CHAPTER 3

KINETICS OF DEFORMATION

3.1. Cauchy Stress

Consider an internal surface S within a loaded deformable body. If the resultant force across an infinitesimal surface element $\mathrm{d}S$ with unit normal $\mathbf{n}$ is $\mathrm{d}\mathbf{f}_n$, the corresponding traction vector is (Fig. 3.1)

$$\mathbf{t}_n = \frac{\mathrm{d}\mathbf{f}_n}{\mathrm{d}S}\,. \tag{3.1.1}$$

The Cauchy or true stress is the second-order tensor $\boldsymbol{\sigma}$ related to the traction vector $\mathbf{t}_n$ by

$$\mathbf{t}_n = \mathbf{n}\cdot\boldsymbol{\sigma}. \tag{3.1.2}$$

When $\boldsymbol{\sigma}$ is decomposed on an orthonormal basis in the deformed configuration as

$$\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i\otimes\mathbf{e}_j, \tag{3.1.3}$$

the traction vector over the area with the normal in the coordinate direction $\mathbf{e}_i$ can be written as

$$\mathbf{t}_i = \mathbf{e}_i\cdot\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_j. \tag{3.1.4}$$

From Eqs. (3.1.2) and (3.1.4) we conclude that the traction vector over the surface element with unit normal $\mathbf{n} = n_i\mathbf{e}_i$ can be expressed in terms of the traction vectors $\mathbf{t}_i$ as

$$\mathbf{t}_n = n_i\mathbf{t}_i. \tag{3.1.5}$$

Equation (3.1.5), known as the Cauchy relation, can also be derived directly by applying the balance law of linear momentum to an infinitesimal tetrahedron around a point of the stressed body (e.g., Prager, 1961; Fung, 1965). In Section 3.3 it will be shown that the Cauchy stress $\boldsymbol{\sigma}$ is a symmetric tensor, provided that there are no distributed surface or body couples acting within the body.

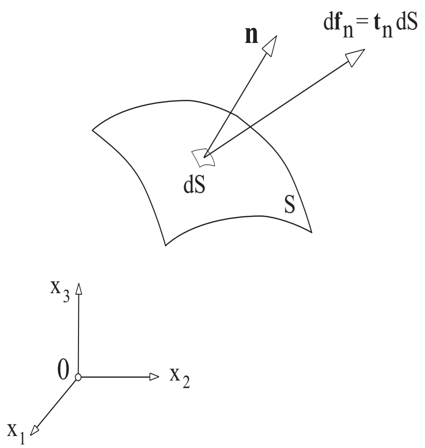

FIGURE 3.1. The traction vector $\mathbf{t}_n$ over the surface element with outward normal $\mathbf{n}$. The total force over $\mathrm{d}S$ is $\mathrm{d}\mathbf{f}_n = \mathbf{t}_n \mathrm{d}S$.

A spherical part of the Cauchy stress is equal to $(\mathrm{tr}\,\boldsymbol{\sigma})\mathbf{I}/3$. The remainder is the deviatoric part,

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \frac{1}{3}(\mathrm{tr}\,\boldsymbol{\sigma})\mathbf{I}. \tag{3.1.6}$$

Since $\boldsymbol{\sigma}'$ is a traceless tensor ($\mathrm{tr}\,\boldsymbol{\sigma}' = 0$), there are in general only two nonvanishing invariants of $\boldsymbol{\sigma}'$. These are, from Eqs. (1.3.4) and (1.3.5),

$$J_2 = \frac{1}{2}\,\mathrm{tr}\,(\boldsymbol{\sigma}'^{\,2}), \quad J_3 = \frac{1}{3}\,\mathrm{tr}\,(\boldsymbol{\sigma}'^{\,3}). \tag{3.1.7}$$

If I_1, I_2 and I_3 are the invariants of $\boldsymbol{\sigma}$, we have the relationships

$$J_2 = I_2 + \frac{1}{3}\,I_1^2, \quad J_3 = I_3 + \frac{1}{3}\,I_1 I_2 + \frac{2}{27}\,I_1^3. \tag{3.1.8}$$

Physically, J_2 can be related to shear stress on the octahedral plane ($n_i = \pm 1/\sqrt{3}$ with respect to principal stress directions), since $J_2 = (3/2)\tau_{\mathrm{oct}}^2$. The octahedral planes are shown in Fig. 3.2. The normal stress on the octahedral plane is $\sigma_{\mathrm{oct}} = I_1/3$. In two-dimensional plane stress problems, the third invariant of the stress tensor $I_3 = 0$, so that in three-dimensional problems I_3 can be viewed as a measure of the stress state triaxiality. For later use, it is also noted that

$$\frac{\partial J_2}{\partial \boldsymbol{\sigma}} = \boldsymbol{\sigma}', \quad \frac{\partial J_3}{\partial \boldsymbol{\sigma}} = \boldsymbol{\sigma}'^{\,2} - \frac{2}{3}\,J_2\,\mathbf{I}, \quad \frac{\partial \boldsymbol{\sigma}'}{\partial \boldsymbol{\sigma}} = \boldsymbol{I} - \frac{1}{3}\,\mathbf{I} \otimes \mathbf{I}, \tag{3.1.9}$$

where $\mathbf{I}$ is the second-order, and $\boldsymbol{I}$ is the fourth-order unit tensor.

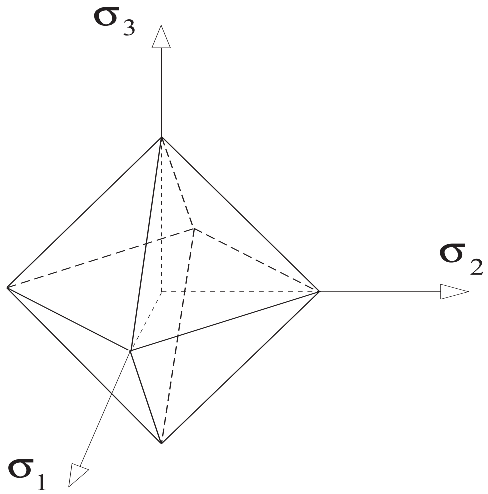

FIGURE 3.2. Octahedral planes in the coordinate system of principal stresses.

3.2. Continuity Equation

If $\rho = \rho(\mathbf{x}, t)$ is a continuous mass density function, the conservation of mass requires that $\mathrm{d}m = \rho\, \mathrm{d}V$ is constant during the deformation process. Since $\mathrm{d}V = (\det \mathbf{F})\, \mathrm{d}V^0$, this implies that

$$\rho\,(\det \mathbf{F}) = \text{const.} \tag{3.2.1}$$

By differentiating we obtain the continuity equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\,(\boldsymbol{\nabla} \cdot \mathbf{v}) = 0, \tag{3.2.2}$$

where $\mathbf{v}$ is the velocity of the particle in the position $\mathbf{x}$ at time t. Recall from Eq. (2.4.12) that the time rate

$$(\det \mathbf{F})^{\cdot} = (\det \mathbf{F})(\boldsymbol{\nabla} \cdot \mathbf{v}). \tag{3.2.3}$$

In view of Eq. (2.1.4) for the total time rate of a spatial field, Eq. (3.2.2) can be rewritten as

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \mathbf{v}) = 0. \tag{3.2.4}$$

If the deformation process is volume preserving (isochoric), so that $\det \mathbf{F} = 1$ and $\rho = \text{const.}$, the continuity equation reduces to

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{3.2.5}$$

i.e., the velocity field is a divergence free vector field.

The Reynolds transport theorem states that for any continuously differentiable tensor field $\mathbf{T} = \mathbf{T}(\mathbf{x}, t)$ within the volume V bounded by surface S,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathbf{T}\,\mathrm{d}V = \int_V \frac{\partial}{\partial t}(\rho\,\mathbf{T})\,\mathrm{d}V + \int_S \rho\,\mathbf{T}(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}S, \tag{3.2.6}$$

where ρ is the mass density, and $\mathbf{n}$ is the unit normal to S (e.g., Malvern, 1969; Gurtin, 1981). By applying the Gauss theorem, Eq. (1.13.16), to convert the surface integral in Eq. (3.2.6) to volume integral, and having in mind the continuity equation (3.2.4), there follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathbf{T}\,\mathrm{d}V = \int_V \rho\,\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\,\mathrm{d}V. \tag{3.2.7}$$

This important formula of continuum mechanics will be frequently utilized in subsequent derivations. For example, by taking $\mathbf{T}$ to be ρ^{-1}, and by using (3.2.2), Eq. (3.2.7) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}V = \int_V (\boldsymbol{\nabla}\cdot\mathbf{v})\,\mathrm{d}V. \tag{3.2.8}$$

3.3. Equations of Motion

Consider an arbitrary portion of a continuous body in the deformed configuration. Denote its volume by V and its bounding surface by S (Fig. 3.3). The rate of change of the linear mass momentum within V is equal to the sum of all surface forces acting on S and all body forces acting in V (first Euler's law of motion), i.e.,

$$\int_S \mathbf{t}_n\,\mathrm{d}S + \int_V \rho\,\mathbf{b}\,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathbf{v}\,\mathrm{d}V. \tag{3.3.1}$$

The body force per unit mass is

$$\mathbf{b} = \frac{\mathrm{d}\mathbf{f}_b}{\mathrm{d}m}, \tag{3.3.2}$$

and $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is the velocity field. Applying the Gauss theorem to convert the surface into volume integral, and incorporating Eq. (3.2.7) in the right-hand side of Eq. (3.3.1), we obtain

$$\int_V \left(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \rho\,\mathbf{b} - \rho\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)\mathrm{d}V = 0. \tag{3.3.3}$$

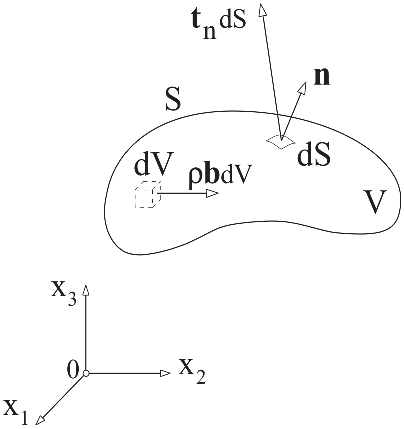

FIGURE 3.3. The volume V of the body bounded by closed surface S. The body force per unit mass is $\mathbf{b}$ and the surface traction over S is $\mathbf{t}_n$.

Since this holds for an arbitrary volume V, the integrand must vanish at each point of the deforming body,

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho\, \mathbf{b} = \rho\, \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,. \tag{3.3.4}$$

These are the Cauchy equations of motion for continuous media that apply at any point $\mathbf{x}$ in the deformed configuration. Equilibrium equations are obtained by setting the acceleration $\mathrm{d}\mathbf{v}/\mathrm{d}t$ equal to zero.

The transition to corresponding equations at points $\mathbf{X}$ in the undeformed configuration is straightforward. We only need to multiply Eq. (3.3.4) with $(\det \mathbf{F})$. Since

$$\rho(\det \mathbf{F}) = \rho^0 \tag{3.3.5}$$

is the density in the undeformed configuration, $\rho^0 = \rho^0(\mathbf{X})$, and since

$$(\det \mathbf{F})\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \boldsymbol{\nabla}^0 \cdot \left(\mathbf{F}^{-1} \cdot \boldsymbol{\tau}\right), \tag{3.3.6}$$

where

$$\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma} \tag{3.3.7}$$

is the Kirchhoff stress, Eq. (3.3.4) becomes

$$\boldsymbol{\nabla}^0 \cdot \left(\mathbf{F}^{-1} \cdot \boldsymbol{\tau}\right) + \rho^0\, \mathbf{b} = \rho^0\, \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\,. \tag{3.3.8}$$

The stress tensor

$$\mathbf{P} = \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \tag{3.3.9}$$

is a nonsymmetric nominal stress, and will be considered in more detail later in Section 3.7.

It is left to prove the identity in Eq. (3.3.6). First, by Eq. (1.13.14),

$$\boldsymbol{\nabla}^0 \cdot \left[(\det \mathbf{F})\mathbf{F}^{-1} \cdot \boldsymbol{\sigma}\right] = \left\{\boldsymbol{\nabla}^0 \cdot \left[(\det \mathbf{F})\mathbf{F}^{-1}\right]\right\} \cdot \boldsymbol{\sigma} + \left[(\det \mathbf{F})\mathbf{F}^{-T} \cdot \boldsymbol{\nabla}^0\right] \cdot \boldsymbol{\sigma}. \tag{3.3.10}$$

The first term on the right-hand side is equal to zero, in view of Eq. (2.2.21). Equation (3.3.8) follows because $\mathbf{F}^{-T} \cdot \boldsymbol{\nabla}^0 = \boldsymbol{\nabla}$, by Eq. (2.4.5).

3.4. Symmetry of Cauchy Stress

The balance law of angular momentum requires that the Cauchy stress is symmetric, if there are no distributed surface or body couples acting on the body (nonpolar case). This is now proven. The rate of change of angular momentum of the mass within V is equal to the sum of the moments of all forces acting on V and S (second Euler's law of motion), i.e.,

$$\int_S (\mathbf{x} \times \mathbf{t}_n)\, \mathrm{d}S + \int_V (\mathbf{x} \times \rho\, \mathbf{b})\, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \int_V (\mathbf{x} \times \rho\, \mathbf{v})\, \mathrm{d}V. \tag{3.4.1}$$

Applying the Gauss theorem to convert the surface integral into the volume integral, we obtain

$$\int_S (\mathbf{x} \times \mathbf{t}_n)\, \mathrm{d}S = -\int_V \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \times \mathbf{x})\, \mathrm{d}V. \tag{3.4.2}$$

The integrand on the right-hand side can be expanded as

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \times \mathbf{x}) = (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \times \mathbf{x} - \boldsymbol{\epsilon} : \boldsymbol{\sigma}, \tag{3.4.3}$$

where $\boldsymbol{\epsilon}$ is the permutation tensor, and : designates the trace product; see Eq. (1.13.15). Thus, Eq. (3.4.1) becomes

$$\int_V \mathbf{x} \times \left(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho\, \mathbf{b} - \rho\, \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) \mathrm{d}V + \int_V (\boldsymbol{\epsilon} : \boldsymbol{\sigma})\, \mathrm{d}V = 0. \tag{3.4.4}$$

The integrand of the first integral in Eq. (3.4.4) vanishes by equations of motion (3.3.4). The second integral has to vanish for all choices of V (the whole body or any part of it), hence

$$\boldsymbol{\epsilon} : \boldsymbol{\sigma} = \mathbf{0} \tag{3.4.5}$$

at each point of the body. Since the permutation tensor $\boldsymbol{\epsilon}$ is antisymmetric with respect to its last two indices, Eq. (3.4.5) requires the Cauchy stress $\boldsymbol{\sigma}$ to be symmetric,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{3.4.6}$$

3.5. Stress Power

The rate at which external surface and body forces are doing work on the mass instantaneously occupying the volume V bounded by S is the power input

$$\mathcal{P} = \int_S \mathbf{t}_n \cdot \mathbf{v} \, \mathrm{d}S + \int_V \rho \, \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}V. \tag{3.5.1}$$

Converting the surface integral into the volume integral, this becomes

$$\mathcal{P} = \int_V \left[(\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \rho \, \mathbf{b}) \cdot \mathbf{v} + \boldsymbol{\sigma} : \mathbf{D} \right] \mathrm{d}V. \tag{3.5.2}$$

The formula (1.13.13) was used, giving

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \mathbf{v}) = (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma}) \cdot \mathbf{v} + \boldsymbol{\sigma} : \mathbf{L}^T. \tag{3.5.3}$$

The symmetry of the Cauchy stress makes

$$\boldsymbol{\sigma} : \mathbf{L}^T = \boldsymbol{\sigma} : \mathbf{D}. \tag{3.5.4}$$

The deformation gradient is

$$\mathbf{L} = \mathbf{v} \otimes \boldsymbol{\nabla}, \tag{3.5.5}$$

and its symmetric part $\mathbf{D}$ is the rate of deformation tensor. Using the Cauchy equations of motion (3.3.4) and Eq. (3.2.7), the rate at which external forces do work is, from Eq. (3.5.2),

$$\mathcal{P} = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{1}{2} \rho \, \mathbf{v} \cdot \mathbf{v} \, \mathrm{d}V + \int_V \boldsymbol{\sigma} : \mathbf{D} \, \mathrm{d}V. \tag{3.5.6}$$

The first term represents the rate of macroscopic kinetic energy of the total mass. The second term is the total stress power expended at the considered instant to deform the material. This contributes to internal energy of the material, and, depending on the nature of deformation, part of it may be dissipated in the form of heat. The scalar quantity $\boldsymbol{\sigma} : \mathbf{D}$ is called the stress power per unit current volume. If it is reckoned with respect to unit initial volume, it becomes $\boldsymbol{\tau} : \mathbf{D}$.

3.6. Conjugate Stress Tensors

3.6.1. Material Stress Tensors

A systematic construction of stress tensors as work conjugates to strain tensors was introduced by Hill (1968). For any material strain $\mathbf{E}_{(n)}$ of Eq. (2.3.1), its work conjugate stress $\mathbf{T}_{(n)}$ is defined such that the stress power per unit reference volume is

$$\mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} = \boldsymbol{\tau} : \mathbf{D}, \tag{3.6.1}$$

where $\boldsymbol{\tau} = (\det \mathbf{F})\boldsymbol{\sigma}$ is the Kirchhoff stress. For $n = 1$, Eq. (3.6.1) gives

$$\mathbf{T}_{(1)} = \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}^{-T} = \mathbf{U}^{-1} \cdot \hat{\boldsymbol{\tau}} \cdot \mathbf{U}^{-1} \quad \Leftrightarrow \quad \mathbf{E}_{(1)} = \frac{1}{2}\left(\mathbf{U}^2 - \mathbf{I}^0\right). \tag{3.6.2}$$

For $n = 1/2$ it follows that

$$\mathbf{T}_{(1/2)} = \frac{1}{2}\left(\mathbf{U}^{-1} \cdot \hat{\boldsymbol{\tau}} + \hat{\boldsymbol{\tau}} \cdot \mathbf{U}^{-1}\right) \quad \Leftrightarrow \quad \mathbf{E}_{(1/2)} = \mathbf{U} - \mathbf{I}^0. \tag{3.6.3}$$

The symbol $\Leftrightarrow$ stands for "conjugate to" and the stress

$$\hat{\boldsymbol{\tau}} = \mathbf{R}^T \cdot \boldsymbol{\tau} \cdot \mathbf{R} \tag{3.6.4}$$

is induced from $\boldsymbol{\tau}$ by the rotation $\mathbf{R}$. Similarly,

$$\mathbf{T}_{(-1)} = \mathbf{F}^T \cdot \boldsymbol{\tau} \cdot \mathbf{F} = \mathbf{U} \cdot \hat{\boldsymbol{\tau}} \cdot \mathbf{U} \quad \Leftrightarrow \quad \mathbf{E}_{(-1)} = \frac{1}{2}\left(\mathbf{I}^0 - \mathbf{U}^{-2}\right), \tag{3.6.5}$$

$$\mathbf{T}_{(-1/2)} = \frac{1}{2}\left(\mathbf{U} \cdot \hat{\boldsymbol{\tau}} + \hat{\boldsymbol{\tau}} \cdot \mathbf{U}\right) \quad \Leftrightarrow \quad \mathbf{E}_{(-1/2)} = \mathbf{I}^0 - \mathbf{U}^{-1}. \tag{3.6.6}$$

In view of Eq. (2.6.5), there is a general relationship

$$\mathbf{T}_{(-n)} = \mathbf{U}^{2n} \cdot \mathbf{T}_{(n)} \cdot \mathbf{U}^{2n}. \tag{3.6.7}$$

Furthermore, for positive n we have

$$\begin{aligned} \dot{\mathbf{E}}_{(n)} = \frac{1}{2n}\Big(&\dot{\mathbf{U}} \cdot \mathbf{U}^{2n-1} + \mathbf{U} \cdot \dot{\mathbf{U}} \cdot \mathbf{U}^{2n-2} + \cdots \\ &+ \mathbf{U}^{2n-2} \cdot \dot{\mathbf{U}} \cdot \mathbf{U} + \mathbf{U}^{2n-1} \cdot \dot{\mathbf{U}}\Big). \end{aligned} \tag{3.6.8}$$

Thus, since

$$\mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} = \mathbf{T}_{(1/2)} : \dot{\mathbf{E}}_{(1/2)}, \tag{3.6.9}$$

it follows that (Ogden, 1984)

$$\begin{aligned} \mathbf{T}_{(1/2)} = \frac{1}{2n}\Big(&\mathbf{U}^{2n-1} \cdot \mathbf{T}_{(n)} + \mathbf{U}^{2n-2} \cdot \mathbf{T}_{(n)} \cdot \mathbf{U} + \cdots \\ &+ \mathbf{U} \cdot \mathbf{T}_{(n)} \cdot \mathbf{U}^{2n-2} + \mathbf{T}_{(n)} \cdot \mathbf{U}^{2n-1}\Big), \quad n > 0. \end{aligned} \tag{3.6.10}$$

Similarly,

$$\mathbf{T}_{(-1/2)} = \frac{1}{2n}\left(\mathbf{U}^{1-2n}\cdot\mathbf{T}_{(-n)} + \mathbf{U}^{2-2n}\cdot\mathbf{T}_{(-n)}\cdot\mathbf{U}^{-1} + \cdots \right. \\ \left. + \mathbf{U}^{-1}\cdot\mathbf{T}_{(-n)}\cdot\mathbf{U}^{2-2n} + \mathbf{T}_{(-n)}\cdot\mathbf{U}^{1-2n}\right), \quad n>0. \tag{3.6.11}$$

If $\mathbf{T}_{(n)}$ and $\mathbf{U}$ are commutative,

$$\mathbf{T}_{(1/2)} = \mathbf{U}^{2n-1}\cdot\mathbf{T}_{(n)}, \quad \mathbf{T}_{(-1/2)} = \mathbf{U}^{1-2n}\cdot\mathbf{T}_{(-n)}. \tag{3.6.12}$$

A derivation of an explicit expression for the stress tensor conjugate to logarithmic strain $\mathbf{E}_{(0)}$ is more involved. The approximate expression can be obtained as follows. From Eq. (2.3.12), by differentiation,

$$\dot{\mathbf{E}}_{(n)} = \dot{\mathbf{E}}_{(0)} + 2n\left(\mathbf{E}_{(0)}\cdot\dot{\mathbf{E}}_{(0)} + \dot{\mathbf{E}}_{(0)}\cdot\mathbf{E}_{(0)}\right) \\ + \frac{2}{3}n^2\left(\mathbf{E}^2_{(0)}\cdot\dot{\mathbf{E}}_{(0)} + \dot{\mathbf{E}}_{(0)}\cdot\mathbf{E}^2_{(0)} + \mathbf{E}_{(0)}\cdot\dot{\mathbf{E}}_{(0)}\cdot\mathbf{E}_{(0)}\right) \\ + \mathcal{O}\left(\mathbf{E}^3_{(0)}\cdot\dot{\mathbf{E}}_{(0)}\right). \tag{3.6.13}$$

Substitution of this into

$$\mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} = \mathbf{T}_{(0)} : \dot{\mathbf{E}}_{(0)} \tag{3.6.14}$$

gives

$$\mathbf{T}_{(0)} = \mathbf{T}_{(n)} + n\left(\mathbf{E}_{(n)}\cdot\mathbf{T}_{(n)} + \mathbf{T}_{(n)}\cdot\mathbf{E}_{(n)}\right) \\ - \frac{1}{3}n^2\left(\mathbf{E}^2_{(n)}\cdot\mathbf{T}_{(n)} + \mathbf{T}_{(n)}\cdot\mathbf{E}^2_{(n)} - 2\mathbf{E}_{(n)}\cdot\mathbf{T}_{(n)}\cdot\mathbf{E}_{(n)}\right) \\ + \mathcal{O}\left(\mathbf{E}^3_{(n)}\cdot\mathbf{T}_{(n)}\right). \tag{3.6.15}$$

Furthermore, from any of Eqs. (3.6.2)–(3.6.6) for the stress $\mathbf{T}_{(n)}$, it can be shown that

$$\mathbf{T}_{(n)} = \hat{\boldsymbol{\tau}} - n\left(\mathbf{E}_{(n)}\cdot\hat{\boldsymbol{\tau}} + \hat{\boldsymbol{\tau}}\cdot\mathbf{E}_{(n)}\right) + \mathcal{O}\left(\mathbf{E}^2_{(n)}\cdot\hat{\boldsymbol{\tau}}\right). \tag{3.6.16}$$

The substitution into Eq. (3.6.15) then yields

$$\mathbf{T}_{(0)} = \hat{\boldsymbol{\tau}} + \mathcal{O}\left(\mathbf{E}^2_{(n)}\cdot\hat{\boldsymbol{\tau}}\right) \quad \Leftrightarrow \quad \mathbf{E}_{(0)} = \ln\mathbf{U}. \tag{3.6.17}$$

The approximation $\mathbf{T}_{(0)} \approx \hat{\boldsymbol{\tau}}$ may be acceptable at moderate strains (Hill, 1978). If deformation is such that the principal directions of $\mathbf{V}$ and $\boldsymbol{\tau}$ are parallel (as in the deformation of isotropic elastic materials), the matrices $\mathbf{E}_{(n)}$ and $\mathbf{T}_{(n)}$ commute, and the term proportional to n^2 in Eq. (3.6.15) vanishes, as well as all other higher-order terms. In that case, therefore,

$\mathbf{T}_{(0)} = \hat{\boldsymbol{\tau}}$ exactly. Also, if principal directions of $\mathbf{U}$ remain fixed during deformation,

$$\dot{\mathbf{E}}_{(0)} = \dot{\mathbf{U}} \cdot \mathbf{U}^{-1} = \hat{\mathbf{D}}, \quad \mathbf{T}_{(0)} = \hat{\boldsymbol{\tau}}. \tag{3.6.18}$$

Additional analysis can be found in the articles by Hoger (1987), Guo and Man (1992), Lehmann and Liang (1993), Heiduschke (1995), and Xiao (1995).

3.6.2. Spatial Stress Tensors

The spatial strain tensors $\boldsymbol{\mathcal{E}}_{(n)}$ in general do not have their conjugate stress tensors $\boldsymbol{\mathcal{T}}_{(n)}$ such that $\mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} = \boldsymbol{\mathcal{T}}_{(n)} : \dot{\boldsymbol{\mathcal{E}}}_{(n)}$. This is clear at the outset, because a spatial stress tensor should be objective ($\boldsymbol{\mathcal{T}}^*_{(n)} = \mathbf{Q} \cdot \boldsymbol{\mathcal{T}}_{(n)} \cdot \mathbf{Q}^T$). Since $\dot{\boldsymbol{\mathcal{E}}}_{(n)}$ is not objective, as seen from Eq. (2.9.13), their trace product cannot in general be equal to an invariant quantity $\mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)}$ (which is independent of the rotation $\mathbf{Q}$ superimposed to the deformed configuration). However, the spatial stress tensors conjugate to strain tensors $\boldsymbol{\mathcal{E}}_{(n)}$ can be introduced by requiring that

$$\mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)} = \boldsymbol{\mathcal{T}}_{(n)} : \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)}, \tag{3.6.19}$$

where the objective, corotational rate of strain $\overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)}$ is defined by Eq. (2.6.19), i.e.,

$$\overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} = \dot{\boldsymbol{\mathcal{E}}}_{(n)} - \boldsymbol{\omega} \cdot \boldsymbol{\mathcal{E}}_{(n)} + \boldsymbol{\mathcal{E}}_{(n)} \cdot \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1}. \tag{3.6.20}$$

In view of the relationship

$$\overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} = \mathbf{R} \cdot \dot{\mathbf{E}}_{(n)} \cdot \mathbf{R}^T, \tag{3.6.21}$$

it follows that

$$\boldsymbol{\mathcal{T}}_{(n)} = \mathbf{R} \cdot \mathbf{T}_{(n)} \cdot \mathbf{R}^T. \tag{3.6.22}$$

This is the conjugate stress to spatial strains $\boldsymbol{\mathcal{E}}_{(n)}$ according to Eq. (3.6.19). Therefore, in this sense we consider

$$\boldsymbol{\mathcal{T}}_{(1)} = \mathbf{F}^{-T} \cdot \hat{\boldsymbol{\tau}} \cdot \mathbf{F}^{-1} = \mathbf{V}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{V}^{-1} \quad \Leftrightarrow \quad \boldsymbol{\mathcal{E}}_{(1)} = \frac{1}{2}\left(\mathbf{V}^2 - \mathbf{I}\right), \tag{3.6.23}$$

$$\boldsymbol{\mathcal{T}}_{(-1)} = \mathbf{F} \cdot \hat{\boldsymbol{\tau}} \cdot \mathbf{F}^T = \mathbf{V} \cdot \boldsymbol{\tau} \cdot \mathbf{V} \quad \Leftrightarrow \quad \boldsymbol{\mathcal{E}}_{(-1)} = \frac{1}{2}\left(\mathbf{I} - \mathbf{V}^{-2}\right), \tag{3.6.24}$$

$$\boldsymbol{\mathcal{T}}_{(1/2)} = \frac{1}{2}\left(\mathbf{V}^{-1} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{V}^{-1}\right) \quad \Leftrightarrow \quad \boldsymbol{\mathcal{E}}_{(1/2)} = \mathbf{V} - \mathbf{I}, \tag{3.6.25}$$

$$\mathcal{T}_{(-1/2)} = \frac{1}{2}(\mathbf{V}\cdot\boldsymbol{\tau}+\boldsymbol{\tau}\cdot\mathbf{V}) \quad \Leftrightarrow \quad \mathcal{E}_{(-1/2)} = \mathbf{I}-\mathbf{V}^{-1}. \tag{3.6.26}$$

It is easy to derive equations dual to Eqs. (3.6.8)–(3.6.12). For example, if $\mathcal{T}$ and $\mathbf{V}$ are coaxial tensors,

$$\mathcal{T}_{(1/2)} = \mathbf{V}^{2n-1}\cdot\mathcal{T}_{(n)}, \quad \mathcal{T}_{(-1/2)} = \mathbf{V}^{1-2n}\cdot\mathcal{T}_{(-n)}. \tag{3.6.27}$$

If the principal directions of $\mathbf{T}_{(n)}$ and $\mathbf{E}_{(n)}$ are parallel (as in the deformation of elastically isotropic materials), so are the principal directions of $\mathcal{T}_{(n)}$ and $\mathcal{E}_{(n)}$. In this case

$$\mathcal{T}_{(n)} : \overset{\bullet}{\mathcal{E}}_{(n)} = \mathcal{T}_{(n)} : \dot{\mathcal{E}}_{(n)}, \tag{3.6.28}$$

because the tensor $\left(\boldsymbol{\Omega}\cdot\mathcal{E}_{(n)} - \mathcal{E}_{(n)}\cdot\boldsymbol{\Omega}\right)$ is orthogonal to $\mathcal{E}_{(n)}$ and thus to $\mathcal{T}_{(n)}$, so that

$$\mathcal{T}_{(n)} : \left(\boldsymbol{\omega}\cdot\mathcal{E}_{(n)} - \mathcal{E}_{(n)}\cdot\boldsymbol{\omega}\right) = 0. \tag{3.6.29}$$

Note that $\mathbf{R}\cdot\boldsymbol{\tau}\cdot\mathbf{R}^T$ is not the work conjugate to any strain measure, since the material stress tensor $\mathbf{T}_{(n)}$ in Eq. (3.6.22) cannot be equal to spatial stress tensor $\boldsymbol{\tau}$. Likewise, although $\hat{\boldsymbol{\tau}} : \hat{\mathbf{D}} = \boldsymbol{\tau} : \mathbf{D}$, the stress tensor $\hat{\boldsymbol{\tau}} = \mathbf{R}^T\cdot\boldsymbol{\tau}\cdot\mathbf{R}$ is not the work conjugate to any strain measure, because $\hat{\mathbf{D}} = \mathbf{R}^T\cdot\mathbf{D}\cdot\mathbf{R}$ is not the rate of any strain. Of course, $\boldsymbol{\tau}$ itself is not the work conjugate to any strain, because $\mathbf{D}$ is not the rate of any strain, either.

If deformation is uniform extension or compression ($\mathbf{F} = \lambda\mathbf{I}$), it can be shown that

$$\dot{\mathbf{E}}_{(n)} = \lambda^{2n}\mathbf{D}, \quad \dot{\mathbf{E}}_{(0)} = \mathbf{D} = \frac{\dot{\lambda}}{\lambda}\,\mathbf{I}, \tag{3.6.30}$$

and in this case

$$\mathbf{T}_{(n)} = \lambda^{-2n}\boldsymbol{\tau}, \quad \mathbf{T}_{(0)} = \boldsymbol{\tau}. \tag{3.6.31}$$

3.7. Nominal Stress

If the element of area $\mathrm{d}\mathbf{S} = \mathrm{d}S\mathbf{n}$ in the deformed configuration carries the force $d\mathbf{f}_n$, the corresponding traction vector is $\mathbf{t}_n = \mathrm{d}\mathbf{f}_n/\mathrm{d}S$. It is related to Cauchy stress by $\mathbf{t}_n = \mathbf{n}\cdot\boldsymbol{\sigma}$. Let $\mathrm{d}\mathbf{S}^0 = \mathrm{d}S^0\,\mathbf{n}^0$ be the element of area in the undeformed configuration, corresponding to $\mathrm{d}\mathbf{S}$ in the deformed configuration. The nominal traction vector is defined as the actual force in the

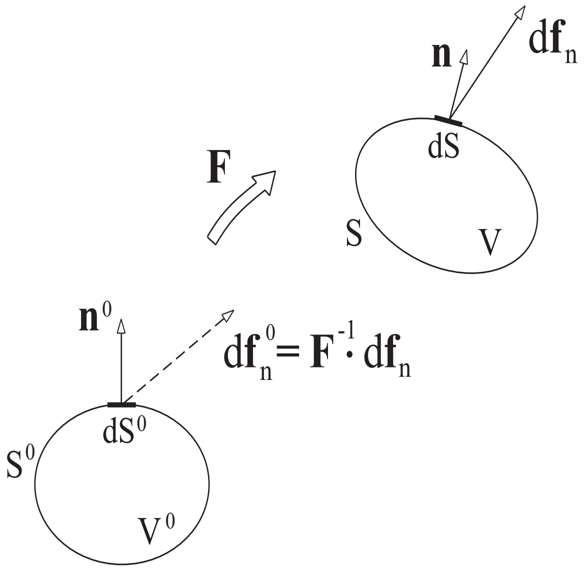

FIGURE 3.4. An infinitesimal surface element $\mathrm{d}S$ with unit normal $\mathbf{n}$ in deformed configuration carries the force $\mathrm{d}\mathbf{f}_n$. The nominal traction vector with respect to undeformed configuration is $\mathrm{d}\mathbf{f}_n/\mathrm{d}S^0$. A pseudo-force vector is $\mathrm{d}\mathbf{f}_n^0$.

deformed configuration divided by the area in the undeformed configuration, i.e.,

$$\mathbf{p}_n = \frac{\mathrm{d}\mathbf{f}_n}{\mathrm{d}S^0}, \tag{3.7.1}$$

so that (Fig. 3.4)

$$\mathbf{p}_n \,\mathrm{d}S^0 = \mathbf{t}_n \,\mathrm{d}S. \tag{3.7.2}$$

The nominal stress tensor $\mathbf{P}$ is introduced by

$$\mathbf{p}_n = \mathbf{n}^0 \cdot \mathbf{P}. \tag{3.7.3}$$

In view of Nanson's relation (2.2.17), it follows that

$$\mathbf{P} = \mathbf{F}^{-1} \cdot \boldsymbol{\tau}. \tag{3.7.4}$$

The nominal stress is a nonsymmetric two-point tensor. Its transpose

$$\mathbf{P}^T = \boldsymbol{\tau} \cdot \mathbf{F}^{-T} \tag{3.7.5}$$

is often referred to as the first or nonsymmetric Piola–Kirchhoff stress tensor, $\mathbf{n}^0 \cdot \mathbf{P} = \mathbf{P}^T \cdot \mathbf{n}^0$; Truesdell and Noll (1965).

Observe that the rate of work can be expressed in terms of the nominal stress as

$$\mathbf{P} \cdot\cdot\, \dot{\mathbf{F}} = \boldsymbol{\tau} : \mathbf{D}. \tag{3.7.6}$$

This, in turn, can serve as a starting point to define $\mathbf{P}$, since

$$\mathbf{P} \cdot\cdot\, \dot{\mathbf{F}} = (\mathbf{F} \cdot \mathbf{P}) \cdot\cdot \left(\dot{\mathbf{F}} \cdot \mathbf{F}^{-1} \right), \quad \mathbf{F} \cdot \mathbf{P} = \boldsymbol{\tau}. \tag{3.7.7}$$

The balance law of linear momentum can be written with respect to undeformed geometry as

$$\int_{S^0} \mathbf{p}_n \, \mathrm{d}S^0 + \int_{V^0} \rho^0 \, \mathbf{b} \, \mathrm{d}V^0 = \frac{\mathrm{d}}{\mathrm{d}t} \int_{V^0} \rho^0 \, \mathbf{v} \, \mathrm{d}V^0, \tag{3.7.8}$$

which, in view of Eq. (3.7.3) and the Gauss theorem, reproduces the equations of motion (3.3.8), written at points of the undeformed configuration.

3.7.1. Piola–Kirchhoff Stress

The second or symmetric Piola–Kirchhoff stress tensor is the stress tensor $\mathbf{T}_{(1)}$, introduced previously as the work conjugate to the Lagrangian strain $\mathbf{E}_{(1)}$. An alternative construction of this stress tensor is as follows. A pseudo-force vector $\mathrm{d}\mathbf{f}_n^0$ in the undeformed configuration is introduced such that

$$\mathrm{d}\mathbf{f}_n = \mathbf{F} \cdot \mathrm{d}\mathbf{f}_n^0. \tag{3.7.9}$$

The associated pseudo-traction is (Fig. 3.4)

$$\mathbf{t}_n^0 = \frac{\mathrm{d}\mathbf{f}_n^0}{\mathrm{d}S^0}\,. \tag{3.7.10}$$

The second Piola–Kirchhoff stress tensor satisfies

$$\mathbf{t}_n^0 = \mathbf{n}^0 \cdot \mathbf{T}_{(1)}. \tag{3.7.11}$$

This gives

$$\mathbf{T}_{(1)} = \mathbf{F}^{-1} \cdot \boldsymbol{\tau} \cdot \mathbf{F}^{-T}, \tag{3.7.12}$$

which is symmetric whenever $\boldsymbol{\tau}$ is symmetric (nonpolar case). The connection with the nominal stress is

$$\mathbf{P} = \mathbf{T}_{(1)} \cdot \mathbf{F}^T, \tag{3.7.13}$$

so that $\mathbf{F} \cdot \mathbf{P}$ is symmetric. It is also noted that

$$\mathbf{T}_{(1/2)} = (\mathbf{P} \cdot \mathbf{R})_{\mathrm{s}}, \tag{3.7.14}$$

which is referred to as the Biot stress (Biot, 1965; Hill, 1968; Ogden, 1984).

Returning to expression (3.7.6) for the rate of work, it is noted that in the case when there is momentarily no rate of stretching tensor $\mathbf{U}$ (from the polar decomposition of deformation gradient $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$), i.e., when $\dot{\mathbf{U}} = \mathbf{0}$, we have

$$\mathbf{P} \cdot\cdot\, \dot{\mathbf{F}} = \mathbf{P} \cdot\cdot\, \dot{\mathbf{R}} \cdot \mathbf{U} = \mathbf{T}_{(1)} \cdot \mathbf{F}^T \cdot\cdot\, \dot{\mathbf{R}} \cdot \mathbf{U} = (\mathbf{F} \cdot \mathbf{T}_{(1)} \cdot \mathbf{F}^T) : (\dot{\mathbf{R}} \cdot \mathbf{R}^{-1}) = 0. \tag{3.7.15}$$

The last trace product vanishes because $\mathbf{F} \cdot \mathbf{T}_{(1)} \cdot \mathbf{F}^T$ is symmetric, while $\dot{\mathbf{R}} \cdot \mathbf{R}^{-1}$ is antisymmetric tensor. The result was expected because there can not be any rate of work associated with instantaneous rigid-body spin of an already stretched body.

3.8. Stress Rates

The material stress tensors $\mathbf{T}_{(n)}$ can be decomposed in four different ways on the primary and reciprocal bases in the undeformed configuration. Likewise, the spatial stress tensors $\boldsymbol{\mathcal{T}}_{(n)}$ can be decomposed on the bases in the deformed configuration. For example, the contravariant decompositions are

$$\mathbf{T}_{(n)} = T^{IJ}_{(n)} \mathbf{e}^0_I \otimes \mathbf{e}^0_J, \quad \boldsymbol{\mathcal{T}}_{(n)} = \mathcal{T}^{ij}_{(n)} \mathbf{e}_i \otimes \mathbf{e}_j. \tag{3.8.1}$$

Since

$$\dot{\mathbf{e}}_i = \mathbf{L} \cdot \mathbf{e}_i, \quad \dot{\mathbf{e}}^i = -\mathbf{L}^T \cdot \mathbf{e}^i, \tag{3.8.2}$$

by Eq. (2.5.1), there are four types of the convected derivatives of the spatial stress tensors. They are given by Eqs. (2.5.8)–(2.5.11), if $\mathbf{A}$ is there replaced by $\boldsymbol{\mathcal{T}}_{(n)}$. In view of Eq. (3.6.22), there is a connection between the rates of material and spatial stress tensors,

$$\overset{\bullet}{\boldsymbol{\mathcal{T}}}_{(n)} = \mathbf{R} \cdot \dot{\mathbf{T}}_{(n)} \cdot \mathbf{R}^T, \quad \overset{\bullet}{\boldsymbol{\mathcal{T}}}_{(n)} = \dot{\boldsymbol{\mathcal{T}}}_{(n)} - \boldsymbol{\omega} \cdot \boldsymbol{\mathcal{T}}_{(n)} + \boldsymbol{\mathcal{T}}_{(n)} \cdot \boldsymbol{\omega}. \tag{3.8.3}$$

Here, $\boldsymbol{\omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1}$ is the spin due to $\mathbf{R}$.

The rates of material stress tensors $\mathbf{T}_{(1)}$ and $\mathbf{T}_{(-1)}$ are related to convected rates of the Kirchhoff stress by

$$\dot{\mathbf{T}}_{(1)} = \mathbf{F}^{-1} \cdot \overset{\triangle}{\boldsymbol{\tau}} \cdot \mathbf{F}^{-T}, \quad \overset{\triangle}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{L} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot \mathbf{L}^T, \tag{3.8.4}$$

$$\dot{\mathbf{T}}_{(-1)} = \mathbf{F}^T \cdot \overset{\triangledown}{\boldsymbol{\tau}} \cdot \mathbf{F}, \quad \overset{\triangledown}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} + \mathbf{L}^T \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{L}. \tag{3.8.5}$$

The rate of stress conjugate to logarithmic strain is obtained from Eq. (3.6.15) by differentiation, and is given by

$$\dot{\mathbf{T}}_{(0)} = \dot{\mathbf{T}}_{(n)} + n\left(\dot{\mathbf{E}}_{(n)} \cdot \mathbf{T}_{(n)} + \mathbf{T}_{(n)} \cdot \dot{\mathbf{E}}_{(n)}\right) + \mathcal{O}\left(\mathbf{E}_{(n)}\right). \qquad (3.8.6)$$

3.8.1. Rate of Nominal Stress

The nominal stress tensor, being a two-point tensor, has four kinds of decompositions

$$\mathbf{P} = P^{Ji}\mathbf{e}_J^0 \otimes \mathbf{e}_i = P_{Ji}\mathbf{e}_0^J \otimes \mathbf{e}^i = P^J{}_i\mathbf{e}_J^0 \otimes \mathbf{e}^i = P_J{}^i\mathbf{e}_0^J \otimes \mathbf{e}_i, \qquad (3.8.7)$$

but only two different convected derivatives result. They are

$$\overset{\triangle}{\mathbf{P}} = \overset{\triangleright}{\mathbf{P}} = \dot{\mathbf{P}} - \mathbf{P} \cdot \mathbf{L}^T, \quad \overset{\nabla}{\mathbf{P}} = \overset{\triangleleft}{\mathbf{P}} = \dot{\mathbf{P}} + \mathbf{P} \cdot \mathbf{L}. \qquad (3.8.8)$$

The Jaumann derivative of the nominal stress is

$$\overset{\circ}{\mathbf{P}} = \dot{\mathbf{P}} + \mathbf{P} \cdot \mathbf{W}. \qquad (3.8.9)$$

Observe the difference in the structure of the expressions (3.8.8) and (3.8.9) for the convected and Jaumann derivatives of a two-point nominal stress tensor, and the corresponding expressions (2.5.6) for objective derivatives of a two-point deformation gradient tensor. This is because, for example,

$$\mathbf{P} = P^{Ji}\mathbf{e}_J^0 \otimes \mathbf{e}_i, \quad \text{while} \quad \mathbf{F} = F^{iJ}\mathbf{e}_i \otimes \mathbf{e}_J^0. \qquad (3.8.10)$$

The transpose tensor $\mathbf{P}^T$ has the convected and the Jaumann derivatives defined according to Eqs. (2.5.6).

The rate of the nominal stress is

$$\dot{\mathbf{P}} = \mathbf{F}^{-1} \cdot (\dot{\boldsymbol{\tau}} - \mathbf{L} \cdot \boldsymbol{\tau}). \qquad (3.8.11)$$

The following relationships are easily established between the objective rates of the nominal and Kirchhoff stress

$$\overset{\triangle}{\mathbf{P}} = \mathbf{F}^{-1} \cdot \overset{\triangle}{\boldsymbol{\tau}}, \quad \overset{\nabla}{\mathbf{P}} = \mathbf{F}^{-1} \cdot \overset{\triangleleft}{\boldsymbol{\tau}}, \qquad (3.8.12)$$

and

$$\overset{\circ}{\mathbf{P}} = \mathbf{F}^{-1} \cdot \left(\overset{\circ}{\boldsymbol{\tau}} - \mathbf{D} \cdot \boldsymbol{\tau}\right), \quad \overset{\circ}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - \mathbf{W} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot \mathbf{W}. \qquad (3.8.13)$$

Furthermore, the rates of the material stress tensors can be expressed as

$$\dot{\mathbf{T}}_{(1)} = \overset{\triangle}{\mathbf{P}} \cdot \mathbf{F}^{-T}, \quad \dot{\mathbf{T}}_{(1/2)} = \left(\overset{\bullet}{\mathbf{P}} \cdot \mathbf{R}\right)_s, \qquad (3.8.14)$$

where

$$\overset{\bullet}{\mathbf{P}} = \dot{\mathbf{P}} + \mathbf{P} \cdot \boldsymbol{\omega}. \tag{3.8.15}$$

Finally, the rates of nominal and true tractions are related by

$$\dot{\mathbf{p}}_n \, \mathrm{d}S^0 = \left[\dot{\mathbf{t}}_n + (\operatorname{tr}\mathbf{D} - \mathbf{n} \cdot \mathbf{D} \cdot \mathbf{n})\, \mathbf{t}_{\mathrm{n}}\right] \mathrm{d}S. \tag{3.8.16}$$

This follows by differentiation of $\mathbf{p}_n \, \mathrm{d}S^0 = \mathbf{t}_n \, \mathrm{d}S$, having in mind the connection (2.4.17).

Higher rates of stress can be investigated similarly, but will not be needed in this book. They are used in modeling certain viscoelastic-type materials. A paper by Prager (1962) and a treatise by Truesdell and Noll (1965) can be consulted in this respect.

3.9. Stress Rates with Current Configuration as Reference

If the current configuration is chosen as the reference configuration ($\mathbf{F} = \mathbf{I}$), all strain measures vanish, and all corresponding stresses are equal to Cauchy stress. All material strain rates are equal to the rate of deformation tensor,

$$\underline{\dot{\mathbf{E}}}_{(n)} = \mathbf{D}. \tag{3.9.1}$$

Since

$$\mathbf{D} = \underline{\dot{\mathbf{U}}}, \quad \mathbf{W} = \underline{\dot{\mathbf{R}}} = \underline{\boldsymbol{\omega}}, \tag{3.9.2}$$

from Eq. (2.6.19) it follows that

$$\overset{\circ}{\underline{\boldsymbol{\mathcal{E}}}}_{(n)} = \overset{\bullet}{\underline{\boldsymbol{\mathcal{E}}}}_{(n)} = \underline{\dot{\mathbf{E}}}_{(n)} = \mathbf{D}. \tag{3.9.3}$$

The underline indicates that the current configuration is used as the reference configuration.

The rate of stress $\underline{\mathbf{T}}_{(0)}$ is, from Eq. (8.5),

$$\underline{\dot{\mathbf{T}}}_{(0)} = \underline{\dot{\mathbf{T}}}_{(n)} + n(\mathbf{D} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbf{D}). \tag{3.9.4}$$

Any $\mathbf{T}_{(n)}$ can be used in Eq. (3.9.4) to evaluate $\underline{\dot{\mathbf{T}}}_{(0)}$. For example, for $n = 1$ we have from Eq. (3.8.4)

$$\underline{\dot{\mathbf{T}}}_{(1)} = \overset{\triangle}{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \operatorname{tr}\mathbf{D}, \quad \overset{\triangle}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{L} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \mathbf{L}^T. \tag{3.9.5}$$

Substitution into Eq. (3.9.4) gives

$$\underline{\dot{\mathbf{T}}}_{(0)} = \overset{\circ}{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \operatorname{tr}\mathbf{D}, \quad \overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{W} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot \mathbf{W}. \tag{3.9.6}$$

The rate of stress $\dot{\underline{\mathbf{T}}}_{(n)}$ for an arbitrary n can be deduced from Eq. (3.9.4) by inserting $\dot{\underline{\mathbf{T}}}_{(0)}$ from Eq. (3.9.6). The result is

$$\dot{\underline{\mathbf{T}}}_{(n)} = \overset{\circ}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D} - n(\mathbf{D}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{D}). \tag{3.9.7}$$

This is also equal to the Jaumann rate of the spatial stress

$$\overset{\circ}{\underline{\boldsymbol{\mathcal{T}}}}_{(n)} = \dot{\underline{\mathbf{T}}}_{(n)}, \tag{3.9.8}$$

again, of course, with the current configuration taken as the reference configuration. Recall that

$$\overset{\bullet}{\underline{\boldsymbol{\mathcal{T}}}}_{(n)} = \overset{\circ}{\underline{\boldsymbol{\mathcal{T}}}}_{(n)}, \tag{3.9.9}$$

since $\underline{\boldsymbol{\omega}} = \mathbf{W}$. Finally, the rate of nominal stress, momentarily equal to $\boldsymbol{\sigma}$, is

$$\dot{\underline{\mathbf{P}}} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D} - \mathbf{L}\cdot\boldsymbol{\sigma}, \tag{3.9.10}$$

which can be rewritten as either of

$$\overset{\triangle}{\underline{\mathbf{P}}} = \overset{\triangle}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D}, \quad \overset{\circ}{\underline{\mathbf{P}}} = \overset{\circ}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D} - \mathbf{D}\cdot\boldsymbol{\sigma}. \tag{3.9.11}$$

The stress rate

$$\overset{\circ}{\underline{\boldsymbol{\tau}}} = \overset{\circ}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D} \tag{3.9.12}$$

repeatedly appears in the above equations. It is the rate of Kirchhoff stress when the current configuration is taken for the reference configuration. Similarly,

$$\overset{\triangle}{\underline{\boldsymbol{\tau}}} = \overset{\triangle}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D}, \quad \overset{\triangledown}{\underline{\boldsymbol{\tau}}} = \overset{\triangledown}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D}, \tag{3.9.13}$$

with the connections

$$\overset{\circ}{\boldsymbol{\tau}} = (\det\mathbf{F})\overset{\circ}{\underline{\boldsymbol{\tau}}}, \quad \overset{\triangle}{\boldsymbol{\tau}} = (\det\mathbf{F})\overset{\triangle}{\underline{\boldsymbol{\tau}}}, \quad \overset{\triangledown}{\boldsymbol{\tau}} = (\det\mathbf{F})\overset{\triangledown}{\underline{\boldsymbol{\tau}}}. \tag{3.9.14}$$

The stress rate $\overset{\triangle}{\underline{\boldsymbol{\tau}}}$ is also known as the Truesdell rate of the Cauchy stress $\boldsymbol{\sigma}$ (the Oldroyd rate of the Cauchy stress plus $\boldsymbol{\sigma}\,\mathrm{tr}\,\mathbf{D}$). Evidently,

$$\dot{\underline{\mathbf{T}}}_{(0)} = \overset{\circ}{\underline{\boldsymbol{\tau}}}, \quad \dot{\underline{\mathbf{T}}}_{(1)} = \overset{\triangle}{\underline{\boldsymbol{\tau}}}, \quad \dot{\underline{\mathbf{T}}}_{(-1)} = \overset{\triangledown}{\underline{\boldsymbol{\tau}}}, \quad \dot{\underline{\mathbf{P}}} = \dot{\underline{\boldsymbol{\tau}}} - \mathbf{L}\cdot\boldsymbol{\sigma}, \tag{3.9.15}$$

and

$$\dot{\mathbf{T}}_{(1)} = (\det\mathbf{F})\mathbf{F}^{-1}\cdot\dot{\underline{\mathbf{T}}}_{(1)}\cdot\mathbf{F}^{-T}, \quad \dot{\mathbf{T}}_{(-1)} = (\det\mathbf{F})\mathbf{F}^{T}\cdot\dot{\underline{\mathbf{T}}}_{(-1)}\cdot\mathbf{F}, \tag{3.9.16}$$

$$\dot{\mathbf{P}} = (\det\mathbf{F})\mathbf{F}^{-1}\cdot\dot{\underline{\mathbf{P}}}. \tag{3.9.17}$$

Lastly, it is noted that at the current state as reference, the rates of nominal and true tractions are related by

$$\dot{\underline{\mathbf{p}}}_n = \dot{\mathbf{t}}_n + (\operatorname{tr}\mathbf{D} - \mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n})\,\mathbf{t}_{\mathrm{n}}. \tag{3.9.18}$$

This follows directly from Eq. (3.8.16), since $\mathrm{d}S^0 = \mathrm{d}S$ at the current state as reference.

3.10. Behavior under Superimposed Rotation

If a time-dependent rotation $\mathbf{Q}$ is superimposed to the deformed configuration at time t, the material stress tensors $\mathbf{T}_{(n)}$ do not change,

$$\mathbf{T}^*_{(n)} = \mathbf{T}_{(n)}, \tag{3.10.1}$$

because the strain rates $\dot{\mathbf{E}}_{(n)}$ remain unchanged ($\mathbf{E}^*_{(n)} = \mathbf{E}_{(n)}$), and

$$\dot{w} = \mathbf{T}^*_{(n)} : \dot{\mathbf{E}}^*_{(n)} = \mathbf{T}_{(n)} : \dot{\mathbf{E}}_{(n)}. \tag{3.10.2}$$

In view of Eq. (3.6.22), the spatial stress tensors change into

$$\boldsymbol{\mathcal{T}}^*_{(n)} = \mathbf{Q}\cdot\boldsymbol{\mathcal{T}}_{(n)}\cdot\mathbf{Q}^T. \tag{3.10.3}$$

The same transformation rule applies to Cauchy and Kirchhoff stress. Since the nominal stress is defined by $\mathbf{P} = \mathbf{F}^{-1}\cdot\boldsymbol{\tau}$, it becomes

$$\mathbf{P}^* = \mathbf{P}\cdot\mathbf{Q}^T. \tag{3.10.4}$$

The transformation rule for the Cauchy stress can be independently deduced from the basic relation $\mathbf{t}_n = \mathbf{n}\cdot\boldsymbol{\sigma}$. Under rotation $\mathbf{Q}$ of the deformed configuration, the traction vector changes into

$$\mathbf{t}^*_n = \mathbf{Q}\cdot\mathbf{t}_n, \tag{3.10.5}$$

and the unit normal becomes $\mathbf{n}^* = \mathbf{Q}\cdot\mathbf{n}$. Hence, the transformation

$$\boldsymbol{\sigma}^* = \mathbf{Q}\cdot\boldsymbol{\sigma}\cdot\mathbf{Q}^T. \tag{3.10.6}$$

Likewise,

$$\boldsymbol{\tau}^* = \mathbf{Q}\cdot\boldsymbol{\tau}\cdot\mathbf{Q}^T. \tag{3.10.7}$$

On the other hand,

$$\hat{\boldsymbol{\tau}}^* = \hat{\boldsymbol{\tau}}, \quad \hat{\boldsymbol{\tau}} = \mathbf{R}^T\cdot\boldsymbol{\tau}\cdot\mathbf{R}. \tag{3.10.8}$$

The following transformation rules apply for the rates of material and spatial stress tensors

$$\dot{\mathbf{T}}^*_{(n)} = \dot{\mathbf{T}}_{(n)}, \quad \dot{\boldsymbol{\mathcal{T}}}^*_{(n)} = \mathbf{Q} \cdot \left(\dot{\boldsymbol{\mathcal{T}}}_{(n)} + \hat{\boldsymbol{\Omega}} \cdot \boldsymbol{\mathcal{T}}_{(n)} - \boldsymbol{\mathcal{T}}_{(n)} \cdot \hat{\boldsymbol{\Omega}} \right) \cdot \mathbf{Q}^T, \tag{3.10.9}$$

where $\hat{\boldsymbol{\Omega}} = \mathbf{Q}^T \cdot \boldsymbol{\Omega} \cdot \mathbf{Q}$ and $\boldsymbol{\Omega} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1}$. The rate of nominal stress becomes

$$\dot{\mathbf{P}}^* = \left(\dot{\mathbf{P}} - \mathbf{P} \cdot \hat{\boldsymbol{\Omega}} \right) \cdot \mathbf{Q}^T. \tag{3.10.10}$$

The objective spatial stress rates change according to

$$\overset{\bullet}{\boldsymbol{\mathcal{T}}}{}^*_{(n)} = \mathbf{Q} \cdot \overset{\bullet}{\boldsymbol{\mathcal{T}}}_{(n)} \cdot \mathbf{Q}^T, \quad \overset{\triangle}{\boldsymbol{\tau}}{}^* = \mathbf{Q} \cdot \overset{\triangle}{\boldsymbol{\tau}} \cdot \mathbf{Q}^T, \quad \overset{\circ}{\boldsymbol{\tau}}{}^* = \mathbf{Q} \cdot \overset{\circ}{\boldsymbol{\tau}} \cdot \mathbf{Q}^T, \tag{3.10.11}$$

while objective rates of the nominal stress transform as

$$\overset{\triangle}{\mathbf{P}}{}^* = \overset{\triangle}{\mathbf{P}} \cdot \mathbf{Q}^T, \quad \overset{\circ}{\mathbf{P}}{}^* = \overset{\circ}{\mathbf{P}} \cdot \mathbf{Q}^T. \tag{3.10.12}$$

3.11. Principle of Virtual Velocities

Kinematically admissible velocity field is one possessing continuous first partial derivatives in the interior of the body (analytically admissible), and satisfying prescribed kinematic (velocity) boundary conditions. Kinetically admissible stress and acceleration fields satisfy equations of motion and prescribed kinetic (traction) boundary conditions. Statically admissible stress field satisfies equations of equilibrium and prescribed traction boundary conditions.

Principle of virtual velocities: If the stress and acceleration fields are kinetically admissible, the rate of work of external and inertial forces on any kinematically admissible virtual velocity field is equal to

$$\int_V \boldsymbol{\sigma} : \delta\mathbf{D} \, \mathrm{d}V. \tag{3.11.1}$$

Conversely, if the rate of work of external and inertial forces is equal to (3.11.1), for the assumed stress and acceleration fields and for every kinematically admissible virtual velocity field, then the stress and acceleration fields are kinetically admissible.

Proof: The rate of work of the surface traction $\mathbf{t}_n$ on an analytically admissible virtual velocity field $\delta\mathbf{v}$ vanishing on S_v is

$$\int_S \mathbf{t}_n \cdot \delta\mathbf{v} \, \mathrm{d}S = \int_V \boldsymbol{\nabla} \cdot (\boldsymbol{\sigma} \cdot \delta\mathbf{v}) \, \mathrm{d}V. \tag{3.11.2}$$

If the traction is applied only on the S_t part of S, while velocity is prescribed on the remainder S_v of the boundary, then $\delta\mathbf{v} = \mathbf{0}$ on S_v by definition of the kinematically admissible virtual velocity field. Thus, the integral on the left-hand side of Eq. (3.11.2) can always be taken over the total S. Applying Eq. (1.13.13) to the integrand on the right-hand side of Eq. (3.11.2), and by the symmetry of $\boldsymbol{\sigma}$, we obtain

$$\int_S \mathbf{t}_n \cdot \delta\mathbf{v}\,\mathrm{d}S - \int_V (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{v}\,\mathrm{d}V = \int_V \boldsymbol{\sigma} : \delta\mathbf{D}\,\mathrm{d}V. \tag{3.11.3}$$

If $\boldsymbol{\sigma}$ and $\mathrm{d}\mathbf{v}/\mathrm{d}t$ are kinetically admissible, from equations of motion (3.3.4) it follows that

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \rho\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} - \mathbf{b}\right). \tag{3.11.4}$$

Substitution into Eq. (3.11.3) gives the desired expression

$$\int_S \mathbf{t}_n \cdot \delta\mathbf{v}\,\mathrm{d}S + \int_V \rho\left(\mathbf{b} - \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)\cdot\delta\mathbf{v}\,\mathrm{d}V = \int_V \boldsymbol{\sigma} : \delta\mathbf{D}\,\mathrm{d}V. \tag{3.11.5}$$

Conversely, assume that Eq. (3.11.5) holds for a prescribed traction on S_t, given body forces in V, and for assumed stress and acceleration fields. Subtracting from both sides of Eq. (3.11.5) the integral of $(\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{v}$ over the surface S, we have

$$\int_S (\mathbf{t}_n - \mathbf{n}\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{v}\,\mathrm{d}S + \int_V \left[\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \rho\left(\mathbf{b} - \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)\right]\cdot\delta\mathbf{v}\,\mathrm{d}V = 0. \tag{3.11.6}$$

This is identically satisfied if $\boldsymbol{\sigma}$ and $\mathrm{d}\mathbf{v}/\mathrm{d}t$ are kinetically admissible, satisfying equations of motion (3.3.4) and the boundary conditions $\mathbf{n}\cdot\boldsymbol{\sigma} = \mathbf{t}_n$ on S_t.

If integrals are written with respect to undeformed geometry, Eq. (3.11.5) is replaced with

$$\int_{S^0} \mathbf{p}_n \cdot \delta\mathbf{v}\,\mathrm{d}S^0 + \int_{V^0} \rho^0\left(\mathbf{b} - \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)\cdot\delta\mathbf{v}\,\mathrm{d}V^0 = \int_{V^0} \mathbf{P}\cdot\cdot\,\delta\dot{\mathbf{F}}\,\mathrm{d}V^0, \tag{3.11.7}$$

where $\delta\dot{\mathbf{F}} = \delta\mathbf{v}\otimes\boldsymbol{\nabla}^0$. If Eq. (3.11.7) holds, the nominal stress $\mathbf{P}$ and the acceleration field satisfy equations of motion (3.3.8), and the boundary conditions $\mathbf{n}^0\cdot\mathbf{P} = \mathbf{p}_n$ on S_t^0.

A straightforward extension of the previous result is obtained by using the rates of nominal stress and traction. Indeed, if

$$\int_{S^0} \dot{\mathbf{p}}_n \cdot \delta\mathbf{v}\,\mathrm{d}S^0 + \int_{V^0} \rho^0\left(\dot{\mathbf{b}} - \frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}t^2}\right)\cdot\delta\mathbf{v}\,\mathrm{d}V^0 = \int_{V^0} \dot{\mathbf{P}}\cdot\cdot\,\delta\dot{\mathbf{F}}\,\mathrm{d}V^0, \tag{3.11.8}$$

for all analytically admissible $\delta\mathbf{v}$ vanishing on S_v^0, the rates of nominal stress $\dot{\mathbf{P}}$ and the rate of acceleration field satisfy the rate-type equations

$$\boldsymbol{\nabla}^0 \cdot \dot{\mathbf{P}} + \rho^0 \, \dot{\mathbf{b}} = \rho^0 \, \frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}t^2} \,, \tag{3.11.9}$$

and the rate-type boundary conditions

$$\mathbf{n}^0 \cdot \dot{\mathbf{P}} = \dot{\mathbf{p}}_n \quad \text{on} \quad S_t^0. \tag{3.11.10}$$

The rate-type equations (3.11.9) also follow from equations of motion (3.3.8) by differentiation.

For static problems, $\mathrm{d}\mathbf{v}/\mathrm{d}t$ and $\mathrm{d}^2\mathbf{v}/\mathrm{d}t^2$ are equal to zero in Eqs. (3.11.4)–(3.11.9), so that

$$\boldsymbol{\nabla}^0 \cdot \dot{\mathbf{P}} + \rho^0 \, \dot{\mathbf{b}} = \mathbf{0}. \tag{3.11.11}$$

If $\dot{\mathbf{P}}$ satisfies Eq. (3.11.11), by Gauss divergence theorem it also follows that

$$\int_{V^0} \dot{\mathbf{P}} \cdot\cdot \, \dot{\mathbf{F}}' \, \mathrm{d}V^0 = \int_{V^0} \rho^0 \, \dot{\mathbf{b}} \cdot \mathbf{v}' \, \mathrm{d}V^0 + \int_{S^0} \mathbf{n}^0 \cdot \dot{\mathbf{P}} \cdot \mathbf{v}' \, \mathrm{d}S^0, \tag{3.11.12}$$

for any analytically admissible velocity field $\mathbf{v}'$. A direct consequence is a Kirchhoff type identity

$$\begin{aligned} \int_{V^0} (\dot{\mathbf{P}} - \dot{\mathbf{P}}') \cdot\cdot \, (\dot{\mathbf{F}} - \dot{\mathbf{F}}') \, \mathrm{d}V^0 &= \int_{V^0} \rho^0 (\dot{\mathbf{b}} - \dot{\mathbf{b}}') \cdot (\mathbf{v} - \mathbf{v}') \, \mathrm{d}V^0 \\ &+ \int_{S^0} \mathbf{n}^0 \cdot (\dot{\mathbf{P}} - \dot{\mathbf{P}}') \cdot (\mathbf{v} - \mathbf{v}') \, \mathrm{d}S^0, \end{aligned} \tag{3.11.13}$$

where $\dot{\mathbf{P}}'$ and $\dot{\mathbf{b}}'$ are related by Eq. (3.11.11).

If $\mathbf{v}' = \mathbf{v}$, the surface integral in Eq. (3.11.12) is

$$\int_{S^0} \mathbf{n}^0 \cdot \dot{\mathbf{P}} \cdot \mathbf{v} \, \mathrm{d}S^0 = \int_{S_t^0} \dot{\mathbf{p}}_n \cdot \mathbf{v} \, \mathrm{d}S_t^0 + \int_{S_v^0} \mathbf{n}^0 \cdot \dot{\mathbf{P}} \cdot \mathbf{v} \, \mathrm{d}S_v^0, \tag{3.11.14}$$

with $\mathbf{v}$ prescribed on S_v^0, and $\mathbf{n}^0 \cdot \dot{\mathbf{P}} = \dot{\mathbf{p}}_n$ prescribed on S_t^0. If $\mathbf{v} \neq \mathbf{v}'$ in Eq. (3.11.13), but both correspond to the same data ($\dot{\mathbf{b}}$ in V^0, $\dot{\mathbf{p}}_n$ on S_t^0, and $\mathbf{v} = \mathbf{v}'$ on S_v^0), the right-hand side of Eq. (3.11.13) vanishes.

3.12. Principle of Virtual Work

If displacement rather than velocity field is used, we arrive at the principle of virtual displacement (or virtual work). Displacement field is $\mathbf{u} = \mathbf{x} - \mathbf{X}$ (with the same coordinate origin for both $\mathbf{x}$ and $\mathbf{X}$). Geometrically admissible displacement field is one possessing continuous first partial derivatives in the interior of the body, and satisfying prescribed geometric (displacement) boundary conditions. Statically admissible stress field satisfies equations of equilibrium and prescribed static (traction) boundary conditions. Thus, if

$$\int_{S^0} \mathbf{p}_n \cdot \delta\mathbf{u}\, \mathrm{d}S^0 + \int_{V^0} \rho^0\, \mathbf{b} \cdot \delta\mathbf{u}\, \mathrm{d}V^0 = \int_{V^0} \mathbf{P} \cdot\cdot\, \delta\mathbf{F}\, \mathrm{d}V^0, \tag{3.12.1}$$

for all analytically admissible virtual displacements $\delta\mathbf{u}$ vanishing on S_u^0, the nominal stress $\mathbf{P}$ satisfies the equilibrium equations

$$\boldsymbol{\nabla}^0 \cdot \mathbf{P} + \rho^0\, \mathbf{b} = \mathbf{0}, \tag{3.12.2}$$

and the traction boundary conditions

$$\mathbf{n}^0 \cdot \mathbf{P} = \mathbf{p}_n \quad \text{on} \quad S_t^0. \tag{3.12.3}$$

In general, the nominal traction $\mathbf{p}_n$ applied at $\mathbf{X}$ depends on the deformation $\mathbf{x}$ and its gradient $\mathbf{F}$. A particular type of loading for which $\mathbf{p}_n$ depends only on $\mathbf{X}$ is known as dead loading. During dead loading an increase in load deforms the body, but the resulting changes in surface geometry do not modify the load.

If $\mathbf{P}$ satisfies Eq. (3.12.2), by Gauss divergence theorem it follows that

$$\int_{V^0} \mathbf{P} \cdot\cdot\, \mathbf{F}'\, \mathrm{d}V^0 = \int_{V^0} \rho^0\, \mathbf{b} \cdot \mathbf{x}'\, \mathrm{d}V^0 + \int_{S^0} \mathbf{n}^0 \cdot \mathbf{P} \cdot \mathbf{x}'\, \mathrm{d}S^0, \tag{3.12.4}$$

for any analytically admissible deformation field $\mathbf{x}'$. A direct consequence is the Kirchhoff identity

$$\begin{aligned} \int_{V^0} (\mathbf{P} - \mathbf{P}') \cdot\cdot\, (\mathbf{F} - \mathbf{F}')\, \mathrm{d}V^0 &= \int_{V^0} \rho^0 (\mathbf{b} - \mathbf{b}') \cdot (\mathbf{x} - \mathbf{x}')\, \mathrm{d}V^0 \\ &\quad + \int_{S^0} \mathbf{n}^0 \cdot (\mathbf{P} - \mathbf{P}') \cdot (\mathbf{x} - \mathbf{x}')\, \mathrm{d}S^0, \end{aligned} \tag{3.12.5}$$

where $\mathbf{P}'$ and $\mathbf{b}'$ are related by Eq. (3.12.2).

If $\mathbf{x}' = \mathbf{x}$, the surface integral in Eq. (3.12.4) becomes

$$\int_{S^0} \mathbf{n}^0 \cdot \mathbf{P} \cdot \mathbf{x}\, \mathrm{d}S^0 = \int_{S_t^0} \mathbf{p}_n \cdot \mathbf{x}\, \mathrm{d}S_t^0 + \int_{S_u^0} \mathbf{n}^0 \cdot \mathbf{P} \cdot \mathbf{x}\, \mathrm{d}S_u^0, \tag{3.12.6}$$

with $\mathbf{x}$ prescribed on S_u^0, and $\mathbf{n}^0 \cdot \mathbf{P} = \mathbf{p}_n$ prescribed on S_t^0.

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