CHAPTER 7

ELASTIC STABILITY

7.1. Principle of Stationary Potential Energy

Denote by $\delta\mathbf{F}$ the variation of the deformation gradient $\mathbf{F}$. Since for Green elasticity $\mathbf{P} = \partial\Psi/\partial\mathbf{F}$, where $\Psi = \Psi(\mathbf{F})$ is the strain energy per unit initial volume, we can write

$$\mathbf{P} \cdot\cdot\, \delta\mathbf{F} = \frac{\partial\Psi}{\partial\mathbf{F}} \cdot\cdot\, \delta\mathbf{F} = \delta\Psi, \tag{7.1.1}$$

and the principle of virtual work of Eq. (3.12.1) becomes

$$\int_{V^0} \delta\Psi \,\mathrm{d}V^0 = \int_{V^0} \rho^0\, \mathbf{b} \cdot \delta\mathbf{u} \,\mathrm{d}V^0 + \int_{S_t^0} \mathbf{p}_n \cdot \delta\mathbf{u} \,\mathrm{d}S_t^0. \tag{7.1.2}$$

In general, for arbitrary loading there is no true variational principle associated with Eq. (7.1.2), because the variation δ affects the applied body force $\mathbf{b}$ and the surface traction $\mathbf{p}_{(n)}$. However, if the loading is conservative, as in the case of dead loading, then

$$\mathbf{b} \cdot \delta\mathbf{u} = \delta(\mathbf{b} \cdot \mathbf{u}), \quad \mathbf{p}_n \cdot \delta\mathbf{u} = \delta(\mathbf{p}_n \cdot \mathbf{u}), \tag{7.1.3}$$

and Eq. (7.1.2) can be recast in the variational form

$$\delta\mathcal{P} = 0, \tag{7.1.4}$$

where

$$\mathcal{P} = \int_{V^0} \Psi \,\mathrm{d}V^0 - \int_{V^0} \rho^0\, \mathbf{b} \cdot \mathbf{u} \,\mathrm{d}V^0 - \int_{S_t^0} \mathbf{p}_n \cdot \mathbf{u} \,\mathrm{d}S_t^0. \tag{7.1.5}$$

Among all geometrically admissible displacement fields, the actual displacement field (whether unique or not) of the considered boundary-value problem makes stationary the potential energy functional $\mathcal{P}(\mathbf{u})$ given by Eq. (7.1.5). See also Nemat-Nasser (1974) and Washizu (1982).

7.2. Uniqueness of Solution

Consider a finite elasticity problem described by the equilibrium equations

$$\boldsymbol{\nabla}^0 \cdot \mathbf{P} + \rho^0 \, \mathbf{b} = \mathbf{0}, \tag{7.2.1}$$

and the mixed boundary conditions

$$\mathbf{u} = \mathbf{u}(\mathbf{X}) \quad \text{on} \quad S_u^0, \qquad \mathbf{n}^0 \cdot \mathbf{P} = \mathbf{p}_n(\mathbf{X}) \quad \text{on} \quad S_t^0. \tag{7.2.2}$$

For simplicity, restrict attention to dead loading on S_t^0, and dead body forces $\mathbf{b} = \mathbf{b}(\mathbf{X})$ in V^0. Suppose that there are two different solutions of Eqs. (7.2.1) and (7.2.2), $\mathbf{u}$ and $\mathbf{u}^*$ (i.e., $\mathbf{x}$ and $\mathbf{x}^*$). The corresponding deformation gradients are $\mathbf{F}$ and $\mathbf{F}^*$, and the nominal stresses $\mathbf{P}$ and $\mathbf{P}^*$. The equilibrium fields $(\mathbf{P}, \mathbf{F})$ and $(\mathbf{P}^*, \mathbf{F}^*)$ necessarily satisfy the condition

$$\int_{V^0} (\mathbf{P}^* - \mathbf{P}) \cdot\cdot (\mathbf{F}^* - \mathbf{F}) \, \mathrm{d}V^0 = 0, \tag{7.2.3}$$

which follows from Eq. (3.12.5). Consequently, the solution $\mathbf{x} = \mathbf{x}(\mathbf{X})$ is unique if

$$\int_{V^0} (\mathbf{P}^* - \mathbf{P}) \cdot\cdot \, (\mathbf{F}^* - \mathbf{F}) \, \mathrm{d}V^0 \neq 0, \tag{7.2.4}$$

for all geometrically admissible $\mathbf{x}^*$ giving rise to

$$\mathbf{F}^* = \frac{\partial \mathbf{x}^*}{\partial \mathbf{X}}, \quad \mathbf{P}^* = \frac{\partial \Psi}{\partial \mathbf{F}^*}. \tag{7.2.5}$$

The stress field $\mathbf{P}^*$ in (7.2.4) need not be statically admissible, so even if equality sign applies in (7.2.4) for some $\mathbf{x}^*$, the uniqueness is not lost unless that $\mathbf{x}^*$ gives rise to statically admissible stress field $\mathbf{P}^*$. Therefore, a sufficient condition for $\mathbf{x}$ to be unique solution is that for all geometrically admissible deformation fields $\mathbf{x}^*$,

$$\int_{V^0} (\mathbf{P}^* - \mathbf{P}) \cdot\cdot \, (\mathbf{F}^* - \mathbf{F}) \, \mathrm{d}V^0 > 0. \tag{7.2.6}$$

The reversed inequality could also serve as a sufficient condition for uniqueness. The solution $\mathbf{x}$ which obeys such inequality for all geometrically admissible $\mathbf{x}^*$ would define unique, but unstable equilibrium configuration, as will be discussed in Section 7.3.

A stronger (more restrictive) condition for uniqueness is

$$(\mathbf{P}^* - \mathbf{P}) \cdot\cdot (\mathbf{F}^* - \mathbf{F}) > 0, \tag{7.2.7}$$

which clearly implies (7.2.6). However, unique solution in finite elasticity is not expected in general (particularly under dead loading), so that inequalities such as (7.2.6) and (7.2.7) are too strong restrictions on elastic constitutive relation. In fact, a nonuniqueness in finite elasticity is certainly anticipated whenever the stress-deformation relation $\mathbf{P} = \partial\Psi/\partial\mathbf{F}$ is not uniquely invertible. For example, Ogden (1984) provides examples in which two, four or more possible states of deformation correspond to a given state of nominal stress. See also Antman (1995). A study of the existence of solutions to boundary-value problems in finite strain elasticity is more difficult, with only few results presently available (e.g., Ball, 1977; Hanyga, 1985; Ciarlet, 1988). A comprehensive account of the uniqueness theorems in linear elasticity is given by Knops and Payne (1971).

7.3. Stability of Equilibrium

Consider the inequality

$$\Psi\left(\mathbf{F}^*\right) - \Psi\left(\mathbf{F}\right) - \mathbf{P} \cdot\cdot \left(\mathbf{F}^* - \mathbf{F}\right) > 0, \tag{7.3.1}$$

where $(\mathbf{P}, \mathbf{F})$ correspond to equilibrium configuration $\mathbf{x}$, and $(\mathbf{P}^*, \mathbf{F}^*)$ to any geometrically admissible configuration $\mathbf{x}^*$ (Coleman and Noll, 1959). This inequality implies (7.2.7), so that (7.3.1) also represents a sufficient condition for uniqueness. (To see that (7.3.1) implies (7.2.7), write another inequality by reversing the role of $\mathbf{F}$ and $\mathbf{F}^*$ in (7.3.1), and add the results; Ogden, *op. cit.*). Inequality (7.3.1) is particularly appealing because it directly leads to stability criterion. To that goal, integrate (7.3.1) to obtain

$$\int_{V^0} \left[\Psi\left(\mathbf{F}^*\right) - \Psi(\mathbf{F})\right] \mathrm{d}V^0 > \int_{V^0} \mathbf{P} \cdot\cdot \left(\mathbf{F}^* - \mathbf{F}\right) \mathrm{d}V^0. \tag{7.3.2}$$

Using Eq. (3.12.4) to express the integral on the right-hand side gives

$$\begin{aligned} \int_{V^0} \left[\Psi\left(\mathbf{F}^*\right) - \Psi(\mathbf{F})\right] \mathrm{d}V^0 > & \int_{V^0} \rho^0\, \mathbf{b} \cdot \left(\mathbf{x}^* - \mathbf{x}\right) \mathrm{d}V^0 \\ & + \int_{S_t^0} \mathbf{p}_n \cdot \left(\mathbf{x}^* - \mathbf{x}\right) \mathrm{d}S_t^0. \end{aligned} \tag{7.3.3}$$

This means that the increase of the strain energy in moving from the configuration $\mathbf{x}$ to $\mathbf{x}^*$ exceeds the work done by the prescribed dead loading on that transition. According to the classical energy criterion of stability this means that $\mathbf{x}$ is a stable equilibrium configuration (Hill, 1957; Pearson, 1959).

Recalling the expression for the potential energy from Eq. (7.1.5), and the identity

$$\mathbf{x}^* - \mathbf{x} = \mathbf{u}^* - \mathbf{u}, \tag{7.3.4}$$

the inequality (7.3.3) can be rewritten as

$$\mathcal{P}(\mathbf{u}^*) > \mathcal{P}(\mathbf{u}). \tag{7.3.5}$$

Consequently, among all geometrically admissible configurations the potential energy is minimized in the configuration of stable equilibrium.

In a broader sense, stability of equilibrium at $\mathbf{x}$ is stable if for some geometrically admissible $\mathbf{x}^*$, $\mathcal{P}(\mathbf{u}^*) = \mathcal{P}(\mathbf{u})$, while for all others $\mathcal{P}(\mathbf{u}^*) > \mathcal{P}(\mathbf{u})$. In this situation, however, equilibrium configuration $\mathbf{x}$ is not necessarily unique, because $\mathbf{x}^*$ for which $\mathcal{P}(\mathbf{u}^*) = \mathcal{P}(\mathbf{u})$ may give rise to statically admissible stress field (in which case $\mathbf{x}^*$ is also an equilibrium configuration). Therefore, stability in the sense $\mathcal{P}(\mathbf{u}^*) \geq \mathcal{P}(\mathbf{u})$ does not in general imply uniqueness. Conversely, unique configuration need not be stable. It is unstable if $\mathcal{P}(\mathbf{u}^*) < \mathcal{P}(\mathbf{u})$ for at least one $\mathbf{u}^*$, and $\mathcal{P}(\mathbf{u}^*) > \mathcal{P}(\mathbf{u})$ for all other geometrically admissible $\mathbf{x}^*$.

In summary, the inequality

$$\mathcal{P}(\mathbf{u}^*) \geq \mathcal{P}(\mathbf{u}) \tag{7.3.6}$$

is a global sufficient condition for stability of equilibrium configuration $\mathbf{x}$. It is, however, too restrictive criterion, because it is formulated relative to all geometrically admissible configurations around $\mathbf{x}$.

7.4. Incremental Uniqueness and Stability

Physically more appealing stability criterion is obtained if $\mathbf{x}^*$ is confined to adjacent configurations, in the neighborhood of $\mathbf{x}$. In that case we talk about local or incremental (infinitesimal) stability (Truesdell and Noll, 1965). We start from the inequality (7.3.1). If $\mathbf{F}^*$ is near $\mathbf{F}$ (corresponding to an equilibrium configuration), so that

$$\mathbf{F}^* = \mathbf{F} + \delta\mathbf{F}, \tag{7.4.1}$$

the Taylor expansion gives

$$\Psi(\mathbf{F} + \delta\mathbf{F}) = \Psi(\mathbf{F}) + \mathbf{P} \cdot\cdot\, \delta\mathbf{F} + \frac{1}{2}\,\mathbf{\Lambda} \cdot\cdot\cdot\cdot\, (\delta\mathbf{F} \otimes \delta\mathbf{F}) + \cdots\ . \tag{7.4.2}$$

Consequently, to second-order terms, the inequality (7.3.1) becomes

$$\frac{1}{2}\,\boldsymbol{\Lambda}\cdot\cdot\cdot\cdot(\delta\mathbf{F}\otimes\delta\mathbf{F})>0. \tag{7.4.3}$$

This is a sufficient condition for incremental (infinitesimal) uniqueness, or uniqueness in the small neighborhood of $\mathbf{F}$. An integration over the volume V^0 yields

$$\frac{1}{2}\int_{V^0}\boldsymbol{\Lambda}\cdot\cdot\cdot\cdot(\delta\mathbf{F}\otimes\delta\mathbf{F})\,\mathrm{d}V^0>0. \tag{7.4.4}$$

Using (7.4.2), Eq. (7.1.5) gives in the case of dead loading

$$\begin{aligned}\mathcal{P}(\mathbf{u}+\delta\mathbf{u})-\mathcal{P}(\mathbf{u})=&\frac{1}{2}\int_{V^0}\boldsymbol{\Lambda}\cdot\cdot\cdot\cdot(\delta\mathbf{F}\otimes\delta\mathbf{F})\,\mathrm{d}V^0\\+\int_{V^0}\mathbf{P}\cdot\cdot\,\delta\mathbf{F}\,\mathrm{d}V^0-&\int_{V^0}\rho^0\,\mathbf{b}\cdot\delta\mathbf{u}\,\mathrm{d}V^0-\int_{S_t^0}\mathbf{p}_n\cdot\delta\mathbf{u}\,\mathrm{d}S^0,\end{aligned} \tag{7.4.5}$$

where $\delta\mathbf{u}=\delta\mathbf{x}$. The last three integrals on the right-hand side of Eq. (7.4.5) cancel each other by Gauss theorem, equilibrium equations, and the condition $\delta\mathbf{u}=\mathbf{0}$ on S_u^0; see Eq. (3.12.1). Thus,

$$\mathcal{P}(\mathbf{u}+\delta\mathbf{u})-\mathcal{P}(\mathbf{u})=\frac{1}{2}\int_{V^0}\boldsymbol{\Lambda}\cdot\cdot\cdot\cdot(\delta\mathbf{F}\otimes\delta\mathbf{F})\,\mathrm{d}V^0. \tag{7.4.6}$$

If equilibrium configuration $\mathbf{x}$ is incrementally unique, so that (7.4.3) applies, then from (7.4.6) it follows that

$$\mathcal{P}(\mathbf{u}+\delta\mathbf{u})>\mathcal{P}(\mathbf{u}), \tag{7.4.7}$$

which means that equilibrium configuration $\mathbf{x}$ is locally or incrementally stable. If for some $\delta\mathbf{u}$, $\mathcal{P}(\mathbf{u}+\delta\mathbf{u})=\mathcal{P}(\mathbf{u})$, while for other $\delta\mathbf{u}$, $\mathcal{P}(\mathbf{u}+\delta\mathbf{u})>\mathcal{P}(\mathbf{u})$, the configuration $\mathbf{x}$ is a state of neutral incremental stability, although the configuration may not be incrementally unique. The strict inequality (7.4.7) is sometimes referred to as the criterion of local (incremental) superstability. See also Knops and Wilkes (1973), and Gurtin (1982).

7.5. Rate-Potentials and Variational Principle

In this section we examine the existence of the variational principle, and the uniqueness and stability of the boundary-value problem of the rate-type elasticity considered in Chapter 6. First, we recall that from Eq. (6.4.2) the rate of nominal stress is

$$\dot{\mathbf{P}}=\boldsymbol{\Lambda}\cdot\cdot\,\dot{\mathbf{F}},\quad \boldsymbol{\Lambda}=\frac{\partial^2\Psi}{\partial\mathbf{F}\otimes\partial\mathbf{F}}\,. \tag{7.5.1}$$

Since the tensor of elastic pseudomoduli $\boldsymbol{\Lambda}$ obeys the reciprocal symmetry, Eq. (7.5.1) can be rephrased by introducing the rate-potential function χ as

$$\dot{\mathbf{P}} = \frac{\partial \chi}{\partial \dot{\mathbf{F}}}, \quad \chi = \frac{1}{2}\, \boldsymbol{\Lambda} \cdots (\dot{\mathbf{F}} \otimes \dot{\mathbf{F}}). \tag{7.5.2}$$

Its Cartesian component representation is

$$\dot{P}_{Ji} = \frac{\partial \chi}{\partial \dot{F}_{iJ}}, \quad \chi = \frac{1}{2}\, \Lambda_{JiLk} \dot{F}_{iJ} \dot{F}_{kL}. \tag{7.5.3}$$

Consequently, we have

$$\dot{\mathbf{P}} \cdot\cdot\, \delta\dot{\mathbf{F}} = \frac{\partial \chi}{\partial \dot{\mathbf{F}}} \cdot\cdot\, \delta\dot{\mathbf{F}} = \delta\chi, \tag{7.5.4}$$

and the principle of virtual velocity from Eq. (3.11.8) becomes, for static problems,

$$\int_{V^0} \delta\chi \, \mathrm{d}V^0 = \int_{V^0} \rho^0 \, \dot{\mathbf{b}} \cdot \delta\mathbf{v} \, \mathrm{d}V^0 + \int_{S_t^0} \dot{\mathbf{p}}_n \cdot \delta\mathbf{v} \, \mathrm{d}S_t^0, \tag{7.5.5}$$

for any analytically admissible virtual velocity field $\delta\mathbf{v}$ vanishing on S_v^0.

For general, nonconservative loading there is no true variational principle associated with Eq. (7.5.5), because the variation δ affects $\dot{\mathbf{b}}$ and $\dot{\mathbf{p}}_{(n)}$. However, if the rates of loading are deformation insensitive (remain unaltered during the variation $\delta\mathbf{v}$), there is a variational principle

$$\delta\Xi = 0, \tag{7.5.6}$$

with

$$\Xi = \int_{V^0} \chi \, \mathrm{d}V^0 - \int_{V^0} \rho^0 \, \dot{\mathbf{b}} \cdot \mathbf{v} \, \mathrm{d}V^0 - \int_{S_t^0} \dot{\mathbf{p}}_n \cdot \mathbf{v} \, \mathrm{d}S_t^0. \tag{7.5.7}$$

Among all kinematically admissible velocity fields, the actual velocity field (whether unique or not) of the considered rate boundary-value problem renders stationary the functional $\Xi(\mathbf{v})$.

There is also a variational principle associated with Eq. (7.5.5) if the rates of prescribed tractions and body forces are self-adjoint in the sense that (Hill, 1978)

$$\int_{S_t^0} (\dot{\mathbf{p}}_n \cdot \delta\mathbf{v} - \mathbf{v} \cdot \delta\dot{\mathbf{p}}_n) \, \mathrm{d}S_t^0 = 0, \tag{7.5.8}$$

and similarly for the body forces, since then

$$\begin{aligned}\delta \int_{S_t^0} (\dot{\mathbf{p}}_n \cdot \mathbf{v})\, \mathrm{d}S_t^0 &= 2 \int_{S_t^0} (\dot{\mathbf{p}}_n \cdot \delta\mathbf{v})\, \mathrm{d}S_t^0,\\ \delta \int_{V^0} \left(\dot{\mathbf{b}} \cdot \mathbf{v}\right) \mathrm{d}V^0 &= 2 \int_{V^0} \left(\dot{\mathbf{b}} \cdot \delta\mathbf{v}\right) \mathrm{d}V^0.\end{aligned} \tag{7.5.9}$$

In this case the variational integral is

$$\Xi = \int_{V^0} \chi \, \mathrm{d}V^0 - \frac{1}{2} \int_{V^0} \rho^0 \, \dot{\mathbf{b}} \cdot \mathbf{v} \, \mathrm{d}V^0 - \frac{1}{2} \int_{S_t^0} \dot{\mathbf{p}}_n \cdot \mathbf{v} \, \mathrm{d}S_t^0. \tag{7.5.10}$$

A loading that is partly controllable (independent of $\mathbf{v}$), and partly deformation sensitive but self-adjoint in the above sense also allows the variational principle. Detailed analysis is available in Hill (*op. cit.*).

7.5.1. Betti's Theorem and Clapeyron's Formula

Let

$$\mathbf{v} = \dot{\mathbf{x}}, \quad \dot{\mathbf{F}} = \frac{\partial \mathbf{v}}{\partial \mathbf{X}}, \quad \dot{\mathbf{P}} = \mathbf{\Lambda} : \dot{\mathbf{F}} \tag{7.5.11}$$

be a solution of the boundary-value problem associated with the prescribed rates of body forces $\dot{\mathbf{b}}$ in V^0, surface tractions $\dot{\mathbf{p}}_n$ on S_t^0, and velocities $\mathbf{v}$ on S_v^0. Similarly, let

$$\mathbf{v}^* = \dot{\mathbf{x}}^*, \quad \dot{\mathbf{F}}^* = \frac{\partial \mathbf{v}^*}{\partial \mathbf{X}}, \quad \dot{\mathbf{P}}^* = \mathbf{\Lambda} : \dot{\mathbf{F}}^* \tag{7.5.12}$$

be a solution of the boundary-value problem associated with the prescribed rates of body forces $\dot{\mathbf{b}}^*$ in V^0, surface tractions $\dot{\mathbf{p}}_n^*$ on S_t^0, and velocities $\mathbf{v}^*$ on S_v^0. By reciprocal symmetry of pseudomoduli $\mathbf{\Lambda}$ we have the reciprocal relation

$$\dot{\mathbf{P}} \cdot\cdot\, \dot{\mathbf{F}}^* = \dot{\mathbf{P}}^* \cdot\cdot\, \dot{\mathbf{F}}. \tag{7.5.13}$$

Upon integration over the volume V^0, and by using Eq. (3.11.12), it follows that

$$\begin{aligned}\int_{V^0} \rho^0 \, \dot{\mathbf{b}} \cdot \mathbf{v}^* \, \mathrm{d}V^0 &+ \int_{S^0} \mathbf{n}^0 \cdot \dot{\mathbf{P}} \cdot \mathbf{v}^* \, \mathrm{d}S^0 \\ &= \int_{V^0} \rho^0 \, \dot{\mathbf{b}}^* \cdot \mathbf{v} \, \mathrm{d}V^0 + \int_{S^0} \mathbf{n}^0 \cdot \dot{\mathbf{P}}^* \cdot \mathbf{v} \, \mathrm{d}S^0.\end{aligned} \tag{7.5.14}$$

This is analogous to Betti's reciprocal theorem of classical elasticity. Also, by incorporating $\dot{\mathbf{P}} = \mathbf{\Lambda} : \dot{\mathbf{F}}$ in the integral on the left-hand side of Eq. (3.11.12), there follows

$$\int_{V^0} \chi \, \mathrm{d}V^0 = \frac{1}{2} \int_{V^0} \rho^0 \dot{\mathbf{b}} \cdot \mathbf{v} \, \mathrm{d}V^0 + \frac{1}{2} \int_{S^0} \mathbf{n}^0 \cdot \dot{\mathbf{P}} \cdot \mathbf{v} \, \mathrm{d}S^0, \tag{7.5.15}$$

which is analogous to Clapeyron's formula from linear elasticity (Hill, 1978).

7.5.2. Other Rate-Potentials

The rate potential χ was introduced in Eq. (7.5.2) for the rate of nominal stress $\dot{\mathbf{P}}$. We can also introduce the rate-potentials for the rates of material and spatial stress tensors, such that

$$\dot{\mathbf{T}}_{(n)} = \frac{\partial \chi_{(n)}}{\partial \dot{\mathbf{E}}_{(n)}}, \quad \chi_{(n)} = \frac{1}{2}\,\boldsymbol{\Lambda}_{(n)} :: \left(\dot{\mathbf{E}}_{(n)} \otimes \dot{\mathbf{E}}_{(n)}\right), \tag{7.5.16}$$

$$\overset{\bullet}{\boldsymbol{\mathcal{T}}}_{(n)} = \frac{\partial \bar{\chi}_{(n)}}{\partial \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)}}, \quad \bar{\chi}_{(n)} = \frac{1}{2}\,\bar{\boldsymbol{\Lambda}}_{(n)} :: \left(\overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} \otimes \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)}\right). \tag{7.5.17}$$

7.5.3. Current Configuration as Reference

If the current configuration is taken as the reference configuration, we have

$$\dot{\underline{\mathbf{P}}} = \frac{\partial \underline{\chi}}{\partial \mathbf{L}}, \quad \underline{\chi} = \frac{1}{2}\,\underline{\boldsymbol{\Lambda}} \cdots (\mathbf{L} \otimes \mathbf{L}), \tag{7.5.18}$$

since $\dot{\underline{\mathbf{F}}} = \mathbf{L}$ (see Section 6.4). Substituting Eq. (6.4.16) for $\underline{\boldsymbol{\Lambda}}$, there follows

$$\underline{\chi} = \frac{1}{2}\,\underline{\boldsymbol{\mathcal{L}}}_{(1)} :: (\mathbf{D} \otimes \mathbf{D}) + \frac{1}{2}\,\boldsymbol{\sigma} : \left(\mathbf{L}^T \cdot \mathbf{L}\right). \tag{7.5.19}$$

Alternatively, in view of Eq. (6.3.14),

$$\underline{\chi} = \frac{1}{2}\,\underline{\boldsymbol{\mathcal{L}}}_{(0)} :: (\mathbf{D} \otimes \mathbf{D}) + \frac{1}{2}\,\boldsymbol{\sigma} : \left(\mathbf{L}^T \cdot \mathbf{L} - 2\mathbf{D}^2\right). \tag{7.5.20}$$

The symmetry of the instantaneous elastic moduli $\underline{\boldsymbol{\mathcal{L}}}_{(1)}$ was used in arriving at Eq. (7.5.19). With the current configuration as the reference, the variational integral of Eq. (7.5.7) becomes

$$\underline{\Xi} = \int_V \underline{\chi}\,\mathrm{d}V - \int_V \rho\,\dot{\mathbf{b}} \cdot \mathbf{v}\,\mathrm{d}V - \int_{S_t} \dot{\underline{\mathbf{p}}}_n \cdot \delta\mathbf{v}\,\mathrm{d}S_t, \tag{7.5.21}$$

where $\mathbf{n} \cdot \dot{\underline{\mathbf{P}}} = \dot{\underline{\mathbf{p}}}_n$ on S_t. The traction rate $\dot{\underline{\mathbf{p}}}_n$ is related to the rate of Cauchy traction $\dot{\mathbf{t}}_n$ by Eq. (3.9.18).

The rate potentials $\underline{\chi}_{(n)}$ are introduced such that

$$\dot{\underline{\mathbf{T}}}_{(n)} = \frac{\partial \underline{\chi}_{(n)}}{\partial \mathbf{D}}, \quad \underline{\chi}_{(n)} = \frac{1}{2}\,\underline{\boldsymbol{\mathcal{L}}}_{(n)} :: (\mathbf{D} \otimes \mathbf{D}). \tag{7.5.22}$$

In view of Eqs. (6.3.10) and (6.3.13), the various rate potentials are related by

$$\underline{\chi}_{(n)} = \underline{\chi}_{(0)} - n\boldsymbol{\sigma} : \mathbf{D}^2 = \underline{\chi}_{(1)} + (1-n)\boldsymbol{\sigma} : \mathbf{D}^2, \tag{7.5.23}$$

and

$$\underline{\chi} = \underline{\chi}_{(n)} + \frac{1}{2}\boldsymbol{\sigma} : \left[\mathbf{L}^T \cdot \mathbf{L} - 2(1-n)\mathbf{D}^2\right]. \tag{7.5.24}$$

Using the results from Section 3.9 for the rates $\dot{\underline{\mathbf{T}}}_{(n)}$, Eq. (7.5.22) gives, for $n = 0$ and $n = \pm 1$,

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} = \frac{\partial \underline{\chi}_{(0)}}{\partial \mathbf{D}}, \quad \overset{\triangle}{\underline{\boldsymbol{\tau}}} = \frac{\partial \underline{\chi}_{(1)}}{\partial \mathbf{D}}, \quad \overset{\triangledown}{\underline{\boldsymbol{\tau}}} = \frac{\partial \underline{\chi}_{(-1)}}{\partial \mathbf{D}}. \tag{7.5.25}$$

7.6. Uniqueness of Solution to Rate Problem

We examine now the uniqueness of solution to the boundary-value problem described by the rate equilibrium equations

$$\boldsymbol{\nabla}^0 \cdot \dot{\mathbf{P}} + \rho^0\, \dot{\mathbf{b}} = \mathbf{0}, \tag{7.6.1}$$

and the boundary conditions

$$\mathbf{v} = \mathbf{v}_0 \quad \text{on} \quad S_v^0, \qquad \mathbf{n}^0 \cdot \dot{\mathbf{P}} = \dot{\mathbf{p}}_n \quad \text{on} \quad S_t^0. \tag{7.6.2}$$

It is assumed that incremental loading is deformation insensitive, so that $\dot{\mathbf{b}}$ in V^0 and $\dot{\mathbf{p}}_n$ on S_t^0 do not depend on the velocity.

Suppose that there are two different solutions of Eqs. (7.6.1) and (7.6.2), $\mathbf{v}$ and $\mathbf{v}^*$. The corresponding rates of deformation gradients are $\dot{\mathbf{F}}$ and $\dot{\mathbf{F}}^*$, with the rates of nominal stresses $\dot{\mathbf{P}}$ and $\dot{\mathbf{P}}^*$. The equilibrium fields $\left(\dot{\mathbf{P}}, \dot{\mathbf{F}}\right)$ and $\left(\dot{\mathbf{P}}^*, \dot{\mathbf{F}}^*\right)$ necessarily satisfy the condition

$$\int_{V^0} (\dot{\mathbf{P}}^* - \dot{\mathbf{P}}) \cdot\cdot (\dot{\mathbf{F}}^* - \dot{\mathbf{F}})\, \mathrm{d}V^0 = 0, \tag{7.6.3}$$

which follows from Eq. (3.11.13). Consequently, from Eq. (7.6.3), the velocity field $\mathbf{v}$ is unique if

$$\begin{aligned} \int_{V^0} \left(\dot{\mathbf{P}}^* - \dot{\mathbf{P}}\right) \cdot\cdot \left(\dot{\mathbf{F}}^* - \dot{\mathbf{F}}\right) \mathrm{d}V^0 \\ = \int_{V^0} \boldsymbol{\Lambda} \cdot\cdot\cdot\cdot \left(\dot{\mathbf{F}}^* - \dot{\mathbf{F}}\right) \otimes \left(\dot{\mathbf{F}}^* - \dot{\mathbf{F}}\right) \mathrm{d}V^0 \neq 0, \end{aligned} \tag{7.6.4}$$

for all kinematically admissible $\mathbf{v}^*$ giving rise to

$$\dot{\mathbf{F}}^* = \frac{\partial \mathbf{v}^*}{\partial \mathbf{X}}, \quad \dot{\mathbf{P}}^* = \boldsymbol{\Lambda} : \dot{\mathbf{F}}^*. \tag{7.6.5}$$

The stress rate $\dot{\mathbf{P}}^*$ in (7.6.4) need not be statically admissible, so even if the equality sign applies in (7.6.4) for some $\mathbf{v}^*$, the uniqueness is lost only if $\mathbf{v}^*$ gives rise to statically admissible stress-rate field $\dot{\mathbf{P}}^*$. Therefore, a sufficient condition for $\mathbf{v}$ to be unique solution is that for all kinematically admissible velocity fields $\mathbf{v}^*$,

$$\int_{V^0} \boldsymbol{\Lambda} \cdot \cdot \cdot \cdot \left(\dot{\mathbf{F}}^* - \dot{\mathbf{F}}\right) \otimes \left(\dot{\mathbf{F}}^* - \dot{\mathbf{F}}\right) \mathrm{d}V^0 > 0. \tag{7.6.6}$$

The reversed inequality could also serve as a sufficient condition for uniqueness. The solution $\mathbf{v}$ which obeys such inequality for all kinematically admissible $\mathbf{v}^*$ would define unique, but unstable equilibrium configuration, analogous to the consideration in Section 7.3.

A more restrictive condition for uniqueness is evidently

$$\boldsymbol{\Lambda} \cdot \cdot \cdot \cdot \left(\dot{\mathbf{F}} - \dot{\mathbf{F}}^*\right) \otimes \left(\dot{\mathbf{F}} - \dot{\mathbf{F}}^*\right) > 0, \tag{7.6.7}$$

which implies (7.6.6), and which states that $\boldsymbol{\Lambda}$ is positive definite. However, since unique solution to a finite elasticity rate problem cannot be expected in general, the inequality (7.6.7) may fail at certain states of deformation. A nonuniqueness of the rate problem is certainly a possibility if the state of deformation is reached when $\boldsymbol{\Lambda}$ becomes singular, so that $\boldsymbol{\Lambda} \cdot \cdot \dot{\mathbf{F}} = \mathbf{0}$ has nontrivial solutions for $\dot{\mathbf{F}}$. Details of the calculations for isotropic materials can be found in Ogden (1984).

If a sufficient condition for uniqueness (7.6.6) applies, then

$$\Xi(\mathbf{v}^*) > \Xi(\mathbf{v}), \tag{7.6.8}$$

and the variational principle is strengthened to a minimum principle: among all kinematically admissible velocity fields, the actual field renders Ξ the minimum. Indeed, from Eq. (7.5.7) it follows that

$$\Xi(\mathbf{v}^*) - \Xi(\mathbf{v}) = \frac{1}{2} \int_{V^0} \left(\dot{\mathbf{P}}^* - \dot{\mathbf{P}}\right) \cdot \cdot \left(\dot{\mathbf{F}}^* - \dot{\mathbf{F}}\right) \mathrm{d}V^0. \tag{7.6.9}$$

In the derivation, Eq. (3.11.12) was used, and the reciprocity relation

$$\dot{\mathbf{P}} \cdot \cdot \dot{\mathbf{F}}^* = \dot{\mathbf{P}}^* \cdot \cdot \dot{\mathbf{F}}. \tag{7.6.10}$$

A useful identity, resulting from the reciprocity of $\boldsymbol{\Lambda}$, is

$$\dot{\mathbf{P}}^* \cdot \cdot \dot{\mathbf{F}}^* - \dot{\mathbf{P}} \cdot \cdot \dot{\mathbf{F}} = \left(\dot{\mathbf{P}}^* - \dot{\mathbf{P}}\right) \cdot \cdot \left(\dot{\mathbf{F}}^* - \dot{\mathbf{F}}\right) + 2\,\dot{\mathbf{P}} \cdot \cdot \left(\dot{\mathbf{F}}^* - \dot{\mathbf{F}}\right). \tag{7.6.11}$$

7.7. Bifurcation Analysis

It was shown in the previous section, if displacement fields $\mathbf{v}$ and $\mathbf{v}^*$ are both solutions of incrementally linear inhomogeneous rate problem described by

Eqs. (7.6.1) and (7.6.2), then

$$\frac{1}{2}\int_{V^0}\left(\Delta\dot{\mathbf{P}}\cdot\cdot\,\Delta\dot{\mathbf{F}}\right)\mathrm{d}V^0=\frac{1}{2}\int_{V^0}\mathbf{\Lambda}\cdot\cdot\cdot\cdot\left(\Delta\dot{\mathbf{F}}\otimes\Delta\dot{\mathbf{F}}\right)\mathrm{d}V^0=0, \tag{7.7.1}$$

with

$$\Delta\dot{\mathbf{F}}=\dot{\mathbf{F}}-\dot{\mathbf{F}}^*,\quad \Delta\dot{\mathbf{P}}=\dot{\mathbf{P}}-\dot{\mathbf{P}}^*. \tag{7.7.2}$$

Consider the associated homogeneous rate problem, described by

$$\boldsymbol{\nabla}^0\cdot\dot{\boldsymbol{P}}=\mathbf{0}, \tag{7.7.3}$$

and the boundary conditions

$$\mathbf{w}=\mathbf{0}\quad\text{on}\quad S^0_v,\qquad \mathbf{n}^0\cdot\dot{\boldsymbol{P}}=\mathbf{0}\quad\text{on}\quad S^0_t, \tag{7.7.4}$$

where

$$\dot{\boldsymbol{F}}=\frac{\partial\mathbf{w}}{\partial\mathbf{X}},\quad \dot{\boldsymbol{P}}=\mathbf{\Lambda}\cdot\cdot\,\dot{\boldsymbol{F}}. \tag{7.7.5}$$

The bold face italic notation is used for the fields associated with the displacement field $\mathbf{w}$. The rate problem described by (7.7.3) and (7.7.4) has always a nul solution $\mathbf{w}=\mathbf{0}$. If the homogeneous problem also has a nontrivial solution

$$\mathbf{w}\neq\mathbf{0}, \tag{7.7.6}$$

then by Eq. (7.7.1)

$$\frac{1}{2}\int_{V^0}\left(\dot{\boldsymbol{P}}\cdot\cdot\,\dot{\boldsymbol{F}}\right)\mathrm{d}V^0=\frac{1}{2}\int_{V^0}\mathbf{\Lambda}\cdot\cdot\cdot\cdot\left(\dot{\boldsymbol{F}}\otimes\dot{\boldsymbol{F}}\right)\mathrm{d}V^0=0. \tag{7.7.7}$$

This condition places the same restrictions on the moduli $\mathbf{\Lambda}$ as does (7.7.1), as expected, since (7.7.7) follows directly from (7.7.1) by taking

$$\mathbf{w}=\mathbf{v}-\mathbf{v}^*. \tag{7.7.8}$$

The examination of the uniqueness of solution to incrementally linear inhomogeneous rate problem (7.6.1) and (7.6.2) is thus equivalent to the examination of the uniqueness of solution to the associated homogeneous rate problem (7.7.3) and (7.7.4).

7.7.1. Exclusion Functional

If for all kinematically admissible $\mathbf{w}$ giving rise to $\dot{\boldsymbol{F}}=\partial\mathbf{w}/\partial\mathbf{X}$,

$$\int_{V^0}\chi(\mathbf{w})\,\mathrm{d}V^0=\frac{1}{2}\int_{V^0}\mathbf{\Lambda}\cdot\cdot\cdot\cdot\left(\dot{\boldsymbol{F}}\otimes\dot{\boldsymbol{F}}\right)\mathrm{d}V^0>0, \tag{7.7.9}$$

from Eq. (7.6.6) it follows that $\mathbf{w}=\mathbf{0}$ is the only solution of the homogeneous rate problem. Furthermore, by Eq. (7.4.4) it follows that the underlying equilibrium configuration $\mathbf{x}$ is incrementally stable (and thus incrementally unique), under a considered dead loading. At some states of deformation, however, there may exist a nontrivial solution $\mathbf{w} \neq \mathbf{0}$ to the homogeneous rate problem. This $\mathbf{w}$ then satisfies Eq. (7.7.7), implying nonuniqueness of the homogeneous rate problem, and from Section 7.4 nonuniqueness and neutral incremental stability of the underlying equilibrium configuration $\mathbf{x}$. The deformation state at which this happens is called an eigenstate. A nontrivial solution to the homogeneous rate problem is called an eigenmode (Hill, 1978). Therefore, since inhomogeneous rate problem with an incrementally linear stress-deformation response is linear, its solution is unique if and only if the current configuration is not an eigenstate for the associated homogeneous rate problem. If the current configuration is an eigenstate, than any multiple of an eigenmode ($k\mathbf{w}$) could be added to one solution of inhomogeneous rate problem ($\mathbf{v}$) to generate others ($\mathbf{v} + k\mathbf{w}$). Thus, to guarantee uniqueness it is enough to exclude the possibility of eigenmodes. Consequently, following Hill (1978), introduce the exclusion functional

$$\mathcal{F} = \int_{V^0} \chi(\mathbf{w})\,\mathrm{d}V^0, \quad \chi(\mathbf{w}) = \frac{1}{2}\,\mathbf{\Lambda} \cdots \left(\dot{\boldsymbol{F}} \otimes \dot{\boldsymbol{F}}\right), \tag{7.7.10}$$

for any kinematically admissible $\mathbf{w}$ giving rise to $\dot{\boldsymbol{F}} = \partial\mathbf{w}/\partial\mathbf{X}$. Starting the deformation from a stable reference configuration, a state is reached where the exclusion functional becomes positive semidefinite ($\mathcal{F} \geq 0$), vanishing for some kinematically admissible $\mathbf{w}$. The state at which

$$\mathcal{F} = 0 \tag{7.7.11}$$

is first reached for some $\mathbf{w}$ is called a primary eigenstate. In this state the uniqueness fails, and the deformation path branches (usually by infinitely many eigenmodes). The phenomenon is referred to as bifurcation. (Beyond the region $\mathcal{F} \geq 0$, the exclusion functional is indefinite. If a kinematically admissible $\mathbf{w}$ makes $\mathcal{F} = 0$ for some configuration in this region, the configuration is an eigenstate, but $\mathbf{w}$ is not an eigenmode unless it gives rise to statically admissible stress rate field $\dot{\boldsymbol{F}}$. Since this region is unstable, it will not be considered further).

In any eigenstate at the boundary $\mathcal{F} \geq 0$, an eigenmode $\mathbf{w}$ makes the exclusion functional stationary within the class of kinematically admissible variations $\delta\mathbf{w}$. Indeed, for homogeneous data

$$\frac{1}{2}\int_{V^0} \boldsymbol{\Lambda} \cdots \left(\dot{\boldsymbol{F}} \otimes \delta\dot{\boldsymbol{F}}\right) \mathrm{d}V^0 = \frac{1}{2}\int_{V^0} \dot{\boldsymbol{P}} \cdot\cdot\, \delta\dot{\boldsymbol{F}}\, \mathrm{d}V^0 = 0, \tag{7.7.12}$$

by Eq. (3.11.12), since the stress rate $\dot{\boldsymbol{P}}$, associated with an eigenmode $\mathbf{w}$, is statically admissible field for the homogeneous rate problem. Since $\boldsymbol{\Lambda}$ possesses reciprocal symmetry, Eq. (7.7.12) implies

$$\delta\mathcal{F} = 0. \tag{7.7.13}$$

Conversely, any kinematically admissible velocity field $\mathbf{w}$ that makes $\mathcal{F}$ stationary is an eigenmode. This is so because for homogeneous problem the variational integral of Eq. (7.5.7) is equal to the exclusion functional ($\Xi = \mathcal{F}$).

As previously indicated, from Eq (7.4.6) it follows that

$$\mathcal{P}(\mathbf{u} + \delta\mathbf{u}) = \mathcal{P}(\mathbf{u}) \tag{7.7.14}$$

for any eigenmode $\mathbf{w}$ giving rise to displacement increment $\delta\mathbf{u} = \mathbf{w}\,\delta t$. Thus, the potential energies are equal in any two adjacent equilibrium states differing under dead load by an eigenmode deformation. These states are neutrally stable, within the second-order approximations used in deriving Eq. (7.4.6). To assess stability of an eigenmode more accurately, higher order terms in the expansion (7.4.2), leading to (7.4.6), would have to be retained.

The criticality of the exclusion functional is independent of the incepient loading rates (inhomogeneous data) in the current configuration. However, inhomogeneous data cannot be prescribed freely in an eigenstate, if the inhomogeneous rate problem is to admit a solution. Indeed, when the reciprocal theorem (7.5.14) is applied to the fields $\left(\mathbf{v}, \dot{\mathbf{P}}\right)$ and $(\mathbf{w}, \mathbf{0})$, it follows that

$$\int_{V^0} \rho^0\, \dot{\mathbf{b}} \cdot \mathbf{w}\, \mathrm{d}V^0 + \int_{S_t^0} \dot{\mathbf{p}}_n \cdot \mathbf{w}\, \mathrm{d}S_t^0 = 0, \tag{7.7.15}$$

for every distinct eigenmode. This may be regarded as a generalized orthogonality between the rates of loading (inhomogeneous data) and the eigenmodes (Hill, 1978; Ogden, 1984).

In the case of homogeneous material and homogeneous deformation, Eq. (7.7.9) implies that $\boldsymbol{\Lambda}$ is positive definite. A primary eigenstate is characterized by positive semidefinite $\boldsymbol{\Lambda}$, i.e.,

$$\chi(\mathbf{w}) = \frac{1}{2}\,\boldsymbol{\Lambda} \cdot\cdot\cdot\cdot (\dot{\boldsymbol{F}} \otimes \dot{\boldsymbol{F}}) \geq 0, \tag{7.7.16}$$

with equality sign for some $\dot{\boldsymbol{F}}$ (uniform throughout the body). The corresponding eigenmode is subject to stationary condition $\delta\mathcal{F} = 0$, which gives $\dot{\boldsymbol{P}} \cdot\cdot\, \delta\dot{\boldsymbol{F}} = 0$ for all $\delta\dot{\boldsymbol{F}}$ from kinematically admissible $\delta\mathbf{w}$. Thus,

$$\dot{\boldsymbol{P}} = \boldsymbol{\Lambda} \cdot\cdot\, \dot{\boldsymbol{F}} = \mathbf{0} \tag{7.7.17}$$

in a primary (uniformly deformed) eigenstate, as anticipated since $\boldsymbol{\Lambda}$ becomes singular in this state.

In the case of deformation sensitive loading rates, the exclusion condition is

$$\mathcal{F} > 0 \tag{7.7.18}$$

for all kinematically admissible fields $\mathbf{w}$, where

$$\mathcal{F} = \int_{V^0} \chi(\mathbf{w})\,\mathrm{d}V^0 - \frac{1}{2}\int_{V^0} \rho^0\,\dot{\mathbf{b}} \cdot \mathbf{w}\,\mathrm{d}V^0 - \frac{1}{2}\int_{S_t^0} \dot{\mathbf{p}}_n \cdot \mathbf{w}\,\mathrm{d}S_t^0. \tag{7.7.19}$$

If the loading rates are self-adjoint in the sense of Eq. (7.5.8), both the exclusion functional and its first variation vanish for an eigenmode. Detailed analysis is given by Hill (1978).

7.8. Localization Bifurcation

Consider a homogeneous elastic body in the state of uniform deformation. For prescribed velocities on the boundary which give rise to uniform $\dot{\mathbf{F}}$ throughout the body, conditions are sought under which bifurcation by localization of deformation within a planar band can occur. This is associated with a primary eigenmode

$$\mathbf{w} = f(\mathbf{N} \cdot \mathbf{X})\,\boldsymbol{\eta}, \quad \dot{\boldsymbol{F}} = f'\,\boldsymbol{\eta} \otimes \mathbf{N}. \tag{7.8.1}$$

For $\dot{\boldsymbol{F}}$ to be discontinuous across the band, the gradient f' is piecewise constant across the band, whose unit normal in the undeformed configuration is $\mathbf{N}$. The localization vector is $\boldsymbol{\eta}$. For example, in the case of shear band, $\mathbf{n}\cdot\boldsymbol{\eta} = 0$, where $\mathbf{n} = \mathbf{N}\cdot\boldsymbol{F}^{-1}$ is the band normal in the deformed configuration (Fig. 7.1). (Although shear and necking instabilities are usually associated

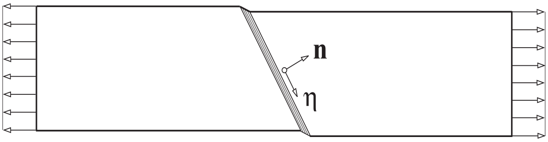

FIGURE 7.1. A shear band with normal $\mathbf{n}$ and localization vector $\boldsymbol{\eta}$ in a homogeneously deformed specimen under plane strain tension.

with plastic response, they can also occur in certain nonlinearly elastic materials; Silling, 1988; Antman, 1974,1995). The stress rate associated with Eq. (7.8.1) is

$$\dot{\boldsymbol{P}} = f'\,\boldsymbol{\Lambda}\cdot\cdot(\boldsymbol{\eta}\otimes\mathbf{N}) = f'\,\boldsymbol{\Lambda} : (\mathbf{N}\otimes\boldsymbol{\eta}). \tag{7.8.2}$$

Substituting this into equilibrium equation (7.7.3) gives

$$f''\,\mathbf{N}\cdot\boldsymbol{\Lambda} : (\mathbf{N}\otimes\boldsymbol{\eta}) = \mathbf{0}. \tag{7.8.3}$$

Thus,

$$\mathbf{N}\cdot\boldsymbol{\Lambda} : (\mathbf{N}\otimes\boldsymbol{\eta}) = \mathbf{A}(\mathbf{N})\cdot\boldsymbol{\eta} = \mathbf{0}. \tag{7.8.4}$$

The second-order tensor

$$\mathbf{A}(\mathbf{N}) = \boldsymbol{\Lambda} : (\mathbf{N}\otimes\mathbf{N}), \quad A_{ij}(\mathbf{N}) = \Lambda_{KiLj}N_K N_L \tag{7.8.5}$$

is a symmetric tensor, obeying the symmetry $\Lambda_{KiLj} = \Lambda_{LjKi}$. For a nontrivial $\boldsymbol{\eta}$ to be determined from the condition

$$\mathbf{A}(\mathbf{N})\cdot\boldsymbol{\eta} = \mathbf{0}, \tag{7.8.6}$$

the matrix $\mathbf{A}(\mathbf{N})$ has to be singular, i.e.,

$$\det\mathbf{A}(\mathbf{N}) = 0. \tag{7.8.7}$$

Note that Eq. (7.8.4) implies

$$\dot{\boldsymbol{p}}_n = \mathbf{N}\cdot\dot{\boldsymbol{P}} = \mathbf{0}, \tag{7.8.8}$$

which is obtained by multiplying Eq. (7.8.2) with $\mathbf{N}$. This means that the rate of nominal traction across the localization band vanishes.

Constitutive law and equilibrium equations are said to be elliptic in any state where

$$\det\mathbf{A}(\mathbf{N}) \neq 0, \quad \text{for all } \mathbf{N}. \tag{7.8.9}$$

Thus, if uniform deformation bifurcates by a band localization eigenmode, the constitutive law and governing equilibrium equations loose their ellipticity. Since there is a correspondence between the conditions for a localization bifurcation and the occurrence of stationary body waves (waves with vanishing wave speeds), the latter is briefly discussed in the next section.

7.9. Acoustic Tensor

Consider a homogeneous elastic body in a state of homogeneous deformation. Its response to small amplitude wave disturbances is examined. Solutions to the rate equations

$$\boldsymbol{\nabla}^0 \cdot \dot{\mathbf{P}} = \rho^0 \, \frac{\mathrm{d}^2 \mathbf{v}}{\mathrm{d}t^2} \tag{7.9.1}$$

are sought in the form of a plane wave propagating with a speed c in the direction $\mathbf{N}$,

$$\mathbf{v} = \boldsymbol{\eta} f(\mathbf{N} \cdot \mathbf{X} - ct). \tag{7.9.2}$$

The unit vector $\boldsymbol{\eta}$ defines the polarization of the wave. On substituting (7.9.2) into (7.9.1), the propagation condition is found to be

$$\mathbf{A}(\mathbf{N}) \cdot \boldsymbol{\eta} = \rho^0 \, c^2 \, \boldsymbol{\eta}. \tag{7.9.3}$$

The second-order tensor $\mathbf{A}(\mathbf{N})$ is referred to as the acoustic tensor. It is explicitly defined by Eq. (7.8.5). From Eq. (7.9.3) we conclude that $\rho^0 c^2$ is an eigenvalue and $\boldsymbol{\eta}$ is an eigenvector of the acoustic tensor $\mathbf{A}(\mathbf{N})$. Since $\mathbf{A}(\mathbf{N})$ is real and symmetric, c^2 must be real. If $c^2 > 0$, there is a stability with respect to propagation of small disturbances. For stationary waves (stationary discontinuity) $c = 0$, which signifies the transition from stability to instability. The instability is associated with $c^2 < 0$, and a divergent growth of an initial disturbance.

Taking a scalar product of Eq. (7.9.3) with $\boldsymbol{\eta}$ gives

$$\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{N}) \cdot \boldsymbol{\eta} = \rho^0 \, c^2. \tag{7.9.4}$$

Therefore, if $\mathbf{A}(\mathbf{N})$ is positive definite,

$$\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{N}) \cdot \boldsymbol{\eta} > 0 \tag{7.9.5}$$

for all $\boldsymbol{\eta}$, we have $c^2 > 0$, and Eq. (7.9.1) admits three linearly independent plane progressive waves for each direction of propagation $\mathbf{N}$. In this case, small amplitude elastic plane waves can propagate along a given direction in three distinct, mutually orthogonal modes. These modes are generally neither longitudinal nor transverse. We say that the wave is longitudinal if $\boldsymbol{\eta}$ and $\mathbf{n} = \mathbf{N} \cdot \mathbf{F}^{-1}$ are parallel, and transverse if $\boldsymbol{\eta}$ and $\mathbf{n}$ are perpendicular. Brugger (1965) calculated directions of propagation of pure mode, longitudinal and transverse waves for most anisotropic crystal classes in their undeformed state. See also Hill (1975) and Milstein (1982).

7.9.1. Strong Ellipticity Condition

If the condition holds

$$\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{N}) \cdot \boldsymbol{\eta} = \boldsymbol{\Lambda} :: [(\mathbf{N} \otimes \boldsymbol{\eta}) \otimes (\mathbf{N} \otimes \boldsymbol{\eta})] > 0 \tag{7.9.6}$$

for each $\mathbf{N} \otimes \boldsymbol{\eta}$, the system of equations (7.9.1) with zero acceleration is said to be strongly elliptic. Clearly, strong ellipticity implies ellipticity, since for positive definite acoustic tensor

$$\det \mathbf{A}(\mathbf{N}) > 0. \tag{7.9.7}$$

Not every strain energy function will yield an acoustic tensor satisfying the conditions of strong ellipticity in every configuration. For example, in the case of undeformed isotropic elastic material, the strong ellipticity requires that the Lamé constants satisfy

$$\lambda + 2\mu > 0, \quad \mu > 0. \tag{7.9.8}$$

This does not imply that the corresponding Ψ is positive definite. The conditions for the latter are

$$\lambda + \frac{2}{3}\,\mu > 0, \quad \mu > 0. \tag{7.9.9}$$

Thus, while the strong ellipticity condition is strong enough to preclude occurrence of shear band localization, it is not strong enough to ensure the physically observed behavior with necessarily positive value of the elastic bulk modulus ($\kappa = \lambda + 2\mu/3$).

A weaker inequality

$$\boldsymbol{\eta} \cdot \mathbf{A}(\mathbf{N}) \cdot \boldsymbol{\eta} = \boldsymbol{\Lambda} :: [(\mathbf{N} \otimes \boldsymbol{\eta}) \otimes (\mathbf{N} \otimes \boldsymbol{\eta})] \geq 0 \tag{7.9.10}$$

for all $\mathbf{N} \otimes \boldsymbol{\eta}$, is known as the Hadamard condition of stability. This condition does not exclude nonpropagating or stationary waves (discontinuities, singular surfaces). The condition is further discussed by Truesdell and Noll (1965), and Marsden and Hughes (1983).

If the current configuration is taken as the reference, Eq. (7.9.1) becomes

$$\boldsymbol{\nabla} \cdot \underline{\dot{\mathbf{P}}} = \rho \, \frac{\mathrm{d}^2 \mathbf{v}}{\mathrm{d}t^2}\,, \tag{7.9.11}$$

where

$$\underline{\dot{\mathbf{P}}} = \underline{\boldsymbol{\Lambda}} \cdot\cdot\, \mathbf{L}, \quad \underline{\Lambda}_{jilk} = \underline{\mathcal{L}}^{(1)}_{jilk} + \sigma_{jl}\delta_{ik}. \tag{7.9.12}$$

The propagation condition is

$$\underline{\mathbf{A}}(\mathbf{n}) \cdot \boldsymbol{\eta} = \rho\, c^2\, \boldsymbol{\eta}, \quad \underline{A}_{ij}(\mathbf{n}) = \underline{\Lambda}_{kilj} n_k n_l, \tag{7.9.13}$$

while the strong ellipticity requires that

$$\boldsymbol{\eta} \cdot \underline{\mathbf{A}}(\mathbf{n}) \cdot \boldsymbol{\eta} = \underline{\boldsymbol{\Lambda}} :: [(\mathbf{n} \otimes \boldsymbol{\eta}) \otimes (\mathbf{n} \otimes \boldsymbol{\eta})] > 0. \tag{7.9.14}$$

Since the moduli $\boldsymbol{\Lambda}$ and $\underline{\boldsymbol{\Lambda}}$ are related by Eq. (6.4.14), and since $\mathbf{n} = \mathbf{N} \cdot \mathbf{F}^{-1}$, there is a connection

$$\mathbf{A}(\mathbf{N}) = (\det \mathbf{F})\, \underline{\mathbf{A}}(\mathbf{n}). \tag{7.9.15}$$

7.10. Constitutive Inequalities

A significant amount of research was devoted to find a constitutive inequality for elastic materials under finite deformation that would hold irrespective of the geometry of the boundary value problem, or prescribed displacement and traction boundary conditions. For example, in the range of infinitesimal deformation such an inequality is $\boldsymbol{\sigma} : \boldsymbol{\varepsilon} > 0$, where $\boldsymbol{\varepsilon}$ is an infinitesimal strain. This is a consequence of positive definiteness of the strain energy function $\Psi = (1/2)\, \boldsymbol{\sigma} : \boldsymbol{\varepsilon}$. For finite elastic deformation, Caprioli (1955) proposed that the elastic work is non-negative on any path, open or closed, from the ground state. This implies the existence of Ψ, which must have an absolute minimum in the ground (unstressed) state.

Constitutive inequalities must be objective, i.e., independent of a superimposed rotation to the deformed configuration. For example, the inequality

$$\dot{\mathbf{P}} \cdot\cdot\, \dot{\mathbf{F}} = \boldsymbol{\Lambda} \cdot\cdot\cdot\cdot \left(\dot{\mathbf{F}} \otimes \dot{\mathbf{F}}\right) > 0, \tag{7.10.1}$$

derived from the considerations of uniqueness and stability of the rate boundary value problem, is not objective, since under the rotation $\mathbf{Q}$,

$$\dot{\mathbf{P}}^* \cdot\cdot\, \dot{\mathbf{F}}^* = \left(\dot{\mathbf{P}} - \mathbf{P} \cdot \hat{\boldsymbol{\Omega}}\right) \cdot\cdot \left(\dot{\mathbf{F}} + \hat{\boldsymbol{\Omega}} \cdot \mathbf{F}\right) \neq \dot{\mathbf{P}} \cdot\cdot\, \dot{\mathbf{F}}. \tag{7.10.2}$$

There is no universal constitutive inequality applicable to all types of finite elastic deformation. Instead, various inequalities have been proposed to hold in certain domains of deformation around the reference state, and for particular types of elastic materials (e.g., Truesdell and Noll, 1965; Hill, 1968, 1970; Ogden, 1970). Such an inequality is

$$\left(\mathbf{T}^*_{(n)} - \mathbf{T}_{(n)}\right) : \left(\mathbf{E}^*_{(n)} - \mathbf{E}_{(n)}\right) = \left(\frac{\partial \Psi}{\partial \mathbf{E}^*_{(n)}} - \frac{\partial \Psi}{\partial \mathbf{E}_{(n)}}\right) : \left(\mathbf{E}^*_{(n)} - \mathbf{E}_{(n)}\right) > 0, \tag{7.10.3}$$

for all $\mathbf{E}_{(n)} \neq \mathbf{E}^*_{(n)}$. If the strain domain in which (7.10.3) holds is convex, the inequality implies that $\Psi(\mathbf{E}_{(n)})$ is globally strictly convex in that domain. It also implies that $\partial\Psi/\partial\mathbf{E}_{(n)}$ is one-to-one in that domain. For different n, (7.10.3) represents different physical requirements, so that inequality may hold for some n, and fail for others.

Another inequality is obtained by requiring that

$$\dot{\mathbf{T}}_{(n)} : \dot{\mathbf{E}}_{(n)} = \boldsymbol{\Lambda}_{(n)} :: (\dot{\mathbf{E}}_{(n)} \otimes \dot{\mathbf{E}}_{(n)}) > 0, \quad \boldsymbol{\Lambda}_{(n)} = \frac{\partial^2 \Psi}{\partial \mathbf{E}_{(n)} \otimes \partial \mathbf{E}_{(n)}}. \tag{7.10.4}$$

This means that $\boldsymbol{\Lambda}_{(n)}$, the Hessian of Ψ with respect to $\mathbf{E}_{(n)}$, is positive definite, i.e., that the strain energy Ψ is locally strictly convex in a considered strain domain. It can be shown that in a convex strain domain local convexity implies global convexity, and *vice versa*. To demonstrate former, for instance, we can choose the strain rate in (7.10.4) to be directed along the line from $\mathbf{E}_{(n)}$ to $\mathbf{E}^*_{(n)}$; integration from $\mathbf{E}_{(n)}$ to $\mathbf{E}^*_{(n)}$ leads (7.10.3). As in the case of (7.10.3), the inequality (7.10.4) represents different physical requirements for different choices of n. Convexity of Ψ is not an invariant property, so that convexity in the space of one strain measure may be lost in the space of another strain measure.

If the current configuration is taken as the reference, (7.10.4) becomes

$$\underline{\dot{\mathbf{T}}}_{(n)} : \mathbf{D} = \underline{\boldsymbol{\mathcal{L}}}_{(n)} :: (\mathbf{D} \otimes \mathbf{D}) = 2\,\underline{\chi}_{(n)} > 0. \tag{7.10.5}$$

This in general imposes different restrictions on the constitutive law than (7.10.4) does. In view of Eqs. (6.3.12) and (6.3.13), we can rewrite (7.10.5) as

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} : \mathbf{D} > 2n\left(\boldsymbol{\sigma} : \mathbf{D}^2\right), \quad \overset{\circ}{\underline{\boldsymbol{\tau}}} = \overset{\circ}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D}. \tag{7.10.6}$$

Hill (1968) proposed that the most appealing inequality is obtained from (7.10.6) for $n = 0$, so that

$$\overset{\circ}{\underline{\boldsymbol{\mathcal{T}}}} : \mathbf{D} > 0. \tag{7.10.7}$$

This inequality is found to be in best agreement with the anticipated features of elastic response. See also Leblond (1992).

Alternative representation of the inequalities (7.10.3) and (7.10.4) is obtained by using spatial tensor measures. They are

$$\left(\boldsymbol{\mathcal{T}}^*_{(n)} - \boldsymbol{\mathcal{T}}_{(n)}\right) : \left(\boldsymbol{\mathcal{E}}^*_{(n)} - \boldsymbol{\mathcal{E}}_{(n)}\right) = \left(\frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}^*_{(n)}} - \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)}}\right) : \left(\boldsymbol{\mathcal{E}}^*_{(n)} - \boldsymbol{\mathcal{E}}_{(n)}\right) > 0, \tag{7.10.8}$$

$$\overset{\bullet}{\boldsymbol{\mathcal{T}}}_{(n)} : \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} = \bar{\boldsymbol{\Lambda}}_{(n)} :: \left(\overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} \otimes \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)}\right) > 0, \quad \bar{\boldsymbol{\Lambda}}_{(n)} = \frac{\partial^2 \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)} \otimes \partial \boldsymbol{\mathcal{E}}_{(n)}}. \tag{7.10.9}$$

Inequality (7.10.5) remains the same, because $\underline{\overset{\bullet}{\boldsymbol{\mathcal{T}}}}_{(n)} = \underline{\dot{\mathbf{T}}}_{(n)}$ and $\underline{\bar{\boldsymbol{\Lambda}}}_{(n)} = \underline{\boldsymbol{\mathcal{L}}}_{(n)}$.

If $\mathbf{E}^*_{(n)}$ is nearby $\mathbf{E}_{(n)}$, so that

$$\mathbf{E}^*_{(n)} = \mathbf{E}_{(n)} + \delta \mathbf{E}_{(n)}, \tag{7.10.10}$$

by Taylor expansion of $\partial \Psi / \partial \mathbf{E}^*_{(n)}$ we obtain

$$\delta \mathbf{T}_{(n)} = \boldsymbol{\Lambda}_{(n)} : \delta \mathbf{E}_{(n)} + \frac{1}{2} \frac{\partial \boldsymbol{\Lambda}_{(n)}}{\partial \mathbf{E}_{(n)}} :: \left(\delta \mathbf{E}_{(n)} \otimes \delta \mathbf{E}_{(n)}\right) + \cdots . \tag{7.10.11}$$

Thus,

$$\begin{aligned} \delta \mathbf{T}_{(n)} : \delta \mathbf{E}_{(n)} &= \boldsymbol{\Lambda}_{(n)} :: \left(\delta \mathbf{E}_{(n)} \otimes \delta \mathbf{E}_{(n)}\right) \\ &+ \frac{1}{2} \frac{\partial \boldsymbol{\Lambda}_{(n)}}{\partial \mathbf{E}_{(n)}} ::: \left(\delta \mathbf{E}_{(n)} \otimes \delta \mathbf{E}_{(n)} \otimes \delta \mathbf{E}_{(n)}\right) + \cdots . \end{aligned} \tag{7.10.12}$$

The sixth-order tensor

$$\frac{\partial \boldsymbol{\Lambda}_{(n)}}{\partial \mathbf{E}_{(n)}} = \frac{\partial^2 \mathbf{T}_{(n)}}{\partial \mathbf{E}_{(n)} \otimes \partial \mathbf{E}_{(n)}} = \frac{\partial^3 \Psi}{\partial \mathbf{E}_{(n)} \otimes \partial \mathbf{E}_{(n)} \otimes \partial \mathbf{E}_{(n)}} \tag{7.10.13}$$

is a tensor of the third-order elastic moduli, previously encountered in Section 5.11 within the context of higher-order elastic constants of cubic crystals. The third-order pseudomoduli are similarly defined as $\partial \boldsymbol{\Lambda} / \partial \mathbf{F}$. These tensors play an important role in assessing the true nature of stability of equilibrium in the cases when the second-order expansions, such as those used in Section 7.4, lead to an assessment of neutral stability. Details are available in Hill (1982) and Ogden (1984).

References

Antman, S. S. (1974), Qualitative theory of the ordinary differential equations of nonlinear elasticity, in *Mechanics Today*, Vol. 1, ed. S. Nemat-Nasser, pp. 58–101, Pergamon Press, New York.

Antman, S. S. (1995), *Nonlinear Problems of Elasticity*, Springer-Verlag, New York.

Ball, J. M. (1977), Convexity conditions and existence theorems in non-linear elasticity, *Arch. Rat. Mech. Anal.*, Vol. 63, pp. 337–403.

Beatty, M. F. (1996), Introduction to nonlinear elasticity, in *Nonlinear Effects in Fluids and Solids*, eds. M. M. Carroll and M. A. Hayes, pp. 13–112, Plenum Press, New York.

Brugger, K. (1965), Pure modes for elastic waves in crystals, *J. Appl. Phys.*, Vol. 36, pp. 759–768.

Caprioli, L. (1965), Su un criterio per l'esistenza dell'energia di deformazione, *Boll. Un. Mat. Ital.*, Vol. 10, pp. 481–483 (1955); English translation in *Foundations of Elasticity Theory*, Intl. Sci. Rev. Ser., Gordon & Breach, New York.

Ciarlet, P. G. (1988), *Mathematical Elasticity, Volume I: Three-Dimensional Elasticity*, North-Holland, Amsterdam.

Coleman, B. and Noll, W. (1959), On the thermostatics of continuous media, *Arch. Rat. Mech. Anal.*, Vol. 4, pp. 97–128.

Ericksen, J. L. (1977), Special topics in elastostatics, *Adv. Appl. Mech.*, Vol. 17, pp. 189–244.

Gurtin, M. E. (1982), On uniquensess in finite elasticity, in *Finite Elasticity*, eds. D. E. Carlson and R. T. Shield, pp. 191–199, Martinus Nijhoff Publishers, The Hague.

Hanyga, A. (1985), *Mathematical Theory of Non-Linear Elasticity*, Ellis Horwood, Chichester, England, and PWN–Polish Scientific Publishers, Warsaw, Poland.

Hill, R. (1957), On uniqueness and stability in the theory of finite elastic strain, *J. Mech. Phys. Solids*, Vol. 5, pp. 229–241.

Hill, R. (1968), On constitutive inequalities for simple materials – I, *J. Mech. Phys. Solids*, Vol. 16, pp. 229–242.

Hill, R. (1970), Constitutive inequalities for isotropic elastic solids under finite strain, *Proc. Roy. Soc. London A*, Vol. 314, pp. 457–472.

Hill, R. (1975), On the elasticity and stability of perfect crystals at finite strain, *Math. Proc. Camb. Phil. Soc.*, Vol. 77, pp. 225–240.

Hill, R. (1978), Aspects of invariance in solid mechanics, *Adv. Appl. Mech.*, Vol. 18, pp. 1–75.

Hill, R. (1982), Constitutive branching in elastic materials, *Math. Proc. Cambridge Philos. Soc.*, Vol. 92, pp. 167–181.

Knops, R. J. and Payne, L. E. (1971), *Uniqueness Theorems in Linear Elasticity*, Springer-Verlag, New York.

Knops, R. J. and Wilkes, E. W. (1973), Theory of elastic stability, in *Handbuch der Physik*, ed. C. Truesdell, Band VIa/3, pp. 125–302, Springer-Verlag, Berlin.

Leblond, J. B. (1992), A constitutive inequality for hyperelastic materials in finite strain, *Eur. J. Mech., A/Solids*, Vol. 11, pp. 447–466.

Marsden, J. E. and Hughes, T. J. R. (1983), *Mathematical Foundations of Elasticity*, Prentice Hall, Englewood Cliffs, New Jersey.

Milstein, F. (1982), Crystal elasticity, in *Mechanics of Solids – The Rodney Hill 60th Anniversary Volume*, eds. H. G. Hopkins and M. J. Sewell, pp. 417–452, Pergamon Press, Oxford.

Nemat-Nasser, S. (1974), General variational principles in nonlinear and linear elasticity with applications, in *Mechanics Today*, Vol. 1, ed. S. Nemat-Nasser, pp. 214–261, Pergamon Press, New York.

Ogden, R. W. (1970), Compressible isotropic elastic solids under finite strain: Constitutive inequalities, *Quart. J. Mech. Appl. Math.*, Vol. 23, pp. 457–468.

Ogden, R. W. (1984), *Non-Linear Elastic Deformations*, Ellis Horwood Ltd., Chichester, England (2nd ed., Dover, 1997).

Person, C. E. (1959), *Theoretical Elasticity*, Harvard University Press, Cambridge, Massachusetts.

Silling, S. A. (1988), Two-dimensional effects in the necking of elastic bars, *J. Appl. Mech.*, Vol. 55, pp. 530–535.

Truesdell, C. (1985), *The Elements of Continuum Mechanics*, Springer-Verlag, New York.

Truesdell, C. and Noll, W. (1965), The nonlinear field theories of mechanics, in *Handbuch der Physik*, ed. S. Flügge, Band III/3, Springer-Verlag, Berlin (2nd ed. 1992).

Washizu, K. (1982), *Variational Methods in Elasticity and Plasticity*, 3rd ed., Pergamon Press, Oxford.