^{\mathrm{p}} = \frac{\mathbf{D}^{\mathrm{p}}}{\|\mathbf{D}^{\mathrm{p}}\|} \tag{9.10.30}$$

has, in general, four independent components (since $\|\mathbf{d}^{\mathrm{p}}\| = 1$). The representation $\mathbf{D}^{\mathrm{p}} = \|\mathbf{D}^{\mathrm{p}}\|\,\mathbf{d}^{\mathrm{p}}$ is referred to as the polar representation of $\mathbf{D}^{\mathrm{p}}$ (Van Houtte, 1994).

More general expressions for the plastic rate of deformation have also been employed in the studies of viscoplastic response. Representative references include Chaboche (1989,1993,1996), Bammann (1990), McDowell (1992), and Freed and Walker (1991,1993). Nonassociative viscoplastic flow rules were considered by Marin and McDowell (1996), and for geomaterials by Cristescu (1994), who also gives the reference to other related work.

Generalized Duvaut–Lions Formulation

According to this model, the viscoplastic rate of deformation is postulated to be

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{t_{\mathrm{d}}}\,\boldsymbol{\mathcal{M}} : (\boldsymbol{\sigma}' - \boldsymbol{\beta}), \quad f(\boldsymbol{\sigma}') \geq 0, \tag{9.10.31}$$

where t_{d} is the relaxation time, and $\boldsymbol{\mathcal{M}}$ is the elastic compliance tensor. For an isotropic material,

$$\boldsymbol{\mathcal{M}} = \frac{1}{2\mu}\,\boldsymbol{J} + \frac{1}{3\kappa}\,\boldsymbol{K}, \tag{9.10.32}$$

where μ and κ are the elastic shear and bulk moduli. The base tensors $\boldsymbol{J}$ and $\boldsymbol{K}$ sum to give the fourth-order unit tensor, $\boldsymbol{J} + \boldsymbol{K} = \boldsymbol{I}$, as discussed following Eq. (9.1.36). The deviatoric rest stress $\boldsymbol{\beta}$ in Eq. (9.10.31) is the stress corresponding to the inviscid solution, which satisfies the static yield condition $f(\boldsymbol{\beta}) = 0$. The rest stress is determined from the actual stress $\boldsymbol{\sigma}$ by the closest-point projection

$$\boldsymbol{\beta} = \boldsymbol{P} : \boldsymbol{\sigma}. \tag{9.10.33}$$

For example, if the operator $\boldsymbol{P}$ is defined by

$$\boldsymbol{P} = \sqrt{\frac{2}{3}}\,\sigma_Y\,\frac{\boldsymbol{J}}{\|\boldsymbol{\sigma}'\|}, \quad \|\boldsymbol{\sigma}'\| = (\boldsymbol{\sigma}' : \boldsymbol{\sigma}')^{1/2}, \tag{9.10.34}$$

there follows

$$\boldsymbol{\beta} = \sqrt{\frac{2}{3}}\,\sigma_Y\,\frac{\boldsymbol{\sigma}'}{\|\boldsymbol{\sigma}'\|}. \tag{9.10.35}$$

This corresponds to the static yield condition of the J_2 perfect plasticity, which is

$$f(\boldsymbol{\beta}) = \|\boldsymbol{\beta}\| - \sqrt{\frac{2}{3}}\,\sigma_Y = 0, \quad \sigma_Y = \text{const}. \tag{9.10.36}$$

The substitution of Eq. (9.10.32) into Eq. (9.10.31) gives the constitutive structure

$$\mathbf{D}^{\mathrm{p}} = \frac{1}{\zeta_{\mathrm{d}}}\,(\boldsymbol{\sigma}' - \boldsymbol{\beta}), \tag{9.10.37}$$

where

$$\zeta_{\mathrm{d}} = 2\mu t_{\mathrm{d}} > 0 \tag{9.10.38}$$

is the viscosity coefficient. Further analysis of the generalized Duvaut–Lions model and its numerical implementation can be found in the book by Simo and Hughes (1998). See also Krempl (1996), and Lubarda and Benson (2001).

Viscosity Tensor

The second-order viscosity tensor can be introduced as

$$\boldsymbol{Z} = \zeta_{\mathrm{d}}\, \boldsymbol{J} + \zeta_{\mathrm{v}}\, \boldsymbol{K}, \tag{9.10.39}$$

where ζ_{d} and ζ_{v} are the shear and bulk viscosities. The plastic rate of deformation of the generalized Duvaut–Lions model is then

$$\mathbf{D}^{\mathrm{p}} = \boldsymbol{Z}^{-1} : (\boldsymbol{\sigma}' - \boldsymbol{\beta}). \tag{9.10.40}$$

Introducing further the relaxation time tensor,

$$\boldsymbol{T} = t_{\mathrm{d}}\, \boldsymbol{J} + t_{\mathrm{v}}\, \boldsymbol{K}, \tag{9.10.41}$$

we have the connection

$$\boldsymbol{Z}^{-1} = \boldsymbol{T}^{-1} : \boldsymbol{\mathcal{M}}. \tag{9.10.42}$$

In particular, the relaxation time and viscosity coefficients are related by

$$\zeta_{\mathrm{d}} = 2\mu t_{\mathrm{d}}, \quad \zeta_{\mathrm{v}} = 3\kappa t_{\mathrm{v}}. \tag{9.10.43}$$

9.11. Deformation Theory of Plasticity

Simple plasticity theory has been suggested for proportional loading and small deformation by Hencky (1924) and Ilyushin (1947,1963). A large deformation version of this theory is here presented. It is convenient to cast the formulation by using the logarithmic strain

$$\mathbf{E}_{(0)} = \ln \mathbf{U}, \tag{9.11.1}$$

and its conjugate stress $\mathbf{T}_{(0)}$. The left stretch tensor is $\mathbf{U}$. Assume that the loading is such that all stress components increase proportionally, i.e.,

$$\mathbf{T}_{(0)} = c(t)\, \mathbf{T}^{*}_{(0)}, \tag{9.11.2}$$

where $\mathbf{T}^{*}_{(0)}$ is the stress tensor at an instant t^*, and $c(t)$ is a monotonically increasing function of t, with $c(t^*) = 1$. Evidently, Eq. (9.11.2) implies that the principal directions of $\mathbf{T}_{(0)}$ remain fixed during the deformation process, and parallel to those of $\mathbf{T}^{*}_{(0)}$.

Since the stress components proportionally increase, and no elastic unloading takes place, it is reasonable to assume that elastoplastic response can be described macroscopically by the constitutive structure of nonlinear elasticity, in which the total strain is a function of the total stress. Thus, we decompose the total strain tensor into elastic and plastic parts,

$$\mathbf{E}_{(0)} = \mathbf{E}^{\mathrm{e}}_{(0)} + \mathbf{E}^{\mathrm{p}}_{(0)}, \tag{9.11.3}$$

and assume that

$$\mathbf{E}^{\mathrm{e}}_{(0)} = \mathbf{M}_{(0)} : \mathbf{T}_{(0)}, \quad \mathbf{M}_{(0)} = \frac{1}{2\mu}\,\boldsymbol{J} + \frac{1}{3\kappa}\,\boldsymbol{K}, \tag{9.11.4}$$

$$\mathbf{E}^{\mathrm{p}}_{(0)} = \varphi\,\mathbf{T}'_{(0)}. \tag{9.11.5}$$

The shear and bulk moduli are μ and κ, the fourth-order tensors $\boldsymbol{J}$ and $\boldsymbol{K}$ are defined following Eq. (9.1.36), and φ is an appropriate scalar function to be determined in accord with experimental data. The prime designates a deviatoric part, so that plastic strain tensor is assumed to be traceless. More generally, a gradient of an isotropic function of $\mathbf{T}_{(0)}$ could be used in Eq. (9.11.5), in place of $\mathbf{T}'_{(0)}$ (Lubarda, 2000). This ensures that principal directions of plastic strain are parallel to those of $\mathbf{T}_{(0)}$. Since $\mathbf{M}_{(0)}$ in Eq. (9.11.4) corresponds to elastically isotropic material, principal directions of total strain $\mathbf{E}_{(0)}$ are also parallel to those of $\mathbf{T}_{(0)}$. Consequently, the stretch tensor $\mathbf{U}$ has its principal directions fixed during the deformation process, the matrix $\dot{\mathbf{U}}$ commutes with $\mathbf{U}$ and, by Eq. (3.6.18),

$$\dot{\mathbf{E}}_{(0)} = \dot{\mathbf{U}} \cdot \mathbf{U}^{-1}, \quad \mathbf{T}_{(0)} = \mathbf{R}^T \cdot \boldsymbol{\tau} \cdot \mathbf{R}. \tag{9.11.6}$$

The Kirchhoff stress is $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$, and $\mathbf{R}$ is the rotation tensor from the polar decomposition of deformation gradient $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$.

The requirement for the fixed principal directions of $\mathbf{U}$ severely restricts the class of admissible deformations. This is not surprising, because the premise of the deformation theory, the proportional stressing, imposes from outset the strong restrictions on the applicability of the analysis.

Introducing the spatial strain (see Subsection 2.3.2),

$$\boldsymbol{\mathcal{E}}_{(0)} = \mathbf{R}^T \cdot \mathbf{E}_{(0)} \cdot \mathbf{R}, \tag{9.11.7}$$

Equations (9.11.3)–(9.11.5) can be rewritten as

$$\boldsymbol{\mathcal{E}}_{(0)} = \boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(0)} + \boldsymbol{\mathcal{E}}^{\mathrm{p}}_{(0)}, \tag{9.11.8}$$

$$\boldsymbol{\mathcal{E}}^{\mathrm{e}}_{(0)} = \mathbf{M}_{(0)} : \boldsymbol{\tau}, \tag{9.11.9}$$

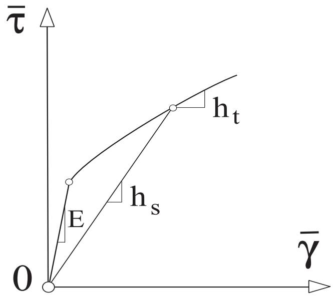

FIGURE 9.23. Nonlinear stress-strain response in pure shear. Indicated are the initial elastic modulus E, the secant modulus h_{s}, and the tangent modulus h_{t}.

$$\boldsymbol{\mathcal{E}}^{\mathrm{p}}_{(0)} = \varphi\, \boldsymbol{\tau}'. \tag{9.11.10}$$

It is noted that

$$\mathbf{T}'_{(0)} = \mathbf{R}^T \cdot \boldsymbol{\tau}' \cdot \mathbf{R}. \tag{9.11.11}$$

Suppose that a nonlinear relationship

$$\overline{\tau} = \overline{\tau}\,(\overline{\gamma}) \tag{9.11.12}$$

between the Kirchhoff stress and the logarithmic strain is available from the elastoplastic pure shear test ($E^{11}_{(0)} = \ln \upsilon$, $E^{22}_{(0)} = -\ln \upsilon$, all other $E^{ij}_{(0)}$ components being equal to zero; υ is the amount of extension and compression in the two fixed principal directions 1 and 2). Let the secant and tangent moduli be defined by (Fig. 9.23)

$$h_{\mathrm{s}} = \frac{\overline{\tau}}{\overline{\gamma}}\,, \quad h_{\mathrm{t}} = \frac{\mathrm{d}\overline{\tau}}{\mathrm{d}\overline{\gamma}}\,, \tag{9.11.13}$$

and let

$$\overline{\tau} = \left(\frac{1}{2}\,\boldsymbol{\tau}' : \boldsymbol{\tau}'\right)^{1/2} = \left(\frac{1}{2}\,\mathbf{T}'_{(0)} : \mathbf{T}'_{(0)}\right)^{1/2}, \tag{9.11.14}$$

$$\overline{\gamma} = \left(2\,\boldsymbol{\mathcal{E}}'_{(0)} : \boldsymbol{\mathcal{E}}'_{(0)}\right)^{1/2} = \left(2\,\mathbf{E}'_{(0)} : \mathbf{E}'_{(0)}\right)^{1/2}. \tag{9.11.15}$$

Since, from Eqs. (9.11.9) and (9.11.10),

$$\boldsymbol{\mathcal{E}}'_{(0)} = \left(\frac{1}{2\mu} + \varphi\right)\boldsymbol{\tau}', \tag{9.11.16}$$

the substitution into Eq. (9.11.15) gives

$$\varphi = \frac{1}{2h_{\mathrm{s}}} - \frac{1}{2\mu}\,. \tag{9.11.17}$$

Rate-Type Formulation of Deformation Theory

Although the deformation theory of plasticity is a total strain theory, the deformation theory can be cast in the rate-type form. This is important for later comparison with the flow theory of plasticity, and for extending the application of the resulting constitutive equations beyond the proportional loading. The rate-type formulation is also needed whenever the considered boundary value problem is being solved in an incremental manner.

Since $\dot{\mathbf{U}} \cdot \mathbf{U}^{-1}$ is symmetric, from the results in Section 2.6 we have

$$\mathbf{D} = \mathbf{R} \cdot \dot{\mathbf{E}}_{(0)} \cdot \mathbf{R}^T, \quad \mathbf{W} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1}. \tag{9.11.18}$$

Thus,

$$\dot{\mathbf{T}}_{(0)} = \mathbf{R}^T \cdot \overset{\circ}{\boldsymbol{\tau}} \cdot \mathbf{R}, \quad \overset{\circ}{\boldsymbol{\mathcal{E}}}_{(0)} = \mathbf{D}. \tag{9.11.19}$$

By differentiating Eqs. (9.11.3)–(9.11.5), or by applying the Jaumann derivative to Eqs. (9.11.8)–(9.11.10), there follows

$$\mathbf{D} = \mathbf{D}^{\mathrm{e}} + \mathbf{D}^{\mathrm{p}}, \tag{9.11.20}$$

$$\mathbf{D}^{\mathrm{e}} = \mathbf{M}_{(0)} : \overset{\circ}{\boldsymbol{\tau}}, \tag{9.11.21}$$

$$\mathbf{D}^{\mathrm{p}} = \dot{\varphi}\, \boldsymbol{\tau}' + \varphi\, \overset{\circ}{\boldsymbol{\tau}}{}'. \tag{9.11.22}$$

The deviatoric and spherical parts of the total rate of deformation tensor are accordingly

$$\mathbf{D}' = \dot{\varphi}\, \boldsymbol{\tau}' + \left(\frac{1}{2\mu} + \varphi\right) \overset{\circ}{\boldsymbol{\tau}}{}', \tag{9.11.23}$$

$$\operatorname{tr} \mathbf{D} = \frac{1}{3\kappa} \operatorname{tr} \overset{\circ}{\boldsymbol{\tau}}. \tag{9.11.24}$$

In order to derive an expression for the rate $\dot{\varphi}$, we differentiate Eqs. (9.11.14) and (9.11.15) to obtain

$$\overline{\tau}\,\dot{\overline{\tau}} = \frac{1}{2}\, \boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}, \quad \overline{\gamma}\,\dot{\overline{\gamma}} = 2\, \boldsymbol{\mathcal{E}}'_{(0)} : \mathbf{D}. \tag{9.11.25}$$

In view of Eqs. (9.11.13), (9.11.16), and (9.11.17), this gives

$$\frac{1}{2}\, \boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}} = 2h_{\mathrm{s}}h_{\mathrm{t}}\, \boldsymbol{\mathcal{E}}'_{(0)} : \mathbf{D}' = h_{\mathrm{t}}\, \boldsymbol{\tau}' : \mathbf{D}'. \tag{9.11.26}$$

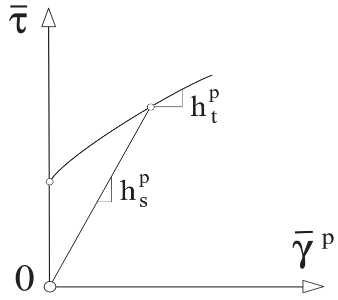

FIGURE 9.24. Shear stress vs. plastic shear strain. The plastic secant modulus is $h_{\rm s}^{\rm p}$, and the plastic tangent modulus is $h_{\rm t}^{\rm p}$.

When Eq. (9.11.23) is incorporated into Eq. (9.11.26), there follows

$$\dot{\varphi} = \frac{1}{2}\left(\frac{1}{h_{\rm t}} - \frac{1}{h_{\rm s}}\right)\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}\,. \tag{9.11.27}$$

Substituting Eq. (9.11.27) into Eq. (9.11.23), the deviatoric part of the total rate of deformation becomes

$$\mathbf{D}' = \frac{1}{2h_{\rm s}}\left[\overset{\circ}{\boldsymbol{\tau}}{}' + \left(\frac{h_{\rm s}}{h_{\rm t}} - 1\right)\frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}\right]. \tag{9.11.28}$$

Equation (9.11.28) can be inverted to express the deviatoric part of $\overset{\circ}{\boldsymbol{\tau}}$ as

$$\overset{\circ}{\boldsymbol{\tau}}{}' = 2h_{\rm s}\left[\mathbf{D}' - \left(1 - \frac{h_{\rm t}}{h_{\rm s}}\right)\frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \mathbf{D}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}\right]. \tag{9.11.29}$$

During initial, purely elastic stage of deformation, $h_{\rm t} = h_{\rm s} = \mu$. The onset of plasticity, beyond which Eqs. (9.11.28) and (9.11.29) apply, occurs when $\overline{\tau}$, defined by the second invariant of the deviatoric stress in Eq. (9.11.14), reaches the initial yield stress in shear. The resulting theory is often referred to as the J_2 deformation theory of plasticity.

If plastic secant and tangent moduli are used (Fig. 9.24), related to secant and tangent moduli with respect to total strain by

$$\frac{1}{h_{\rm t}} - \frac{1}{h_{\rm t}^{\rm p}} = \frac{1}{h_{\rm s}} - \frac{1}{h_{\rm s}^{\rm p}} = \frac{1}{\mu}\,, \tag{9.11.30}$$

the plastic part of the rate of deformation can be rewritten as

$$\mathbf{D}^{\rm p} = \frac{1}{2h_{\rm s}^{\rm p}}\,\overset{\circ}{\boldsymbol{\tau}}{}' + \left(\frac{1}{2h_{\rm t}^{\rm p}} - \frac{1}{2h_{\rm s}^{\rm p}}\right)\frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}\,. \tag{9.11.31}$$

9.11.1. Deformation vs. Flow Theory of Plasticity

For proportional loading, defined by Eq. (9.11.2), the stress rates are

$$\dot{\mathbf{T}}_{(0)} = \frac{\dot{c}}{c}\,\mathbf{T}_{(0)}, \quad \overset{\circ}{\boldsymbol{\tau}} = \frac{\dot{c}}{c}\,\boldsymbol{\tau}. \tag{9.11.32}$$

Consequently,

$$\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'} = \frac{\dot{c}}{c}, \tag{9.11.33}$$

and from Eq. (9.11.27) we have

$$\dot{\varphi} = \frac{1}{2}\left(\frac{1}{h_{\rm t}} - \frac{1}{h_{\rm s}}\right)\frac{\dot{c}}{c} = \frac{1}{2}\left(\frac{1}{h_{\rm t}^{\rm p}} - \frac{1}{h_{\rm s}^{\rm p}}\right)\frac{\dot{c}}{c}. \tag{9.11.34}$$

The plastic part of the rate of deformation reduces to

$$\mathbf{D}^{\rm p} = \frac{1}{2h_{\rm t}^{\rm p}}\,\frac{\dot{c}}{c}\,\boldsymbol{\tau}'. \tag{9.11.35}$$

On the other hand, in the case of the flow theory of plasticity,

$$\dot{\mathbf{E}}_{(0)} = \dot{\mathbf{E}}^{\rm e}_{(0)} + \dot{\mathbf{E}}^{\rm p}_{(0)}, \tag{9.11.36}$$

$$\dot{\mathbf{E}}^{\rm e}_{(0)} = \mathbf{M}_{(0)} : \dot{\mathbf{T}}_{(0)}, \quad \dot{\mathbf{E}}^{\rm p}_{(0)} = \dot{\gamma}\,\mathbf{T}'_{(0)}. \tag{9.11.37}$$

The yield surface is defined by

$$\frac{1}{2}\,\mathbf{T}'_{(0)} : \mathbf{T}'_{(0)} - \hat{k}^2(\vartheta) = 0, \quad \vartheta = \int_0^t \left(2\,\dot{\mathbf{E}}^{\rm p}_{(0)} : \dot{\mathbf{E}}^{\rm p}_{(0)}\right)^{1/2} {\rm d}t, \tag{9.11.38}$$

so that the consistency condition gives

$$\dot{\gamma} = \frac{1}{2h_{\rm t}^{\rm p}}\,\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}. \tag{9.11.39}$$

The plastic tangent modulus is $h_{\rm t}^{\rm p} = {\rm d}\hat{k}/{\rm d}\vartheta$. The parameter $\hat{k}$ is related to k of Subsection 9.4.1 by $\hat{k} = (\det \mathbf{F})\,k$. Since

$$\mathbf{T}_{(0)} = \mathbf{R}^T \cdot \boldsymbol{\tau} \cdot \mathbf{R}, \quad \dot{\mathbf{E}}_{(0)} = \mathbf{R}^T \cdot \mathbf{D} \cdot \mathbf{R}, \tag{9.11.40}$$

the plastic part of the rate of deformation becomes

$$\mathbf{D}^{\rm p} = \frac{1}{2h_{\rm t}^{\rm p}}\,\frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}. \tag{9.11.41}$$

In the case of proportional loading, Eq. (9.11.41) reduces to Eq. (9.11.35). Illustrative examples can be found in Kachanov (1971), and Neale and Shrivastava (1990). Also, note the connection

$$\dot{\gamma} - \dot{\varphi} = \varphi\,\frac{\dot{c}}{c}. \tag{9.11.42}$$

A study of variational principles within the framework of deformation theory of plasticity is presented by Martin (1975), Temam (1985), Gao and Strang (1989), Ponte Castañeda (1992), and Han and Reddy (1999).

9.11.2. Application beyond Proportional Loading

Deformation theory agrees with flow theory of plasticity only under proportional loading, since then specification of the final state of stress also specifies the stress history. For general (nonproportional) loading, more accurate and physically appropriate is the flow theory of plasticity, particularly with an accurate modeling of the yield surface and the hardening characteristics. Budiansky (1959), however, indicated that deformation theory can be successfully used for certain nearly proportional loading paths, as well. The stress rate $\overset{\circ}{\boldsymbol{\tau}}'$ in Eq. (9.11.31) then does not have to be codirectional with $\boldsymbol{\tau}'$. The first and third term (both proportional to $1/2h_{\rm s}^{\rm p}$) in Eq. (9.11.31) do not cancel each other in this case (as they do for proportional loading), and the plastic part of the rate of deformation depends on both components of the stress rate $\overset{\circ}{\boldsymbol{\tau}}'$, one in the direction of $\boldsymbol{\tau}'$ and the other normal to it. In contrast, according to flow theory with the von Mises smooth yield surface, the component of the stress rate $\overset{\circ}{\boldsymbol{\tau}}'$ normal to $\boldsymbol{\tau}'$ (thus tangential to the yield surface) does not affect the plastic part of the rate of deformation. Physical theories of plasticity (Batdorf and Budiansky, 1954; Sanders, 1954; Hill, 1967b) indicate that the yield surface of a polycrystalline aggregate develops a vertex at its loading stress point, so that infinitesimal increments of stress in the direction normal to $\boldsymbol{\tau}'$ indeed cause further plastic flow ("vertex softening"). Since the structure of the deformation theory of plasticity under proportional loading does not use any notion of the yield surface, Budiansky (*op. cit.*) suggested that Eq. (9.11.31) can be adopted to describe the response when the yield surface develops a vertex. If Eq. (9.11.31) is rewritten in the form

$$\mathbf{D}^{\rm p} = \frac{1}{2h_{\rm s}^{\rm p}}\left[\overset{\circ}{\boldsymbol{\tau}}' - \frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}'):\overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}':\boldsymbol{\tau}'}\right] + \frac{1}{2h_{\rm t}^{\rm p}}\,\frac{(\boldsymbol{\tau}'\otimes\boldsymbol{\tau}'):\overset{\circ}{\boldsymbol{\tau}}}{\boldsymbol{\tau}':\boldsymbol{\tau}'}\,, \qquad (9.11.43)$$

the first term on the right-hand side gives the response to component of the stress increment normal to $\boldsymbol{\tau}'$. The associated plastic modulus is $h_{\rm s}^{\rm p}$. The plastic modulus associated with the component of the stress increment in the direction of $\boldsymbol{\tau}'$ is $h_{\mathrm{t}}^{\mathrm{p}}$. Therefore, for continued plastic flow with small deviations from proportional loading (so that all yield segments which intersect at the vertex are active – fully active loading), Eq. (9.11.43) can be used as a model of a pointed vertex (Stören and Rice, 1975). The idea was used by Rudnicki and Rice (1975) in modeling the inelastic behavior of fissured rocks, as discussed in Subsection 9.8.2. See also Gotoh (1985), and Goya and Ito (1991).

For the full range of directions of the stress increment, the relationship between the rates of stress and plastic deformation is not necessarily linear, although it is homogeneous in these rates, in the absence of time-dependent (creep) effects. A corner theory that predicts continuous variation of the stiffness and allows increasingly nonproportional increments of stress was formulated by Christoffersen and Hutchinson (1979). This is discussed in the next subsection. When applied to the analysis of necking in thin sheets under biaxial stretching, the results were in better agreement with experiments than those obtained from the theory with a smooth yield characterization. Similar observations were long known in the field of elastoplastic buckling. Deformation theory predicts the buckling loads better than flow theory with a smooth yield surface (Hutchinson, 1974).

9.11.3. $\mathbf{J_2}$ Corner Theory

In phenomenological J_2 corner theory of plasticity, proposed by Christoffersen and Hutchinson (1979), the instantaneous elastoplastic moduli for nearly proportional loading are chosen equal to the J_2 deformation theory moduli, while for increasing deviation from proportional loading the moduli increase smoothly until they coincide with elastic moduli for stress increments directed along or within the corner of the yield surface. The yield surface in the neighborhood of the loading point in deviatoric stress space (Fig. 9.25) is a cone around the axis

$$\boldsymbol{l} = \frac{\boldsymbol{\tau}'}{(\boldsymbol{\tau}' : \mathbf{M}_{\mathrm{def}}^{\mathrm{p}} : \boldsymbol{\tau}')^{1/2}}\,, \tag{9.11.44}$$

where $\mathbf{M}_{\mathrm{def}}^{\mathrm{p}}$ is the plastic compliance tensor of the deformation theory. The angular measure θ of the stress rate direction, relative to the cone axis, is defined by

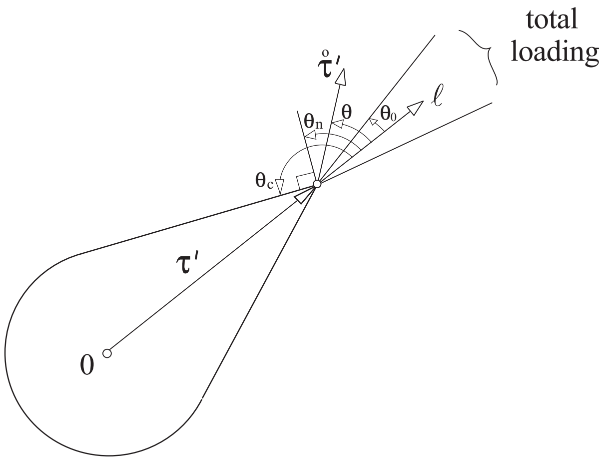

FIGURE 9.25. Near proportional or total loading range at the yield vertex of J_2 corner theory is a cone with the angle θ_0 around the axis $l \sim \boldsymbol{\tau}'$. The vertex cone is defined by the angle $\theta_{\rm c}$, and $\theta_n = \theta_{\rm c} - \pi/2$.

$$\cos\theta = \frac{\boldsymbol{l} : \mathbf{M}^{\rm p}_{\rm def} : \overset{\circ}{\boldsymbol{\tau}}}{(\overset{\circ}{\boldsymbol{\tau}} : \mathbf{M}^{\rm p}_{\rm def} : \overset{\circ}{\boldsymbol{\tau}})^{1/2}} \,. \tag{9.11.45}$$

The conical surface separating elastic unloading and plastic loading is $\theta = \theta_{\rm c}$, so that plastic rate of deformation falls within the range $0 \le \theta \le \theta_n$, where $\theta_n = \theta_{\rm c} - \pi/2$. The range of near proportional loading is $0 \le \theta \le \theta_0$. The angle θ_0 is a suitable fraction of θ_n. The range of near proportional loading is the range of stress-rate directions for which no elastic unloading takes place on any of the yield vertex segments. This range is also called fully active or total loading range.

The stress-rate potential at the corner is defined by

$$\Pi = \Pi^{\rm e} + \Pi^{\rm p}, \quad \Pi^{\rm p} = f(\theta)\Pi^{\rm p}_{\rm def}. \tag{9.11.46}$$

The elastic contribution to the stress-rate potential is

$$\Pi^{\rm e} = \frac{1}{2}\,\overset{\circ}{\boldsymbol{\tau}} : \mathbf{M}^{\rm e} : \overset{\circ}{\boldsymbol{\tau}}, \quad \mathbf{M}^{\rm e} = \frac{1}{2\mu}\,\boldsymbol{J} + \frac{1}{3\kappa}\,\boldsymbol{K}. \tag{9.11.47}$$

The plastic stress-rate potential of the J_2 deformation theory can be written, from Eq. (9.11.28), as

$$\Pi^{\rm p}_{\rm def} = \frac{1}{2}\,\overset{\circ}{\boldsymbol{\tau}} : \mathbf{M}^{\rm p}_{\rm def} : \overset{\circ}{\boldsymbol{\tau}}, \quad \mathbf{M}^{\rm p}_{\rm def} = \frac{1}{2h_{\rm s}}\left[\left(1 - \frac{h_{\rm s}}{\mu}\right)\boldsymbol{J} + \left(\frac{h_{\rm s}}{h_{\rm t}} - 1\right)\frac{\boldsymbol{\tau}' \otimes \boldsymbol{\tau}'}{\boldsymbol{\tau}' : \boldsymbol{\tau}'}\right]. \tag{9.11.48}$$

The plastic stress-rate potential $\Pi^{\mathrm{p}}_{\mathrm{def}}$ is weighted by the cone transition function $f(\theta)$ to obtain the plastic stress-rate potential Π^{p} of the J_2 corner theory.

In the range of near proportional loading

$$0 \le \theta \le \theta_0, \quad f(\theta) = 1, \tag{9.11.49}$$

while in the elastic unloading range

$$\theta_{\mathrm{c}} \le \theta \le \pi, \quad f(\theta) = 0. \tag{9.11.50}$$

In the transition region $\theta_0 \le \theta \le \theta_{\mathrm{c}}$, the function $f(\theta)$ decreases monotonically and smoothly from one to zero in a way which ensures convexity of the plastic-rate potential,

$$\Pi^{\mathrm{p}}(\overset{\circ}{\boldsymbol{\tau}}_2) - \Pi^{\mathrm{p}}(\overset{\circ}{\boldsymbol{\tau}}_1) \ge \frac{\partial \Pi^{\mathrm{p}}}{\partial \overset{\circ}{\boldsymbol{\tau}}_1} : (\overset{\circ}{\boldsymbol{\tau}}_2 - \overset{\circ}{\boldsymbol{\tau}}_1). \tag{9.11.51}$$

A simple choice of $f(\theta)$ meeting these requirements is

$$f(\theta) = \cos^2\left(\frac{\pi}{2}\,\frac{\theta - \theta_0}{\theta_{\mathrm{c}} - \theta_0}\right), \quad \theta_0 \le \theta \le \theta_{\mathrm{c}}. \tag{9.11.52}$$

The specification of the angles θ_{c} and θ_0 in terms of the current stress measure is discussed by Christoffersen and Hutchinson (1979).

The rate-independence of the material response requires

$$\mathbf{D}^{\mathrm{p}} = \frac{\partial \Pi^{\mathrm{p}}}{\partial \overset{\circ}{\boldsymbol{\tau}}} = \frac{\partial^2 \Pi^{\mathrm{p}}}{\partial \overset{\circ}{\boldsymbol{\tau}} \otimes \partial \overset{\circ}{\boldsymbol{\tau}}} : \overset{\circ}{\boldsymbol{\tau}} = \mathbf{M}^{\mathrm{p}} : \overset{\circ}{\boldsymbol{\tau}} \tag{9.11.53}$$

to be a homogeneous function of degree one, and Π^{p} to be a homogeneous function of degree two in the stress rate $\overset{\circ}{\boldsymbol{\tau}}$. The function $\Pi^{\mathrm{p}}(\overset{\circ}{\boldsymbol{\tau}})$ is quadratic in the region of nearly proportional loading, but highly nonlinear in the transition region, due to nonlinearity associated with $f(\theta)$. The plastic rate of deformation is accordingly a linear function of $\overset{\circ}{\boldsymbol{\tau}}$ in the region of nearly proportional loading, but a nonlinear function in the transition region. Further details on the structure of J_2 corner theory, with its application to the study of sheet necking, are given in the Christoffersen and Hutchinson's paper. See also Needleman and Tvergaard (1982).

9.11.4. Pressure-Dependent Deformation Theory

To include pressure dependence and allow inelastic volume changes in deformation theory of plasticity, assume that, in place of Eq. (9.11.5), the plastic strain is related to stress by

$$\mathbf{E}^{\mathrm{p}}_{(0)} = \varphi \left[\mathbf{T}'_{(0)} + \frac{2}{3}\,\beta \left(\frac{1}{2}\,\mathbf{T}'_{(0)} : \mathbf{T}'_{(0)} \right)^{1/2} \mathbf{I}^0 \right], \tag{9.11.54}$$

where β is a material parameter, and $\mathbf{I}^0$ is the second-order unit tensor. It follows that the deviatoric and spherical parts of the plastic rate of deformation are

$$\mathbf{D}^{\mathrm{p}\prime} = \dot{\varphi}\,\boldsymbol{\tau}' + \varphi\,\overset{\circ}{\boldsymbol{\tau}}{}', \tag{9.11.55}$$

$$\operatorname{tr}\mathbf{D}^{\mathrm{p}} = 2\beta\,J_2^{1/2} \left(\dot{\varphi} + \varphi\,\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2} \right). \tag{9.11.56}$$

The invariant

$$J_2 = \frac{1}{2}\,\boldsymbol{\tau}' : \boldsymbol{\tau}' \tag{9.11.57}$$

here represents the second invariant of the deviatoric part of the Kirchhoff stress.

Suppose that a nonlinear relationship $\overline{\tau} = \overline{\tau}\,(\overline{\gamma}^{\mathrm{p}})$ between the Kirchhoff stress and the plastic part of the logarithmic strain is available from the elastoplastic shear test. Let the plastic secant and tangent moduli be defined by

$$h^{\mathrm{p}}_{\mathrm{s}} = \frac{\overline{\tau}}{\overline{\gamma}^{\mathrm{p}}}\,, \quad h^{\mathrm{p}}_{\mathrm{t}} = \frac{\mathrm{d}\overline{\tau}}{\mathrm{d}\overline{\gamma}^{\mathrm{p}}}\,, \tag{9.11.58}$$

and let, in three-dimensional problems of overall compressive states of stress,

$$\overline{\tau} = J_2^{1/2} + \frac{1}{3}\,\mu_*\operatorname{tr}\boldsymbol{\tau}, \tag{9.11.59}$$

$$\overline{\gamma}^{\mathrm{p}} = \left(2\,\boldsymbol{\mathcal{E}}^{\mathrm{p}}_{(0)}{}' : \boldsymbol{\mathcal{E}}^{\mathrm{p}}_{(0)}{}' \right)^{1/2} = 2\,\varphi\,J_2^{1/2}. \tag{9.11.60}$$

Observe, from Eq. (9.11.54), that

$$\boldsymbol{\mathcal{E}}^{\mathrm{p}}_{(0)}{}' = \varphi\,\boldsymbol{\tau}'. \tag{9.11.61}$$

The friction-type coefficient in Eq. (9.11.59) is denoted by μ_*. By using the first of Eq. (9.11.58), therefore,

$$\varphi = \frac{1}{2h^{\mathrm{p}}_{\mathrm{s}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\,. \tag{9.11.62}$$

In order to derive an expression for the rate $\dot{\varphi}$, differentiate Eqs. (9.11.59) and (9.11.60) to obtain

$$\dot{\overline{\tau}} = \frac{1}{2}\,J_2^{-1/2}(\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}) + \frac{1}{3}\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}}, \tag{9.11.63}$$

$$\dot{\bar{\gamma}}^{\mathrm{p}} = 2\left[\dot{\varphi}\, J_2^{1/2} + \frac{1}{2}\,\varphi\, J_2^{-1/2}(\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}})\right]. \tag{9.11.64}$$

Combining this with the second of Eq. (9.11.58) gives

$$\dot{\varphi} = \frac{1}{2}\left(\frac{1}{h_{\mathrm{t}}^{\mathrm{p}}} - \frac{1}{h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\right)\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2} + \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{1}{3}\,\mu_*\,\frac{\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}}}{J_2^{1/2}}\,. \tag{9.11.65}$$

Consequently, by substituting Eqs. (9.11.62) and (9.11.65) into Eqs. (9.11.55) and (9.11.56), there follows

$$\begin{aligned}\mathbf{D}^{\mathrm{p}\prime} = {} & \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\,\overset{\circ}{\boldsymbol{\tau}}{}' + \frac{1}{2}\left(\frac{1}{h_{\mathrm{t}}^{\mathrm{p}}} - \frac{1}{h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\right)\frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2} \\ & + \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{1}{3}\,\mu_*\,\frac{\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}}}{J_2^{1/2}}\,\boldsymbol{\tau}',\end{aligned} \tag{9.11.66}$$

$$\operatorname{tr}\mathbf{D}^{\mathrm{p}} = \frac{\beta}{h_{\mathrm{t}}^{\mathrm{p}}}\left(\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2^{1/2}} + \frac{1}{3}\,\mu_*\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}}\right). \tag{9.11.67}$$

In the case when

$$\mu_* = 0, \quad \overline{\tau} = J_2^{1/2}, \tag{9.11.68}$$

Equation (9.11.66) simplifies and the deviatoric part of the plastic rate of deformation becomes

$$\mathbf{D}^{\mathrm{p}\prime} = \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\left[\overset{\circ}{\boldsymbol{\tau}}{}' + \left(\frac{h_{\mathrm{s}}^{\mathrm{p}}}{h_{\mathrm{t}}^{\mathrm{p}}} - 1\right)\frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2}\right], \tag{9.11.69}$$

while from Eq. (9.11.67) the volumetric part of the plastic rate of deformation is

$$\operatorname{tr}\mathbf{D}^{\mathrm{p}} = \frac{\beta}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{J_2^{1/2}}\,. \tag{9.11.70}$$

Noncoaxiality Factor

Equation (9.11.66) can be rewritten in an alternative form as

$$\mathbf{D}^{\mathrm{p}\prime} = \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}}\,\frac{\boldsymbol{\tau}'}{J_2^{1/2}}\left(\frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2^{1/2}} + \frac{1}{3}\,\mu_*\operatorname{tr}\overset{\circ}{\boldsymbol{\tau}}\right) + \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}}\,\frac{\overline{\tau}}{J_2^{1/2}}\left[\overset{\circ}{\boldsymbol{\tau}}{}' - \frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{2\,J_2}\right]. \tag{9.11.71}$$

The first part of $\mathbf{D}^{\mathrm{p}\,\prime}$ is coaxial with $\boldsymbol{\tau}'$. The second part is in the direction of the component of stress rate $\overset{\circ}{\boldsymbol{\tau}}{}'$ that is normal to $\boldsymbol{\tau}'$. There is no work associated with this part of the plastic rate of deformation, so that

$$\boldsymbol{\tau} : \mathbf{D}^{\mathrm{p}\prime} = \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}} \left(\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}} + \frac{2}{3}\, \mu_* \, J_2^{1/2} \operatorname{tr} \overset{\circ}{\boldsymbol{\tau}} \right). \tag{9.11.72}$$

Observe from Eqs. (9.11.67) and (9.11.72) that

$$\operatorname{tr} \mathbf{D}^{\mathrm{p}} = \beta \, \frac{\boldsymbol{\tau} : \mathbf{D}^{\mathrm{p}\,\prime}}{J_2^{1/2}} \,, \tag{9.11.73}$$

which offers a simple physical interpretation of the parameter β.

The coefficient

$$\varsigma = \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}} \, \frac{\overline{\tau}}{J_2^{1/2}} = \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}} \left(1 + \frac{1}{3}\, \mu_* \, \frac{\operatorname{tr} \boldsymbol{\tau}}{J_2^{1/2}} \right) \tag{9.11.74}$$

in Eq. (9.11.71) is a stress-dependent noncoaxiality factor. Other definitions of this factor have also been used in the literature (e.g., Nemat-Nasser, 1983).

Inverse Constitutive Equations

The deviatoric and volumetric part of the total rate of deformation are

$$\mathbf{D}' = \left(\frac{1}{2\mu} + \frac{1}{2h_{\mathrm{s}}^{\mathrm{p}}} \, \frac{\overline{\tau}}{J_2^{1/2}} \right) \overset{\circ}{\boldsymbol{\tau}}{}' + \frac{1}{2} \left(\frac{1}{h_{\mathrm{t}}^{\mathrm{p}}} - \frac{1}{h_{\mathrm{s}}^{\mathrm{p}}} \, \frac{\overline{\tau}}{J_2^{1/2}} \right) \frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \overset{\circ}{\boldsymbol{\tau}}}{2\, J_2} + \frac{1}{2h_{\mathrm{t}}^{\mathrm{p}}} \, \frac{1}{3}\, \mu_* \, \frac{\operatorname{tr} \overset{\circ}{\boldsymbol{\tau}}}{J_2^{1/2}} \, \boldsymbol{\tau}', \tag{9.11.75}$$

$$\operatorname{tr} \mathbf{D} = \frac{1}{3} \left(\frac{1}{\kappa} + \frac{\mu_* \beta}{h_{\mathrm{t}}^{\mathrm{p}}} \right) \operatorname{tr} \overset{\circ}{\boldsymbol{\tau}} + \frac{\beta}{2h_{\mathrm{t}}^{\mathrm{p}}} \, \frac{\boldsymbol{\tau}' : \overset{\circ}{\boldsymbol{\tau}}}{J_2^{1/2}} \,. \tag{9.11.76}$$

The inverse relations are

$$\overset{\circ}{\underline{\boldsymbol{\tau}}}{}' = 2\mu \left[\frac{1}{b} \, \mathbf{D}' - \frac{a}{bc} \, \frac{(\boldsymbol{\tau}' \otimes \boldsymbol{\tau}') : \mathbf{D}}{2\, J_2} - \frac{1}{c}\, \mu_* \, \frac{\kappa}{2\mu} \, \frac{\boldsymbol{\tau}'}{J_2^{1/2}} \operatorname{tr} \mathbf{D} \right], \tag{9.11.77}$$

$$\operatorname{tr} \overset{\circ}{\underline{\boldsymbol{\tau}}} = \frac{3\kappa}{c} \left[\left(1 + \frac{h_{\mathrm{t}}^{\mathrm{p}}}{\mu} \right) \operatorname{tr} \mathbf{D} - \beta \, \frac{\boldsymbol{\tau}' : \mathbf{D}}{J_2^{1/2}} \right], \tag{9.11.78}$$

where

$$a = 1 - \frac{h_{\mathrm{t}}^{\mathrm{p}}}{h_{\mathrm{s}}^{\mathrm{p}}} \, \frac{\overline{\tau}}{J_2^{1/2}} \left(1 + \mu_* \beta \, \frac{\kappa}{h_{\mathrm{t}}^{\mathrm{p}}} \right), \quad b = 1 + \frac{\mu}{h_{\mathrm{s}}^{\mathrm{p}}} \, \frac{\overline{\tau}}{J_2^{1/2}} \,, \tag{9.11.79}$$

and

$$c = 1 + \frac{h_{\mathrm{t}}^{\mathrm{p}}}{\mu} + \mu_* \beta \, \frac{\kappa}{\mu} \,. \tag{9.11.80}$$

Comparing Eq. (9.8.36) of the modified flow theory with Eq. (9.11.71) of the pressure-dependent deformation theory of plasticity, it can be recognized that the two constitutive structures are equivalent, provided that the identification is made

$$H = h_{\mathrm{t}}^{\mathrm{p}}, \quad H_1 = h_{\mathrm{s}}^{\mathrm{p}} \frac{J_2^{1/2}}{\overline{\tau}} = \frac{1}{2\varsigma}. \tag{9.11.81}$$

With these connections, Eqs. (9.8.37) and (9.8.38) are also equivalent to Eqs. (9.11.77) and (9.11.78). The relationship between the two theories have been further discussed by Rudnicki (1982) and Nemat-Nasser (1982).

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