Part 3

THEORY OF PLASTICITY

CHAPTER 8

ELASTOPLASTIC CONSTITUTIVE FRAMEWORK

This chapter provides a basic framework for the constitutive analysis of elastoplastic materials. Such materials are capable of exhibiting, under certain loadings, purely elastic response at any stage of deformation. The development is originally due to Hill and Rice (1973). Rate-independent and rate-dependent plastic materials are both encompassed by this framework. For rate-independent materials, purely elastic response results when stress variations are directed within the current yield surface, which is introduced for such materials. For rate-dependent materials, the response may be purely elastic only in the limit, when stress variations are sufficiently rapid compared to fastest rates at which inelastic processes can take place. We start the analysis by defining elastic and plastic increments of stress and strain tensors. Normality properties are then discussed for rate-independent plastic materials which admit the yield surface. Formulations in both stress and strain space are given. Plasticity postulates of Ilyushin and Drucker are studied in detail. Conditions for the existence of flow potential for rate-dependent materials are also examined.

8.1. Elastic and Plastic Increments

An introductory thermodynamic analysis of inelastic deformation process within the framework of thermodynamics with internal state variables was presented in Sections 4.4–4.6. We proceed in this chapter with the analysis of elastoplastic deformation under isothermal conditions only. Basic physical mechanisms of such deformation are described in standard texts, such as Cottrell (1961, 1964) and Honeycombe (1984). We shall assume that there is a set of variables ξ_j that, in some approximate sense, represent internal rearrangements of the material due to plastic deformation. These variables are not necessarily state variables in the sense that the free or complementary energy is not a point function of ξ_j but, instead, depends on their path history (Rice, 1971). Denoting the pattern of internal rearrangements symbolically by $\mathcal{H}$ (the set of internal variables ξ_j together with the path history by which they were achieved), the free energy per unit reference volume can be expressed as

$$\Psi = \Psi\left(\mathbf{E}_{(n)}, \mathcal{H}\right). \tag{8.1.1}$$

At any given state of deformation, an infinitesimal change of $\mathcal{H}$ is assumed to be fully described by a set of scalar infinitesimals $d\xi_j$, such that the change in Ψ due to $\mathrm{d}\mathbf{E}_{(n)}$ and $d\xi_j$ is, to first order,

$$\mathrm{d}\Psi = \frac{\partial \Psi}{\partial \mathbf{E}_{(n)}} : \mathrm{d}\mathbf{E}_{(n)} - \rho^0 f_j\, d\xi_j = \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} - \rho^0 f_j\, d\xi_j. \tag{8.1.2}$$

The reference density is ρ^0, and $f_j\, d\xi_j$ is an increment of dissipative work per unit mass. It is not necessary that any variable ξ_j exists such that $d\xi_j$ represents an infinitesimal change of ξ_j. The use of an italic d in $d\xi_j$ is intended to indicate this. The stress response is

$$\mathbf{T}_{(n)} = \frac{\partial \Psi}{\partial \mathbf{E}_{(n)}}, \tag{8.1.3}$$

evaluated from Ψ at fixed values of $\mathcal{H}$. The energetic forces f_j are associated with the infinitesimals $d\xi_j$, so that plastic change of the free energy, due to change of $\mathcal{H}$ alone,

$$\mathrm{d}^{\mathrm{p}}\Psi = \Psi\left(\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \Psi\left(\mathbf{E}_{(n)}, \mathcal{H}\right), \tag{8.1.4}$$

is equal to

$$\mathrm{d}^{\mathrm{p}}\Psi = -\rho^0 f_j\, d\xi_j = -\rho^0 \bar{f}_j\left(\mathbf{E}_{(n)}, \mathcal{H}\right) d\xi_j. \tag{8.1.5}$$

Higher-order terms, such as $(1/2) df_j\, d\xi_j$, associated with infinitesimal changes of f_j during the variations $d\xi_j$, are neglected.

8.1.1. Plastic Stress Increment

The plastic part of stress increment is defined by Hill and Rice (1973) as

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = \mathbf{T}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \mathbf{T}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right). \tag{8.1.6}$$

In view of Eqs. (8.1.3) and (8.1.4), this gives

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = \frac{\partial}{\partial \mathbf{E}_{(n)}} \left(\mathrm{d}^{\mathrm{p}}\Psi\right). \tag{8.1.7}$$

Thus, the plastic increment of free energy $\mathrm{d}^{\mathrm{p}}\Psi$ can be viewed as a potential for the plastic part of stress increment $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$. From Eqs. (8.1.5) and (8.1.7), we also have

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = -\rho^0 \frac{\partial \bar{f}_j}{\partial \mathbf{E}_{(n)}} \, d\xi_j. \tag{8.1.8}$$

Furthermore, by considering the function

$$\mathbf{T}_{(n)} = \mathbf{T}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right), \tag{8.1.9}$$

we deduce from Eq. (8.1.6) that

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = \mathrm{d}\mathbf{T}_{(n)} - \frac{\partial \mathbf{T}_{(n)}}{\partial \mathbf{E}_{(n)}} : \mathrm{d}\mathbf{E}_{(n)} = \mathrm{d}\mathbf{T}_{(n)} - \boldsymbol{\Lambda}_{(n)} : \mathrm{d}\mathbf{E}_{(n)}. \tag{8.1.10}$$

The fourth-order tensor

$$\boldsymbol{\Lambda}_{(n)} = \frac{\partial \mathbf{T}_{(n)}}{\partial \mathbf{E}_{(n)}} = \frac{\partial^2 \Psi}{\partial \mathbf{E}_{(n)} \otimes \partial \mathbf{E}_{(n)}} \tag{8.1.11}$$

is the tensor of elastic moduli corresponding to the selected strain measure $\mathbf{E}_{(n)}$.

In a rate-independent elastoplastic material, the only way to vary $\mathcal{H}$ but not $\mathbf{E}_{(n)}$ is to consider a cycle of strain $\mathbf{E}_{(n)}$ that involves $d\mathcal{H}$. Suppose that the cycle emanates from the state $A\left(\mathbf{E}_{(n)}, \mathcal{H}\right)$, it goes through $B\left(\mathbf{E}_{(n)} + \mathrm{d}\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}\right)$, and ends at the state $C\left(\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}\right)$, as shown in Fig. 8.1. If the stress at A was $\mathbf{T}_{(n)}$, in the state B it is $\mathbf{T}_{(n)} + \mathrm{d}\mathbf{T}_{(n)}$. After the strain is returned to its value at the beginning of the cycle by elastic unloading, the state C is reached. The stress there is $\mathbf{T}_{(n)} + \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$. The stress difference $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ is then the stress decrement left after the cycle of strain that involves $\mathrm{d}\mathcal{H}$.

8.1.2. Plastic Strain Increment

Dually, consider a complementary energy defined by the Legendre transform of the free energy as

$$\Phi_{(n)}\left(\mathbf{T}_{(n)}, \mathcal{H}\right) = \mathbf{T}_{(n)} : \mathbf{E}_{(n)} - \Psi\left(\mathbf{E}_{(n)}, \mathcal{H}\right). \tag{8.1.12}$$

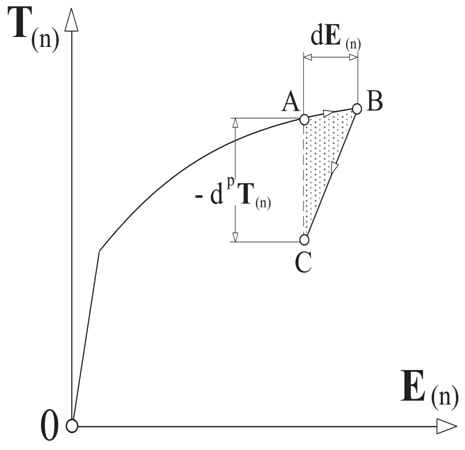

FIGURE 8.1. Strain cycle ABC involving plastic deformation along an infinitesimal segment AB.

The change of complementary energy due to $\mathrm{d}\mathbf{T}_{(n)}$ and $d\xi_j$ is

$$\mathrm{d}\Phi_{(n)} = \frac{\partial \Phi_{(n)}}{\partial \mathbf{T}_{(n)}} : \mathrm{d}\mathbf{T}_{(n)} + \rho^0 \, f_j \, d\xi_j = \mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} + \rho^0 \, f_j \, d\xi_j . \tag{8.1.13}$$

The strain response is accordingly

$$\mathbf{E}_{(n)} = \frac{\partial \Phi_{(n)}}{\partial \mathbf{T}_{(n)}} , \tag{8.1.14}$$

evaluated from $\Phi_{(n)}$ at fixed values of $\mathcal{H}$. The plastic change of complementary energy, due to change of $\mathcal{H}$ alone,

$$\mathrm{d}^{\mathrm{p}}\Phi_{(n)} = \Phi_{(n)} \left(\mathbf{T}_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \Phi_{(n)} \left(\mathbf{T}_{(n)}, \mathcal{H}\right) , \tag{8.1.15}$$

is equal to

$$\mathrm{d}^{\mathrm{p}}\Phi_{(n)} = \rho^0 \, f_j \, d\xi_j = \rho^0 \, \hat{f}_j \left(\mathbf{T}_{(n)}, \mathcal{H}\right) d\xi_j . \tag{8.1.16}$$

The plastic part of strain increment is defined by

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = \mathbf{E}_{(n)} \left(\mathbf{T}_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \mathbf{E}_{(n)} \left(\mathbf{T}_{(n)}, \mathcal{H}\right) . \tag{8.1.17}$$

In view of Eqs. (8.1.14) and (8.1.15), this gives

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = \frac{\partial}{\partial \mathbf{T}_{(n)}} \left(\mathrm{d}^{\mathrm{p}}\Phi_{(n)}\right) . \tag{8.1.18}$$

Thus, the plastic increment of complementary energy $\mathrm{d}^{\mathrm{p}}\Phi_{(n)}$ can be viewed as a potential for the plastic part of strain increment $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$. From Eqs.

(8.1.16) and (8.1.18), we also have

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = \rho^0 \, \frac{\partial \hat{f}_j}{\partial \mathbf{T}_{(n)}} \, d\xi_j. \tag{8.1.19}$$

Furthermore, by taking a differential of the function

$$\mathbf{E}_{(n)} = \mathbf{E}_{(n)}\left(\mathbf{T}_{(n)}, \mathcal{H}\right), \tag{8.1.20}$$

and by employing Eq. (8.1.17), we have

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = \mathrm{d}\mathbf{E}_{(n)} - \frac{\partial \mathbf{E}_{(n)}}{\partial \mathbf{T}_{(n)}} : \mathrm{d}\mathbf{T}_{(n)} = \mathrm{d}\mathbf{E}_{(n)} - \mathbf{M}_{(n)} : \mathrm{d}\mathbf{T}_{(n)}. \tag{8.1.21}$$

The fourth-order tensor

$$\mathbf{M}_{(n)} = \frac{\partial \mathbf{E}_{(n)}}{\partial \mathbf{T}_{(n)}} = \frac{\partial^2 \Phi}{\partial \mathbf{T}_{(n)} \otimes \partial \mathbf{T}_{(n)}} \tag{8.1.22}$$

is the tensor of elastic compliances corresponding to selected stress measure $\mathbf{T}_{(n)}$.

In a rate-independent elastoplastic material, the only way to vary $\mathcal{H}$ but not $\mathbf{T}_{(n)}$ is to consider a cycle of stress $\mathbf{T}_{(n)}$ that involves $d\mathcal{H}$. Consider a cycle $A \to B \to D$; see Fig. 8.2. In state D the stress is returned to its value before the cycle, i.e., $A\left(\mathbf{T}_{(n)}, \mathcal{H}\right)$, $B\left(\mathbf{T}_{(n)} + \mathrm{d}\mathbf{T}_{(n)}, \mathcal{H} + d\mathcal{H}\right)$ and $D\left(\mathbf{T}_{(n)}, \mathcal{H} + d\mathcal{H}\right)$. The strains in the states A and B are $\mathbf{E}_{(n)}$ and $\mathbf{E}_{(n)} + \mathrm{d}\mathbf{E}_{(n)}$, respectively. After stress is returned to its value before the cycle by elastic unloading, the state D is reached, where the strain is $\mathbf{E}_{(n)} + \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$. The strain difference $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ is the strain increment left after the cycle of stress that involves $d\mathcal{H}$.

For a rate-dependent material, $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ is the difference between the strains when $\mathbf{T}_{(n)}$ is instantaneously applied after inelastic histories $\mathcal{H}$ and $\mathcal{H} + d\mathcal{H}$, respectively.

8.1.3. Relationship between Plastic Increments

Equations (8.1.4) and (8.1.16) show that

$$\mathrm{d}^{\mathrm{p}}\Psi + \mathrm{d}^{\mathrm{p}}\Phi_{(n)} = 0, \tag{8.1.23}$$

within the order of accuracy used in Eqs. (8.1.4) and (8.1.16). The relationship between the plastic increments $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ and $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ is easily established

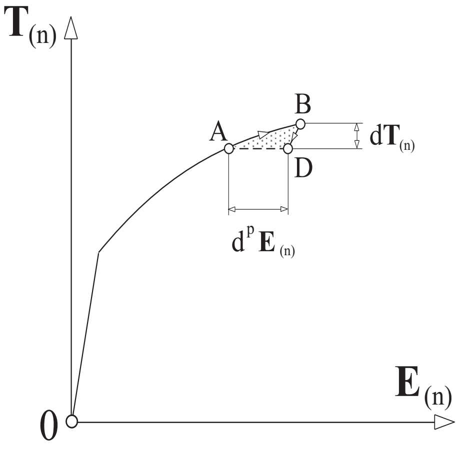

FIGURE 8.2. Stress cycle ABD involving plastic deformation along an infinitesimal segment AB.

from Eqs. (8.1.8) and (8.1.19). This is

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = -\rho^0 \,\frac{\partial \bar{f}_j}{\partial \mathbf{E}_{(n)}}\, d\xi_j = -\rho^0 \left(\frac{\partial \hat{f}_j}{\partial \mathbf{T}_{(n)}} : \frac{\partial \mathbf{T}_{(n)}}{\partial \mathbf{E}_{(n)}} \right) d\xi_j = -\frac{\partial \mathbf{T}_{(n)}}{\partial \mathbf{E}_{(n)}} : \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}. \tag{8.1.24}$$

Similarly,

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = \rho^0 \,\frac{\partial \hat{f}_j}{\partial \mathbf{T}_{(n)}}\, d\xi_j = \rho^0 \left(\frac{\partial \bar{f}_j}{\partial \mathbf{E}_{(n)}} : \frac{\partial \mathbf{E}_{(n)}}{\partial \mathbf{T}_{(n)}} \right) d\xi_j = -\frac{\partial \mathbf{E}_{(n)}}{\partial \mathbf{T}_{(n)}} : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}. \tag{8.1.25}$$

Therefore, the plastic increments are related by

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = -\boldsymbol{\Lambda}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}, \quad \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = -\mathbf{M}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}. \tag{8.1.26}$$

These expressions also follow directly from Eqs. (8.1.10) and (8.1.21), since $\boldsymbol{\Lambda}_{(n)}$ and $\mathbf{M}_{(n)}$ are mutual inverses.

Note that purely elastic increment of strain is related to the corresponding increment of stress by

$$\delta\mathbf{T}_{(n)} = \boldsymbol{\Lambda}_{(n)} : \delta\mathbf{E}_{(n)}, \quad \delta\mathbf{E}_{(n)} = \mathbf{M}_{(n)} : \delta\mathbf{T}_{(n)}. \tag{8.1.27}$$

The variation of the free energy associated with $\delta\mathbf{E}_{(n)}$ is

$$\delta\Psi = \frac{\partial \Psi}{\partial \mathbf{E}_{(n)}} : \delta\mathbf{E}_{(n)} = \mathbf{T}_{(n)} : \delta\mathbf{E}_{(n)}. \tag{8.1.28}$$

8.2. Yield Surface for Rate-Independent Materials

Rate-independent plastic materials have an elastic range within which they respond in a purely elastic manner. The boundary of this range, in either stress or strain space, is called the yield surface. The shape of the yield surface depends on the entire history of deformation from the reference state. During plastic deformation the states of stress or strain remain on the subsequent yield surfaces. The yield surfaces for actual materials are experimentally found to be mainly smooth, although they may develop pyramidal or conical vertices, or regions of high curvature (Hill, 1978). If elasticity within the yield surface is linear and unaffected by plastic flow, the yield surfaces for metals are convex in the Cauchy stress space. General discussion regarding the geometry and experimental determination of the yield surfaces can be found in Drucker (1960), Naghdi (1960), and Hecker (1976).

8.2.1. Yield Surface in Strain Space

Consider the yield surface in strain space defined by

$$g_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right) = 0, \tag{8.2.1}$$

where $\mathcal{H}$ represents the pattern of internal rearrangements due to plastic deformation. The strain $\mathbf{E}_{(n)}$ is defined relative to an arbitrary reference state. The shape of the yield surface at each stage of deformation is different for different choices of $\mathbf{E}_{(n)}$, so that different functions $g_{(n)}$ correspond to different n. It is assumed that elastic response within the yield surface is Green-elastic, associated with the strain energy

$$\Psi = \Psi\left(\mathbf{E}_{(n)}, \mathcal{H}\right) \tag{8.2.2}$$

per unit reference volume, such that

$$\mathbf{T}_{(n)} = \frac{\partial \Psi}{\partial \mathbf{E}_{(n)}}\,. \tag{8.2.3}$$

Let the state of strain $\mathbf{E}_{(n)}$ be on the current yield surface. An increment of strain $\mathrm{d}\mathbf{E}_{(n)}$ directed inside the yield surface constitutes an elastic unloading. The corresponding incremental elastic response is governed by the rate-type equation

$$\dot{\mathbf{T}}_{(n)} = \boldsymbol{\Lambda}_{(n)} : \dot{\mathbf{E}}_{(n)}, \quad \boldsymbol{\Lambda}_{(n)} = \frac{\partial^2 \Psi}{\partial \mathbf{E}_{(n)} \otimes \partial \mathbf{E}_{(n)}}\,, \tag{8.2.4}$$

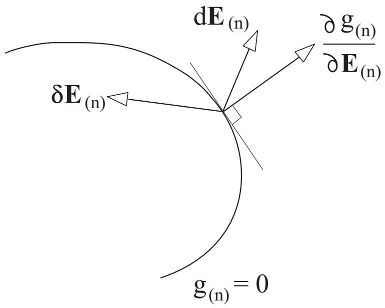

FIGURE 8.3. Strain increment associated with plastic loading $\mathrm{d}\mathbf{E}_{(n)}$ is directed outside the current yield surface in strain space. Strain increment of elastic unloading $\delta\mathbf{E}_{(n)}$ is directed inside the current yield surface.

where $\boldsymbol{\Lambda}_{(n)} = \boldsymbol{\Lambda}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right)$ is the tensor of instantaneous elastic moduli of the material at the considered state of strain and internal structure.

An increment of strain directed outside the current yield surface constitutes plastic loading. The resulting increment of stress consists of elastic and plastic parts, such that

$$\dot{\mathbf{T}}_{(n)} = \dot{\mathbf{T}}^{\mathrm{e}}_{(n)} + \dot{\mathbf{T}}^{\mathrm{p}}_{(n)} = \boldsymbol{\Lambda}_{(n)} : \dot{\mathbf{E}}_{(n)} + \dot{\mathbf{T}}^{\mathrm{p}}_{(n)}. \tag{8.2.5}$$

During plastic loading increment, the yield surface locally expands, while the strain state remains on the yield surface. The consistency condition assuring this is

$$g_{(n)}\left(\mathbf{E}_{(n)} + \mathrm{d}\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}\right) = 0. \tag{8.2.6}$$

The elastic stress decrement $\mathrm{d}^{\mathrm{e}}\mathbf{T}_{(n)}$ is associated with the elastic removal of the strain increment $\mathrm{d}\mathbf{E}_{(n)}$ from the state of strain $\mathbf{E}_{(n)} + \mathrm{d}\mathbf{E}_{(n)}$, where the elastic moduli are $\boldsymbol{\Lambda}_{(n)} + \mathrm{d}\boldsymbol{\Lambda}_{(n)}$. This is $\mathrm{d}^{\mathrm{e}}\mathbf{T}_{(n)} = \left(\boldsymbol{\Lambda}_{(n)} + \mathrm{d}\boldsymbol{\Lambda}_{(n)}\right) : \mathrm{d}\mathbf{E}_{(n)}$, which is, to first order, equal to $\boldsymbol{\Lambda}_{(n)} : \mathrm{d}\mathbf{E}_{(n)}$. Thus, in the limit

$$\dot{\mathbf{T}}^{\mathrm{e}}_{(n)} = \boldsymbol{\Lambda}_{(n)} : \dot{\mathbf{E}}_{(n)}. \tag{8.2.7}$$

The plastic part of the stress rate $\dot{\mathbf{T}}^{\mathrm{p}}_{(n)}$ corresponds to residual stress decrement $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ in the considered infinitesimal strain cycle (Fig. 8.3). A transition between elastic unloading and plastic loading is a neutral loading. In this case an infinitesimal strain increment is tangential to the yield surface and represents purely elastic deformation. Therefore,

$$\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} \begin{cases} > 0, & \text{for plastic loading,} \\ = 0, & \text{for neutral loading,} \\ < 0, & \text{for elastic unloading.} \end{cases} \tag{8.2.8}$$

The gradient $\partial g_{(n)}/\partial \mathbf{E}_{(n)}$ is codirectional with the outward normal to a locally smooth yield surface $g_{(n)} = 0$ at the state of strain $\mathbf{E}_{(n)}$. For incrementally linear response, all infinitesimal increments $\mathrm{d}\mathbf{E}_{(n)}$, which have equal projections on the normal $\partial g_{(n)}/\partial \mathbf{E}_{(n)}$ (thus forming a cone around $\partial g_{(n)}/\partial \mathbf{E}_{(n)}$), produce the same plastic increment of stress $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$. The components obtained by projecting $\mathrm{d}\mathbf{E}_{(n)}$ on the plane tangential to the yield surface represent elastic deformation only (Fig. 8.4).

8.2.2. Yield Surface in Stress Space

If the yield surface is introduced in stress space, it can be generally expressed as

$$f_{(n)}\left(\mathbf{T}_{(n)}, \mathcal{H}\right) = 0. \tag{8.2.9}$$

The stress $\mathbf{T}_{(n)}$ is conjugate to strain $\mathbf{E}_{(n)}$, and the function $f_{(n)}$ corresponds to $g_{(n)}$ such that

$$f_{(n)}\left[\mathbf{T}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right), \mathcal{H}\right] = g_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right) = 0. \tag{8.2.10}$$

This implies that physically identical yield conditions are imposed in both stress and strain spaces. The shape of the yield surface is at each stage of deformation different for different choices of $\mathbf{T}_{(n)}$, so that different functions $f_{(n)}$ correspond to different n. It will be assumed that elastic response within the yield surface is Green-elastic, associated with the complementary strain energy

$$\Phi_{(n)} = \Phi_{(n)}\left(\mathbf{T}_{(n)}, \mathcal{H}\right) \tag{8.2.11}$$

per unit reference volume. Since $\Phi_{(n)}$ is not measure invariant (see Section 4.3), the index (n) is attached to Φ. We assume here that at any given $\mathcal{H}$ there is a one-to-one relationship between $\mathbf{T}_{(n)}$ and $\mathbf{E}_{(n)}$, such that

$$\mathbf{E}_{(n)} = \frac{\partial \Phi_{(n)}}{\partial \mathbf{T}_{(n)}}\,. \tag{8.2.12}$$

Let the stress state $\mathbf{T}_{(n)}$ be on the current yield surface. If material is in the hardening range relative to the pair $\mathbf{E}_{(n)}$ and $\mathbf{T}_{(n)}$ (precise definition

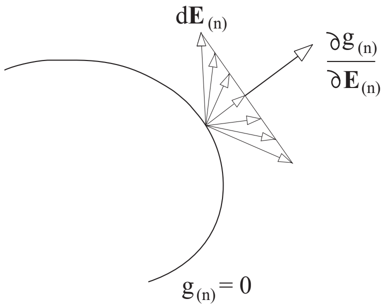

FIGURE 8.4. All strain increments $\mathrm{d}\mathbf{E}_{(n)}$ within a cone around the yield surface normal in strain space, which have the same projection on the axis of the cone, give rise to the same plastic stress increment $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$.

of hardening is given in Sections 8.8 and 9.2), an increment of stress $\mathrm{d}\mathbf{T}_{(n)}$ directed inside the yield surface will cause purely elastic deformation ($d\mathcal{H} = 0$). This constitutes an elastic unloading from the current yield surface. The corresponding incremental elastic response is governed by the rate-type equation

$$\dot{\mathbf{E}}_{(n)} = \mathbf{M}_{(n)} : \dot{\mathbf{T}}_{(n)}, \quad \mathbf{M}_{(n)} = \frac{\partial^2 \Phi_{(n)}}{\partial \mathbf{T}_{(n)} \otimes \partial \mathbf{T}_{(n)}}. \tag{8.2.13}$$

The tensor $\mathbf{M}_{(n)} = \mathbf{M}_{(n)}\left(\mathbf{T}_{(n)}, \mathcal{H}\right)$ is the tensor of instantaneous elastic compliance of the material at the considered state of stress and internal structure.

An increment of stress directed outside the current yield surface constitutes plastic loading in the hardening range of the material response. The resulting increment of strain consists of elastic and plastic parts, such that

$$\dot{\mathbf{E}}_{(n)} = \dot{\mathbf{E}}^{\mathrm{e}}_{(n)} + \dot{\mathbf{E}}^{\mathrm{p}}_{(n)} = \mathbf{M}_{(n)} : \dot{\mathbf{T}}_{(n)} + \dot{\mathbf{E}}^{\mathrm{p}}_{(n)}. \tag{8.2.14}$$

During plastic loading, the yield surface of a hardening material locally expands, while the stress state remains on it. The consistency condition that assures this is

$$f\left(\mathbf{T}_{(n)} + \mathrm{d}\mathbf{T}_{(n)}, \mathcal{H} + d\mathcal{H}\right) = 0. \tag{8.2.15}$$

The elastic increment of strain $\mathrm{d}^{\mathrm{e}}\mathbf{E}_{(n)}$ is recovered upon elastic unloading of the stress increment $\mathrm{d}\mathbf{T}_{(n)}$. Since elastic unloading takes place from the state of stress $\mathbf{T}_{(n)} + \mathrm{d}\mathbf{T}_{(n)}$, where the elastic compliance is $\mathbf{M}_{(n)} + \mathrm{d}\mathbf{M}_{(n)}$,

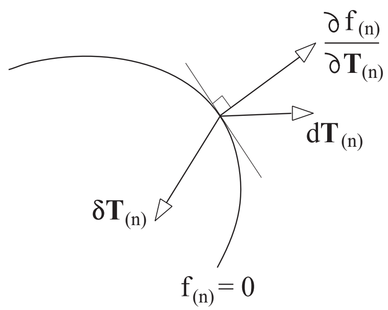

FIGURE 8.5. Stress increment associated with plastic loading $d\mathbf{T}_{(n)}$ is directed outside the current yield surface in stress space. Stress increment of elastic unloading $\delta\mathbf{T}_{(n)}$ is directed inside the current yield surface.

the removal of the stress increment $d\mathbf{T}_{(n)}$ recovers the elastic deformation $d^e\mathbf{E}_{(n)} = \left(\mathbf{M}_{(n)} + d\mathbf{M}_{(n)}\right) : d\mathbf{T}_{(n)}$. To first order this is equal to $\mathbf{M}_{(n)} : d\mathbf{T}_{(n)}$, and in the limit we have

$$\dot{\mathbf{E}}^{e}_{(n)} = \mathbf{M}_{(n)} : \dot{\mathbf{T}}_{(n)}, \tag{8.2.16}$$

as used in Eq. (8.2.14). The plastic part of the strain rate $\dot{\mathbf{E}}^{p}_{(n)}$ corresponds to residual increment of strain $d^p\mathbf{E}_{(n)}$, left upon removal of the stress increment $d\mathbf{T}_{(n)}$ (Fig. 8.5).

A transition between elastic unloading and plastic loading is a neutral loading. Here, an infinitesimal stress increment is tangential to the yield surface and produces only elastic deformation. Thus, we have in the hardening range

$$\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} : \dot{\mathbf{T}}_{(n)} \begin{cases} > 0, & \text{for plastic loading,} \\ = 0, & \text{for neutral loading,} \\ < 0, & \text{for elastic unloading.} \end{cases} \tag{8.2.17}$$

The gradient $\partial f_{(n)}/\partial \mathbf{T}_{(n)}$ is codirectional with the outward normal to a locally smooth yield surface $f_{(n)} = 0$ at the state of stress $\mathbf{T}_{(n)}$. Assuming incrementally linear response, it follows that all infinitesimal increments $d\mathbf{T}_{(n)}$, which have equal projection on $\partial f_{(n)}/\partial \mathbf{T}_{(n)}$, thus forming a cone around $\partial f_{(n)}/\partial \mathbf{T}_{(n)}$, produce the same plastic increment of deformation $d^p\mathbf{E}_{(n)}$. The components obtained by projecting $d\mathbf{T}_{(n)}$ on the plane tangential to the yield surface give rise to elastic deformation only. This is schematically depicted in Fig. 8.6.

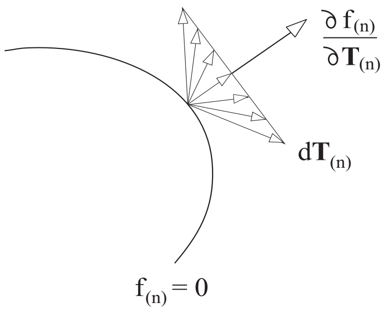

FIGURE 8.6. All stress increments $d\mathbf{T}_{(n)}$ within a cone around the yield surface normal in stress space, which have the same projection on the axis of the cone, give rise to the same plastic strain increment $d^p\mathbf{E}_{(n)}$.

In the softening range of material response, Eq. (8.2.14) still holds, although elastic and plastic parts of the strain rate have purely formal significance, because in the softening range it is not physically possible to perform an infinitesimal cycle of stress starting from the stress point on the yield surface. It should be noted, however, that the hardening is a relative term: the material may be in the hardening range relative to one pair of stress and strain measures, and in the softening range relative to another pair (Hill, 1978).

There are theories of plasticity proposed for rate-independent response which do not use the concept of the yield surface, such as the endochronic theory of Valanis (1971,1975), and a generalized theory of plasticity by Lubliner (1974,1984,1991). They are not discussed in this book, but we refer to original papers, and to Bažant (1978), Murakami and Read (1987), and Hüttel and Matzenmiller (1999). Gurtin (1983) developed a hypoelastic formulation of plasticity in which the existence of the yield surface is a consequence rather than an initial assumption of the theory. Pipkin and Rivlin (1965) earlier proposed a functional-type theory for rate-independent plasticity in which the strain history was defined as a function of the arc length along the strain path. See also Ilyushin (1954) for his geometric theory of plasticity, and Mróz (1966) for his nonlinear formulation of the rate-type theory. The so-called deformation theory of plasticity for proportional or nearly proportional loading paths is presented separately in Section 9.11.

8.3. Normality Rules

Let $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ be the plastic increment of strain produced by the stress increment $\mathrm{d}\mathbf{T}_{(n)}$ applied from the state of stress $\mathbf{T}_{(n)}$ on the current yield surface. Denote by $\delta\mathbf{T}_{(n)}$ an arbitrary stress variation emanating from the same $\mathbf{T}_{(n)}$ and directed inside the yield surface. If

$$\delta\mathbf{T}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} < 0, \tag{8.3.1}$$

for every such $\delta\mathbf{T}_{(n)}$, the material obeys the normality rule: the plastic strain increment must be codirectional with the outward normal to a locally smooth yield surface in stress space (Fig. 8.7), whereas at the vertex it must lie within or on the cone of limiting outward normals (Hill and Rice, 1973).

Since

$$\delta\mathbf{T}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = -\delta\mathbf{E}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}, \tag{8.3.2}$$

Equation (8.3.1) implies

$$\delta\mathbf{E}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} > 0, \tag{8.3.3}$$

for all strain variations $\delta\mathbf{E}_{(n)}$ emanating from the same $\mathbf{E}_{(n)}$ on the yield surface in strain space and directed inside the yield surface. This expresses a dual normality, requiring that $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ must be codirectional with the inward normal to a locally smooth yield surface in strain space (Fig. 8.8), with an appropriate generalization at a vertex. Further discussion of normality rules for rate-independent plastic materials is presented in Sections 8.5 and 8.6.

8.3.1. Invariance of Normality Rules

The normality rules (8.3.1) and (8.3.3) are invariant to reference configuration and strain measure, i.e., they apply for every choice of reference configuration and strain measure, or for none. In proof, we first observe that from Eqs. (8.1.7) and (8.1.18),

$$\delta\mathbf{E}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = \delta\mathbf{E}_{(n)} : \frac{\partial}{\partial\mathbf{E}_{(n)}}(\mathrm{d}^{\mathrm{p}}\Psi) = \delta(\mathrm{d}^{\mathrm{p}}\Psi), \tag{8.3.4}$$

$$\delta\mathbf{T}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = \delta\mathbf{T}_{(n)} : \frac{\partial}{\partial\mathbf{T}_{(n)}}(\mathrm{d}^{\mathrm{p}}\Phi) = \delta(\mathrm{d}^{\mathrm{p}}\Phi). \tag{8.3.5}$$

For example, $\delta(\mathrm{d}^{\mathrm{p}}\Psi)$ represents the difference between the values of $\mathrm{d}^{\mathrm{p}}\Psi$ evaluated at $\mathbf{E}_{(n)} + \delta\mathbf{E}_{(n)}$ and $\mathbf{E}_{(n)}$, for the same $\mathcal{H}$ and $\mathcal{H} + d\mathcal{H}$. Thus,

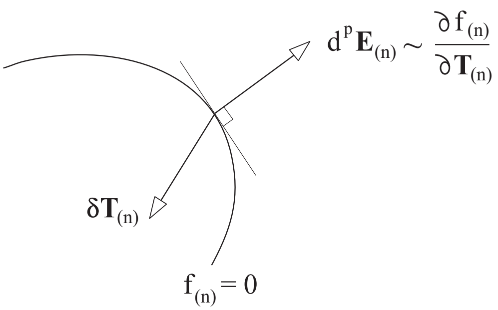

FIGURE 8.7. Normality rule in stress space. The plastic strain increment $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ is codirectional with the outward normal to a locally smooth yield surface, so that $\delta\mathbf{T}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} < 0$, where $\delta\mathbf{T}_{(n)}$ is a stress increment associated with elastic unloading.

either from (8.3.2) or (8.1.23), we have

$$\delta(\mathrm{d}^{\mathrm{p}}\Psi) = -\delta(\mathrm{d}^{\mathrm{p}}\Phi). \tag{8.3.6}$$

On the other hand,

$$\delta(\mathrm{d}^{\mathrm{p}}\Psi) = (\mathrm{d}^{\mathrm{p}}\Psi)_{\mathbf{E}_{(n)}+\mathrm{d}\mathbf{E}_{(n)}} - (\mathrm{d}^{\mathrm{p}}\Psi)_{\mathbf{E}_{(n)}} = (\delta\Psi)_{\mathcal{H}+d\mathcal{H}} - (\delta\Psi)_{\mathcal{H}} = \mathrm{d}^{\mathrm{p}}(\delta\Psi). \tag{8.3.7}$$

Since elastic work per unit mass (at fixed $\mathcal{H}$),

$$\frac{1}{\rho}\,\delta\Psi = \frac{1}{\rho}\,\mathbf{T}_{(n)} : \delta\mathbf{E}_{(n)}, \tag{8.3.8}$$

is invariant to choice of reference state and strain measure (provided that all strains define the same geometry change), it follows that

$$\frac{1}{\rho}\,\mathrm{d}^{\mathrm{p}}(\delta\Psi) = \frac{1}{\rho}\,\delta\mathbf{E}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} \tag{8.3.9}$$

is also the reference and strain measure invariant. Therefore, since the mass density of the reference state is positive ($\rho > 0$), we conclude that both normality rules (8.3.1) and (8.3.3) are invariant to choice of reference configuration and strain measure.

It is noted that

$$\begin{aligned}\frac{1}{\rho}\left(\delta\mathbf{T}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\right) &= \frac{1}{\rho}\,\delta\mathbf{T}_{(n)} : \left(\mathrm{d}\mathbf{E} - \mathbf{M}_{(n)} : \mathrm{d}\mathbf{T}_{(n)}\right) \\ &= \frac{1}{\rho}\left(\delta\mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} - \delta\mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)}\right),\end{aligned} \tag{8.3.10}$$

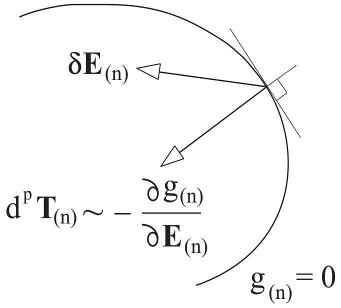

FIGURE 8.8. Normality rule in strain space. The plastic stress increment $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ is codirectional with the inward normal to a locally smooth yield surface, so that $\delta\mathbf{E}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} > 0$, where $\delta\mathbf{E}_{(n)}$ is a strain increment associated with elastic unloading.

which demonstrates that the combination on the far right-hand side is invariant. This is a particular type of Hill's (1972) invariant bilinear form.

Normality rules can be expressed in terms of internal variables and conjugate energetic forces by recalling that, from Eq. (8.1.5),

$$\frac{1}{\rho}\,\mathrm{d}^{\mathrm{p}}\Psi = -f_j\,d\xi_j. \tag{8.3.11}$$

This implies that

$$-\frac{1}{\rho}\,\delta\mathbf{T}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = \frac{1}{\rho}\,\delta(\mathrm{d}^{\mathrm{p}}\Psi) = -\delta f_j\,d\xi_j. \tag{8.3.12}$$

Thus, the normality rule (8.3.1) is obeyed if

$$\delta f_j\,d\xi_j < 0. \tag{8.3.13}$$

The inequality is, for example, guaranteed if each increment $d\xi_j$ is, at given $\mathcal{H}$, governed only by its own energetic force f_j. Indeed, the yield criterion for the j-th variable is solely expressed in terms of f_j as

$$f_j^{\mathrm{L}} < f_j < f_j^{\mathrm{U}}. \tag{8.3.14}$$

The yield emanating from the lower bound f_j^{L} involves $d\xi_j < 0$, while any elastic variation δf_j must be positive. The yield emanating from the upper bound f_j^{U} involves $d\xi_j > 0$, while any elastic variation δf_j must be negative. Thus, for each j the product $\delta f_j\,d\xi_j$ is negative, and so is the sum over all j (Rice, 1971).

8.4. Flow Potential for Rate-Dependent Materials

The constitutive framework of Sections 8.2 and 8.3 applies to rate-dependent plastic materials which exhibit elastic response to sufficiently rapid loading or straining (instantaneous elasticity). For the plastic part of strain rate we take

$$\frac{\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{E}_{(n)}}{\mathrm{d}t} - \mathbf{M}_{(n)} : \frac{\mathrm{d}\mathbf{T}_{(n)}}{\mathrm{d}t}\,, \tag{8.4.1}$$

where t is the physical time. The plastic part of strain rate is a function of the current stress and accumulated inelastic history $\mathcal{H}$,

$$\frac{\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}}{\mathrm{d}t} = \frac{\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}}{\mathrm{d}t}\left(\mathbf{T}_{(n)}, \mathcal{H}\right). \tag{8.4.2}$$

Therefore, an instantaneous change of stress $\delta\mathbf{T}_{(n)}$ causes an instantaneous change of the plastic part of strain rate, but not the change of $\mathcal{H}$ or the plastic strain itself. We can thus examine the functional dependence of $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}/\mathrm{d}t$ on $\mathbf{T}_{(n)}$ at any fixed $\mathcal{H}$. If this is such that $\delta\mathbf{T}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}/\mathrm{d}t$ is a perfect differential at fixed $\mathcal{H}$, i.e., if

$$\delta\mathbf{T}_{(n)} : \frac{\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}}{\mathrm{d}t} = \delta\hat{\Omega}\left(\mathbf{T}_{(n)}, \mathcal{H}\right), \tag{8.4.3}$$

then (Hill and Rice, 1973)

$$\frac{\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}}{\mathrm{d}t} = \frac{\partial\hat{\Omega}\left(\mathbf{T}_{(n)}, \mathcal{H}\right)}{\partial\mathbf{T}_{(n)}}\,. \tag{8.4.4}$$

This establishes the existence of a scalar flow potential for the plastic part of strain rate in rate-dependent materials,

$$\Omega = \hat{\Omega}\left(\mathbf{T}_{(n)}, \mathcal{H}\right). \tag{8.4.5}$$

Since

$$\delta\mathbf{T}_{(n)} : \frac{\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}}{\mathrm{d}t} = -\delta\mathbf{E}_{(n)} : \frac{\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}}{\mathrm{d}t}\,, \tag{8.4.6}$$

there follows

$$\frac{\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}}{\mathrm{d}t} = -\frac{\partial\bar{\Omega}\left(\mathbf{E}_{(n)}, \mathcal{H}\right)}{\partial\mathbf{E}_{(n)}}\,. \tag{8.4.7}$$

This shows that Ω, when expressed in terms of strain and inelastic history,

$$\Omega = \bar{\Omega}\left(\mathbf{E}_{(n)}, \mathcal{H}\right), \tag{8.4.8}$$

is also a flow potential for the plastic part of stress rate.

The normality rules (8.4.4) and (8.4.7) are clearly invariant to choice of reference configuration and strain measure. Deduction of the normality rules for rate-independent materials as singular limits of the normality rules for rate-dependent materials has been demonstrated by Rice (1970, 1971).

If it is assumed that, at a given $\mathcal{H}$, each $d\xi_j/\mathrm{d}t$ depends only on its own energetic force,

$$\frac{d\xi_j}{\mathrm{d}t} = \mathrm{function}(f_j, \mathcal{H}), \tag{8.4.9}$$

then

$$\delta\mathbf{T}_{(n)} : \frac{\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}}{\mathrm{d}t} = \delta f_j \, \frac{d\xi_j}{\mathrm{d}t} \tag{8.4.10}$$

is a perfect differential, because each term in the sum on the right-hand side is a perfect differential. This, for example, establishes the existence of flow potential in rate-dependent crystal plasticity, in which it is assumed that the crystallographic slip on each slip system is governed by the resolved shear stress on that system. A study of crystal plasticity is presented in Chapter 12.

8.5. Ilyushin's Postulate

The remaining sections in this chapter deal with the so-called plasticity postulates of rate-independent plasticity. These postulates are in the form of constitutive inequalities, proposed for certain types of materials undergoing plastic deformation. The two most well-known are by Drucker (1951) and Ilyushin (1961). They are discussed here within the framework of conjugate stress and strain measures, following the presentations by Hill (1968), and Hill and Rice (1973). Particular attention is given to the relationship between these postulates and the plastic normality rules. We begin with the Ilyushin postulate, and consider the Drucker postulate in Section 8.6. Other postulates are discussed in Section 8.9.

Ilyushin (1961) proposed that the net work in an isothermal cycle of strain must be positive,

$$\oint_E \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} > 0, \tag{8.5.1}$$

if a cycle involves plastic deformation at some stage. The integral in (8.5.1) over an elastic strain cycle is equal to zero, which implies the existence of elastic potential, such that $\mathbf{T}_{(n)} = \partial\Psi/\partial\mathbf{E}_{(n)}$. Since the cycle of strain that includes plastic deformation in general does not return the material to its state at the beginning of the cycle, the inequality (8.5.1) is not a law of

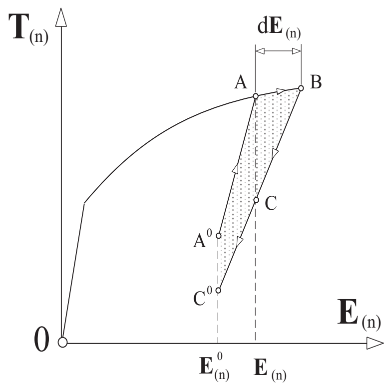

FIGURE 8.9. A strain cycle A^0ABCC^0 involving plastic deformation along an infinitesimal segment AB.

thermodynamics. For example, it does not apply to materials which dissipate energy by friction (Drucker, 1964; Rice, 1971; Dafalias, 1977; Chandler, 1985).

The inequality (8.5.1) is invariant to change of the reference configuration and strain measure, because it is based on an invariant work quantity. The value of the integral over a strain cycle that involves plastic deformation, and that begins and ends at the state of identical geometry, is independent of n and the reference state used to define $\mathbf{E}_{(n)}$. This has been examined in detail by Hill (1968).

The Ilyushin postulate imposes constitutive restrictions on the materials to which it applies. To elaborate, let $A^0\left(\mathbf{E}^0_{(n)}, \mathcal{H}\right)$ be an arbitrary state within the yield surface in strain space. Consider a strain cycle that starts from A^0, includes an elastic segment from A^0 to the state $A\left(\mathbf{E}_{(n)}, \mathcal{H}\right)$ on the current yield surface, followed by an infinitesimal elastoplastic segment from A to $B\left(\mathbf{E}_{(n)} + \mathrm{d}\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}\right)$, and elastic unloading segments from B to $C(\mathbf{E}_{(n)}, \mathcal{H}+d\mathcal{H})$, and from C to $C^0\left(\mathbf{E}^0_{(n)}, \mathcal{H} + d\mathcal{H}\right)$, as shown in Fig. 8.9. By using Eq. (8.1.3), the work done along the segment A^0A is readily evaluated to be

$$\int_{A^0}^{A} \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = \int_{A^0}^{A} \frac{\partial \Psi}{\partial \mathbf{E}_{(n)}} : \mathrm{d}\mathbf{E}_{(n)} \\ = \Psi\left(\mathbf{E}_{(n)}, \mathcal{H}\right) - \Psi\left(\mathbf{E}^0_{(n)}, \mathcal{H}\right), \tag{8.5.2}$$

while along the segment CC^0,

$$\begin{aligned}\int_{C}^{C^0} \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} &= \int_{C}^{C^0} \frac{\partial \Psi}{\partial \mathbf{E}_{(n)}} : \mathrm{d}\mathbf{E}_{(n)} \\ &= \Psi\left(\mathbf{E}^0_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \Psi\left(\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}\right).\end{aligned} \tag{8.5.3}$$

The work done along the segments AB and BC is, by the trapezoidal rule of quadrature,

$$\int_{A}^{B} \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} + \frac{1}{2}\,\mathrm{d}\mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)}, \tag{8.5.4}$$

$$\int_{B}^{C} \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = -\mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} - \frac{1}{2}\left(\mathrm{d}\mathbf{T}_{(n)} + \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}\right) : \mathrm{d}\mathbf{E}_{(n)}, \tag{8.5.5}$$

accurate to second-order terms. The plastic stress increment $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ is introduced following Eq. (8.2.7), and is indicated schematically in Fig. 8.1. Consequently, the net work in the considered strain cycle is

$$\oint_{E} \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = -\frac{1}{2}\,\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} + (\mathrm{d}^{\mathrm{p}}\Psi)^0 - \mathrm{d}^{\mathrm{p}}\Psi, \tag{8.5.6}$$

where

$$\mathrm{d}^{\mathrm{p}}\Psi = \Psi\left(\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \Psi\left(\mathbf{E}_{(n)}, \mathcal{H}\right), \tag{8.5.7}$$

$$(\mathrm{d}^{\mathrm{p}}\Psi)^0 = \Psi\left(\mathbf{E}^0_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \Psi\left(\mathbf{E}^0_{(n)}, \mathcal{H}\right). \tag{8.5.8}$$

8.5.1. Normality Rule in Strain Space

If the strain cycle emanates from the state on the yield surface, i.e., if $A^0 = A$ and $\mathbf{E}^0_{(n)} = \mathbf{E}_{(n)}$, Eq. (8.5.6) reduces to

$$\oint_{E} \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = -\frac{1}{2}\,\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)}. \tag{8.5.9}$$

By Ilyushin's postulate this must be positive, so that

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} < 0. \tag{8.5.10}$$

Since during plastic loading the strain increment $\mathrm{d}\mathbf{E}_{(n)}$ is directed outward from the yield surface, and since the same $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ is associated with a fan of infinitely many $\mathrm{d}\mathbf{E}_{(n)}$ around the normal $\partial g_{(n)}/\partial \mathbf{E}_{(n)}$, all having the same projection on that normal, the inequality (8.5.10) requires that $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ is codirectional with the inward normal to a locally smooth yield surface in strain $\mathbf{E}_{(n)}$ space, i.e.,

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = -\mathrm{d}\gamma_{(n)}\,\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}}\,. \tag{8.5.11}$$

The scalar multiplier

$$\mathrm{d}\gamma_{(n)} > 0 \tag{8.5.12}$$

is referred to as the loading index. At the vertex of the yield surface, $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ must lie within the cone of limiting inward normals.

The inequality (8.5.10) and the normality rule (8.5.11) hold for all pairs of conjugate stress and strain measures, irrespective of the nature of elastic changes caused by plastic deformation, or possible elastic nonlinearities within the yield surface. Also, (8.5.11) applies regardless of whether the material is in the hardening or softening range.

8.5.2. Convexity of the Yield Surface in Strain Space

If elastic response is nonlinear, we can not conclude from (8.5.6) that the yield surface is necessarily convex. Consider, instead, a linear elastic response within the yield surface, for which the strain energy can be expressed as

$$\Psi(\mathbf{E}_{(n)}, \mathcal{H}) = \frac{1}{2}\,\boldsymbol{\Lambda}_{(n)}(\mathcal{H}) :: \left[\left(\mathbf{E}_{(n)} - \mathbf{E}^{\mathrm{p}}_{(n)}(\mathcal{H})\right) \otimes \left(\mathbf{E}_{(n)} - \mathbf{E}^{\mathrm{p}}_{(n)}(\mathcal{H})\right)\right], \tag{8.5.13}$$

so that

$$\mathbf{T}_{(n)} = \frac{\partial \Psi}{\partial \mathbf{E}_{(n)}} = \boldsymbol{\Lambda}_{(n)}(\mathcal{H}) : \left(\mathbf{E}_{(n)} - \mathbf{E}^{\mathrm{p}}_{(n)}(\mathcal{H})\right). \tag{8.5.14}$$

The tensor $\mathbf{E}^{\mathrm{p}}_{(n)}(\mathcal{H})$ represents a residual or plastic strain that is left upon (actual or conceptual) unloading to zero stress, at fixed values of the internal structure $\mathcal{H}$. Incorporating (8.5.13) and (8.5.14) into (8.5.6) gives

$$\begin{aligned}\oint_E \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = &-\frac{1}{2}\,\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} + \left(\mathbf{E}^{0}_{(n)} - \mathbf{E}_{(n)}\right) : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} \\ &+ \frac{1}{2}\,\mathrm{d}\boldsymbol{\Lambda}_{(n)} :: \left[\left(\mathbf{E}_{(n)} - \mathbf{E}^{0}_{(n)}\right) \otimes \left(\mathbf{E}_{(n)} - \mathbf{E}^{0}_{(n)}\right)\right],\end{aligned} \tag{8.5.15}$$

where

$$\mathrm{d}\boldsymbol{\Lambda}_{(n)} = \boldsymbol{\Lambda}_{(n)}(\mathbf{E}_{(n)}, \mathcal{H} + d\mathcal{H}) - \boldsymbol{\Lambda}_{(n)}(\mathbf{E}_{(n)}, \mathcal{H}). \tag{8.5.16}$$

In the derivation, the following relationship was used

$$\begin{aligned}\mathbf{\Lambda}_{(n)}(\mathcal{H}+d\mathcal{H}) : \mathbf{E}^{\mathrm{p}}_{(n)}(\mathcal{H}+d\mathcal{H}) - \mathbf{\Lambda}_{(n)}(\mathcal{H}) : \mathbf{E}^{\mathrm{p}}_{(n)}(\mathcal{H}) \\ = \mathrm{d}\mathbf{\Lambda}_{(n)} : \mathbf{E}_{(n)} - \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}.\end{aligned} \tag{8.5.17}$$

By taking the strain cycle with a sufficiently small $\mathrm{d}\mathbf{E}_{(n)}$ comparing to $\mathbf{E}_{(n)} - \mathbf{E}^0_{(n)}$, the first term in Eq. (8.5.15) can be neglected, and for such cycles

$$\begin{aligned}\oint_E \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = \left(\mathbf{E}^0_{(n)} - \mathbf{E}_{(n)}\right) : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} \\ + \frac{1}{2}\,\mathrm{d}\mathbf{\Lambda}_{(n)} :: \left[\left(\mathbf{E}_{(n)} - \mathbf{E}^0_{(n)}\right) \otimes \left(\mathbf{E}_{(n)} - \mathbf{E}^0_{(n)}\right)\right] > 0,\end{aligned} \tag{8.5.18}$$

i.e.,

$$\left(\mathbf{E}^0_{(n)} - \mathbf{E}_{(n)}\right) : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} > -\frac{1}{2}\,\mathrm{d}\mathbf{\Lambda}_{(n)} :: \left[\left(\mathbf{E}_{(n)} - \mathbf{E}^0_{(n)}\right) \otimes \left(\mathbf{E}_{(n)} - \mathbf{E}^0_{(n)}\right)\right]. \tag{8.5.19}$$

Thus, if the change of elastic stiffness caused by plastic deformation is such that $\mathrm{d}\mathbf{\Lambda}_{(n)}$ is negative semi-definite, or if there is no change in elastic stiffness, from (8.5.19) it follows that (Fig. 8.10)

$$\left(\mathbf{E}^0_{(n)} - \mathbf{E}_{(n)}\right) : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} > 0. \tag{8.5.20}$$

Since $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ is codirectional with the inward normal to a locally smooth yield surface in strain $\mathbf{E}_{(n)}$ space, (8.5.20) implies that the yield surface is convex. It should be observed, however, that for some $\mathbf{E}_{(n)}$ and $\mathbf{T}_{(n)}$ the stiffness change $\mathrm{d}\mathbf{\Lambda}_{(n)}$ can be negative definite, but not for others, so that convexity of the yield surface is not invariant to change of stress and strain measures.

Returning to Eq. (8.5.15), we can write

$$\begin{aligned}\oint_E \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = -\frac{1}{2}\,\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} \\ + \frac{1}{2}\left(\mathbf{E}^0_{(n)} - \mathbf{E}_{(n)}\right) : \left[\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} + \left(\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}\right)^0\right],\end{aligned} \tag{8.5.21}$$

where

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = \mathbf{T}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}+d\mathcal{H}\right) - \mathbf{T}_{(n)}\left(\mathbf{E}_{(n)}, \mathcal{H}\right), \tag{8.5.22}$$

$$\left(\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}\right)^0 = \mathbf{T}_{(n)}\left(\mathbf{E}^0_{(n)}, \mathcal{H}+d\mathcal{H}\right) - \mathbf{T}_{(n)}\left(\mathbf{E}^0_{(n)}, \mathcal{H}\right). \tag{8.5.23}$$

If there is no change in elastic stiffness,

$$\left(\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}\right)^0 = \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}. \tag{8.5.24}$$

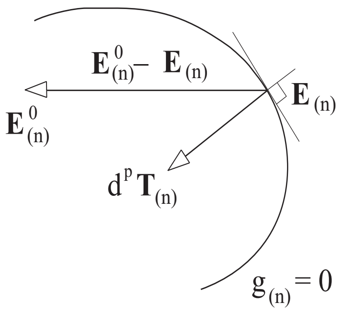

FIGURE 8.10. The plastic stress increment $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ is codirectional with the inward normal to locally smooth yield surface in strain space, so that $(\mathbf{E}^0_{(n)} - \mathbf{E}_{(n)}) : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} > 0$, where $\mathbf{E}_{(n)}$ is the strain state on the current yield surface and $\mathbf{E}^0_{(n)}$ is the strain state within the yield surface.

8.5.3. Normality Rule in Stress Space

By taking a trace product of Eq. (8.2.14) with $\mathbf{\Lambda}_{(n)} = \mathbf{M}^{-1}_{(n)}$, we obtain

$$\mathbf{\Lambda}_{(n)} : \dot{\mathbf{E}}_{(n)} = \dot{\mathbf{T}}_{(n)} + \mathbf{\Lambda}_{(n)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(n)}, \tag{8.5.25}$$

and comparison with Eq. (8.2.13) establishes

$$\dot{\mathbf{T}}^{\mathrm{p}}_{(n)} = -\mathbf{\Lambda}_{(n)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(n)}. \tag{8.5.26}$$

Thus, to first order,

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = -\mathbf{\Lambda}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}. \tag{8.5.27}$$

Since for any elastic strain increment $\delta\mathbf{E}_{(n)}$, emanating from a point on the yield surface in strain space and directed inside of it,

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} : \delta\mathbf{E}_{(n)} > 0, \tag{8.5.28}$$

the substitution of (8.5.27) into (8.5.28) gives

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} : \mathbf{\Lambda}_{(n)} : \delta\mathbf{E}_{(n)} = \mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} : \delta\mathbf{T}_{(n)} < 0. \tag{8.5.29}$$

Here,

$$\delta\mathbf{T}_{(n)} = \mathbf{\Lambda}_{(n)} : \delta\mathbf{E}_{(n)} \tag{8.5.30}$$

is the stress increment from a point on the yield surface in stress space, directed inside of the yield surface (elastic unloading increment associated with elastic strain increment $\delta\mathbf{E}_{(n)}$). Inequality (8.5.29) holds for any such

$\delta \mathbf{T}_{(n)}$ and, consequently, $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ must be codirectional with the outward normal to a locally smooth yield surface in stress $\mathbf{T}_{(n)}$ space, i.e.,

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = \mathrm{d}\gamma_{(n)} \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}, \quad \mathrm{d}\gamma_{(n)} > 0. \tag{8.5.31}$$

At a vertex of the yield surface, $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ must lie within the cone of limiting outward normals. Inequality (8.5.29) and the normality rule (8.5.31) hold for all pairs of conjugate stress and strain measures.

If material is in the hardening range relative to $\mathbf{E}_{(n)}$ and $\mathbf{T}_{(n)}$, the stress increment $\mathrm{d}\mathbf{T}_{(n)}$ producing plastic deformation $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ is directed outside the yield surface, and satisfies the condition

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} > 0. \tag{8.5.32}$$

If material is in the softening range, the stress increment $\mathrm{d}\mathbf{T}_{(n)}$ producing plastic deformation $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ is directed inside the yield surface, and satisfies the reversed inequality in (8.5.32). The normality rule (8.5.31) applies to both hardening and softening. Inequality (8.5.32) is not measure invariant, since the material may be in the hardening range relative to one pair of conjugate stress and strain measures, but in the softening range relative to another pair.

In view of (8.5.11), (8.5.27), and (8.5.31), the yield surface normals in stress and strain space are related by

$$\frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}} = \mathbf{\Lambda}_{(n)} : \frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}. \tag{8.5.33}$$

This also follows directly from Eq. (8.2.10) by partial differentiation.

8.5.4. Additional Inequalities for Strain Cycles

Additional inequalities can be derived as follows. First, by partial differentiation we have

$$\mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = \mathrm{d}\left(\mathbf{T}_{(n)} : \mathbf{E}_{(n)}\right) - \mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)}. \tag{8.5.34}$$

The substitution of Eq. (8.5.34) into the integral of (8.5.1) gives, for the strain cycle A^0ABCC^0,

$$\begin{aligned} \oint_E \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = &\left[\mathbf{T}_{(n)}\left(\mathbf{E}^0_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \mathbf{T}_{(n)}\left(\mathbf{E}^0_{(n)}, \mathcal{H}\right)\right] : \mathbf{E}^0_{(n)} \\ &- \oint \mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)}. \end{aligned} \tag{8.5.35}$$

This must be positive by Ilyushin's postulate, so that

$$\oint_E \mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} < \left[\mathbf{T}_{(n)}\left(\mathbf{E}^0_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \mathbf{T}_{(n)}\left(\mathbf{E}^0_{(n)}, \mathcal{H}\right)\right] : \mathbf{E}^0_{(n)}. \tag{8.5.36}$$

Alternatively, the inequality (8.5.36) can be written as

$$\oint_E \left(\mathbf{E}_{(n)} - \mathbf{E}^0_{(n)}\right) : \mathrm{d}\mathbf{T}_{(n)} < 0, \tag{8.5.37}$$

for all strain cycles that at some stage involve plastic deformation (not necessarily infinitesimal). Since (8.5.1) is invariant, the inequality (8.5.37) holds irrespective of the reference state and strain measure. In particular, if we choose a reference state for strain measure $\mathbf{E}_{(n)}$ to be the state A^0, the strain $\mathbf{E}^0_{(n)}$ vanishes and (8.5.37) gives

$$\oint_E \mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} < 0. \tag{8.5.38}$$

This applies for all strain measures defined relative to A^0, and for all strain cycles that involve plastic deformation at some stage. Further discussion can be found in Hill (1968, 1978) and Nemat-Nasser (1983).

8.6. Drucker's Postulate

Drucker (1951) introduced a postulate by considering the work done in stress cycles. His original formulation was in the context of infinitesimal strain and is presented in Subsection 8.6.3. We consider here a (noninvariant) dual inequality to (8.5.1), which is

$$\oint_T \mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} < 0. \tag{8.6.1}$$

This means that a net complementary work (relative to measures $\mathbf{E}_{(n)}$ and $\mathbf{T}_{(n)}$) in an isothermal cycle of stress is negative, if a cycle involves plastic deformation at some stage. Inequality (8.6.1) is noninvariant because the value of the integral in (8.6.1) depends on the selected measures $\mathbf{E}_{(n)}$ and $\mathbf{T}_{(n)}$, and the reference state with respect to which they are defined. This is so because $\mathbf{T}_{(n)}$ is introduced as a conjugate stress to $\mathbf{E}_{(n)}$ such that, for the same geometry change, $\mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)}$, and not $\mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)}$, is measure invariant. Physically, cycling one stress measure does not necessarily imply cycling of another stress measure. Thus, for different n the integral in (8.6.1) corresponds to different physical cycles, and has different values.

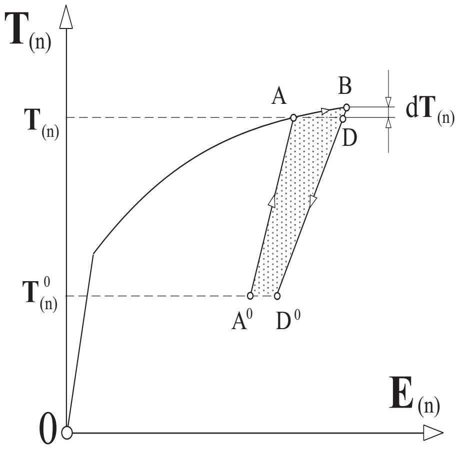

FIGURE 8.11. A stress cycle A^0ABDD^0 involving plastic deformation along an infinitesimal segment AB.

Constitutive inequalities which depend on the choice of reference configuration are not well suited for plastically deforming materials, for which no preferred state can be single out (Hill, 1968). Nevertheless, we proceed with the analysis of (8.6.1) and examine its consequences for different choices of strain measure and reference state.

First, since the cycle of stress that involves plastic deformation in general does not return the material to its state at the beginning of the cycle, the inequality (8.6.1) does not represent a law of thermodynamics for any n. If the integral in (8.6.1) vanishes for stress cycles that give rise to elastic deformation only (for a selected pair $\mathbf{E}_{(n)}$ and $\mathbf{T}_{(n)}$), the material admits a complementary strain energy $\Phi_{(n)} = \Phi_{(n)}\left(\mathbf{T}_{(n)}, \mathcal{H}\right)$, such that $\mathbf{E}_{(n)} = \partial\Phi_{(n)}/\partial\mathbf{T}_{(n)}$. In contrast to measure invariant strain energy Ψ, the complementary energy is in general not measure invariant. However, if the integral in (8.6.1) over an elastic cycle vanishes for some n, it vanishes for other n, as well.

Consider a yield surface in stress $\mathbf{T}_{(n)}$ space. Assume that within the yield surface there is one-to-one relationship between the stress $\mathbf{T}_{(n)}$ and strain $\mathbf{E}_{(n)}$, at a given state of internal structure $\mathcal{H}$. Let $A^0\left(\mathbf{T}^0_{(n)}, \mathcal{H}\right)$ be an arbitrary state within the yield surface. Consider a stress cycle that starts from A^0, includes an elastic segment from A^0 to $A\left(\mathbf{T}_{(n)}, \mathcal{H}\right)$ on the current yield surface, followed by an infinitesimal elastoplastic segment from A to $B\left(\mathbf{T}_{(n)}+\mathrm{d}\mathbf{T}_{(n)},\mathcal{H}+d\mathcal{H}\right)$, and elastic unloading segments from B to $D\left(\mathbf{T}_{(n)},\mathcal{H}+d\mathcal{H}\right)$, and from D to $D^0\left(\mathbf{T}^0_{(n)},\mathcal{H}+d\mathcal{H}\right)$; see Fig. 8.11. The complementary work along the segment A^0A is

$$\begin{aligned}\int_{A^0}^{A}\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}&=\int_{A^0}^{A}\frac{\partial\Phi_{(n)}}{\partial\mathbf{T}_{(n)}}:\mathrm{d}\mathbf{T}_{(n)}\\&=\Phi_{(n)}\left(\mathbf{T}_{(n)},\mathcal{H}\right)-\Phi_{(n)}\left(\mathbf{T}^0_{(n)},\mathcal{H}\right),\end{aligned}\tag{8.6.2}$$

while along the segment DD^0,

$$\begin{aligned}\int_{D}^{D^0}\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}&=\int_{D}^{D^0}\frac{\partial\Phi_{(n)}}{\partial\mathbf{T}_{(n)}}:\mathrm{d}\mathbf{T}_{(n)}\\&=\Phi_{(n)}\left(\mathbf{T}^0_{(n)},\mathcal{H}+d\mathcal{H}\right)-\Phi_{(n)}\left(\mathbf{T}_{(n)},\mathcal{H}+d\mathcal{H}\right).\end{aligned}\tag{8.6.3}$$

The complementary work along the segments AB and BD is, by the trapezoidal rule of quadrature,

$$\int_{A}^{B}\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}=\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}+\frac{1}{2}\,\mathrm{d}\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)},\tag{8.6.4}$$

$$\int_{B}^{C}\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}=-\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}-\frac{1}{2}\left(\mathrm{d}\mathbf{E}_{(n)}+\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\right):\mathrm{d}\mathbf{T}_{(n)},\tag{8.6.5}$$

accurate to second-order terms. The plastic strain increment $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ is defined following Eq. (8.2.13), and is indicated schematically in Fig. 8.2. Consequently,

$$\oint_T\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}=-\frac{1}{2}\,\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}+\left(\mathrm{d}^{\mathrm{p}}\Phi_{(n)}\right)^0-\mathrm{d}^{\mathrm{p}}\Phi_{(n)},\tag{8.6.6}$$

where

$$\mathrm{d}^{\mathrm{p}}\Phi_{(n)}=\Phi_{(n)}\left(\mathbf{T}_{(n)},\mathcal{H}+d\mathcal{H}\right)-\Phi_{(n)}\left(\mathbf{T}_{(n)},\mathcal{H}\right),\tag{8.6.7}$$

$$\left(\mathrm{d}^{\mathrm{p}}\Phi_{(n)}\right)^0=\Psi\left(\mathbf{T}^0_{(n)},\mathcal{H}+d\mathcal{H}\right)-\Phi_{(n)}\left(\mathbf{T}^0_{(n)},\mathcal{H}\right).\tag{8.6.8}$$

8.6.1. Normality Rule in Stress Space

Assume that material is in the hardening range relative to $\mathbf{E}_{(n)}$ and $\mathbf{T}_{(n)}$. An infinitesimal stress cycle can be performed starting from the point on the yield surface. Thus, taking $A^0=A$ and $\mathbf{T}^0_{(n)}=\mathbf{T}_{(n)}$, Eq. (8.6.6) reduces to

$$\oint_T\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}=-\frac{1}{2}\,\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}.\tag{8.6.9}$$

If the inequality (8.6.1) applies to conjugate pair $\mathbf{E}_{(n)}$, $\mathbf{T}_{(n)}$, the integral in (8.6.9) must be negative, so that

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} > 0. \tag{8.6.10}$$

During plastic loading in the hardening range relative to $\mathbf{E}_{(n)}$ and $\mathbf{T}_{(n)}$, the stress increment $\mathrm{d}\mathbf{T}_{(n)}$ is directed outward from the yield surface. Since one $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ is associated with a fan of infinitely many $\mathrm{d}\mathbf{T}_{(n)}$ around the normal $\partial f_{(n)}/\partial \mathbf{T}_{(n)}$ (all having the same projection on the normal), the inequality (8.6.10) requires that $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ is codirectional with the outward normal to a locally smooth yield surface in stress $\mathbf{T}_{(n)}$ space, i.e.,

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = \mathrm{d}\gamma_{(n)}\,\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}}\,, \quad \mathrm{d}\gamma_{(n)} > 0. \tag{8.6.11}$$

At the vertex of the yield surface, $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ must lie within the cone of limiting outward normals.

The inequality (8.6.10) and the normality rule (8.6.11) apply to a conjugate pair of stress and strain which obey (8.6.1), irrespective of the nature of elastic changes caused by plastic deformation, or possible elastic nonlinearities within the yield surface. If inequality (8.6.1) holds for all pairs of conjugate stress and strain measures, then (8.6.10) and (8.6.11) also hold with respect to all conjugate stress and strain measures.

When material is in the softening range, relative to a considered pair of stress and strain measures, it is physically impossible to perform a cycle of stress starting from a point on the yield surface. In this case, however, we can choose an infinitesimal stress cycle A^0AB, where $A^0\left(\mathbf{T}_{(n)} + \mathrm{d}\mathbf{T}_{(n)}, \mathcal{H}\right)$ is inside the yield surface, while $A\left(\mathbf{T}_{(n)}, \mathcal{H}\right)$ and $B\left(\mathbf{T}_{(n)} + \mathrm{d}\mathbf{T}_{(n)}, \mathcal{H} + d\mathcal{H}\right)$ are on the current and subsequent yield surfaces. Then,

$$\oint_T \mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} = \frac{1}{2}\,\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} < 0. \tag{8.6.12}$$

Since in the softening range $\mathrm{d}\mathbf{T}_{(n)}$ is directed inside the current yield surface, (8.6.12) requires that $\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}$ is codirectional with the outward normal to a locally smooth yield surface in stress $\mathbf{T}_{(n)}$ space.

8.6.2. Convexity of the Yield Surface in Stress Space

Returning to (8.6.6), if elastic response is nonlinear we can not conclude from it that the yield surface in stress space is necessarily convex. In fact, a concavity of the yield surface in the Cauchy stress space in the presence of nonlinear elasticity has been demonstrated for a particular material model by Palmer, Maier, and Drucker (1967). Consider, instead, a linear elastic response within the yield surface, for which the complementary energy can be expressed as

$$\Phi_{(n)}\left(\mathbf{T}_{(n)},\mathcal{H}\right)=\mathbf{E}_{(n)}(0,\mathcal{H}):\mathbf{T}_{(n)}+\frac{1}{2}\,\mathbf{M}_{(n)}(\mathcal{H})::\left(\mathbf{T}_{(n)}\otimes\mathbf{T}_{(n)}\right),\tag{8.6.13}$$

so that

$$\mathbf{E}_{(n)}=\frac{\partial\Phi_{(n)}}{\partial\mathbf{T}_{(n)}}=\mathbf{E}_{(n)}(0,\mathcal{H})+\mathbf{M}_{(n)}(\mathcal{H}):\mathbf{T}_{(n)}.\tag{8.6.14}$$

The tensor $\mathbf{E}_{(n)}(0,\mathcal{H})$, which is equal to $\mathbf{E}^{\mathrm{p}}_{(n)}(\mathcal{H})$ in the notation of Section 8.5, represents a residual or plastic strain, left upon elastic unloading to zero stress at the fixed values of internal structure $\mathcal{H}$. Incorporating (8.6.13) and (8.6.14) into (8.6.6) gives

$$\begin{aligned}\oint_T\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}=&-\frac{1}{2}\,\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}-\left(\mathbf{T}_{(n)}-\mathbf{T}^0_{(n)}\right):\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\\&+\frac{1}{2}\,\mathrm{d}\mathbf{M}_{(n)}::\left[\left(\mathbf{T}_{(n)}-\mathbf{T}^0_{(n)}\right)\otimes\left(\mathbf{T}_{(n)}-\mathbf{T}^0_{(n)}\right)\right],\end{aligned}\tag{8.6.15}$$

where

$$\mathrm{d}\mathbf{M}_{(n)}=\mathbf{M}_{(n)}(\mathbf{T}_{(n)},\mathcal{H}+d\mathcal{H})-\mathbf{M}_{(n)}(\mathbf{T}_{(n)},\mathcal{H}).\tag{8.6.16}$$

In the derivation, the following expression was used

$$\mathbf{E}_{(n)}(0,\mathcal{H}+d\mathcal{H})-\mathbf{E}_{(n)}(0,\mathcal{H})=-\mathrm{d}\mathbf{M}_{(n)}:\mathbf{T}_{(n)}+\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}.\tag{8.6.17}$$

By taking the stress cycle with a sufficiently small $\mathrm{d}\mathbf{T}_{(n)}$ comparing to $\mathbf{T}_{(n)}-\mathbf{T}^0_{(n)}$, the first term in Eq. (8.6.15) can be neglected, and for such cycles

$$\begin{aligned}\oint_T\mathbf{E}_{(n)}:\mathrm{d}\mathbf{T}_{(n)}=&-\left(\mathbf{T}_{(n)}-\mathbf{T}^0_{(n)}\right):\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\\&+\frac{1}{2}\,\mathrm{d}\mathbf{M}_{(n)}::\left[\left(\mathbf{T}_{(n)}-\mathbf{T}^0_{(n)}\right)\otimes\left(\mathbf{T}_{(n)}-\mathbf{T}^0_{(n)}\right)\right]<0.\end{aligned}\tag{8.6.18}$$

This gives

$$\left(\mathbf{T}_{(n)}-\mathbf{T}^0_{(n)}\right):\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}>\frac{1}{2}\,\mathrm{d}\mathbf{M}_{(n)}::\left[\left(\mathbf{T}_{(n)}-\mathbf{T}^0_{(n)}\right)\otimes\left(\mathbf{T}_{(n)}-\mathbf{T}^0_{(n)}\right)\right].\tag{8.6.19}$$

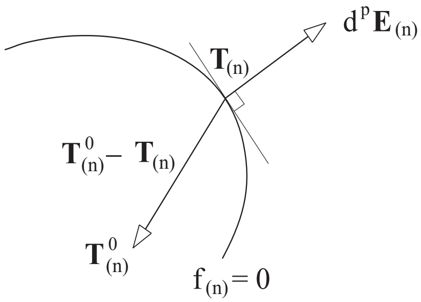

FIGURE 8.12. The plastic strain increment $\mathrm{d^p}\mathbf{E}_{(n)}$ is codirectional with the outward normal to locally smooth yield surface in stress space, so that $(\mathbf{T}^0_{(n)} - \mathbf{T}_{(n)}) : \mathrm{d^p}\mathbf{E}_{(n)} < 0$, where $\mathbf{T}_{(n)}$ is the stress state on the current yield surface and $\mathbf{T}^0_{(n)}$ is the stress state within the yield surface.

Thus, if the change of elastic stiffness caused by plastic deformation is such that $\mathrm{d}\mathbf{M}_{(n)}$ is positive semi-definite, or if there is no change in $\mathbf{M}_{(n)}$, from (8.6.19) it follows that (Fig. 8.12)

$$\left(\mathbf{T}_{(n)} - \mathbf{T}^0_{(n)}\right) : \mathrm{d^p}\mathbf{E}_{(n)} > 0. \tag{8.6.20}$$

Since $\mathrm{d^p}\mathbf{E}_{(n)}$ is codirectional with the outward normal to a locally smooth yield surface in stress $\mathbf{T}_{(n)}$ space, (8.6.20) implies that the yield surface is convex in the considered stress space.

Returning to (8.6.15), it is noted that it can be rewritten as

$$\begin{aligned}\oint_T \mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} = &-\frac{1}{2}\,\mathrm{d^p}\mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} \\ &-\frac{1}{2}\left(\mathbf{T}_{(n)} - \mathbf{T}^0_{(n)}\right) : \left[\mathrm{d^p}\mathbf{E}_{(n)} + \left(\mathrm{d^p}\mathbf{E}_{(n)}\right)^0\right],\end{aligned} \tag{8.6.21}$$

where

$$\mathrm{d^p}\mathbf{E}_{(n)} = \mathbf{E}_{(n)}(\mathbf{T}_{(n)}, \mathcal{H} + d\mathcal{H}) - \mathbf{E}_{(n)}(\mathbf{T}_{(n)}, \mathcal{H}), \tag{8.6.22}$$

$$\left(\mathrm{d^p}\mathbf{E}_{(n)}\right)^0 = \mathbf{E}_{(n)}\left(\mathbf{T}^0_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \mathbf{E}_{(n)}\left(\mathbf{T}^0_{(n)}, \mathcal{H}\right). \tag{8.6.23}$$

8.6.3. Normality Rule in Strain Space

The normality rule for the yield surface in strain space can be deduced from the results based on the inequality (8.6.1) in stress space. By taking a trace product of Eq. (8.2.5) with $\mathbf{M}_{(n)} = \mathbf{\Lambda}_{(n)}^{-1}$, we obtain

$$\mathbf{M}_{(n)} : \dot{\mathbf{T}}_{(n)} = \dot{\mathbf{E}}_{(n)} + \mathbf{M}_{(n)} : \dot{\mathbf{T}}^{\mathrm{p}}_{(n)}, \tag{8.6.24}$$

and comparison with Eq. (8.2.14) yields

$$\dot{\mathbf{E}}^{\mathrm{p}}_{(n)} = -\mathbf{M}_{(n)} : \dot{\mathbf{T}}^{\mathrm{p}}_{(n)}, \tag{8.6.25}$$

in accord with Eq. (8.5.26). Thus, to first order,

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} = -\mathbf{M}_{(n)} : \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}. \tag{8.6.26}$$

Since for any elastic strain increment $\delta\mathbf{T}_{(n)}$, emanating from a point on the yield surface in stress space and directed inside of it,

$$\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)} : \delta\mathbf{T}_{(n)} < 0, \tag{8.6.27}$$

substitution of (8.6.26) into (8.6.27) gives

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} : \mathbf{M}_{(n)} : \delta\mathbf{T}_{(n)} = \mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} : \delta\mathbf{E}_{(n)} > 0. \tag{8.6.28}$$

Here,

$$\delta\mathbf{E}_{(n)} = \mathbf{M}_{(n)} : \delta\mathbf{T}_{(n)} \tag{8.6.29}$$

is the elastic strain increment from a point on the yield surface in strain space, associated with the stress increment $\delta\mathbf{T}_{(n)}$, and directed inside the yield surface. Inequality (8.6.28) holds for any such $\delta\mathbf{E}_{(n)}$ and, therefore, $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ must be codirectional with the inward normal to a locally smooth yield surface in strain $\mathbf{E}_{(n)}$ space,

$$\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)} = -\mathrm{d}\gamma_{(n)} \frac{\partial g_{(n)}}{\partial \mathbf{E}_{(n)}}, \quad \mathrm{d}\gamma_{(n)} > 0. \tag{8.6.30}$$

At the vertex of the yield surface, $\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}$ must lie within the cone of limiting inward normals.

In view of (8.6.11), (8.5.28), and (8.6.30), the yield surface normals in stress and strain space are related by

$$\frac{\partial f_{(n)}}{\partial \mathbf{T}_{(n)}} = \mathbf{M}_{(n)} : \frac{\partial f_{(n)}}{\partial \mathbf{E}_{(n)}}, \tag{8.6.31}$$

in agreement with Eq. (8.5.33).

8.6.4. Additional Inequalities for Stress Cycles

Dually to the analysis from Subsection 8.5.2, we can write

$$\mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} = \mathrm{d}\left(\mathbf{E}_{(n)} : \mathbf{T}_{(n)}\right) - \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)}, \tag{8.6.32}$$

and substitution into (8.6.1) gives, for the stress cycle A^0ABDD^0,

$$\begin{aligned} \oint_T \mathbf{E}_{(n)} : \mathrm{d}\mathbf{T}_{(n)} = &\left[\mathbf{E}_{(n)}\left(\mathbf{T}^0_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \mathbf{E}_{(n)}\left(\mathbf{T}^0_{(n)}, \mathcal{H}\right)\right] : \mathbf{T}^0_{(n)} \\ &- \oint_T \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)}. \end{aligned} \tag{8.6.33}$$

If this is assumed to be negative by (8.6.1), there follows

$$\oint_T \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} > \left[\mathbf{E}_{(n)}\left(\mathbf{T}^0_{(n)}, \mathcal{H} + d\mathcal{H}\right) - \mathbf{E}_{(n)}\left(\mathbf{T}^0_{(n)}, \mathcal{H}\right)\right] : \mathbf{T}^0_{(n)}. \tag{8.6.34}$$

Alternatively, (8.6.34) can be written as

$$\oint_T \left(\mathbf{T}_{(n)} - \mathbf{T}^0_{(n)}\right) : \mathrm{d}\mathbf{E}_{(n)} > 0. \tag{8.6.35}$$

Since (8.6.1) is not invariant, neither is (8.6.35). For example, if we choose a reference state for the strain measure $\mathbf{E}_{(n)}$ to be the state A^0, we have

$$\mathbf{E}_{(n)}\left(\mathbf{T}^0_{(n)}, \mathcal{H}\right) = \mathbf{0}, \quad \mathbf{T}^0_{(n)} = \boldsymbol{\sigma}^0, \tag{8.6.36}$$

where $\boldsymbol{\sigma}^0$ is the Cauchy stress at A^0. Thus, (8.6.35) gives

$$\oint_T \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} > \boldsymbol{\sigma}^0 : \mathbf{E}_{(n)}\left(\boldsymbol{\sigma}^0, \mathcal{H} + d\mathcal{H}\right). \tag{8.6.37}$$

This shows that the bound on the work done in a stress cycle that involves plastic deformation (the right-hand side of the above inequality) depends on the selected strain measure. This was expected on physical grounds, because cycling one stress measure does not necessarily cycle another stress measure, and different amounts of work are done in cycles of different stress measures. These cycles are different cycles; they involve the same plastic, but not elastic deformation of the material.

8.6.5. Infinitesimal Strain Formulation

In the infinitesimal strain theory all stress measures reduce to the Cauchy stress $\boldsymbol{\sigma}$, and (8.6.35) becomes

$$\oint_\sigma \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^0\right) : \mathrm{d}\boldsymbol{\varepsilon} > 0. \tag{8.6.38}$$

This is the original postulate of Drucker (1951, 1959). The net work of added stresses in all physically possible stress cycles originating and terminating at some initial stress state $\boldsymbol{\sigma}^0$ within the yield surface is positive, if plastic deformation occurred at some stage of the cycle. In the hardening range $\boldsymbol{\sigma}^0$ can be inside or on the current yield surface, while in the softening range $\boldsymbol{\sigma}^0$ must be inside the current yield surface. If Drucker's postulate is restricted to stress cycles that involve only infinitesimal increment of plastic deformation, (8.6.38) becomes

$$\frac{1}{2}\,\mathrm{d}\boldsymbol{\sigma} : \mathrm{d}^{\mathrm{p}}\boldsymbol{\varepsilon} + (\boldsymbol{\sigma} - \boldsymbol{\sigma}^0) : \mathrm{d}^{\mathrm{p}}\boldsymbol{\varepsilon} > 0, \tag{8.6.39}$$

to terms of second order (assuming that there is no change in elastic properties due to plastic deformation). If the stress state $\boldsymbol{\sigma}^0$ is well inside the current yield surface, or on the yield surface far from the state of stress $\boldsymbol{\sigma}$, the first term in (8.6.39) can be neglected, and

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^0) : \mathrm{d}^{\mathrm{p}}\boldsymbol{\varepsilon} > 0. \tag{8.6.40}$$

The inequality is referred to as the principle of maximum plastic work. It was introduced in continuum plasticity by Hill (1948), and in crystalline plasticity by Bishop and Hill (1951) (see Chapter 12). Detailed discussion of the inequality can be found in Hill (1950), Johnson and Mellor (1973), Martin (1975), and Lubliner (1990). It assures both normality and convexity. Its other implications in mathematical theory of plasticity are examined by Duvaut and Lions (1976), Temam (1985), and Han and Reddy (1998).

In the hardening range, the initial state can be chosen to be on the yield surface, so that $\boldsymbol{\sigma}^0 = \boldsymbol{\sigma}$ and (8.6.39) gives

$$\mathrm{d}\boldsymbol{\sigma} : \mathrm{d}^{\mathrm{p}}\boldsymbol{\varepsilon} > 0. \tag{8.6.41}$$

In the softening range, the initial state

$$\boldsymbol{\sigma}^0 = \boldsymbol{\sigma} + \mathrm{d}\boldsymbol{\sigma} \tag{8.6.42}$$

is chosen to be inside the yield surface, and (8.6.39) gives

$$\mathrm{d}\boldsymbol{\sigma} : \mathrm{d}^{\mathrm{p}}\boldsymbol{\varepsilon} < 0. \tag{8.6.43}$$

Both, (8.6.41) and (8.6.43), imply that $\mathrm{d}^{\mathrm{p}}\boldsymbol{\varepsilon}$ is codirectional with the outward normal to a locally smooth yield surface in the Cauchy stress space. Further discussion is given in the paper by Palgen and Drucker (1983).

8.7. Relationship between Work in Stress and Strain Cycles

The Ilyushin work in the cycle of strain $A^0ABD^0C^0$ can be written as

$$W_I = \oint_E \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} = \oint_T \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)} + \int_{D^0}^{C^0} \mathbf{T}_{(n)} : \mathrm{d}\mathbf{E}_{(n)}. \tag{8.7.1}$$

Denoting the work of added stresses in the cycle of stress A^0ABD^0 as

$$W_D = \oint_T \left(\mathbf{T}_{(n)} - \mathbf{T}^0_{(n)}\right) : \mathrm{d}\mathbf{E}_{(n)}, \tag{8.7.2}$$

and recalling that $\mathbf{T}_{(n)} = \partial\Psi/\partial\mathbf{E}_{(n)}$, we rewrite Eq. (8.7.1) as

$$\begin{aligned} W_I - W_D = \mathbf{T}^0_{(n)} &: \left(\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\right)^0 \\ &+ \Psi\left(\mathbf{E}^0_{(n)}, \mathcal{H} + \mathrm{d}\mathcal{H}\right) - \Psi\left[\mathbf{E}^0_{(n)} + \left(\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\right)^0, \mathcal{H} + d\mathcal{H}\right]. \end{aligned} \tag{8.7.3}$$

Furthermore,

$$\begin{aligned} &\Psi\left[\mathbf{E}^0_{(n)} + \left(\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\right)^0, \mathcal{H} + d\mathcal{H}\right] - \Psi\left(\mathbf{E}^0_{(n)}, \mathcal{H} + d\mathcal{H}\right) \\ &\quad = \left(\frac{\partial\Psi}{\partial\mathbf{E}_{(n)}}\right)_{C^0} : \left(\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\right)^0 \\ &\quad + \frac{1}{2}\left(\frac{\partial^2\Psi}{\partial\mathbf{E}_{(n)} \otimes \partial\mathbf{E}_{(n)}}\right)_{C^0} : \left[\left(\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\right)^0 \otimes \left(\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\right)^0\right] \\ &\quad = \left[\mathbf{T}^0_{(n)} + \frac{1}{2}\left(\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}\right)^0\right] : \left(\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\right)^0, \end{aligned} \tag{8.7.4}$$

neglecting the higher-order infinitesimals. The subscript C^0 in Eq. (8.7.4) indicates that partial derivatives are evaluated in the state C^0, where the stress is $\mathbf{T}_{(n)} + \left(\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}\right)^0$. Substitution of (8.7.4) into (8.7.3) gives

$$W_I - W_D = -\frac{1}{2}\left(\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}\right)^0 : \left(\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\right)^0. \tag{8.7.5}$$

Here,

$$\left(\mathrm{d}^{\mathrm{p}}\mathbf{T}_{(n)}\right)^0 = -\boldsymbol{\Lambda}_{(n)} : \left(\mathrm{d}^{\mathrm{p}}\mathbf{E}_{(n)}\right)^0 \tag{8.7.6}$$

is the stress decrement from A^0 to C^0 caused by infinitesimal plastic deformation along AB (Fig. 8.13). Therefore, if elastic stiffness tensor $\boldsymbol{\Lambda}_{(n)}$ is positive definite, (8.7.5) implies that

$$W_I > W_D. \tag{8.7.7}$$

It is recalled that W_I is independent of the reference state and strain measure, while W_D is not. Thus, the right-hand side of (8.7.5) is dependent on the reference state and measure. However, if $\boldsymbol{\Lambda}_{(n)}$ is positive definite in each

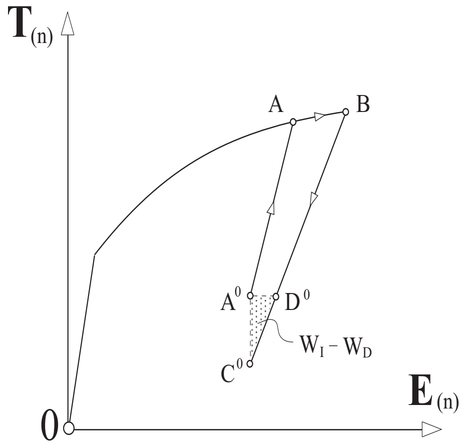

FIGURE 8.13. The dotted area represents the difference between the work done in the Ilyushin and Drucker closed cycles of strain and stress, indicating that $W_I > W_D$.

case, the inequality (8.7.7) holds for all pairs of conjugate stress and strain measures, and for any reference state.

Since $W_I > W_D$, the class of materials obeying inequality (8.5.1) is broader than that obeying (8.6.1). For example, it may happen that material behavior is such that over some stress cycles $W_D < 0$, while $W_I > 0$ for every strain cycle. Since Ilyushin's postulate (8.5.1) is a sufficient condition for the normality rule, it follows that plastic part of strain increment can be normal to a locally smooth yield surface in stress space, although the material does not satisfy (8.6.1) for some stress cycles. Thus, although sufficient, (8.6.1) is not a necessary condition for the normality. This was anticipated, because (8.6.1) places strong restrictions on material behavior, when imposed on all cycles of stress, involving infinitesimal or large plastic deformation. Weaker restrictions on material response are placed by requiring (8.6.1) to hold for stress cycles that involve only infinitesimal plastic deformation, such as cycle A^0ABD^0 considered in Section 8.6.

Returning to Ilyushin's postulate (8.5.1), although it imposes less restrictions than (8.6.1), it is not a necessary condition for the normality rule, either. For example, Palmer, Maier, and Drucker (1967) provide an example of negative work in certain strain cycles for materials that have experienced enormous cyclic work-softening. Yet, normality rule can be used to describe behavior of such materials in a satisfactory manner. For an analysis of plasticity postulates and nonassociative flow rules, considered in Chapter 9, the papers by Nicholson (1987), Lade, Bopp, and Peters (1993), and Lubarda, Mastilovic, and Knap (1996) can be consulted. See also Dougill (1975) and Lee (1994).

8.8. Further Inequalities

If the material obeys Ilyushin's postulate, we have from (8.5.10) a measure-invariant inequality

$$\dot{\mathbf{T}}^{\mathrm{p}}_{(n)} : \dot{\mathbf{E}}_{(n)} < 0. \tag{8.8.1}$$

Since

$$\dot{\mathbf{T}}^{\mathrm{p}}_{(n)} = -\boldsymbol{\Lambda}_{(n)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(n)}, \quad \dot{\mathbf{T}}^{\mathrm{e}}_{(n)} = \boldsymbol{\Lambda}_{(n)} : \dot{\mathbf{E}}_{(n)}, \tag{8.8.2}$$

the inequality (8.8.1) is equivalent to

$$\dot{\mathbf{T}}^{\mathrm{e}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(n)} > 0. \tag{8.8.3}$$

By taking a trace product of the first of (8.8.2) with $\dot{\mathbf{E}}^{\mathrm{p}}_{(n)}$, and of the second with $\dot{\mathbf{E}}_{(n)}$, it follows that

$$\dot{\mathbf{T}}^{\mathrm{p}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(n)} < 0, \quad \dot{\mathbf{T}}^{\mathrm{e}}_{(n)} : \dot{\mathbf{E}}_{(n)} > 0, \tag{8.8.4}$$

provided that $\boldsymbol{\Lambda}_{(n)}$ is positive definite. Both inequalities in (8.8.4) are measure-invariant. Furthermore, since

$$\dot{\mathbf{E}}^{\mathrm{e}}_{(n)} = \mathbf{M}_{(n)} : \dot{\mathbf{T}}_{(n)}, \tag{8.8.5}$$

a trace product with $\dot{\mathbf{T}}_{(n)}$ yields another measure-invariant inequality

$$\dot{\mathbf{T}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{e}}_{(n)} > 0. \tag{8.8.6}$$

In view of (8.8.2) and (8.8.3), there is an identity

$$\dot{\mathbf{T}}^{\mathrm{p}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{e}}_{(n)} = -\dot{\mathbf{T}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(n)}. \tag{8.8.7}$$

If material is in the hardening range, relative to a particular pair of stress and strain measures, then for that pair

$$\dot{\mathbf{T}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(n)} > 0, \quad \dot{\mathbf{T}}^{\mathrm{p}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{e}}_{(n)} < 0. \tag{8.8.8}$$

These are not measure-invariant inequalities, so that hardening with respect to one pair of measures may appear as softening relative to another pair.

If (8.8.8) holds for a particular pair of stress and strain measures, we have for that pair

$$\dot{\mathbf{T}}^{\mathrm{e}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{e}}_{(n)} = \dot{\mathbf{T}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{e}}_{(n)} - \dot{\mathbf{T}}^{\mathrm{p}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{e}}_{(n)} > 0, \tag{8.8.9}$$

in view of (8.8.6) and (8.8.8). Since, by (8.8.2) and (8.8.5),

$$\dot{\mathbf{T}}^{\mathrm{e}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{e}}_{(n)} = \dot{\mathbf{T}}_{(n)} : \dot{\mathbf{E}}_{(n)}, \tag{8.8.10}$$

the inequality (8.8.9) gives

$$\dot{\mathbf{T}}_{(n)} : \dot{\mathbf{E}}_{(n)} > 0, \tag{8.8.11}$$

for the same conjugate pair. Neither (8.8.9) nor (8.8.11) is measure-invariant.

In the softening range the directions of inequalities in (8.8.8) are reversed. Since the first term on the right-hand side of the equality sign in (8.8.9) is always positive, by measure-invariant (8.8.6), the direction of inequalities in (8.8.9) and (8.8.11) is uncertain. Thus, in the softening range, corresponding to given n, $\dot{\mathbf{T}}_{(n)} : \dot{\mathbf{E}}_{(n)}$ can be either positive or negative. As a result, (8.8.11) is not a criterion of hardening. A necessary and sufficient condition for hardening, relative to selected stress and strain measures, is given by (8.8.8).

8.8.1. Inequalities with Current State as Reference

If current state is taken as the reference, we have from Section 3.9

$$\underline{\dot{\mathbf{E}}}_{(n)} = \mathbf{D}, \quad \underline{\dot{\mathbf{T}}}_{(n)} = \underline{\overset{\circ}{\boldsymbol{\tau}}} - n\,(\mathbf{D}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{D}). \tag{8.8.12}$$

Equation (8.2.5) consequently becomes

$$\underline{\dot{\mathbf{T}}}_{(n)} = \underline{\dot{\mathbf{T}}}^{\mathrm{e}}_{(n)} + \underline{\dot{\mathbf{T}}}^{\mathrm{p}}_{(n)}, \quad \underline{\dot{\mathbf{T}}}^{\mathrm{e}}_{(n)} = \underline{\boldsymbol{\Lambda}}_{(n)} : \mathbf{D}, \tag{8.8.13}$$

while Eq. (8.2.13) gives

$$\mathbf{D} = \mathbf{D}^{\mathrm{e}}_{(n)} + \mathbf{D}^{\mathrm{p}}_{(n)}, \quad \mathbf{D}^{\mathrm{e}}_{(n)} = \underline{\mathbf{M}}_{(n)} : \underline{\dot{\mathbf{T}}}_{(n)}. \tag{8.8.14}$$

Inequalities (8.8.1) and (8.8.6) yield

$$\underline{\dot{\mathbf{T}}}^{\mathrm{p}}_{(n)} : \mathbf{D} < 0, \quad \underline{\dot{\mathbf{T}}}_{(n)} : \mathbf{D}^{\mathrm{e}}_{(n)} > 0. \tag{8.8.15}$$

In addition, the inequalities in (8.8.4) reduce to

$$\underline{\dot{\mathbf{T}}}^{\mathrm{p}}_{(n)} : \mathbf{D}^{\mathrm{p}}_{(n)} < 0, \quad \underline{\dot{\mathbf{T}}}^{\mathrm{e}}_{(n)} : \mathbf{D} > 0. \tag{8.8.16}$$

For example, for $n = 0, \pm 1$, the inequalities in (8.8.15) give

$$\overset{\circ}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}} : \mathbf{D} < 0, \quad \overset{\triangle}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}} : \mathbf{D} < 0, \quad \overset{\triangledown}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}} : \mathbf{D} < 0, \tag{8.8.17}$$

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} : \mathbf{D}^{\mathrm{e}}_{(0)} > 0, \quad \overset{\triangle}{\underline{\boldsymbol{\tau}}} : \mathbf{D}^{\mathrm{e}}_{(1)} < 0, \quad \overset{\triangledown}{\underline{\boldsymbol{\tau}}} : \mathbf{D}^{\mathrm{e}}_{(-1)} < 0. \tag{8.8.18}$$

Similarly, from (8.8.16), we obtain

$$\overset{\circ}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}}_{(0)} < 0, \quad \overset{\triangle}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}}_{(1)} < 0, \quad \overset{\triangledown}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}} : \mathbf{D}^{\mathrm{p}}_{(-1)} < 0, \tag{8.8.19}$$

$$\overset{\circ}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{e}} : \mathbf{D} > 0, \quad \overset{\triangle}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{e}} : \mathbf{D} > 0, \quad \overset{\triangledown}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{e}} : \mathbf{D} > 0. \tag{8.8.20}$$

It is observed that

$$\underline{\dot{\mathbf{T}}}{}^{\mathrm{e}}_{(n)} = \boldsymbol{\Lambda}_{(n)} : \mathbf{D} = (\underline{\boldsymbol{\Lambda}}_{(0)} - 2n\,\underline{\mathbf{S}}) : \mathbf{D}, \tag{8.8.21}$$

and

$$\underline{\dot{\mathbf{T}}}{}^{\mathrm{p}}_{(n)} = \underline{\dot{\mathbf{T}}}_{(n)} - \underline{\dot{\mathbf{T}}}{}^{\mathrm{e}}_{(n)} = \overset{\circ}{\underline{\boldsymbol{\tau}}} - \overset{\circ}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{e}} = \overset{\circ}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}}, \tag{8.8.22}$$

for all n. In particular,

$$\overset{\circ}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}} = \overset{\triangle}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}} = \overset{\triangledown}{\underline{\boldsymbol{\tau}}}{}^{\mathrm{p}}. \tag{8.8.23}$$

Furthermore,

$$\underline{\dot{\mathbf{T}}}_{(n)} : \underline{\dot{\mathbf{E}}}_{(n)} = \underline{\dot{\mathbf{T}}}_{(n)} : \mathbf{D} = \overset{\circ}{\underline{\boldsymbol{\tau}}} : \mathbf{D} - 2n\,(\boldsymbol{\sigma} : \mathbf{D}^2). \tag{8.8.24}$$

To illustrate that $\underline{\dot{\mathbf{T}}}_{(n)} : \underline{\dot{\mathbf{E}}}_{(n)}$ can have a different sign for different n, consider a tensile test under superposed hydrostatic pressure. The corresponding stress and rate of deformation tensors are

$$\boldsymbol{\sigma} = \sigma\,\mathbf{e}_3 \otimes \mathbf{e}_3 - p\mathbf{I}, \quad \mathbf{D} = \frac{3}{2}\,\frac{\dot{l}}{l}\,\mathbf{e}_3 \otimes \mathbf{e}_3 - \frac{1}{2}\,\frac{\dot{l}}{l}\,\mathbf{I}, \tag{8.8.25}$$

where l is a current length of the specimen under tensile stress σ (in the direction $\mathbf{e}_3$), and under constant superposed pressure p. Substitution into Eq. (8.8.24) yields

$$\underline{\dot{\mathbf{T}}}_{(n)} : \mathbf{D} = \frac{\dot{\sigma}}{\sigma}\,\frac{\dot{l}}{l}\left[\sigma - 2n\left(1 - \frac{3}{2}\,\frac{p}{\sigma}\right)\frac{\dot{l}}{l}\right]. \tag{8.8.26}$$

For $n = 0$ this gives

$$\underline{\dot{\mathbf{T}}}_{(0)} : \mathbf{D} = \dot{\sigma}\,\frac{\dot{l}}{l}\,. \tag{8.8.27}$$

If this is positive, from (8.8.26) it follows that for other n the trace product $\underline{\dot{\mathbf{T}}}_{(n)} : \mathbf{D}$ can be either positive or negative, depending on the magnitude of the superposed pressure p (Hill, 1968).

8.9. Related Postulates

Consider again the measure-invariant inequality (8.8.1), i.e.,

$$\dot{\mathbf{E}}_{(n)} : \dot{\mathbf{T}}^{\mathrm{p}}_{(n)} < 0. \tag{8.9.1}$$

Since plastic parts of the stress and strain rates are related by

$$\dot{\mathbf{T}}^{\mathrm{p}}_{(n)} = -\boldsymbol{\Lambda}_{(n)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(n)}, \tag{8.9.2}$$

there follows

$$\dot{\mathbf{E}}_{(n)} : \boldsymbol{\Lambda}_{(n)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(n)} > 0. \tag{8.9.3}$$

Thus, recalling that

$$\dot{\mathbf{E}}_{(n)} = \dot{\mathbf{T}}_{(n)} : \mathbf{M}_{(n)} + \dot{\mathbf{E}}^{\mathrm{p}}_{(n)}, \tag{8.9.4}$$

the substitution into (8.9.3) gives

$$\dot{\mathbf{T}}_{(n)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(n)} > -\dot{\mathbf{E}}^{\mathrm{p}}_{(n)} : \boldsymbol{\Lambda}_{(n)} : \dot{\mathbf{E}}^{\mathrm{p}}_{(n)}. \tag{8.9.5}$$

An inequality of this type was originally proposed by Nguyen and Bui (1974). See also Lubliner (1986). In particular, with the current state as the reference, and with the logarithmic strain measure, we obtain

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} : \mathbf{D}^{\mathrm{p}}_{(0)} > -\mathbf{D}^{\mathrm{p}}_{(0)} : \underline{\boldsymbol{\Lambda}}_{(0)} : \mathbf{D}^{\mathrm{p}}_{(0)}. \tag{8.9.6}$$

Naghdi and Trapp (1975 a, b) proposed that the external work done on the body by surface tractions and body forces in any smooth spatially homogeneous closed cycle is non-negative, i.e.,

$$\int_{t_1}^{t_2} \mathcal{P}\,\mathrm{d}t \geq 0, \tag{8.9.7}$$

where

$$\mathcal{P} = \int_{S^0} \mathbf{p}_n \cdot \mathbf{v}\,\mathrm{d}S^0 + \int_{V^0} \rho^0\,\mathbf{b} \cdot \mathbf{v}\,\mathrm{d}V^0. \tag{8.9.8}$$

A smooth closed cycle is defined as a closed cycle of deformation which also restores the velocity and thus the kinetic energy,

$$\mathbf{E}_{(n)}(t_2) = \mathbf{E}_{(n)}(t_1), \quad \mathbf{v}(t_2) = \mathbf{v}(t_1). \tag{8.9.9}$$

By rewriting Eq. (8.9.8) as (see Section 3.5)

$$\mathcal{P} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{V^0} \frac{1}{2}\,\rho^0\,\mathbf{v} \cdot \mathbf{v}\,\mathrm{d}V^0 + \int_{V^0} \mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)}\,\mathrm{d}V^0, \tag{8.9.10}$$

substitution into (8.9.7) gives

$$\int_{t_1}^{t_2} \left(\int_{V^0} \mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} \, \mathrm{d}V^0 \right) \mathrm{d}t \geq 0. \tag{8.9.11}$$

Since deformation is assumed to be spatially uniform, this reduces to

$$\int_{t_1}^{t_2} \mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} \, \mathrm{d}t \geq 0, \tag{8.9.12}$$

or

$$\oint_E \mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} \, \mathrm{d}t \geq 0. \tag{8.9.13}$$

This, in fact, is Ilyushin's postulate in the form presented by Hill (1968). Additional discussion can be found in Carroll (1987), Hill and Rice (1987), and Rajagopal and Srinivasa (1998). The work inequalities in plastic fracturing materials were discussed by Bažant (1980), among others, and for elastic-viscoplastic materials by Naghdi (1984).

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