CHAPTER 6

RATE-TYPE ELASTICITY

6.1. Elastic Moduli Tensors

The rate-type constitutive equation for finite deformation elasticity is obtained by differentiating Eq. (5.1.2) with respect to a time-like monotonically increasing parameter t. This gives

$$\dot{\mathbf{T}}_{(n)} = \mathbf{\Lambda}_{(n)} : \dot{\mathbf{E}}_{(n)}, \quad \mathbf{\Lambda}_{(n)} = \frac{\partial^2 \Psi \left(\mathbf{E}_{(n)}\right)}{\partial \mathbf{E}_{(n)} \otimes \partial \mathbf{E}_{(n)}} . \tag{6.1.1}$$

The fourth-order tensor $\mathbf{\Lambda}_{(n)}$ is the tensor of elastic moduli (or tensor of elasticities) associated with a conjugate pair of material tensors $\left(\mathbf{E}_{(n)}, \mathbf{T}_{(n)}\right)$. Its representation in an orthonormal basis in the undeformed configuration is

$$\mathbf{\Lambda}_{(n)} = \Lambda^{(n)}_{IJKL} \mathbf{e}^0_I \otimes \mathbf{e}^0_J \otimes \mathbf{e}^0_K \otimes \mathbf{e}^0_L . \tag{6.1.2}$$

Similarly, by applying to Eq. (5.1.10) the Jaumann derivative with respect to spin $\boldsymbol{\omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1}$, we obtain the rate-type constitutive equation

$$\overset{\bullet}{\boldsymbol{\mathcal{T}}}_{(n)} = \bar{\mathbf{\Lambda}}_{(n)} : \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)}, \quad \bar{\mathbf{\Lambda}}_{(n)} = \frac{\partial^2 \Psi \left(\hat{\boldsymbol{\mathcal{E}}}_{(n)}\right)}{\partial \boldsymbol{\mathcal{E}}_{(n)} \otimes \partial \boldsymbol{\mathcal{E}}_{(n)}} . \tag{6.1.3}$$

The fourth-order tensor $\bar{\mathbf{\Lambda}}_{(n)}$ is the tensor of elastic moduli associated with a conjugate pair of spatial tensors $\left(\boldsymbol{\mathcal{E}}_{(n)}, \boldsymbol{\mathcal{T}}_{(n)}\right)$. This can be represented in an orthonormal basis in the deformed configuration as

$$\bar{\mathbf{\Lambda}}_{(n)} = \bar{\Lambda}^{(n)}_{ijkl} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l . \tag{6.1.4}$$

The relationship between the tensors $\mathbf{\Lambda}_{(n)}$ and $\bar{\mathbf{\Lambda}}_{(n)}$ follows by recalling that

$$\hat{\boldsymbol{\mathcal{E}}}_{(n)} = \mathbf{E}_{(n)}, \quad \dot{\mathbf{T}}_{(n)} = \mathbf{R}^T \cdot \overset{\bullet}{\boldsymbol{\mathcal{T}}}_{(n)} \cdot \mathbf{R}, \quad \dot{\mathbf{E}}_{(n)} = \mathbf{R}^T \cdot \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} \cdot \mathbf{R}, \tag{6.1.5}$$

which gives

$$\bar{\mathbf{\Lambda}}_{(n)} = \mathbf{R}\,\mathbf{R}\,\mathbf{\Lambda}_{(n)}\,\mathbf{R}^T\,\mathbf{R}^T . \tag{6.1.6}$$

The tensor products in Eq. (6.1.6) are defined so that the Cartesian components are related by

$$\bar{\Lambda}^{(n)}_{ijkl} = R_{iM} R_{jN} \Lambda^{(n)}_{MNPQ} R^T_{Pk} R^T_{Ql}. \tag{6.1.7}$$

In performing the Jaumann derivation of Eq. (5.1.10) it should be kept in mind that

$$\overset{\bullet}{\hat{\boldsymbol{\mathcal{E}}}}_{(n)} = \dot{\hat{\boldsymbol{\mathcal{E}}}}_{(n)} = \dot{\mathbf{E}}_{(n)}, \tag{6.1.8}$$

since corotational (and convected) derivatives of the material tensors are equal to ordinary material derivatives (material tensors not being affected by the transformation of the base tensors in the deformed configuration). It is instructive to discuss this point a little further. To be more specific, consider a transversely isotropic material from Section 5.9, for which the strain energy is

$$\Psi = \Psi\left(\mathbf{E}_{(n)}, \mathbf{M}^0\right) = \Psi\left(\boldsymbol{\mathcal{E}}_{(n)}, \overline{\mathbf{M}}\right), \tag{6.1.9}$$

with the spatial stress tensor

$$\boldsymbol{\mathcal{T}}_{(n)} = \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)}}. \tag{6.1.10}$$

The application of the Jaumann derivative with respect to spin $\boldsymbol{\omega}$ to Eq. (6.1.10) gives

$$\begin{aligned} \overset{\bullet}{\boldsymbol{\mathcal{T}}}_{(n)} &= \frac{\partial^2 \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)} \otimes \partial \boldsymbol{\mathcal{E}}_{(n)}} : \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} + \frac{\partial^2 \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)} \otimes \partial \overline{\mathbf{M}}} : \overset{\bullet}{\overline{\mathbf{M}}} \\ &= \frac{\partial^2 \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)} \otimes \partial \boldsymbol{\mathcal{E}}_{(n)}} : \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)}, \end{aligned} \tag{6.1.11}$$

because $\overset{\bullet}{\overline{\mathbf{M}}} = \mathbf{0}$. Recall that $\overline{\mathbf{M}} = \mathbf{R} \cdot \mathbf{M}^0 \cdot \mathbf{R}^T$, so that

$$\dot{\overline{\mathbf{M}}} = \boldsymbol{\omega} \cdot \overline{\mathbf{M}} - \overline{\mathbf{M}} \cdot \boldsymbol{\omega}. \tag{6.1.12}$$

If the ordinary material derivative of Eq. (6.1.10) is taken, we have

$$\dot{\boldsymbol{\mathcal{T}}}_{(n)} = \frac{\partial^2 \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)} \otimes \partial \boldsymbol{\mathcal{E}}_{(n)}} : \dot{\boldsymbol{\mathcal{E}}}_{(n)} + \frac{\partial^2 \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)} \otimes \partial \overline{\mathbf{M}}} : \dot{\overline{\mathbf{M}}}. \tag{6.1.13}$$

This is in accord with Eq. (6.1.11) because the identity holds

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)} \otimes \partial \boldsymbol{\mathcal{E}}_{(n)}} &: \left(\boldsymbol{\mathcal{E}}_{(n)} \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \boldsymbol{\mathcal{E}}_{(n)}\right) + \frac{\partial^2 \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)} \otimes \partial \overline{\mathbf{M}}} \\ &: \left(\overline{\mathbf{M}} \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \overline{\mathbf{M}}\right) = \boldsymbol{\mathcal{T}}_{(n)} \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \boldsymbol{\mathcal{T}}_{(n)}. \end{aligned} \tag{6.1.14}$$

To verify Eq. (6.1.14), we can differentiate both sides of Eq. (6.1.9) to obtain

$$\frac{\partial \Psi}{\partial \mathbf{E}_{(n)}} : \dot{\mathbf{E}}_{(n)} = \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)}} : \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} = \frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)}} : \dot{\boldsymbol{\mathcal{E}}}_{(n)} + \frac{\partial \Psi}{\partial \overline{\mathbf{M}}} : \dot{\overline{\mathbf{M}}}, \tag{6.1.15}$$

which establishes the identity

$$\frac{\partial \Psi}{\partial \boldsymbol{\mathcal{E}}_{(n)}} : \left(\boldsymbol{\mathcal{E}}_{(n)} \cdot \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \boldsymbol{\mathcal{E}}_{(n)}\right) = \frac{\partial \Psi}{\partial \overline{\mathbf{M}}} : \left(\boldsymbol{\omega} \cdot \overline{\mathbf{M}} - \overline{\mathbf{M}} \cdot \boldsymbol{\omega}\right). \tag{6.1.16}$$

Differentiation of Eq. (6.1.16) with respect to $\boldsymbol{\mathcal{E}}_{(n)}$ gives Eq. (6.1.14).

6.2. Elastic Moduli for Conjugate Measures with $\mathbf{n} = \pm 1$

The rates of the conjugate tensors $\mathbf{E}_{(1)}$ and $\mathbf{T}_{(1)}$ are, from Eqs. (2.6.1) and (3.8.4),

$$\dot{\mathbf{E}}_{(1)} = \mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F}, \quad \dot{\mathbf{T}}_{(1)} = \mathbf{F}^{-1} \cdot \overset{\triangle}{\boldsymbol{\tau}} \cdot \mathbf{F}^{-T}. \tag{6.2.1}$$

Substitution into Eq. (6.1.1) gives the Oldroyd rate of the Kirchhoff stress $\boldsymbol{\tau}$ in terms of the rate of deformation $\mathbf{D}$,

$$\overset{\triangle}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_{(1)} : \mathbf{D}. \tag{6.2.2}$$

The corresponding elastic moduli tensor is

$$\boldsymbol{\mathcal{L}}_{(1)} = \mathbf{F}\,\mathbf{F}\,\boldsymbol{\Lambda}_{(1)}\,\mathbf{F}^T\,\mathbf{F}^T. \tag{6.2.3}$$

The products in Eq. (6.2.3) are such that the Cartesian components of the two tensors of elasticities are related by

$$\mathcal{L}^{(1)}_{ijkl} = F_{iM} F_{jN} \Lambda^{(1)}_{MNPQ} F^T_{Pk} F^T_{Ql}. \tag{6.2.4}$$

Equation (6.2.2) can also be derived from the first of Eq. (5.1.11) by applying to it, for example, the convected derivative $\overset{\triangle}{(\;)}$, and by recalling that

$$\overset{\triangle}{\mathbf{F}} = \mathbf{0}, \quad \overset{\triangle}{\mathbf{E}}_{(1)} = \dot{\mathbf{E}}_{(1)}. \tag{6.2.5}$$

See also Truesdell and Noll (1965), and Marsden and Hughes (1983).

Similarly, from Eqs. (2.6.3) and (3.8.5), the rates of the conjugate measures $\mathbf{E}_{(-1)}$ and $\mathbf{T}_{(-1)}$ are

$$\dot{\mathbf{E}}_{(-1)} = \mathbf{F}^{-1} \cdot \mathbf{D} \cdot \mathbf{F}^{-T}, \quad \dot{\mathbf{T}}_{(-1)} = \mathbf{F}^T \cdot \overset{\triangledown}{\boldsymbol{\tau}} \cdot \mathbf{F}. \tag{6.2.6}$$

Substitution into Eq. (6.1.1) gives

$$\overset{\triangledown}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_{(-1)} : \mathbf{D}, \tag{6.2.7}$$

where

$$\boldsymbol{\mathcal{L}}_{(-1)} = \mathbf{F}^{-T}\,\mathbf{F}^{-T}\,\mathbf{\Lambda}_{(-1)}\,\mathbf{F}^{-1}\,\mathbf{F}^{-1}. \tag{6.2.8}$$

This can be alternatively derived by applying the convected derivative $\overset{\triangledown}{(\;)}$ to the second of Eq. (5.1.11), and by recalling that

$$(\mathbf{F}^{-1})^{\triangledown} = \mathbf{0}, \quad \overset{\triangledown}{\mathbf{E}}_{(-1)} = \dot{\mathbf{E}}_{(-1)}. \tag{6.2.9}$$

In view of the connection (6.1.6) between the moduli $\mathbf{\Lambda}_{(n)}$ and $\bar{\mathbf{\Lambda}}_{(n)}$, Eqs. (6.2.3) and (6.2.8) can be rewritten as

$$\boldsymbol{\mathcal{L}}_{(1)} = \mathbf{V}\,\mathbf{V}\,\bar{\mathbf{\Lambda}}_{(1)}\,\mathbf{V}\,\mathbf{V}, \quad \boldsymbol{\mathcal{L}}_{(-1)} = \mathbf{V}^{-1}\,\mathbf{V}^{-1}\,\bar{\mathbf{\Lambda}}_{(-1)}\,\mathbf{V}^{-1}\,\mathbf{V}^{-1}. \tag{6.2.10}$$

Another route to derive Eq. (6.2.2) is by differentiation of Eqs. (5.1.12). For example, by applying the Jaumann derivative $\overset{\bullet}{(\;)}$ to the first of Eqs. (5.1.12) gives

$$\begin{aligned} \overset{\bullet}{\boldsymbol{\tau}} &= \left(\overset{\bullet}{\mathbf{V}}\cdot\mathbf{V}^{-1}\right)\cdot\boldsymbol{\tau} + \boldsymbol{\tau}\cdot\left(\mathbf{V}^{-1}\cdot\overset{\bullet}{\mathbf{V}}\right) \\ &+ \mathbf{V}\left(\frac{\partial^2\Psi}{\partial\boldsymbol{\mathcal{E}}_{(1)}\otimes\partial\hat{\boldsymbol{\mathcal{E}}}_{(1)}} : \dot{\hat{\boldsymbol{\mathcal{E}}}}_{(1)}\right)\mathbf{V}. \end{aligned} \tag{6.2.11}$$

Since

$$\overset{\bullet}{\mathbf{V}}\cdot\mathbf{V}^{-1} = \mathbf{L} - \boldsymbol{\omega}, \quad \dot{\hat{\boldsymbol{\mathcal{E}}}}_{(1)} = \mathbf{R}^T\cdot\overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(1)}\cdot\mathbf{R}, \quad \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(1)} = \mathbf{V}\cdot\mathbf{D}\cdot\mathbf{V}, \tag{6.2.12}$$

Equation (6.2.11) becomes

$$\overset{\triangle}{\boldsymbol{\tau}} = \left(\mathbf{V}\mathbf{V}\,\frac{\partial^2\Psi}{\partial\boldsymbol{\mathcal{E}}_{(1)}\otimes\partial\boldsymbol{\mathcal{E}}_{(1)}}\,\mathbf{V}\mathbf{V}\right) : \mathbf{D} = \boldsymbol{\mathcal{L}}_{(1)} : \mathbf{D}. \tag{6.2.13}$$

The rate-type constitutive Eqs. (6.2.2) and (6.2.7) can be rewritten in terms of the Jaumann rate $\overset{\circ}{\boldsymbol{\tau}}$ as

$$\overset{\circ}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_{(0)} : \mathbf{D}, \tag{6.2.14}$$

where

$$\boldsymbol{\mathcal{L}}_{(0)} = \boldsymbol{\mathcal{L}}_{(1)} + 2\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{L}}_{(-1)} - 2\boldsymbol{\mathcal{S}}. \tag{6.2.15}$$

This follows because of the relationships (see Section 3.8)

$$\overset{\circ}{\boldsymbol{\tau}} = \overset{\triangle}{\boldsymbol{\tau}} + \mathbf{D}\cdot\boldsymbol{\tau} + \boldsymbol{\tau}\cdot\mathbf{D} = \overset{\triangledown}{\boldsymbol{\tau}} - \mathbf{D}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\mathbf{D}. \tag{6.2.16}$$

The Cartesian components of the fourth-order tensor $\boldsymbol{\mathcal{S}}$ are

$$\mathcal{S}_{ijkl} = \tau_{(ik}\delta_{jl)} = \frac{1}{4}\left(\tau_{ik}\delta_{jl} + \tau_{jk}\delta_{il} + \tau_{il}\delta_{jk} + \tau_{jl}\delta_{ik}\right). \tag{6.2.17}$$

The elastic moduli tensors $\boldsymbol{\Lambda}_{(n)}$, $\bar{\boldsymbol{\Lambda}}_{(n)}$ and $\boldsymbol{\mathcal{L}}_{(n)}$ all possess the basic and reciprocal (major) symmetries, e.g.,

$$\mathcal{L}^{(n)}_{ijkl} = \mathcal{L}^{(n)}_{jikl} = \mathcal{L}^{(n)}_{ijlk}, \quad \mathcal{L}^{(n)}_{ijkl} = \mathcal{L}^{(n)}_{klij}. \tag{6.2.18}$$

Further analysis of elastic moduli tensors can be found in Truesdell and Toupin (1960), Ogden (1984), and Holzapfel (2000).

6.3. Instantaneous Elastic Moduli

The instantaneous elastic moduli relate the rates of conjugate stress and strain tensors, when these are evaluated at the current configuration as the reference. Thus, since $\underline{\dot{\mathbf{E}}}_{(n)} = \mathbf{D}$, we write

$$\underline{\dot{\mathbf{T}}}_{(n)} = \underline{\boldsymbol{\Lambda}}_{(n)} : \underline{\dot{\mathbf{E}}}_{(n)} = \underline{\boldsymbol{\Lambda}}_{(n)} : \mathbf{D}. \tag{6.3.1}$$

The tensor of instantaneous elastic moduli $\underline{\boldsymbol{\Lambda}}_{(n)}$ can be related to the corresponding tensor of elastic moduli $\boldsymbol{\Lambda}_{(n)}$ by using the relationship between $\dot{\mathbf{E}}_{(n)}$ and $\underline{\dot{\mathbf{E}}}_{(n)}$. For example, for $n = 1$, from Eq. (3.9.16) we obtain

$$\dot{\mathbf{T}}_{(1)} = (\det \mathbf{F})\, \mathbf{F}^{-1} \cdot \underline{\dot{\mathbf{T}}}_{(1)} \cdot \mathbf{F}^{-T}, \quad \dot{\mathbf{E}}_{(1)} = \mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F}. \tag{6.3.2}$$

The substitution into Eq. (6.1.1) gives

$$\begin{gathered} \underline{\dot{\mathbf{T}}}_{(1)} = \underline{\boldsymbol{\Lambda}}_{(1)} : \mathbf{D}, \\ \underline{\boldsymbol{\Lambda}}_{(1)} = (\det \mathbf{F})^{-1}\, \mathbf{F}\, \mathbf{F}\, \boldsymbol{\Lambda}_{(1)}\, \mathbf{F}^T\, \mathbf{F}^T = (\det \mathbf{F})^{-1}\, \boldsymbol{\mathcal{L}}_{(1)}. \end{gathered} \tag{6.3.3}$$

Recalling from Eq. (3.9.15) that $\underline{\dot{\mathbf{T}}}_{(1)} = \overset{\triangle}{\underline{\boldsymbol{\tau}}}$, Eq. (6.3.3) becomes

$$\overset{\triangle}{\underline{\boldsymbol{\tau}}} = \underline{\boldsymbol{\mathcal{L}}}_{(1)} : \mathbf{D}, \quad \underline{\boldsymbol{\mathcal{L}}}_{(1)} = \underline{\boldsymbol{\Lambda}}_{(1)}. \tag{6.3.4}$$

Similarly,

$$\overset{\triangledown}{\underline{\boldsymbol{\tau}}} = \underline{\boldsymbol{\mathcal{L}}}_{(-1)} : \mathbf{D}, \quad \underline{\boldsymbol{\mathcal{L}}}_{(-1)} = (\det \mathbf{F})^{-1}\, \boldsymbol{\mathcal{L}}_{(-1)}. \tag{6.3.5}$$

Furthermore, from Eq. (3.9.7) we have

$$\underline{\dot{\mathbf{T}}}_{(n)} = \overset{\circ}{\underline{\boldsymbol{\tau}}} - n(\mathbf{D} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbf{D}) = \overset{\triangle}{\underline{\boldsymbol{\tau}}} - (n-1)(\mathbf{D} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbf{D}). \tag{6.3.6}$$

Thus, Eq. (6.3.1) can be recast in the form

$$\overset{\triangle}{\underline{\boldsymbol{\tau}}} - (n-1)(\mathbf{D} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbf{D}) = \underline{\boldsymbol{\mathcal{L}}}_{(n)} : \mathbf{D}, \tag{6.3.7}$$

since, in general,

$$\underline{\boldsymbol{\mathcal{L}}}_{(n)} = \underline{\boldsymbol{\Lambda}}_{(n)} = \underline{\bar{\boldsymbol{\Lambda}}}_{(n)}. \tag{6.3.8}$$

Substituting the expression (6.3.4) for $\overset{\triangle}{\boldsymbol{\tau}}$ into Eq. (6.3.7) gives

$$\underline{\boldsymbol{\mathcal{L}}}_{(1)} : \mathbf{D} - (n-1)(\mathbf{D}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{D}) = \underline{\boldsymbol{\mathcal{L}}}_{(n)} : \mathbf{D}. \tag{6.3.9}$$

This establishes the relationship between the instantaneous elastic moduli $\underline{\boldsymbol{\mathcal{L}}}_{(n)}$ and $\underline{\boldsymbol{\mathcal{L}}}_{(1)}$,

$$\underline{\boldsymbol{\mathcal{L}}}_{(n)} = \underline{\boldsymbol{\mathcal{L}}}_{(1)} - 2(n-1)\underline{\boldsymbol{\mathcal{S}}}. \tag{6.3.10}$$

The Cartesian components of the tensor $\underline{\boldsymbol{\mathcal{S}}}$ are

$$\underline{\mathcal{S}}_{ijkl} = \sigma_{(ik}\delta_{jl)} = \frac{1}{4}\left(\sigma_{ik}\delta_{jl} + \sigma_{jk}\delta_{il} + \sigma_{il}\delta_{jk} + \sigma_{jl}\delta_{ik}\right). \tag{6.3.11}$$

Thus, the difference between the various instantaneous elastic moduli in Eq. (6.3.10) is of the order of the Cauchy stress.

If the logarithmic strain is used, we have

$$\dot{\underline{\mathbf{T}}}_{(0)} = \overset{\circ}{\underline{\boldsymbol{\tau}}} = \underline{\boldsymbol{\mathcal{L}}}_{(0)} : \mathbf{D}, \tag{6.3.12}$$

and comparison with Eq. (6.3.7) gives

$$\underline{\boldsymbol{\mathcal{L}}}_{(n)} = \underline{\boldsymbol{\mathcal{L}}}_{(0)} - 2n\underline{\boldsymbol{\mathcal{S}}}. \tag{6.3.13}$$

In particular,

$$\underline{\boldsymbol{\mathcal{L}}}_{(0)} = \underline{\boldsymbol{\mathcal{L}}}_{(1)} + 2\underline{\boldsymbol{\mathcal{S}}} = \underline{\boldsymbol{\mathcal{L}}}_{(-1)} - 2\underline{\boldsymbol{\mathcal{S}}}, \tag{6.3.14}$$

as expected from Eq. (6.2.15). Further details are available in Hill (1978) and Ogden (1984).

6.4. Elastic Pseudomoduli

The nonsymmetric nominal stress $\mathbf{P}$ is derived from the strain energy function as its gradient with respect to deformation gradient $\mathbf{F}$, such that

$$\mathbf{P} = \frac{\partial\Psi}{\partial\mathbf{F}}, \quad P_{Ji} = \frac{\partial\Psi}{\partial F_{iJ}}. \tag{6.4.1}$$

The rate of the nominal stress is, therefore,

$$\dot{\mathbf{P}} = \boldsymbol{\Lambda}\cdot\cdot\,\dot{\mathbf{F}} = \boldsymbol{\Lambda}\cdot\cdot\,(\mathbf{L}\cdot\mathbf{F}), \quad \boldsymbol{\Lambda} = \frac{\partial^2\Psi}{\partial\mathbf{F}\otimes\partial\mathbf{F}}. \tag{6.4.2}$$

A two-point tensor of elastic pseudomoduli is denoted by $\boldsymbol{\Lambda}$. The Cartesian component representation of Eq. (6.4.2) is

$$\dot{P}_{Ji} = \Lambda_{JiLk}\dot{F}_{kL}, \quad \Lambda_{JiLk} = \frac{\partial^2\Psi}{\partial F_{iJ}\partial F_{kL}}. \tag{6.4.3}$$

The elastic pseudomoduli Λ_{JiLk} are not true moduli since they are partly associated with the material spin. They clearly possess the reciprocal symmetry

$$\Lambda_{JiLk} = \Lambda_{LkJi}. \tag{6.4.4}$$

In view of the connection

$$\mathbf{P} = \mathbf{T}_{(1)} \cdot \mathbf{F}^T, \tag{6.4.5}$$

the differentiation gives

$$\boldsymbol{\Lambda} \cdot\cdot\, \dot{\mathbf{F}} = \left(\boldsymbol{\Lambda}_{(1)} : \dot{\mathbf{E}}_{(1)}\right) \cdot \mathbf{F}^T + \mathbf{T}_{(1)} \cdot \dot{\mathbf{F}}^T. \tag{6.4.6}$$

Upon using

$$\dot{\mathbf{E}}_{(1)} = \frac{1}{2}\left(\dot{\mathbf{F}}^T \cdot \mathbf{F} + \mathbf{F}^T \cdot \dot{\mathbf{F}}\right), \tag{6.4.7}$$

Equation (6.4.6) yields the connection between the elastic moduli $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}_{(1)}$. Their Cartesian components are related by

$$\Lambda_{JiLk} = \Lambda^{(1)}_{JMLN} F_{iM} F_{kN} + T^{(1)}_{JL} \delta_{ik}. \tag{6.4.8}$$

Since $\mathbf{F} \cdot \mathbf{P}$ is a symmetric tensor, i.e.,

$$F_{iK} P_{Kj} = F_{jK} P_{Ki}, \tag{6.4.9}$$

by differentiation and incorporation of Eq. (6.4.3) it follows that

$$F_{jM} \Lambda_{MiLk} - F_{iM} \Lambda_{MjLk} = \delta_{ik} P_{Lj} - \delta_{jk} P_{Li}. \tag{6.4.10}$$

This corresponds to the symmetry in the leading pair of indices of the true elastic moduli

$$\Lambda^{(1)}_{IJKL} = \Lambda^{(1)}_{JIKL}. \tag{6.4.11}$$

The tensor of elastic pseudomoduli $\boldsymbol{\Lambda}$ can be related to the tensor of instantaneous elastic moduli, appearing in the expression

$$\underline{\dot{\mathbf{P}}} = \underline{\boldsymbol{\Lambda}} \cdot\cdot\, \mathbf{L}, \tag{6.4.12}$$

by recalling the relationship

$$\dot{\mathbf{P}} = (\det \mathbf{F}) \mathbf{F}^{-1} \cdot \underline{\dot{\mathbf{P}}}, \tag{6.4.13}$$

from Section 3.9. This gives

$$\underline{\boldsymbol{\Lambda}} = (\det \mathbf{F})^{-1} \mathbf{F}\, \boldsymbol{\Lambda}\, \mathbf{F}^T, \tag{6.4.14}$$

with the Cartesian component representation

$$\underline{\Lambda}_{ijkl} = (\det \mathbf{F})^{-1} F_{iM} \Lambda_{MjNk} F^T_{Nl}. \tag{6.4.15}$$

In addition, from Eq. (6.4.8), we have

$$\underline{\Lambda}_{jilk} = \underline{\Lambda}^{(1)}_{jilk} + \sigma_{jl}\delta_{ik}. \tag{6.4.16}$$

6.5. Elastic Moduli of Isotropic Elasticity

For isotropic elasticity, the strain energy function is an isotropic function of strain, so that

$$\Psi = \Psi\left(\hat{\boldsymbol{\mathcal{E}}}_{(n)}\right) = \Psi\left(\boldsymbol{\mathcal{E}}_{(n)}\right), \tag{6.5.1}$$

and

$$\boldsymbol{\mathcal{T}}_{(n)} = \frac{\partial\Psi\left(\boldsymbol{\mathcal{E}}_{(n)}\right)}{\partial\boldsymbol{\mathcal{E}}_{(n)}} = c_0\mathbf{I} + c_1\boldsymbol{\mathcal{E}}_{(n)} + c_2\boldsymbol{\mathcal{E}}^2_{(n)}. \tag{6.5.2}$$

By definition of the Jaumann derivative, we have

$$\left(\frac{\partial\Psi}{\partial\boldsymbol{\mathcal{E}}_{(n)}}\right)^{\circ} = \left(\frac{\partial\Psi}{\partial\boldsymbol{\mathcal{E}}_{(n)}}\right)^{\cdot} - \mathbf{W}\cdot\frac{\partial\Psi}{\partial\boldsymbol{\mathcal{E}}_{(n)}} + \frac{\partial\Psi}{\partial\boldsymbol{\mathcal{E}}_{(n)}}\cdot\mathbf{W}. \tag{6.5.3}$$

Since Ψ is an isotropic function of $\boldsymbol{\mathcal{E}}_{(n)}$, there is an identity

$$\frac{\partial^2\Psi}{\partial\boldsymbol{\mathcal{E}}_{(n)}\otimes\partial\boldsymbol{\mathcal{E}}_{(n)}} : \left(\mathbf{W}\cdot\boldsymbol{\mathcal{E}}_{(n)} - \boldsymbol{\mathcal{E}}_{(n)}\cdot\mathbf{W}\right) = \mathbf{W}\cdot\frac{\partial\Psi}{\partial\boldsymbol{\mathcal{E}}_{(n)}} - \frac{\partial\Psi}{\partial\boldsymbol{\mathcal{E}}_{(n)}}\cdot\mathbf{W}, \tag{6.5.4}$$

which is easily verified by using Eq. (6.5.2). Thus, we can write

$$\overset{\circ}{\boldsymbol{\mathcal{T}}}_{(n)} = \frac{\partial^2\Psi}{\partial\boldsymbol{\mathcal{E}}_{(n)}\otimes\partial\boldsymbol{\mathcal{E}}_{(n)}} : \overset{\circ}{\boldsymbol{\mathcal{E}}}_{(n)}. \tag{6.5.5}$$

This is one of the constitutive structures of the rate-type isotropic elasticity. It is pointed out that Eq. (6.5.5) also applies if $\overset{\circ}{(\ )}$ is replaced by the material derivative, or the Jaumann derivative with respect to spin $\boldsymbol{\omega}$, or any other spin associated with the deformed configuration.

An appealing rate-type constitutive structure of isotropic elasticity is obtained by using Eq. (5.5.5) to express the Kirchhoff stress in terms of the left Cauchy–Green deformation tensor $\mathbf{B}$. The application of the Jaumann derivative $\overset{\circ}{(\ )}$ gives (e.g., Lubarda, 1986)

$$\begin{aligned}\overset{\circ}{\boldsymbol{\tau}} = \frac{1}{2}\left(\mathbf{D}\cdot\boldsymbol{\tau} + \boldsymbol{\tau}\cdot\mathbf{D}\right) + \frac{1}{2}\left[\mathbf{B}\cdot(\mathbf{D}\cdot\boldsymbol{\tau})\cdot\mathbf{B}^{-1} + \mathbf{B}^{-1}\cdot(\boldsymbol{\tau}\cdot\mathbf{D})\cdot\mathbf{B}\right]\\ + 4\left(\mathbf{B}\,\frac{\partial^2\Psi}{\partial\mathbf{B}\otimes\partial\mathbf{B}}\,\mathbf{B}\right) : \mathbf{D} = \boldsymbol{\mathcal{L}}_{(0)} : \mathbf{D}.\end{aligned} \tag{6.5.6}$$

Recall that

$$\overset{\circ}{\mathbf{B}} = \mathbf{B}\cdot\mathbf{D} + \mathbf{D}\cdot\mathbf{B}, \tag{6.5.7}$$

and that Ψ is an isotropic function of $\mathbf{B}$, which allows us to write

$$\left(\frac{\partial \Psi}{\partial \mathbf{B}}\right)^{\circ} = \frac{\partial^2 \Psi}{\partial \mathbf{B} \otimes \partial \mathbf{B}} : \overset{\circ}{\mathbf{B}}. \tag{6.5.8}$$

The Cartesian components of the elastic moduli tensor $\boldsymbol{\mathcal{L}}_{(0)}$ are

$$\mathcal{L}^{(0)}_{ijkl} = \tau_{(ik}\delta_{jl)} + B_{(ik}\tau_{lm}B^{-1}_{mj)} + B_{(im}\bar{\Lambda}^{(1)}_{mjkn}B_{nl)}, \tag{6.5.9}$$

where

$$\bar{\Lambda}^{(1)}_{mjkn} = \frac{\partial^2 \Psi}{\partial \mathcal{E}^{(1)}_{mj}\partial \mathcal{E}^{(1)}_{kn}} = 4\,\frac{\partial^2 \Psi}{\partial B_{mj}\partial B_{kn}}\,. \tag{6.5.10}$$

The symmetry in i and j, k and l, and ij and kl is ensured by Eq. (6.2.17), and by the symmetrization

$$\begin{aligned} B_{(ik}\tau_{lm}B^{-1}_{mj)} = \frac{1}{4}\big(&B_{ik}\tau_{lm}B^{-1}_{mj} + B_{jk}\tau_{lm}B^{-1}_{mi} \\ &+ B_{il}\tau_{km}B^{-1}_{mj} + B_{jl}\tau_{km}B^{-1}_{mi}\big)\,, \end{aligned} \tag{6.5.11}$$

and

$$\begin{aligned} B_{(im}\bar{\Lambda}^{(1)}_{mjkn}B_{nl)} = \frac{1}{4}\Big(&B_{im}\bar{\Lambda}^{(1)}_{mjkn}B_{nl} + B_{jm}\bar{\Lambda}^{(1)}_{mikn}B_{nl} \\ &+ B_{im}\bar{\Lambda}^{(1)}_{mjln}B_{nk} + B_{jm}\bar{\Lambda}^{(1)}_{miln}B_{nk}\Big)\,. \end{aligned} \tag{6.5.12}$$

Equation (6.5.6) can be recast in terms of the convected derivatives of the Kirchhoff stress as

$$\begin{aligned} \overset{\triangle}{\boldsymbol{\tau}} &= \boldsymbol{\mathcal{L}}_{(0)} : \mathbf{D} - \mathbf{D}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot\mathbf{D} = \boldsymbol{\mathcal{L}}_{(1)} : \mathbf{D}, \\ \overset{\triangledown}{\boldsymbol{\tau}} &= \boldsymbol{\mathcal{L}}_{(0)} : \mathbf{D} + \mathbf{D}\cdot\boldsymbol{\tau} + \boldsymbol{\tau}\cdot\mathbf{D} = \boldsymbol{\mathcal{L}}_{(-1)} : \mathbf{D}. \end{aligned} \tag{6.5.13}$$

By using the instantaneous elastic moduli, these become

$$\begin{aligned} \underline{\overset{\triangle}{\boldsymbol{\tau}}} &= (\underline{\boldsymbol{\mathcal{L}}}_{(0)} - 2\underline{\boldsymbol{\mathcal{S}}}) : \mathbf{D} = \underline{\boldsymbol{\mathcal{L}}}_{(1)} : \mathbf{D}, \\ \underline{\overset{\triangledown}{\boldsymbol{\tau}}} &= (\underline{\boldsymbol{\mathcal{L}}}_{(0)} + 2\underline{\boldsymbol{\mathcal{S}}}) : \mathbf{D} = \underline{\boldsymbol{\mathcal{L}}}_{(-1)} : \mathbf{D}. \end{aligned} \tag{6.5.14}$$

The tensor $\underline{\boldsymbol{\mathcal{S}}}$ is defined by Eq. (6.3.11), and

$$\underline{\boldsymbol{\mathcal{L}}}_{(0)} = (\det\mathbf{F})^{-1}\,\boldsymbol{\mathcal{L}}_{(0)}, \quad \underline{\boldsymbol{\mathcal{L}}}_{(\pm 1)} = (\det\mathbf{F})^{-1}\,\boldsymbol{\mathcal{L}}_{(\pm 1)}. \tag{6.5.15}$$

To obtain the elastic pseudomoduli we can proceed from the general expressions given in Section 3.4, or alternatively use Eq. (3.8.12) to express the rate of nominal stress as

$$\dot{\mathbf{P}} = \overset{\triangle}{\mathbf{P}} + \mathbf{P}\cdot\mathbf{L}^T = \mathbf{F}^{-1}\cdot\overset{\triangle}{\boldsymbol{\tau}} + \mathbf{P}\cdot\mathbf{L}^T. \tag{6.5.16}$$

Since, from Eq. (6.5.13),

$$\overset{\triangle}{\boldsymbol{\tau}} = \boldsymbol{\mathcal{L}}_{(1)} : \mathbf{D} = \boldsymbol{\mathcal{L}}_{(1)} : \mathbf{L}, \tag{6.5.17}$$

by the reciprocal symmetry of $\boldsymbol{\mathcal{L}}_{(1)}$, the substitution into Eq. (6.5.16) gives

$$\dot{P}_{Ji} = \Lambda_{JiLk}\dot{F}_{kL}, \quad \Lambda_{JiLk} = F^{-1}_{Jm}\mathcal{L}^{(1)}_{mikn}F^{-T}_{nL} + P_{Jm}F^{-T}_{mL}\delta_{ik}. \tag{6.5.18}$$

The instantaneous elastic pseudomoduli $\underline{\Lambda}_{jilk}$ follow from Eq. (6.5.18) by setting $\mathbf{F} = \mathbf{I}$,

$$\underline{\Lambda}_{jilk} = \underline{\mathcal{L}}^{(1)}_{jilk} + \sigma_{jl}\delta_{ik}. \tag{6.5.19}$$

This is in agreement with Eq. (6.4.16), because $\underline{\boldsymbol{\mathcal{L}}}_{(1)} = \underline{\boldsymbol{\Lambda}}_{(1)}$.

6.5.1. Components of Elastic Moduli in Terms of C

When the Lagrangian strain and its conjugate Piola–Kirchhoff stress are used, the rate-type constitutive structure of isotropic elasticity is

$$\begin{aligned}\mathbf{T}_{(1)} = \frac{\partial \Psi}{\partial \mathbf{E}_{(1)}} = 2\,\frac{\partial \Psi}{\partial \mathbf{C}} = 2\left[\left(\frac{\partial \Psi}{\partial I_C} - I_C\,\frac{\partial \Psi}{\partial II_C}\right)\mathbf{I}^0 + \left(\frac{\partial \Psi}{\partial II_C}\right)\mathbf{C}\right.\\ \left. + \left(III_C\,\frac{\partial \Psi}{\partial III_C}\right)\mathbf{C}^{-1}\right].\end{aligned} \tag{6.5.20}$$

The strain energy function $\Psi = \Psi\,(I_C, II_C, III_C)$ is here expressed in terms of the principal invariants of the right Cauchy–Green deformation tensor $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = \mathbf{I}^0 + 2\mathbf{E}_{(1)}$. The corresponding elastic moduli tensor is

$$\boldsymbol{\Lambda}_{(1)} = \frac{\partial \mathbf{T}_{(1)}}{\partial \mathbf{E}_{(1)}} = \frac{\partial^2 \Psi}{\partial \mathbf{E}_{(1)} \otimes \partial \mathbf{E}_{(1)}} = 4\,\frac{\partial^2 \Psi}{\partial \mathbf{C} \otimes \partial \mathbf{C}}\,, \tag{6.5.21}$$

which is thus defined by the fully symmetric tensor $\partial^2\Psi/(\partial\mathbf{C} \otimes \partial\mathbf{C})$. Since

$$\begin{aligned}&\frac{\partial I_C}{\partial \mathbf{C}} = \mathbf{I}^0, \quad \frac{\partial II_C}{\partial \mathbf{C}} = \mathbf{C} - I_C\mathbf{I}^0,\\ &\frac{\partial III_C}{\partial \mathbf{C}} = \mathbf{C}^2 - I_C\mathbf{C} - II_C\mathbf{I}^0 = III_C\mathbf{C}^{-1},\end{aligned} \tag{6.5.22}$$

and in view of the symmetry $C_{ij} = C_{ji}$, we obtain

$$\begin{aligned}\frac{\partial^2 \Psi}{\partial C_{ij}\partial C_{kl}} &= c_1\delta_{ij}\delta_{kl} + c_2\left(\delta_{ij}C_{kl} + C_{ij}\delta_{kl}\right) + c_3C_{ij}C_{kl}\\ &+ c_4\left(\delta_{ij}C^{-1}_{kl} + C^{-1}_{ij}\delta_{kl}\right) + c_5\left(C_{ij}C^{-1}_{kl} + C^{-1}_{ij}C_{kl}\right)\\ &+ c_6C^{-1}_{ij}C^{-1}_{kl} + c_7\left(C^{-1}_{ik}C^{-1}_{jl} + C^{-1}_{il}C^{-1}_{jk}\right)\\ &+ c_8\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right).\end{aligned} \tag{6.5.23}$$

The parameters c_i $(i = 1, 2, \ldots, 8)$ are (e.g., Lubarda and Lee, 1981)

$$c_1 = \frac{\partial^2 \Psi}{\partial I_C^2} - 2I_C\,\frac{\partial^2 \Psi}{\partial I_C \partial II_C} + I_C^2\,\frac{\partial^2 \Psi}{\partial II_C^2} - \frac{\partial \Psi}{\partial II_C}\,, \tag{6.5.24}$$

$$c_2 = \frac{\partial^2 \Psi}{\partial I_C \partial II_C} - I_C \, \frac{\partial^2 \Psi}{\partial II_C^2}\,, \tag{6.5.25}$$

$$c_3 = \frac{\partial^2 \Psi}{\partial II_C^2}\,, \quad c_5 = III_C \, \frac{\partial^2 \Psi}{\partial II_C \partial III_C}\,, \tag{6.5.26}$$

$$c_4 = III_C \, \frac{\partial^2 \Psi}{\partial III_C \partial I_C} - III_C I_C \, \frac{\partial^2 \Psi}{\partial II_C \partial III_C}\,, \tag{6.5.27}$$

$$c_6 = III_C^2 \, \frac{\partial^2 \Psi}{\partial III_C^2} + III_C \, \frac{\partial \Psi}{\partial III_C}\,, \tag{6.5.28}$$

$$c_7 = -\frac{1}{2}\, III_C \, \frac{\partial \Psi}{\partial III_C}\,, \quad c_8 = \frac{1}{2}\, \frac{\partial \Psi}{\partial II_C}\,. \tag{6.5.29}$$

6.5.2. Elastic Moduli in Terms of Principal Stretches

For isotropic elastic material the principal directions $\mathbf{N}_i$ of the right Cauchy–Green deformation tensor

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \, \mathbf{N}_i \otimes \mathbf{N}_i, \quad C_i = \lambda_i^2, \tag{6.5.30}$$

where λ_i are the principal stretches, are parallel to those of the symmetric Piola–Kirchhoff stress $\mathbf{T}_{(1)}$. Thus, the spectral representation of $\mathbf{T}_{(1)}$ is

$$\mathbf{T}_{(1)} = \sum_{i=1}^{3} T_i^{(1)} \, \mathbf{N}_i \otimes \mathbf{N}_i. \tag{6.5.31}$$

From the analysis presented in Section 2.8 it readily follows that

$$\dot{\mathbf{C}} = \sum_{i=1}^{3} 2\lambda_i \dot{\lambda}_i \, \mathbf{N}_i \otimes \mathbf{N}_i + \sum_{i \neq j} \Omega_{ij}^0 \left(\lambda_j^2 - \lambda_i^2\right) \mathbf{N}_i \otimes \mathbf{N}_j, \tag{6.5.32}$$

and

$$\dot{\mathbf{T}}_{(1)} = \sum_{i=1}^{3} \dot{T}_i^{(1)} \, \mathbf{N}_i \otimes \mathbf{N}_i + \sum_{i \neq j} \Omega_{ij}^0 \left(T_j^{(1)} - T_i^{(1)}\right) \mathbf{N}_i \otimes \mathbf{N}_j. \tag{6.5.33}$$

The components of the spin tensor $\boldsymbol{\Omega}_0 = \dot{\boldsymbol{\mathcal{R}}}_0 \cdot \boldsymbol{\mathcal{R}}_0^{-1}$ on the axes $\mathbf{N}_i$ are denoted by Ω_{ij}^0. The rotation tensor $\boldsymbol{\mathcal{R}}_0$ maps the reference triad of unit vectors $\mathbf{e}_i$ into the Lagrangian triad $\mathbf{N}_i = \boldsymbol{\mathcal{R}}_0 \cdot \mathbf{e}_i^0$. For elastically isotropic material the strain energy can be expressed as a function of the principal stretches, $\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3)$, so that

$$T_i^{(1)} = \frac{\partial \Psi}{\partial E_i^{(1)}} = \frac{1}{\lambda_i} \frac{\partial \Psi}{\partial \lambda_i}\,. \tag{6.5.34}$$

$$\dot{T}_i^{(1)} = \sum_{j=1}^{3} \frac{\partial T_i^{(1)}}{\partial \lambda_j}\, \dot{\lambda}_j, \quad \frac{\partial T_i^{(1)}}{\partial \lambda_j} = -\delta_{ij}\, \frac{1}{\lambda_i^2} \frac{\partial \Psi}{\partial \lambda_i} + \frac{1}{\lambda_i} \frac{\partial^2 \Psi}{\partial \lambda_i \partial \lambda_j}\,. \tag{6.5.35}$$

Thus, Eq. (6.5.33) can be rewritten as

$$\dot{\mathbf{T}}_{(1)} = \sum_{i,j=1}^{3} \frac{\partial T_i^{(1)}}{\partial \lambda_j}\, \dot{\lambda}_j\, \mathbf{N}_i \otimes \mathbf{N}_i + \sum_{i \neq j} \Omega_{ij}^0 \left(\lambda_j^2 - \lambda_i^2\right) \frac{T_j^{(1)} - T_i^{(1)}}{\lambda_j^2 - \lambda_i^2}\, \mathbf{N}_i \otimes \mathbf{N}_j. \tag{6.5.36}$$

Since

$$\dot{\mathbf{T}}_{(1)} = \mathbf{\Lambda}_{(1)} : \dot{\mathbf{E}}_{(1)} = \frac{1}{2}\, \mathbf{\Lambda}_{(1)} : \dot{\mathbf{C}}, \tag{6.5.37}$$

we recognize from Eqs. (6.5.32) and (6.5.36) by inspection (Chadwick and Ogden, 1971; Ogden, 1984) that

$$\begin{aligned} \mathbf{\Lambda}_{(1)} &= \sum_{i,j=1}^{3} \frac{1}{\lambda_j}\, \frac{\partial T_i^{(1)}}{\partial \lambda_j}\, \mathbf{N}_i \otimes \mathbf{N}_i \otimes \mathbf{N}_j \otimes \mathbf{N}_j \\ &+ \sum_{i \neq j} \frac{T_j^{(1)} - T_i^{(1)}}{\lambda_j^2 - \lambda_i^2}\, \mathbf{N}_i \otimes \mathbf{N}_j \otimes (\mathbf{N}_i \otimes \mathbf{N}_j + \mathbf{N}_j \otimes \mathbf{N}_i)\,. \end{aligned} \tag{6.5.38}$$

Note also

$$\frac{\partial T_i^{(1)}}{\partial E_j^{(1)}} = \frac{1}{\lambda_j}\, \frac{\partial T_i^{(1)}}{\partial \lambda_j}\,, \quad \frac{T_j^{(1)} - T_i^{(1)}}{E_j^{(1)} - E_i^{(1)}} = 2\, \frac{T_j^{(1)} - T_i^{(1)}}{\lambda_j^2 - \lambda_i^2}\,. \tag{6.5.39}$$

If $\lambda_j \to \lambda_i$, i.e., $E_j^{(1)} \to E_i^{(1)}$, then by the l'Hopital rule

$$\lim_{E_j \to E_i} \frac{T_j^{(1)} - T_i^{(1)}}{E_j^{(1)} - E_i^{(1)}} = \frac{\partial (T_j^{(1)} - T_i^{(1)})}{\partial E_j^{(1)}}\,, \tag{6.5.40}$$

so that the representation of the elastic moduli tensor in Eq. (6.5.38) holds regardless of the relative magnitude of the principal stretches.

6.6. Hypoelasticity

The material is hypoelastic if its rate-type constitutive equation can be expressed in the form (Truesdell, 1955; Truesdell and Noll, 1965)

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{f}(\boldsymbol{\sigma}, \mathbf{D}). \tag{6.6.1}$$

Under rigid-body rotation $\mathbf{Q}$ of the deformed configuration, Eq. (6.6.1) transforms according to

$$\mathbf{Q} \cdot \overset{\circ}{\boldsymbol{\sigma}} \cdot \mathbf{Q}^T = \mathbf{f}\left(\mathbf{Q} \cdot \boldsymbol{\sigma} \cdot \mathbf{Q}^T, \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^T\right), \tag{6.6.2}$$

which requires the second-order tensor function $\mathbf{f}$ to be an isotropic function of both of its arguments. Such a function can be expressed by Eq. (1.11.10) as

$$\begin{aligned} \overset{\circ}{\boldsymbol{\sigma}} &= a_1\mathbf{I} + a_2\boldsymbol{\sigma} + a_3\boldsymbol{\sigma}^2 + a_4\mathbf{D} + a_5\mathbf{D}^2 \\ &+ a_6(\boldsymbol{\sigma}\cdot\mathbf{D} + \mathbf{D}\cdot\boldsymbol{\sigma}) + a_7\left(\boldsymbol{\sigma}^2\cdot\mathbf{D} + \mathbf{D}\cdot\boldsymbol{\sigma}^2\right) \\ &+ a_8\left(\boldsymbol{\sigma}\cdot\mathbf{D}^2 + \mathbf{D}^2\cdot\boldsymbol{\sigma}\right) + a_9\left(\boldsymbol{\sigma}^2\cdot\mathbf{D}^2 + \mathbf{D}^2\cdot\boldsymbol{\sigma}^2\right). \end{aligned} \tag{6.6.3}$$

The coefficients a_i are the scalar functions of ten individual and joint invariants of $\boldsymbol{\sigma}$ and $\mathbf{D}$. These are

$$\begin{gathered} \operatorname{tr}(\boldsymbol{\sigma}), \quad \operatorname{tr}\left(\boldsymbol{\sigma}^2\right), \quad \operatorname{tr}\left(\boldsymbol{\sigma}^3\right), \quad \operatorname{tr}(\mathbf{D}), \quad \operatorname{tr}\left(\mathbf{D}^2\right), \quad \operatorname{tr}\left(\mathbf{D}^3\right), \\ \operatorname{tr}(\boldsymbol{\sigma}\cdot\mathbf{D}), \quad \operatorname{tr}\left(\boldsymbol{\sigma}\cdot\mathbf{D}^2\right), \quad \operatorname{tr}\left(\boldsymbol{\sigma}^2\cdot\mathbf{D}\right), \quad \operatorname{tr}\left(\boldsymbol{\sigma}^2\cdot\mathbf{D}^2\right). \end{gathered} \tag{6.6.4}$$

Suppose that the material behavior is time independent, in the sense that any monotonically increasing parameter can serve as a time scale (materials without a natural time; Hill, 1959). The function $\mathbf{f}$ is then a homogeneous function of degree one in the rate of deformation tensor $\mathbf{D}$. Indeed, if two different time scales are used (t and $t' = kt$, $k =$ const.), we have

$$\overset{\circ}{\boldsymbol{\sigma}}_t = k\overset{\circ}{\boldsymbol{\sigma}}_{t'}, \quad \mathbf{D}_t = k\mathbf{D}_{t'}\,, \tag{6.6.5}$$

and

$$\mathbf{f}\left(\boldsymbol{\sigma}, k\mathbf{D}_{t'}\right) = k\mathbf{f}\left(\boldsymbol{\sigma}, \mathbf{D}_{t'}\right). \tag{6.6.6}$$

Consequently, in this case, the constitutive structure of Eq. (6.6.3) does not contain quadratic and higher order terms in $\mathbf{D}$, so that

$$\overset{\circ}{\boldsymbol{\sigma}} = a_1\mathbf{I} + a_2\boldsymbol{\sigma} + a_3\boldsymbol{\sigma}^2 + a_4\mathbf{D} + a_6(\boldsymbol{\sigma}\cdot\mathbf{D} + \mathbf{D}\cdot\boldsymbol{\sigma}) + a_7\left(\boldsymbol{\sigma}^2\cdot\mathbf{D} + \mathbf{D}\cdot\boldsymbol{\sigma}^2\right), \tag{6.6.7}$$

where

$$\begin{aligned} a_1 &= c_1\operatorname{tr}(\mathbf{D}) + c_2\operatorname{tr}(\boldsymbol{\sigma}\cdot\mathbf{D}) + c_3\operatorname{tr}\left(\boldsymbol{\sigma}^2\cdot\mathbf{D}\right), \\ a_2 &= c_4\operatorname{tr}(\mathbf{D}) + c_5\operatorname{tr}(\boldsymbol{\sigma}\cdot\mathbf{D}) + c_6\operatorname{tr}\left(\boldsymbol{\sigma}^2\cdot\mathbf{D}\right), \\ a_3 &= c_7\operatorname{tr}(\mathbf{D}) + c_8\operatorname{tr}(\boldsymbol{\sigma}\cdot\mathbf{D}) + c_9\operatorname{tr}\left(\boldsymbol{\sigma}^2\cdot\mathbf{D}\right), \end{aligned} \tag{6.6.8}$$

and

$$a_4 = c_{10}, \quad a_6 = c_{11}, \quad a_7 = c_{12}. \tag{6.6.9}$$

The coefficients c_i ($i = 1, 2, \ldots, 12$) are the scalar functions of the invariants of $\boldsymbol{\sigma}$ (e.g., $I_\sigma, II_\sigma, III_\sigma$). The structure of the expressions for a_i in Eq. (6.6.8) ensures that $\overset{\circ}{\boldsymbol{\sigma}}$ in Eq. (6.6.7) is linearly dependent on $\mathbf{D}$, i.e.,

$$\overset{\circ}{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{L}} : \mathbf{D}. \tag{6.6.10}$$

The fourth-order tensor $\boldsymbol{\mathcal{L}}$ has the Cartesian components

$$\begin{aligned}\mathcal{L}_{ijkl} = {} & c_1\delta_{ij}\delta_{kl} + c_2\delta_{ij}\sigma_{kl} + c_3\delta_{ij}\sigma^2_{kl} + c_4\sigma_{ij}\delta_{kl} \\ & + c_5\sigma_{ij}\sigma_{kl} + c_6\sigma_{ij}\sigma^2_{kl} + c_7\sigma^2_{ij}\delta_{kl} + c_8\sigma^2_{ij}\sigma_{kl} \\ & + c_9\sigma^2_{ij}\sigma^2_{kl} + c_{10}\delta_{(ik}\delta_{jl)} + c_{11}\sigma_{(ik}\delta_{jl)} + c_{12}\sigma^2_{(ik}\delta_{jl)}.\end{aligned} \tag{6.6.11}$$

If $c_2 = c_4$, $c_3 = c_7$ and $c_6 = c_8$, the tensor $\boldsymbol{\mathcal{L}}$ obeys the reciprocal symmetry $\mathcal{L}_{ijkl} = \mathcal{L}_{klij}$.

A hypoelastic material is of degree N if $\mathbf{f}$ is a polynomial of degree N in the components of $\boldsymbol{\sigma}$. For example, for hypoelastic material of degree one,

$$\begin{aligned} & c_1 = \alpha_1 + \alpha_2 \operatorname{tr}(\boldsymbol{\sigma}), \quad c_{10} = \alpha_3 + \alpha_4 \operatorname{tr}(\boldsymbol{\sigma}), \\ & c_2 = \alpha_5, \quad c_4 = \alpha_6, \quad c_{11} = \alpha_7, \\ & c_3 = c_5 = c_6 = c_7 = c_8 = c_9 = c_{12} = 0, \end{aligned} \tag{6.6.12}$$

where $\alpha_i \, (i = 1, 2, \ldots, 7)$ are seven constants available as material parameters.

In general, elasticity and hypoelasticity are different concepts, although under infinitesimal deformation from an arbitrary stressed configuration, Eq. (6.6.10), with anisotropic tensor $\boldsymbol{\mathcal{L}}$ given by Eq. (6.6.11), corresponds to some type of anisotropic elastic response. However, a hypoelastic constitutive equation cannot describe an anisotropic elastic material in infinitesimal deformation from the unstressed configuration, because the tensor $\boldsymbol{\mathcal{L}}$ becomes an isotropic fourth-order tensor in the unstressed state ($\boldsymbol{\sigma} = \mathbf{0}$).

Furthermore, a general rate-type constitutive equation of anisotropic elasticity, e.g., Eq. (6.2.14), is not of the hypoelastic type, because the anisotropic elastic moduli $\boldsymbol{\mathcal{L}}_{(0)}$ depend on the nine components of the deformation gradient $\mathbf{F}$, which cannot be expressed in terms of the six components of the stress tensor $\boldsymbol{\sigma}$, as required by the hypoelastic constitutive structure. However, a rate-type constitutive equation of finite strain isotropic elasticity (with invertible stress-strain relation) is of hypoelastic type. This follows because $\boldsymbol{\mathcal{L}}_{(0)}$ in Eq. (6.5.9) depends on $\mathbf{V}$, and for isotropic elasticity the six components of $\mathbf{V}$ can be expressed in terms of the six components of $\boldsymbol{\sigma}$, from an invertible type of Eq. (5.5.1). For additional discussion and comparison between elasticity and hypoelasticity, the papers by Pinsky, Ortiz, and Pister (1983), Simo and Pister (1984), and Simo and Ortiz (1985) can be consulted. A majority of hypoelastic solids are inelastic, in the sense that the stress state is generally not recovered upon an arbitrary closed cycle of strain (Hill, 1959). Illustrative examples can be found in Kojić and Bathe (1987), Weber and Anand (1990), Christoffersen (1991), and Bruhns, Xiao, and Meyers (1999). For instance, there is no truly hyperelastic material corresponding to hypoelastic constitutive equation

$$\overset{\circ}{\boldsymbol{\sigma}} = (\lambda \mathbf{I} \otimes \mathbf{I} + 2\mu \boldsymbol{I}) : \mathbf{D}, \tag{6.6.13}$$

where λ and μ are the Lamé type elasticity constants. Integration of Eq. (6.6.13) over a closed cycle of strain gives rise to a small net work left upon a cycle and the hysteresis effects. This is a consequence of the fact that Eq. (6.6.13) is not exactly an integrable equation. As pointed out by Simo and Ortiz (1985), a hypoelastic response with constant components of the fourth-order tensor in Eq. (6.6.13) cannot integrate into a truly hyperelastic response. Further discussion of hypoelastic constitutive equations, particularly regarding the use of different objective stress rates, is given by Dienes (1979), Atluri (1984), Johnson and Bammann (1984), Sowerby and Chu (1984), Metzger and Dubey (1987), and Szabó and Balla (1989).

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