CHAPTER 2

KINEMATICS OF DEFORMATION

2.1. Material and Spatial Description of Motion

The locations of material points of a three-dimensional body in its initial, undeformed configuration are specified by vectors $\mathbf{X}$. Their locations in deformed configuration at time t are specified by vectors $\mathbf{x}$, such that

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t). \tag{2.1.1}$$

The one-to-one deformation mapping from $\mathbf{X}$ to $\mathbf{x}$ is assumed to be twice continuously differentiable. The components of $\mathbf{X}$ are the material coordinates of the particle, while those of $\mathbf{x}$ are the spatial coordinates. They can be referred to the same or different bases. For example, if the orthonormal base vectors in the undeformed configuration are $\mathbf{e}_J^0$, and those in the deformed configuration are $\mathbf{e}_i$, then $\mathbf{X} = X_J \mathbf{e}_J^0$ and $\mathbf{x} = x_i \mathbf{e}_i$. Often, the same basis is used for both configurations (common frame).

If a tensor field $\mathbf{T}$ is expressed as a function of the material coordinates,

$$\mathbf{T} = \mathbf{T}(\mathbf{X}, t), \tag{2.1.2}$$

the description is referred to as the material or Lagrangian description. If the changes of $\mathbf{T}$ are observed at fixed points in space,

$$\mathbf{T} = \mathbf{T}(\mathbf{x}, t), \tag{2.1.3}$$

the description is spatial or Eulerian. The time derivative of $\mathbf{T}$ can be calculated as

$$\dot{\mathbf{T}} = \frac{\partial \mathbf{T}(\mathbf{X}, t)}{\partial t} = \frac{\partial \mathbf{T}(\mathbf{x}, t)}{\partial t} + \mathbf{v} \cdot (\boldsymbol{\nabla} \otimes \mathbf{T}). \tag{2.1.4}$$

The $\boldsymbol{\nabla}$ operator in Eq. (2.1.4) is defined with respect to spatial coordinates $\mathbf{x}$, and

$$\mathbf{v} = \frac{\partial \mathbf{x}(\mathbf{X}, t)}{\partial t} \tag{2.1.5}$$

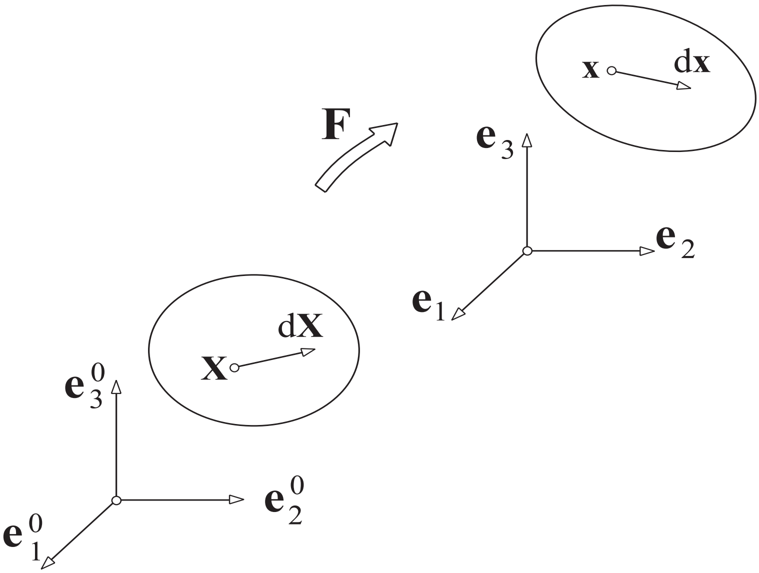

FIGURE 2.1. An infinitesimal material element $d\mathbf{X}$ from the initial configuration becomes $d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}$ in the deformed configuration, where $\mathbf{F}$ is the deformation gradient. The orthonormal base vectors in the undeformed and deformed configurations are $\mathbf{e}_J^0$ and $\mathbf{e}_i$.

is the velocity of a considered material particle at time t. The first term on the right-hand side of Eq. (2.1.4) is the local rate of change of $\mathbf{T}$, while the second term represents the convective rate of change (e.g., Eringen, 1967; Chadwick, 1976).

2.2. Deformation Gradient

An infinitesimal material element $d\mathbf{X}$ from the initial configuration becomes

$$d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}, \quad \mathbf{F} = \mathbf{x} \otimes \boldsymbol{\nabla}^0 = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \tag{2.2.1}$$

in the deformed configuration at time t (Fig. 2.1). The gradient operator $\boldsymbol{\nabla}^0$ is defined with respect to material coordinates. The tensor $\mathbf{F}$ is called the deformation gradient. If the orthonormal base vectors in the undeformed and deformed configurations are $\mathbf{e}_J^0$ and $\mathbf{e}_i$, then

$$\mathbf{F} = F_{iJ}\mathbf{e}_i \otimes \mathbf{e}_J^0, \quad F_{iJ} = \frac{\partial x_i}{\partial X_J}\,. \tag{2.2.2}$$

This represents a two-point tensor: when the base vectors in the deformed configuration are rotated by $\mathbf{Q}$, and those in the undeformed configuration by $\mathbf{Q}^0$, the components F_{iJ} change into $Q_{ki}F_{kL}Q^0_{LJ}$. If $\mathbf{Q}^0$ is the unit tensor, the components of $\mathbf{F}$ transform like those of a vector. Physically possible deformation mappings have the positive Jacobian determinant,

$$\det \mathbf{F} > 0. \tag{2.2.3}$$

Hence, $\mathbf{F}$ is an invertible tensor and $d\mathbf{X}$ can be recovered from $\mathrm{d}\mathbf{x}$ by the inverse operation

$$\mathrm{d}\mathbf{X} = \mathbf{F}^{-1} \cdot \mathrm{d}\mathbf{x}. \tag{2.2.4}$$

The transpose and the inverse of $\mathbf{F}$ have the rectangular representations

$$\mathbf{F}^T = F_{iJ}\mathbf{e}^0_J \otimes \mathbf{e}_i, \quad \mathbf{F}^{-1} = F^{-1}_{Ji}\mathbf{e}^0_J \otimes \mathbf{e}_i. \tag{2.2.5}$$

2.2.1. Polar Decomposition

By the polar decomposition theorem, $\mathbf{F}$ can be decomposed into the product of a proper orthogonal tensor and a positive-definite symmetric tensor, such that (Truesdell and Noll, 1965; Malvern, 1969)

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R}. \tag{2.2.6}$$

The symmetric tensor $\mathbf{U}$ is the right stretch tensor, $\mathbf{V}$ is the left stretch tensor, and $\mathbf{R}$ is the rotation tensor (Fig. 2.2). Evidently,

$$\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T, \tag{2.2.7}$$

so that $\mathbf{V}$ and $\mathbf{U}$ share the same eigenvalues (principal stretches λ_i), while their eigenvectors are related by

$$\mathbf{n}_i = \mathbf{R} \cdot \mathbf{N}_i. \tag{2.2.8}$$

The right and left Cauchy–Green deformation tensors are

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} = \mathbf{U}^2, \quad \mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T = \mathbf{V}^2. \tag{2.2.9}$$

The inverse of the left Cauchy–Green deformation tensor, $\mathbf{B}^{-1}$, is often referred to as the Finger deformation tensor. If there are three distinct principal stretches, $\mathbf{C}$ and $\mathbf{B}$ have the spectral representations

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2\, \mathbf{N}_i \otimes \mathbf{N}_i, \quad \mathbf{B} = \sum_{i=1}^{3} \lambda_i^2\, \mathbf{n}_i \otimes \mathbf{n}_i. \tag{2.2.10}$$

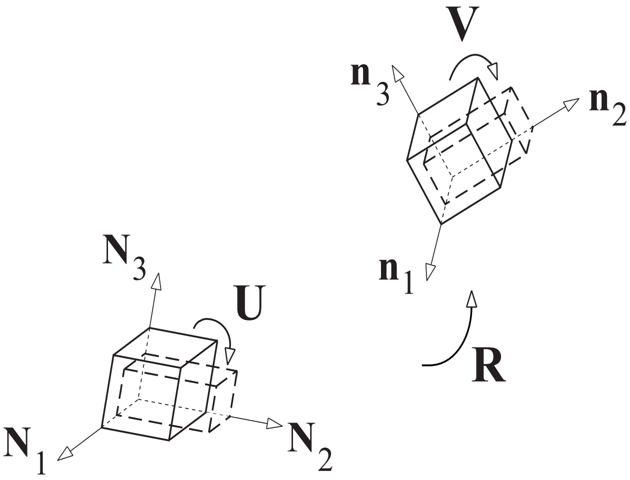

FIGURE 2.2. Schematic representation of the polar decomposition of deformation gradient. Material element is first stretched by $\mathbf{U}$ and then rotated by $\mathbf{R}$, or first rotated by $\mathbf{R}$ and then stretched by $\mathbf{V}$. The principal directions of $\mathbf{U}$ are $\mathbf{N}_i$, and those of $\mathbf{V}$ are $\mathbf{n}_i = \mathbf{R} \cdot \mathbf{N}_i$.

Furthermore,

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \, \mathbf{N}_i \otimes \mathbf{N}_i, \quad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \, \mathbf{n}_i \otimes \mathbf{n}_i, \quad \mathbf{R} = \sum_{i=1}^{3} \mathbf{n}_i \otimes \mathbf{N}_i, \tag{2.2.11}$$

and

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \, \mathbf{n}_i \otimes \mathbf{N}_i. \tag{2.2.12}$$

If

$$j_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad j_2 = -(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1), \quad j_3 = \lambda_1\lambda_2\lambda_3 \tag{2.2.13}$$

are the principal invariants of $\mathbf{U}$, then (Hoger and Carlson, 1984; Simo and Hughes, 1998)

$$\mathbf{U} = \frac{1}{j_1 j_2 + j_3} \left[\mathbf{C}^2 - (j_1^2 + j_2)\mathbf{C} - j_3 j_1 \mathbf{I}^0 \right], \quad \mathbf{U}^{-1} = \frac{1}{j_3} \left(\mathbf{C} - j_1 \mathbf{U} - j_2 \mathbf{I}^0 \right). \tag{2.2.14}$$

The unit second-order tensors $\mathbf{I}^0$ is defined by

$$\mathbf{I}^0 = \sum_{i=1}^{3} \mathbf{N}_i \otimes \mathbf{N}_i. \tag{2.2.15}$$

2.2.2. Nanson's Relation

An infinitesimal volume element $\mathrm{d}V^0$ from the undeformed configuration becomes

$$\mathrm{d}V = (\det \mathbf{F})\mathrm{d}V^0 \tag{2.2.16}$$

in the deformed configuration. An infinitesimal area $\mathrm{d}S^0$ with unit normal $\mathbf{n}^0$ in the undeformed configuration becomes the area $\mathrm{d}S$ with unit normal $\mathbf{n}$ in the deformed configuration, such that (Nanson's relation)

$$\mathbf{n}\,\mathrm{d}S = (\det \mathbf{F})\mathbf{F}^{-T} \cdot \mathbf{n}^0\,\mathrm{d}S^0. \tag{2.2.17}$$

The following is a proof of (2.2.17). Consider a triad of vectors in the undeformed configuration $\mathbf{e}_J^0$, and its reciprocal triad $\mathbf{e}_0^J$. Then, the vector area

$$\mathrm{d}\mathbf{S}^0 = \mathbf{e}_1^0 \times \mathbf{e}_2^0 = D^0\,\mathbf{e}_0^3, \quad D^0 = (\mathbf{e}_1^0 \times \mathbf{e}_2^0) \cdot \mathbf{e}_3^0, \tag{2.2.18}$$

by definition of the reciprocal vectors (Hill, 1978). If the primary vectors are embedded in the material, they become in the deformed configuration $\mathbf{e}_i = \mathbf{F} \cdot \mathbf{e}_i^0$. Their reciprocal vectors are $\mathbf{e}^i = \mathbf{F}^{-T} \cdot \mathbf{e}_0^i$ (Fig. 2.3). Thus, the vector area corresponding to (2.2.18) is in the deformed configuration

$$\mathrm{d}\mathbf{S} = \mathbf{e}_1 \times \mathbf{e}_2 = D\,\mathbf{e}^3 = (\det \mathbf{F})\mathbf{F}^{-T} \cdot \mathrm{d}\mathbf{S}^0, \tag{2.2.19}$$

because

$$\mathbf{e}^3 = \mathbf{F}^{-T} \cdot \mathbf{e}_0^3, \quad D = (\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = (\det \mathbf{F})D^0. \tag{2.2.20}$$

Equation (2.2.19) is the Nanson's relation.

By Eq. (1.13.16) the integral of $\mathbf{n}\,\mathrm{d}S$ over any closed surface S is equal to zero. Therefore, by applying the Gauss theorem to the integral of the right-hand side of Eq. (2.2.17) over the corresponding surface S^0 in the undeformed configuration gives

$$\boldsymbol{\nabla}^0 \cdot \left[(\det \mathbf{F})\mathbf{F}^{-1}\right] = \mathbf{0}. \tag{2.2.21}$$

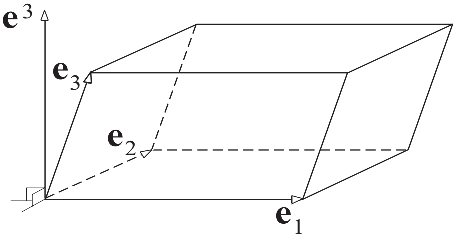

FIGURE 2.3. Deformed primary base vectors define an infinitesimal volume element $\mathrm{d}V$ in the deformed configuration. The reciprocal vector $\mathbf{e}^3 = D^{-1}(\mathbf{e}_1 \times \mathbf{e}_2)$, where $D = \mathrm{d}V = (\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3$.

2.2.3. Simple Shear

This is an isochoric plane deformation in which the planes with unit normal $\mathbf{N}$ slide relative to each other in the direction $\mathbf{M}$ (Fig. 2.4), such that

$$\mathbf{x} = \mathbf{X} + \gamma\, [\mathbf{N} \cdot (\mathbf{X} - \mathbf{X}_0)]\; \mathbf{m}. \tag{2.2.22}$$

The point $\mathbf{X}_0$ is fixed during the deformation, as are all other points within the plane for which $\mathbf{X} - \mathbf{X}_0$ is perpendicular to $\mathbf{N}$. The amount of shear is specified by $\gamma = \tan\varphi$, where φ is the shear angle. The vectors embedded in the planes of shearing preserve their length and orientation, so that $\mathbf{m} = \mathbf{M}$. The deformation gradient corresponding to Eq. (2.2.22), and its inverse are

$$\mathbf{F} = \mathbf{I} + \gamma(\mathbf{m} \otimes \mathbf{N}), \quad \mathbf{F}^{-1} = \mathbf{I} - \gamma(\mathbf{m} \otimes \mathbf{N}). \tag{2.2.23}$$

It is assumed that the same basis is used in both undeformed and deformed configurations. Clearly,

$$\mathbf{m} = \mathbf{F} \cdot \mathbf{M} = \mathbf{M}, \quad \mathbf{n} = \mathbf{N} \cdot \mathbf{F}^{-1} = \mathbf{N}, \tag{2.2.24}$$

where $\mathbf{n}$ is the unit normal to shear plane in the deformed configuration.

If different orthogonal bases are used in the undeformed and deformed configurations, we have

$$\mathbf{F} = g_{iJ}\left(\mathbf{e}_i \otimes \mathbf{e}_J^0\right) + \gamma(\mathbf{m} \otimes \mathbf{N}), \quad \mathbf{F}^{-1} = g_{Ji}\left(\mathbf{e}_J^0 \otimes \mathbf{e}_I\right) - \gamma(\mathbf{M} \otimes \mathbf{n}), \tag{2.2.25}$$

where

$$g_{iJ} = \mathbf{e}_i \cdot \mathbf{e}_J^0. \tag{2.2.26}$$

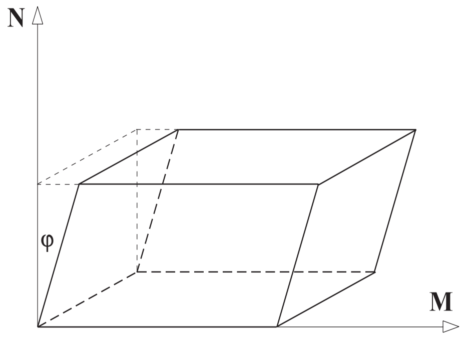

FIGURE 2.4. Simple shear of a rectangular block in the direction $\mathbf{M}$, parallel to the plane with normal $\mathbf{N}$. The shear angle is φ.

These are components of orthogonal matrices such that $g_{Ij}g_{jK} = \delta_{IK}$ and $g_{iJ}g_{Jk} = \delta_{ik}$ represent the components of unit tensors in the undeformed and deformed configurations, respectively, i.e.,

$$\mathbf{I}^0 = \delta_{IK}\mathbf{e}_I^0 \otimes \mathbf{e}_K^0, \quad \mathbf{I} = \delta_{ik}\mathbf{e}_i \otimes \mathbf{e}_k. \tag{2.2.27}$$

The corresponding right and left Cauchy–Green deformation tensors are accordingly

$$\mathbf{C} = \mathbf{I}^0 + \gamma(\mathbf{M} \otimes \mathbf{N} + \mathbf{N} \otimes \mathbf{M}) + \gamma^2(\mathbf{N} \otimes \mathbf{N}), \tag{2.2.28}$$

$$\mathbf{B} = \mathbf{I} + \gamma(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}) + \gamma^2(\mathbf{m} \otimes \mathbf{m}). \tag{2.2.29}$$

2.3. Strain Tensors

2.3.1. Material Strain Tensors

Various tensor measures of strain can be defined. A fairly general definition of material strain measures, reckoned relative to the initial configuration, was introduced by Seth (1964, 1966) and Hill (1968, 1978). This is

$$\mathbf{E}_{(n)} = \frac{1}{2n}\left(\mathbf{U}^{2n} - \mathbf{I}^0\right) = \sum_{i=1}^{3} \frac{1}{2n}\left(\lambda_i^{2n} - 1\right)\mathbf{N}_i \otimes \mathbf{N}_i, \tag{2.3.1}$$

where $2n$ is a positive or negative integer, and λ_i and $\mathbf{N}_i$ are the principal values and directions of the right stretch tensor $\mathbf{U}$. The unit tensor in the initial configuration is $\mathbf{I}^0$. For $n = 1$, Eq. (2.3.1) gives the Lagrangian or

Green strain

$$\mathbf{E}_{(1)} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}^0), \tag{2.3.2}$$

for $n = -1$ the Almansi strain

$$\mathbf{E}_{(-1)} = \frac{1}{2}(\mathbf{I}^0 - \mathbf{U}^{-2}), \tag{2.3.3}$$

and for $n = 1/2$ the Biot strain

$$\mathbf{E}_{(1/2)} = (\mathbf{U} - \mathbf{I}^0). \tag{2.3.4}$$

There is a general connection

$$\mathbf{E}_{(-n)} = \mathbf{U}^{-n} \cdot \mathbf{E}_{(n)} \cdot \mathbf{U}^{-n}. \tag{2.3.5}$$

The logarithmic or Hencky strain is obtained from (2.3.1) in the limit $n \to 0$, and is given by

$$\mathbf{E}_{(0)} = \ln \mathbf{U} = \sum_{i=1}^{3} \ln \lambda_i \, \mathbf{N}_i \otimes \mathbf{N}_i. \tag{2.3.6}$$

For isochoric deformation ($\lambda_1 \lambda_2 \lambda_3 = 1$), $\mathbf{E}_{(0)}$ is a traceless tensor.

Since,

$$\ln \lambda = (\lambda - 1) - \frac{1}{2}(\lambda - 1)^2 + \frac{1}{3}(\lambda - 1)^3 - \cdots \, , \quad 0 < \lambda \leq 2, \tag{2.3.7}$$

$$\begin{aligned} \frac{1}{2n}\left(\lambda^{2n} - 1\right) = (\lambda - 1) &+ \frac{1}{2}(2n - 1)(\lambda - 1)^2 \\ &+ \frac{1}{3}(n - 1)(2n - 1)(\lambda - 1)^3 + \cdots \, , \quad \lambda > 0, \end{aligned} \tag{2.3.8}$$

there follows

$$\mathbf{E}_{(0)} = \mathbf{E}_{(1/2)} - \frac{1}{2}\mathbf{E}^2_{(1/2)} + \frac{1}{3}\mathbf{E}^3_{(1/2)} + \mathcal{O}\left(\mathbf{E}^4_{(1/2)}\right), \tag{2.3.9}$$

$$\mathbf{E}_{(n)} = \mathbf{E}_{(1/2)} + \frac{1}{2}(2n - 1)\mathbf{E}^2_{(1/2)} + \frac{1}{3}(n - 1)(2n - 1)\mathbf{E}^3_{(1/2)} + \mathcal{O}\left(\mathbf{E}^4_{(1/2)}\right). \tag{2.3.10}$$

From this we can deduce the following useful connections

$$\mathbf{E}_{(0)} = \mathbf{E}_{(n)} - n\mathbf{E}^2_{(n)} + \frac{4}{3}n^2\mathbf{E}^3_{(n)} + \mathcal{O}\left(\mathbf{E}^4_{(n)}\right), \tag{2.3.11}$$

$$\mathbf{E}_{(n)} = \mathbf{E}_{(0)} + n\mathbf{E}^2_{(0)} + \frac{2}{3}n^2\mathbf{E}^3_{(0)} + \mathcal{O}\left(\mathbf{E}^4_{(0)}\right). \tag{2.3.12}$$

For the later purposes it is also noted that

$$\mathbf{E}^2_{(0)} = \mathbf{E}^2_{(n)} + \mathcal{O}\left(\mathbf{E}^3_{(n)}\right). \tag{2.3.13}$$

2.3.2. Spatial Strain Tensors

A family of spatial strain measures, reckoned relative to the deformed configuration and corresponding to material strain measures of Eqs. (2.3.1) and (2.3.6), is defined by

$$\boldsymbol{\mathcal{E}}_{(n)} = \frac{1}{2n}\left(\mathbf{V}^{2n} - \mathbf{I}\right) = \sum_{i=1}^{3} \frac{1}{2n}\left(\lambda_i^{2n} - 1\right) \mathbf{n}_i \otimes \mathbf{n}_i, \tag{2.3.14}$$

$$\boldsymbol{\mathcal{E}}_{(0)} = \ln \mathbf{V} = \sum_{i=1}^{3} \ln \lambda_i \, \mathbf{n}_i \otimes \mathbf{n}_i. \tag{2.3.15}$$

The unit tensor in the deformed configuration is $\mathbf{I}$, and $\mathbf{n}_i$ are the principal directions of the left stretch tensor $\mathbf{V}$. For example,

$$\boldsymbol{\mathcal{E}}_{(1)} = \frac{1}{2}(\mathbf{V}^2 - \mathbf{I}), \tag{2.3.16}$$

and

$$\boldsymbol{\mathcal{E}}_{(-1)} = \frac{1}{2}(\mathbf{I} - \mathbf{V}^{-2}), \tag{2.3.17}$$

the latter being known as the Eulerian strain tensor. Since

$$\mathbf{U}^{2n} = \mathbf{R}^T \cdot \mathbf{V}^{2n} \cdot \mathbf{R}, \tag{2.3.18}$$

and $\mathbf{n}_i = \mathbf{R} \cdot \mathbf{N}_i$, the material and spatial strain measures are related by

$$\mathbf{E}_{(n)} = \mathbf{R}^T \cdot \boldsymbol{\mathcal{E}}_{(n)} \cdot \mathbf{R}, \quad \mathbf{E}_{(0)} = \mathbf{R}^T \cdot \boldsymbol{\mathcal{E}}_{(0)} \cdot \mathbf{R}, \tag{2.3.19}$$

i.e., the former are induced from the latter by the rotation $\mathbf{R}$. Also, for any integer m, $\mathbf{E}_{(n)}^m$ is induced from $\boldsymbol{\mathcal{E}}_{(n)}^m$ by the rotation $\mathbf{R}$.

If $\mathrm{d}\mathbf{X}$ and $\delta\mathbf{X}$ are two material line elements in the undeformed configuration, and $\mathrm{d}\mathbf{x}$ and $\delta\mathbf{x}$ are the corresponding elements in the deformed configuration, it follows that

$$\mathrm{d}\mathbf{x} \cdot \delta\mathbf{x} - \mathrm{d}\mathbf{X} \cdot \delta\mathbf{X} = 2\,\mathrm{d}\mathbf{X} \cdot \mathbf{E}_{(1)} \cdot \delta\mathbf{X} = 2\,\mathrm{d}\mathbf{x} \cdot \boldsymbol{\mathcal{E}}_{(-1)} \cdot \delta\mathbf{x}. \tag{2.3.20}$$

Evidently, the Lagrangian and Eulerian strains are related by

$$\mathbf{E}_{(1)} = \mathbf{F}^T \cdot \boldsymbol{\mathcal{E}}_{(-1)} \cdot \mathbf{F}, \tag{2.3.21}$$

so that $\mathbf{E}_{(1)}$ is one of the induced tensors from $\boldsymbol{\mathcal{E}}_{(-1)}$ by the deformation $\mathbf{F}$ (Section 1.8). In the component form, the material and spatial strain tensors can be expressed as

$$\mathbf{E}_{(n)} = E_{(n)}^{IJ} \mathbf{e}_I^0 \otimes \mathbf{e}_J^0, \quad \boldsymbol{\mathcal{E}}_{(n)} = \mathcal{E}_{(n)}^{ij} \mathbf{e}_i \otimes \mathbf{e}_j, \tag{2.3.22}$$

relative to primary bases in the undeformed and deformed configuration, respectively. Covariant and two mixed representations are similarly written.

2.3.3. Infinitesimal Strain and Rotation Tensors

Introducing the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$ such that

$$\mathbf{x} = \mathbf{X} + \mathbf{u}, \tag{2.3.23}$$

the deformation gradient can be written as

$$\mathbf{F} = \mathbf{x} \otimes \boldsymbol{\nabla}^0 = \mathbf{I} + \mathbf{u} \otimes \boldsymbol{\nabla}^0. \tag{2.3.24}$$

The tensor $\mathbf{u} \otimes \boldsymbol{\nabla}^0$ is called the displacement gradient tensor. The right Cauchy–Green deformation tensor is expressed in terms of the displacement gradient tensor as

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T \cdot \mathbf{F} = \mathbf{I} + \mathbf{u} \otimes \boldsymbol{\nabla}^0 + \boldsymbol{\nabla}^0 \otimes \mathbf{u} + (\boldsymbol{\nabla}^0 \otimes \mathbf{u}) \cdot (\mathbf{u} \otimes \boldsymbol{\nabla}^0). \tag{2.3.25}$$

If each component of the displacement gradient tensor is small compared with unity, Eq. (2.3.25) becomes

$$\mathbf{U}^2 \approx \mathbf{I} + \mathbf{u} \otimes \boldsymbol{\nabla}^0 + \boldsymbol{\nabla}^0 \otimes \mathbf{u}, \tag{2.3.26}$$

upon neglecting quadratic terms in the displacement gradient. Consequently,

$$\mathbf{U} \approx \mathbf{I} + \boldsymbol{\varepsilon}, \quad \mathbf{U}^{2n} \approx \mathbf{I} + 2n\boldsymbol{\varepsilon}, \tag{2.3.27}$$

where

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{u} \otimes \boldsymbol{\nabla}^0 + \boldsymbol{\nabla}^0 \otimes \mathbf{u}\right). \tag{2.3.28}$$

The material strain tensors are, therefore,

$$\mathbf{E}_{(n)} = \frac{1}{2n}\left(\mathbf{U}^{2n} - \mathbf{I}\right) \approx \boldsymbol{\varepsilon}, \quad \mathbf{E}_{(0)} = \ln \mathbf{U} \approx \boldsymbol{\varepsilon}, \tag{2.3.29}$$

all being approximately equal to $\boldsymbol{\varepsilon}$. The tensor $\boldsymbol{\varepsilon}$ defined by (2.3.28) is called the infinitesimal strain tensor. This tensor can also be expressed as (Hunter, 1976)

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right) - \mathbf{I}. \tag{2.3.30}$$

If the displacement gradient is decomposed into its symmetric and antisymmetric parts,

$$\mathbf{u} \otimes \boldsymbol{\nabla}^0 = \boldsymbol{\varepsilon} + \boldsymbol{\omega}, \tag{2.3.31}$$

we have

$$\boldsymbol{\omega} = \frac{1}{2}\left(\mathbf{u} \otimes \boldsymbol{\nabla}^0 - \boldsymbol{\nabla}^0 \otimes \mathbf{u}\right) = \frac{1}{2}\left(\mathbf{F} - \mathbf{F}^T\right). \tag{2.3.32}$$

The tensor $\boldsymbol{\omega}$ is the infinitesimal rotation tensor. Its corresponding axial vector is $(1/2)(\boldsymbol{\nabla}^0 \times \mathbf{u})$. When the deformation gradient is decomposed by polar decomposition as $\mathbf{F} = \mathbf{V} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{U}$, it follows that

$$\mathbf{V} \approx \mathbf{U} \approx \mathbf{I} + \boldsymbol{\varepsilon}, \quad \mathbf{R} \approx \mathbf{I} + \boldsymbol{\omega}, \tag{2.3.33}$$

again neglecting quadratic terms in the displacement gradient. Note also that, within the same order of approximation,

$$\det \mathbf{F} \approx 1 + \operatorname{tr} \boldsymbol{\varepsilon}. \tag{2.3.34}$$

If an infinitesimal strain tensor is defined by

$$\hat{\boldsymbol{\varepsilon}} = \frac{1}{2} \left(\mathbf{u} \otimes \boldsymbol{\nabla} + \boldsymbol{\nabla} \otimes \mathbf{u} \right), \tag{2.3.35}$$

then

$$\hat{\boldsymbol{\varepsilon}} = \mathbf{I} - \frac{1}{2} \left(\mathbf{F}^{-1} + \mathbf{F}^{-T} \right). \tag{2.3.36}$$

Since,

$$\mathbf{F}^{-1} = [\mathbf{I} + (\mathbf{F} - \mathbf{I})]^{-1} = \mathbf{I} - (\mathbf{F} - \mathbf{I}) + (\mathbf{F} - \mathbf{I})^2 - \cdots, \tag{2.3.37}$$

it follows that $\hat{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon}$, provided that quadratic and higher-order terms in $(\mathbf{F} - \mathbf{I})$ are neglected. Indeed, in infinitesimal deformation (displacement gradient) theory, no distinction is made between the Lagrangian and Eulerian coordinates. For further details, the texts by Jaunzemis (1967), Spencer (1971), and Chung (1996) can be reviewed.

2.4. Velocity Gradient, Velocity Strain, and Spin Tensors

Consider a material line element $d\mathbf{x}$ in the deformed configuration at time t. If the velocity field is

$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t), \tag{2.4.1}$$

the velocities of the end points of $d\mathbf{x}$ differ by

$$\mathrm{d}\mathbf{v} = (\mathbf{v} \otimes \boldsymbol{\nabla}) \cdot \mathrm{d}\mathbf{x} = \mathbf{L} \cdot \mathrm{d}\mathbf{x}, \tag{2.4.2}$$

where $\boldsymbol{\nabla}$ represents the gradient operator with respect to spatial coordinates (Fig. 2.5). The tensor

$$\mathbf{L} = \mathbf{v} \otimes \boldsymbol{\nabla} \tag{2.4.3}$$

is called the velocity gradient. Its rectangular Cartesian components are

$$L_{ij} = \frac{\partial v_i}{\partial x_j}. \tag{2.4.4}$$

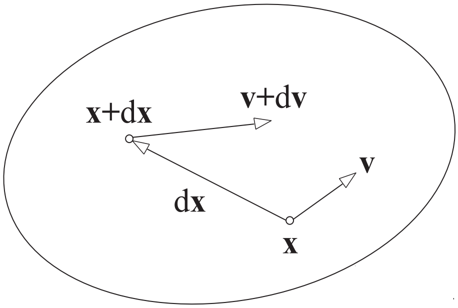

FIGURE 2.5. The velocity vectors of two nearby material points in deformed configuration at time t. The velocity gradient $\mathbf{L}$ is defined such that $\mathrm{d}\mathbf{v} = \mathbf{L} \cdot \mathrm{d}\mathbf{x}$.

The gradient operators with respect to material and spatial coordinates are related by

$$\overleftarrow{\boldsymbol{\nabla}} = \overleftarrow{\boldsymbol{\nabla}}^0 \cdot \mathbf{F}^{-1}, \quad \overrightarrow{\boldsymbol{\nabla}} = \mathbf{F}^{-T} \cdot \overrightarrow{\boldsymbol{\nabla}}^0. \tag{2.4.5}$$

For clarity, the arrows above the nabla operators are attached to indicate the direction in which the operators apply. Since from Eq. (2.2.1), the rate of deformation gradient is

$$\dot{\mathbf{F}} = \mathbf{v} \otimes \overleftarrow{\boldsymbol{\nabla}}^0, \tag{2.4.6}$$

the substitution into Eq. (2.4.3) gives the relationship

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1}. \tag{2.4.7}$$

The symmetric and antisymmetric parts of $\mathbf{L}$ are the velocity strain or rate of deformation tensor, and the spin tensor, i.e.,

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right), \quad \mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right). \tag{2.4.8}$$

For example, the rate of change of the length $\mathrm{d}s$ of the material element $\mathrm{d}\mathbf{x}$ can be calculated from

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}s)^2 = 2\,\mathrm{d}\mathbf{x} \cdot \mathbf{D} \cdot \mathrm{d}\mathbf{x}, \quad \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}s) = (\mathbf{m} \cdot \mathbf{D} \cdot \mathbf{m})\,\mathrm{d}s, \tag{2.4.9}$$

where $\mathbf{m} = \mathrm{d}\mathbf{x}/\mathrm{d}s$. By differentiating $\mathrm{d}\mathbf{x}/\mathrm{d}s$ it also follows that the rate of unit vector $\mathbf{m}$ along the material direction $\mathrm{d}\mathbf{x}$ is

$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} = \mathbf{L} \cdot \mathbf{m} - (\mathbf{m} \cdot \mathbf{D} \cdot \mathbf{m})\mathbf{m}. \tag{2.4.10}$$

If $\mathbf{m}$ is an eigenvector of $\mathbf{D}$, then

$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} = \mathbf{W} \cdot \mathbf{m}. \tag{2.4.11}$$

Thus, we can interpret $\mathbf{W}$ as the spin of the triad of line elements directed, at the considered instant of deformation, along the principal axes of the rate of deformation $\mathbf{D}$.

The rate of the inverse $\mathbf{F}^{-1}$ and the rate of the Jacobian determinant are

$$\left(\mathbf{F}^{-1}\right)^{\cdot} = -\mathbf{F}^{-1}\cdot\dot{\mathbf{F}}\cdot\mathbf{F}^{-1}, \quad \frac{\mathrm{d}}{\mathrm{d}t}(\det\mathbf{F}) = (\det\mathbf{F})\,\mathrm{tr}\,\mathbf{D}. \tag{2.4.12}$$

The first expression follows by differentiating $\mathbf{F}\cdot\mathbf{F}^{-1} = \mathbf{I}$, and the second from

$$\frac{\mathrm{d}}{\mathrm{d}t}(\det\mathbf{F}) = \mathrm{tr}\left[\frac{\partial(\det\mathbf{F})}{\partial\mathbf{F}}\cdot\dot{\mathbf{F}}\right] = \mathrm{tr}\left[(\det\mathbf{F})\,\mathbf{F}^{-1}\cdot\dot{\mathbf{F}}\right] = (\det\mathbf{F})\,\mathrm{tr}\,\mathbf{D}, \tag{2.4.13}$$

because $\mathrm{tr}\,\mathbf{W} = 0$. Furthermore, since $\mathrm{d}V = (\det\mathbf{F})\mathrm{d}V^0$, the rate of volume change is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}V) = (\mathrm{tr}\,\mathbf{D})\mathrm{d}V. \tag{2.4.14}$$

By differentiating Nanson's relation (2.2.17), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\mathbf{S}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}S\mathbf{n}) = \left[(\mathrm{tr}\,\mathbf{D})\mathbf{n} - (\mathbf{n}\cdot\mathbf{L})\right]\mathrm{d}S. \tag{2.4.15}$$

Since $\dot{\mathbf{n}}\cdot\mathbf{n} = 0$, $\mathbf{n}$ being the unit vector normal to $\mathrm{d}S$, and having in mind that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\mathbf{S}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}S\mathbf{n}) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}S)\,\mathbf{n} + \mathrm{d}S\,\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{n}), \tag{2.4.16}$$

there follows

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}S) = (\mathrm{tr}\,\mathbf{D} - \mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n})\,\mathrm{d}S. \tag{2.4.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{n}) = (\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n})\cdot\mathbf{n} - \mathbf{n}\cdot\mathbf{L}. \tag{2.4.18}$$

In the case of simple shearing deformation considered in Subsection 2.2.2, the velocity gradient can be written as

$$\mathbf{L} = \dot{\gamma}(\mathbf{m}\otimes\mathbf{n}). \tag{2.4.19}$$

2.5. Convected Derivatives

Consider the primary and reciprocal bases in the undeformed configuration, $\mathbf{e}_I^0$ and $\mathbf{e}_0^I$. If the primary basis is embedded in the material, its base vectors in the deformed configuration become $\mathbf{e}_i = \mathbf{F}\cdot\mathbf{e}_I^0$. The associated reciprocal (non-embedded) base vectors are $\mathbf{e}^i = \mathbf{e}_0^I \cdot \mathbf{F}^{-1}$. Thus, by differentiation it follows that

$$\dot{\mathbf{e}}_i = \mathbf{L} \cdot \mathbf{e}_i, \quad \dot{\mathbf{e}}^i = -\mathbf{L}^T \cdot \mathbf{e}^i. \tag{2.5.1}$$

In view of Eq. (1.7.8), the velocity gradient can be expressed as

$$\mathbf{L} = \dot{\mathbf{e}}_i \otimes \mathbf{e}^i. \tag{2.5.2}$$

The rate of change of an arbitrary vector in the deformed configuration, $\mathbf{a} = a^i \mathbf{e}_i = a_i \mathbf{e}^i$, is

$$\dot{\mathbf{a}} = \dot{a}^i \mathbf{e}_i + \mathbf{L} \cdot \mathbf{a} = \dot{a}_i \mathbf{e}^i - \mathbf{L}^T \cdot \mathbf{a}. \tag{2.5.3}$$

The two derivatives,

$$\overset{\triangle}{\mathbf{a}} = \dot{a}^i \mathbf{e}_i = \dot{\mathbf{a}} - \mathbf{L} \cdot \mathbf{a}, \quad \overset{\triangledown}{\mathbf{a}} = \dot{a}_i \mathbf{e}^i = \dot{\mathbf{a}} + \mathbf{L}^T \cdot \mathbf{a}, \tag{2.5.4}$$

are the two convected-type derivatives of the vector $\mathbf{a}$. The first gives the rate of change observed in the embedded basis $\mathbf{e}_i$, which is convected with the deforming material. The second is the rate of change observed in the basis $\mathbf{e}^i$, reciprocal to the embedded basis $\mathbf{e}_i$.

The corotational or Jaumann derivative of $\mathbf{a}$ is

$$\overset{\circ}{\mathbf{a}} = \dot{\mathbf{a}} - \mathbf{W} \cdot \mathbf{a}, \tag{2.5.5}$$

which represents the rate of change observed in the basis that momentarily rotates with the material spin $\mathbf{W}$. Two types of convected, and the Jaumann derivative of a two-point deformation gradient tensor are likewise

$$\overset{\triangle}{\mathbf{F}} = \dot{\mathbf{F}} - \mathbf{L} \cdot \mathbf{F} = \mathbf{0}, \quad \overset{\triangledown}{\mathbf{F}} = \dot{\mathbf{F}} + \mathbf{L}^T \cdot \mathbf{F}, \quad \overset{\circ}{\mathbf{F}} = \dot{\mathbf{F}} - \mathbf{W} \cdot \mathbf{F}. \tag{2.5.6}$$

Therefore,

$$\overset{\triangle}{\mathbf{F}} \cdot \mathbf{F}^{-1} = \mathbf{0}, \quad \overset{\triangledown}{\mathbf{F}} \cdot \mathbf{F}^{-1} = 2\mathbf{D}, \quad \overset{\circ}{\mathbf{F}} \cdot \mathbf{F}^{-1} = \mathbf{D}. \tag{2.5.7}$$

Four kinds of convected derivatives of a second-order tensor $\mathbf{A}$ in the deformed configuration can be similarly introduced. They are given by the following formulas

$$\overset{\triangle}{\mathbf{A}} = \dot{A}^{ij} \mathbf{e}_i \otimes \mathbf{e}_j = \dot{\mathbf{A}} - \mathbf{L} \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{L}^T, \tag{2.5.8}$$

$$\overset{\triangledown}{\mathbf{A}} = \dot{A}_{ij} \mathbf{e}^i \otimes \mathbf{e}^j = \dot{\mathbf{A}} + \mathbf{L}^T \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{L}, \tag{2.5.9}$$

$$\overset{\triangleleft}{\mathbf{A}} = \dot{A}^i{}_j \mathbf{e}_i \otimes \mathbf{e}^j = \dot{\mathbf{A}} - \mathbf{L} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{L}, \tag{2.5.10}$$

$$\overset{\triangleright}{\mathbf{A}} = \dot{A}_i{}^j \mathbf{e}^i \otimes \mathbf{e}_j = \dot{\mathbf{A}} + \mathbf{L}^T \cdot \mathbf{A} - \mathbf{A} \cdot \mathbf{L}^T. \tag{2.5.11}$$

The rate $\overset{\triangle}{\mathbf{A}}$ is often referred to as the Oldroyd, and $\overset{\triangledown}{\mathbf{A}}$ as the Cotter–Rivlin convected rate. Additional discussion can be found in Prager (1961), Truesdell and Noll (1965), Sedov (1966), and Hill (1978). Convected derivatives of the second-order tensors can also be interpreted as the Lie derivatives (Marsden and Hughes, 1983). Note that convected derivatives of the unit tensor in the deformed configuration are

$$\overset{\triangledown}{\mathbf{I}} = -\overset{\triangle}{\mathbf{I}} = 2\mathbf{D}, \quad \overset{\triangleleft}{\mathbf{I}} = \overset{\triangleright}{\mathbf{I}} = \mathbf{0}. \tag{2.5.12}$$

The Jaumann (or Jaumann–Zaremba) derivative of a second-order tensor $\mathbf{A}$ is

$$\overset{\circ}{\mathbf{A}} = \dot{\mathbf{A}} - \mathbf{W}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{W}. \tag{2.5.13}$$

The relationships hold

$$\overset{\circ}{\mathbf{A}} = \frac{1}{2}\left(\overset{\triangle}{\mathbf{A}} + \overset{\triangledown}{\mathbf{A}}\right) = \frac{1}{2}\left(\overset{\triangleleft}{\mathbf{A}} + \overset{\triangleright}{\mathbf{A}}\right). \tag{2.5.14}$$

It is easily verified that

$$\left(\mathbf{F}^{-1}\right)^{\triangle} = -\mathbf{F}^{-1}\cdot\overset{\triangledown}{\mathbf{F}}\cdot\mathbf{F}^{-1} = -2\mathbf{F}^{-1}\cdot\mathbf{D}, \quad \left(\mathbf{F}^{-1}\right)^{\triangledown} = -\mathbf{F}^{-1}\cdot\overset{\triangle}{\mathbf{F}}\cdot\mathbf{F}^{-1} = \mathbf{0}. \tag{2.5.15}$$

Convected derivatives of the higher-order tensors can be introduced analogously.

2.5.1. Convected Derivatives of Tensor Products

Let $\mathbf{F}$ be a two-point tensor such that

$$\mathbf{F} = F^{iJ}\mathbf{e}_i \otimes \mathbf{e}^0_J, \tag{2.5.16}$$

and similarly for the other three decompositions. Its convected and corotational derivatives are

$$\overset{\triangle}{\mathbf{F}} = \overset{\triangleleft}{\mathbf{F}} = \dot{\mathbf{F}} - \mathbf{L}\cdot\mathbf{F}, \quad \overset{\triangledown}{\mathbf{F}} = \overset{\triangleright}{\mathbf{F}} = \dot{\mathbf{F}} + \mathbf{L}^T\cdot\mathbf{F}, \quad \overset{\circ}{\mathbf{F}} = \dot{\mathbf{F}} - \mathbf{W}\cdot\mathbf{F}. \tag{2.5.17}$$

Introduce a two-point tensor $\mathbf{G}$ such that

$$\mathbf{G} = G^{Ji}\mathbf{e}^0_J \otimes \mathbf{e}_i, \tag{2.5.18}$$

and similarly for the other three decompositions. Its convected and corotational derivatives are

$$\overset{\triangle}{\mathbf{G}} = \overset{\triangleright}{\mathbf{G}} = \dot{\mathbf{G}} - \mathbf{G}\cdot\mathbf{L}^T, \quad \overset{\triangledown}{\mathbf{G}} = \overset{\triangleleft}{\mathbf{G}} = \dot{\mathbf{G}} + \mathbf{G}\cdot\mathbf{L}, \quad \overset{\circ}{\mathbf{G}} = \dot{\mathbf{G}} + \mathbf{G}\cdot\mathbf{W}. \tag{2.5.19}$$

The tensor $\mathbf{B} = \mathbf{F} \cdot \mathbf{G}$ is a spatial tensor, whose convected derivatives are defined by Eqs. (2.5.8)–(2.5.11). The following connections hold

$$\overset{\triangle}{\mathbf{B}} = \overset{\triangle}{\mathbf{F}} \cdot \mathbf{G} + \mathbf{F} \cdot \overset{\triangle}{\mathbf{G}}, \quad \overset{\triangledown}{\mathbf{B}} = \overset{\triangledown}{\mathbf{F}} \cdot \mathbf{G} + \mathbf{F} \cdot \overset{\triangledown}{\mathbf{G}}, \quad \overset{\circ}{\mathbf{B}} = \overset{\circ}{\mathbf{F}} \cdot \mathbf{G} + \mathbf{F} \cdot \overset{\circ}{\mathbf{G}}. \tag{2.5.20}$$

The same type of chain rule applies to $\overset{\triangleleft}{\mathbf{B}}$ and $\overset{\triangleright}{\mathbf{B}}$. Two additional identities exist, which are

$$\overset{\triangleleft}{\mathbf{B}} = \overset{\triangle}{\mathbf{F}} \cdot \mathbf{G} + \mathbf{F} \cdot \overset{\triangledown}{\mathbf{G}}, \quad \overset{\triangleright}{\mathbf{B}} = \overset{\triangledown}{\mathbf{F}} \cdot \mathbf{G} + \mathbf{F} \cdot \overset{\triangle}{\mathbf{G}}. \tag{2.5.21}$$

On the other hand, the tensor $\mathbf{C} = \mathbf{G} \cdot \mathbf{F}$ is a material tensor, unaffected by convected operations in the deformed configuration, so that

$$\overset{\triangle}{\mathbf{C}} = \overset{\triangledown}{\mathbf{C}} = \overset{\circ}{\mathbf{C}} = \dot{\mathbf{C}}. \tag{2.5.22}$$

The following identities are easily verified

$$\dot{\mathbf{C}} = \overset{\triangleleft}{\mathbf{G}} \cdot \mathbf{F} + \mathbf{G} \cdot \overset{\triangleleft}{\mathbf{F}} = \overset{\triangleright}{\mathbf{G}} \cdot \mathbf{F} + \mathbf{G} \cdot \overset{\triangleright}{\mathbf{F}} = \overset{\circ}{\mathbf{G}} \cdot \mathbf{F} + \mathbf{G} \cdot \overset{\circ}{\mathbf{F}}. \tag{2.5.23}$$

Furthermore,

$$\dot{\mathbf{C}} = \overset{\triangle}{\mathbf{G}} \cdot \mathbf{F} + \mathbf{G} \cdot \overset{\triangle}{\mathbf{F}} + 2\mathbf{G} \cdot \mathbf{D} \cdot \mathbf{F} = \overset{\triangledown}{\mathbf{G}} \cdot \mathbf{F} + \mathbf{G} \cdot \overset{\triangledown}{\mathbf{F}} - 2\mathbf{G} \cdot \mathbf{D} \cdot \mathbf{F}, \tag{2.5.24}$$

and

$$\dot{\mathbf{C}} = \overset{\triangle}{\mathbf{G}} \cdot \mathbf{F} + \mathbf{G} \cdot \overset{\triangledown}{\mathbf{F}} = \overset{\triangledown}{\mathbf{G}} \cdot \mathbf{F} + \mathbf{G} \cdot \overset{\triangle}{\mathbf{F}}. \tag{2.5.25}$$

If both $\mathbf{A}$ and $\mathbf{B}$ are spatial tensors, then $\mathbf{K} = \mathbf{A} \cdot \mathbf{B}$ is as well. Its convected derivatives are defined by Eqs. (2.5.8)–(2.5.11). It can be shown that

$$\overset{\triangleleft}{\mathbf{K}} = \overset{\triangleleft}{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \overset{\triangleleft}{\mathbf{B}}, \quad \overset{\triangleright}{\mathbf{K}} = \overset{\triangleright}{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \overset{\triangleright}{\mathbf{B}}, \tag{2.5.26}$$

$$\overset{\circ}{\mathbf{K}} = \overset{\circ}{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \overset{\circ}{\mathbf{B}}, \tag{2.5.27}$$

$$\overset{\triangle}{\mathbf{K}} = \overset{\triangle}{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \overset{\triangle}{\mathbf{B}} + 2\mathbf{A} \cdot \mathbf{D} \cdot \mathbf{B}, \tag{2.5.28}$$

$$\overset{\triangledown}{\mathbf{K}} = \overset{\triangledown}{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \overset{\triangledown}{\mathbf{B}} - 2\mathbf{A} \cdot \mathbf{D} \cdot \mathbf{B}, \tag{2.5.29}$$

$$\overset{\triangleleft}{\mathbf{K}} = \overset{\triangle}{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \overset{\triangledown}{\mathbf{B}}, \quad \overset{\triangleright}{\mathbf{K}} = \overset{\triangledown}{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \overset{\triangle}{\mathbf{B}}. \tag{2.5.30}$$

2.6. Rates of Strain

2.6.1. Rates of Material Strains

The rate of the Lagrangian strain is expressed in terms of the rate of deformation tensor as

$$\dot{\mathbf{E}}_{(1)} = \mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F} = \mathbf{U} \cdot \hat{\mathbf{D}} \cdot \mathbf{U}, \tag{2.6.1}$$

where

$$\hat{\mathbf{D}} = \mathbf{R}^T \cdot \mathbf{D} \cdot \mathbf{R}. \tag{2.6.2}$$

The rate of the Almansi strain is similarly

$$\dot{\mathbf{E}}_{(-1)} = \mathbf{F}^{-1} \cdot \mathbf{D} \cdot \mathbf{F}^{-T} = \mathbf{U}^{-1} \cdot \hat{\mathbf{D}} \cdot \mathbf{U}^{-1}. \tag{2.6.3}$$

Evidently, the two strain rates are related by

$$\dot{\mathbf{E}}_{(-1)} = \mathbf{U}^{-2} \cdot \dot{\mathbf{E}}_{(1)} \cdot \mathbf{U}^{-2}. \tag{2.6.4}$$

This is a particular case of the general relationship (Ogden, 1984)

$$\dot{\mathbf{E}}_{(-n)} = \mathbf{U}^{-2n} \cdot \dot{\mathbf{E}}_{(n)} \cdot \mathbf{U}^{-2n}, \quad n \neq 0, \tag{2.6.5}$$

which holds because

$$\left(\mathbf{U}^{-n}\right)^{\cdot} = -\mathbf{U}^{-n} \cdot \left(\mathbf{U}^{n}\right)^{\cdot} \cdot \mathbf{U}^{-n}. \tag{2.6.6}$$

An expression for the rate of the logarithmic strain can be derived as follows. From Eq. (2.3.11), we have

$$\dot{\mathbf{E}}_{(0)} = \dot{\mathbf{E}}_{(n)} - n\left(\mathbf{E}_{(n)} \cdot \dot{\mathbf{E}}_{(n)} + \dot{\mathbf{E}}_{(n)} \cdot \mathbf{E}_{(n)}\right) + \mathcal{O}\left(\mathbf{E}_{(n)}^2 \cdot \dot{\mathbf{E}}_{(n)}\right). \tag{2.6.7}$$

To evaluate $\dot{\mathbf{E}}_{(0)}$, any $\mathbf{E}_{(n)}$ can be used. For example, if $\mathbf{E}_{(1)}$ is used, from Eq. (2.6.1) we have

$$\dot{\mathbf{E}}_{(1)} = \hat{\mathbf{D}} + \mathbf{E}_{(1)} \cdot \hat{\mathbf{D}} + \hat{\mathbf{D}} \cdot \mathbf{E}_{(1)} + \mathcal{O}\left(\mathbf{E}_{(1)}^2 \cdot \hat{\mathbf{D}}\right). \tag{2.6.8}$$

Substitution of Eq. (2.6.8) into Eq. (2.6.7), therefore, gives

$$\dot{\mathbf{E}}_{(0)} = \hat{\mathbf{D}} + \mathcal{O}\left(\mathbf{E}_{(n)}^2 \cdot \hat{\mathbf{D}}\right). \tag{2.6.9}$$

Recall from Eqs. (2.3.11) and (2.3.12) that $\mathbf{E}_{(1)}^2 = \mathbf{E}_{(n)}^2$, neglecting cubic are higher-order terms in strain. If principal directions of $\mathbf{U}$ remain fixed ($\dot{\mathbf{N}}_i = \mathbf{0}$), we have

$$\dot{\mathbf{E}}_{(0)} = \hat{\mathbf{D}}, \tag{2.6.10}$$

exactly. Further analysis can be found in the papers by Fitzgerald (1980), Hoger (1986), and Dui, Ren, and Shen (1999).

2.6.2. Rates of Spatial Strains

The following relationships hold for convected rates of the strains $\boldsymbol{\mathcal{E}}_{(1)}$ and $\boldsymbol{\mathcal{E}}_{(-1)}$,

$$\overset{\triangle}{\boldsymbol{\mathcal{E}}}_{(1)} = \mathbf{D}, \quad \overset{\triangledown}{\boldsymbol{\mathcal{E}}}_{(1)} = \mathbf{D} + 2\left(\boldsymbol{\mathcal{E}}_{(1)} \cdot \mathbf{D} + \mathbf{D} \cdot \boldsymbol{\mathcal{E}}_{(1)}\right), \tag{2.6.11}$$

$$\overset{\triangledown}{\boldsymbol{\mathcal{E}}}_{(-1)} = \mathbf{D}, \quad \overset{\triangle}{\boldsymbol{\mathcal{E}}}_{(-1)} = \mathbf{D} - 2\left(\boldsymbol{\mathcal{E}}_{(-1)} \cdot \mathbf{D} + \mathbf{D} \cdot \boldsymbol{\mathcal{E}}_{(-1)}\right). \tag{2.6.12}$$

The rate of the deformation tensor $\mathbf{B} = \mathbf{F} \cdot \mathbf{F}^T$ is

$$\dot{\mathbf{B}} = \mathbf{L} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{L}^T, \tag{2.6.13}$$

so that

$$\overset{\triangle}{\mathbf{B}} = \mathbf{0}, \quad \overset{\triangledown}{\mathbf{B}} = 2(\mathbf{B} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{B}), \quad \overset{\triangleleft}{\mathbf{B}} = 2\mathbf{B} \cdot \mathbf{D}, \quad \overset{\triangleright}{\mathbf{B}} = 2\mathbf{D} \cdot \mathbf{B}, \tag{2.6.14}$$

and

$$\left(\mathbf{B}^{-1}\right)^{\triangle} = -\mathbf{B}^{-1} \cdot \overset{\triangledown}{\mathbf{B}} \cdot \mathbf{B}^{-1}, \quad \left(\mathbf{B}^{-1}\right)^{\triangledown} = -\mathbf{B}^{-1} \cdot \overset{\triangle}{\mathbf{B}} \cdot \mathbf{B}^{-1}, \tag{2.6.15}$$

$$\left(\mathbf{B}^{-1}\right)^{\triangleleft} = -\mathbf{B}^{-1} \cdot \overset{\triangleleft}{\mathbf{B}} \cdot \mathbf{B}^{-1}, \quad \left(\mathbf{B}^{-1}\right)^{\triangleright} = -\mathbf{B}^{-1} \cdot \overset{\triangleright}{\mathbf{B}} \cdot \mathbf{B}^{-1}. \tag{2.6.16}$$

Furthermore,

$$\overset{\circ}{\mathbf{B}} = \mathbf{B} \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{B}, \quad \overset{\bullet}{\mathbf{B}} = 2\mathbf{V} \cdot \mathbf{D} \cdot \mathbf{V}, \tag{2.6.17}$$

where

$$\overset{\circ}{\mathbf{B}} = \dot{\mathbf{B}} - \mathbf{W} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{W}, \quad \overset{\bullet}{\mathbf{B}} = \dot{\mathbf{B}} - \boldsymbol{\omega} \cdot \mathbf{B} + \mathbf{B} \cdot \boldsymbol{\omega}. \tag{2.6.18}$$

The corotational rate with respect to $\boldsymbol{\omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1}$ is sometimes referred to as the Green–Naghdi–McInnis corotational rate.

The expressions for the rates of other strain measures in terms of $\mathbf{D}$ are more involved. Since $\boldsymbol{\mathcal{E}}_{(n)} = \mathbf{R} \cdot \mathbf{E}_{(n)} \cdot \mathbf{R}^T$, there is a general connection

$$\overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} = \mathbf{R} \cdot \dot{\mathbf{E}}_{(n)} \cdot \mathbf{R}^T, \quad \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} = \dot{\boldsymbol{\mathcal{E}}}_{(n)} - \boldsymbol{\omega} \cdot \boldsymbol{\mathcal{E}}_{(n)} + \boldsymbol{\mathcal{E}}_{(n)} \cdot \boldsymbol{\omega}. \tag{2.6.19}$$

Higher rates of strain can be investigated along similar lines. For example, it can be shown that

$$\ddot{\mathbf{E}}_{(1)} = \mathbf{F}^T \cdot \overset{\triangledown}{\mathbf{D}} \cdot \mathbf{F}, \quad \overset{\triangledown}{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{L}^T \cdot \mathbf{D} + \mathbf{D} \cdot \mathbf{L}. \tag{2.6.20}$$

2.7. Relationship between Spins $\mathbf{W}$ and $\boldsymbol{\omega}$

The velocity gradient $\mathbf{L}$ can be written, in terms of the constituents of the polar decomposition of deformation gradient $\mathbf{F} = \mathbf{V} \cdot \mathbf{R}$, as

$$\mathbf{L} = \dot{\mathbf{V}} \cdot \mathbf{V}^{-1} + \mathbf{V} \cdot \boldsymbol{\omega} \cdot \mathbf{V}^{-1} = \boldsymbol{\omega} + \overset{\bullet}{\mathbf{V}} \cdot \mathbf{V}^{-1}, \tag{2.7.1}$$

where

$$\overset{\bullet}{\mathbf{V}} = \dot{\mathbf{V}} - \boldsymbol{\omega} \cdot \mathbf{V} + \mathbf{V} \cdot \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1}. \tag{2.7.2}$$

By taking symmetric and antisymmetric parts of Eq. (2.7.1), there follows

$$\mathbf{D} = \left(\overset{\bullet}{\mathbf{V}} \cdot \mathbf{V}^{-1}\right)_{\mathrm{s}}, \quad \mathbf{W} = \boldsymbol{\omega} + \left(\overset{\bullet}{\mathbf{V}} \cdot \mathbf{V}^{-1}\right)_{a}. \tag{2.7.3}$$

Similarly, if the decomposition $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$ is used, we obtain

$$\mathbf{L} = \boldsymbol{\omega} + \mathbf{R} \cdot \left(\dot{\mathbf{U}} \cdot \mathbf{U}^{-1}\right) \cdot \mathbf{R}^{T}. \tag{2.7.4}$$

This can be rewritten as

$$\hat{\mathbf{L}} = \hat{\boldsymbol{\omega}} + \dot{\mathbf{U}} \cdot \mathbf{U}^{-1}, \tag{2.7.5}$$

where

$$\hat{\mathbf{L}} = \mathbf{R}^{T} \cdot \mathbf{L} \cdot \mathbf{R}, \quad \hat{\boldsymbol{\omega}} = \mathbf{R}^{T} \cdot \boldsymbol{\omega} \cdot \mathbf{R} \tag{2.7.6}$$

are the tensors induced from $\mathbf{L}$ and $\boldsymbol{\omega}$ by the rotation $\mathbf{R}$. Upon taking symmetric and antisymmetric parts of Eq. (2.7.5),

$$\hat{\mathbf{D}} = \left(\dot{\mathbf{U}} \cdot \mathbf{U}^{-1}\right)_{\mathrm{s}}, \quad \hat{\mathbf{W}} = \hat{\boldsymbol{\omega}} + \left(\dot{\mathbf{U}} \cdot \mathbf{U}^{-1}\right)_{a}. \tag{2.7.7}$$

Since $\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^{T}$, we also have

$$\overset{\bullet}{\mathbf{V}} = \mathbf{R} \cdot \dot{\mathbf{U}} \cdot \mathbf{R}^{T}. \tag{2.7.8}$$

In particular, if $\dot{\mathbf{U}} = \mathbf{0}$, then $\overset{\bullet}{\mathbf{V}} = \mathbf{0}$ and

$$\dot{\mathbf{V}} = \boldsymbol{\omega} \cdot \mathbf{V} - \mathbf{V} \cdot \boldsymbol{\omega}, \quad \boldsymbol{\omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1}. \tag{2.7.9}$$

With these preliminaries, we now derive a relationship between $\mathbf{W}$ and $\boldsymbol{\omega}$ (or $\hat{\mathbf{W}}$ and $\hat{\boldsymbol{\omega}}$). First, observe the identity

$$\mathbf{V}^{-1} \cdot \left(\overset{\bullet}{\mathbf{V}} \cdot \mathbf{V}^{-1}\right) = \left(\overset{\bullet}{\mathbf{V}} \cdot \mathbf{V}^{-1}\right)^{T} \cdot \mathbf{V}^{-1}, \tag{2.7.10}$$

which can be rewritten as

$$\mathbf{V}^{-1} \cdot \left(\overset{\bullet}{\mathbf{V}} \cdot \mathbf{V}^{-1}\right)_{a} + \left(\overset{\bullet}{\mathbf{V}} \cdot \mathbf{V}^{-1}\right)_{a} \cdot \mathbf{V}^{-1} = \mathbf{D} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \mathbf{D}. \tag{2.7.11}$$

This can be solved for $\left(\overset{\bullet}{\mathbf{V}} \cdot \mathbf{V}^{-1}\right)_a$ by using the procedure described in Subsection 1.12.1. The result is

$$\begin{aligned}\left(\overset{\bullet}{\mathbf{V}} \cdot \mathbf{V}^{-1}\right)_a &= K_1 \left(\mathbf{D} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \mathbf{D}\right) \\ &- \left[\left(J_1 \mathbf{I} - \mathbf{V}^{-1}\right)^{-1} \cdot \left(\mathbf{D} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \mathbf{D}\right)\right. \\ &+ \left.\left(\mathbf{D} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \mathbf{D}\right) \cdot \left(J_1 \mathbf{I} - \mathbf{V}^{-1}\right)^{-1}\right],\end{aligned} \tag{2.7.12}$$

where

$$J_1 = \operatorname{tr} \mathbf{V}^{-1}, \quad K_1 = \operatorname{tr} \left(J_1 \mathbf{I} - \mathbf{V}^{-1}\right)^{-1}. \tag{2.7.13}$$

Substitution of Eq. (2.7.12) into the second of Eq. (2.7.3) gives

$$\begin{aligned}\boldsymbol{\omega} = \mathbf{W} &- K_1 \left(\mathbf{D} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \mathbf{D}\right) \\ &+ \left[\left(J_1 \mathbf{I} - \mathbf{V}^{-1}\right)^{-1} \cdot \left(\mathbf{D} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \mathbf{D}\right)\right. \\ &+ \left.\left(\mathbf{D} \cdot \mathbf{V}^{-1} - \mathbf{V}^{-1} \cdot \mathbf{D}\right) \cdot \left(J_1 \mathbf{I} - \mathbf{V}^{-1}\right)^{-1}\right],\end{aligned} \tag{2.7.14}$$

which shows that the spin $\boldsymbol{\omega}$ can be determined at each stage of deformation solely in terms of $\mathbf{V}$, $\mathbf{D}$, and $\mathbf{W}$.

Analogous derivation proceeds to find

$$\begin{aligned}\left(\dot{\mathbf{U}} \cdot \mathbf{U}^{-1}\right)_a &= K_1 \left(\hat{\mathbf{D}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1} \cdot \hat{\mathbf{D}}\right) \\ &- \left[\left(J_1 \mathbf{I} - \mathbf{U}^{-1}\right)^{-1} \cdot \left(\hat{\mathbf{D}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1} \cdot \hat{\mathbf{D}}\right)\right. \\ &+ \left.\left(\hat{\mathbf{D}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1} \cdot \hat{\mathbf{D}}\right) \cdot \left(J_1 \mathbf{I} - \mathbf{U}^{-1}\right)^{-1}\right].\end{aligned} \tag{2.7.15}$$

Substitution into second of Eq. (2.7.7) gives

$$\begin{aligned}\hat{\boldsymbol{\omega}} = \hat{\mathbf{W}} &- K_1 \left(\hat{\mathbf{D}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1} \cdot \hat{\mathbf{D}}\right) \\ &+ \left[\left(J_1 \mathbf{I} - \mathbf{U}^{-1}\right)^{-1} \cdot \left(\hat{\mathbf{D}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1} \cdot \hat{\mathbf{D}}\right)\right. \\ &+ \left.\left(\hat{\mathbf{D}} \cdot \mathbf{U}^{-1} - \mathbf{U}^{-1} \cdot \hat{\mathbf{D}}\right) \cdot \left(J_1 \mathbf{I} - \mathbf{U}^{-1}\right)^{-1}\right],\end{aligned} \tag{2.7.16}$$

as anticipated at the outset from its duality with Eq. (2.7.14). Additional kinematic analysis is provided by Mehrabadi and Nemat-Nasser (1987), and Reinhardt and Dubey (1996).

2.8. Rate of F in Terms of Principal Stretches

From Eq. (2.2.11) the right stretch tensor can be expressed in terms of its eigenvalues – principal stretches λ_i (assumed here to be different), and corresponding eigendirections $\mathbf{N}_i$ as

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \, \mathbf{N}_i \otimes \mathbf{N}_i. \tag{2.8.1}$$

The rate of $\mathbf{U}$ is then

$$\dot{\mathbf{U}} = \sum_{i=1}^{3} \left[\dot{\lambda}_i \, \mathbf{N}_i \otimes \mathbf{N}_i + \lambda_i \left(\dot{\mathbf{N}}_i \otimes \mathbf{N}_i + \mathbf{N}_i \otimes \dot{\mathbf{N}}_i \right) \right]. \tag{2.8.2}$$

If $\mathbf{e}_i^0$ $(i = 1, 2, 3)$ are the fixed reference unit vectors, the unit vectors $\mathbf{N}_i$ of the principal directions of $\mathbf{U}$ can be expressed as

$$\mathbf{N}_i = \boldsymbol{\mathcal{R}}_0 \cdot \mathbf{e}_i^0, \tag{2.8.3}$$

where $\boldsymbol{\mathcal{R}}_0$ is the rotation that carries the orthogonal triad $\{\mathbf{e}_i^0\}$ into the Lagrangian triad $\{\mathbf{N}_i\}$. Defining the spin of the Lagrangian triad by

$$\boldsymbol{\Omega}_0 = \dot{\boldsymbol{\mathcal{R}}}_0 \cdot \boldsymbol{\mathcal{R}}_0^{-1}, \tag{2.8.4}$$

it follows that

$$\dot{\mathbf{N}}_i = \dot{\boldsymbol{\mathcal{R}}}_0 \cdot \mathbf{e}_i^0 = \boldsymbol{\Omega}_0 \cdot \mathbf{N}_i = -\mathbf{N}_i \cdot \boldsymbol{\Omega}_0, \tag{2.8.5}$$

and the substitution into Eq. (2.8.2) gives

$$\dot{\mathbf{U}} = \sum_{i=1}^{3} \dot{\lambda}_i \, \mathbf{N}_i \otimes \mathbf{N}_i + \boldsymbol{\Omega}_0 \cdot \mathbf{U} - \mathbf{U} \cdot \boldsymbol{\Omega}_0. \tag{2.8.6}$$

If the spin tensor $\boldsymbol{\Omega}_0$ is expressed on the axes of the Lagrangian triad as

$$\boldsymbol{\Omega}_0 = \sum_{i \neq j} \Omega_{ij}^0 \, \mathbf{N}_i \otimes \mathbf{N}_j, \tag{2.8.7}$$

it is readily found that

$$\begin{aligned} \boldsymbol{\Omega}_0 \cdot \mathbf{U} = \Omega_{12}^0 (\lambda_2 - \lambda_1) \, \mathbf{N}_1 \otimes \mathbf{N}_2 + \Omega_{23}^0 (\lambda_3 - \lambda_2) \, \mathbf{N}_2 \otimes \mathbf{N}_3 \\ + \Omega_{31}^0 (\lambda_1 - \lambda_3) \, \mathbf{N}_3 \otimes \mathbf{N}_1. \end{aligned} \tag{2.8.8}$$

Consequently,

$$\boldsymbol{\Omega}_0 \cdot \mathbf{U} - \mathbf{U} \cdot \boldsymbol{\Omega}_0 = \boldsymbol{\Omega}_0 \cdot \mathbf{U} + (\boldsymbol{\Omega}_0 \cdot \mathbf{U})^T = \sum_{i \neq j} \Omega_{ij}^0 \, (\lambda_j - \lambda_i) \, \mathbf{N}_i \otimes \mathbf{N}_j. \tag{2.8.9}$$

The substitution into Eq. (2.8.6) yields

$$\dot{\mathbf{U}} = \sum_{i=1}^{3} \dot{\lambda}_i \, \mathbf{N}_i \otimes \mathbf{N}_i + \sum_{i \neq j} \Omega_{ij}^0 \, (\lambda_j - \lambda_i) \, \mathbf{N}_i \otimes \mathbf{N}_j. \tag{2.8.10}$$

Similarly, the rate of the material strain tensor of Eq. (2.3.1) is

$$\dot{\mathbf{E}}_{(n)} = \sum_{i=1}^{3} \lambda_i^{2n-1} \dot{\lambda}_i \, \mathbf{N}_i \otimes \mathbf{N}_i + \sum_{i \neq j} \Omega_{ij}^0 \, \frac{\lambda_j^{2n} - \lambda_i^{2n}}{2n} \, \mathbf{N}_i \otimes \mathbf{N}_j. \tag{2.8.11}$$

The principal directions of the left stretch tensor $\mathbf{V}$, appearing in the spectral representation

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \, \mathbf{n}_i \otimes \mathbf{n}_i, \tag{2.8.12}$$

are related to principal directions $\mathbf{N}_i$ of the right stretch tensor $\mathbf{U}$ by

$$\mathbf{n}_i = \mathbf{R} \cdot \mathbf{N}_i = \boldsymbol{\mathcal{R}} \cdot \mathbf{e}_i^0, \quad \boldsymbol{\mathcal{R}} = \mathbf{R} \cdot \boldsymbol{\mathcal{R}}_0. \tag{2.8.13}$$

The rotation tensor $\mathbf{R}$ is from the polar decomposition of the the deformation gradient $\mathbf{F} = \mathbf{V} \cdot \mathbf{R} = \mathbf{R} \cdot \mathbf{U}$. By differentiating Eq. (2.8.13) there follows

$$\dot{\mathbf{n}}_i = \boldsymbol{\Omega} \cdot \mathbf{n}_i, \tag{2.8.14}$$

where the spin of the Eulerian triad $\{\mathbf{n}_i\}$ is defined by

$$\boldsymbol{\Omega} = \dot{\boldsymbol{\mathcal{R}}} \cdot \boldsymbol{\mathcal{R}}^{-1} = \boldsymbol{\omega} + \mathbf{R} \cdot \boldsymbol{\Omega}_0 \cdot \mathbf{R}^T, \quad \boldsymbol{\omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1}. \tag{2.8.15}$$

On the axes $\mathbf{n}_i$, the spin $\boldsymbol{\Omega}$ can be decomposed as

$$\boldsymbol{\Omega} = \sum_{i \neq j} \Omega_{ij} \, \mathbf{n}_i \otimes \mathbf{n}_j. \tag{2.8.16}$$

By an analogous derivation as used to obtain the rate $\dot{\mathbf{U}}$ it follows that

$$\dot{\mathbf{V}} = \sum_{i=1}^{3} \dot{\lambda}_i \, \mathbf{n}_i \otimes \mathbf{n}_i + \sum_{i \neq j} \Omega_{ij} \left(\lambda_j - \lambda_i\right) \mathbf{n}_i \otimes \mathbf{n}_j. \tag{2.8.17}$$

The rate of the rotation tensor

$$\mathbf{R} = \sum_{i=1}^{3} \mathbf{n}_i \otimes \mathbf{N}_i \tag{2.8.18}$$

is

$$\dot{\mathbf{R}} = \sum_{i=1}^{3} \left(\dot{\mathbf{n}}_i \otimes \mathbf{N}_i + \mathbf{n}_i \otimes \dot{\mathbf{N}}_i \right) = \boldsymbol{\Omega} \cdot \mathbf{R} - \mathbf{R} \cdot \boldsymbol{\Omega}_0, \tag{2.8.19}$$

or

$$\dot{\mathbf{R}} = \sum_{i \neq j} \left(\Omega_{ij} - \Omega_{ij}^0 \right) \mathbf{n}_i \otimes \mathbf{N}_j. \tag{2.8.20}$$

Finally, the rate of the deformation gradient

$$\mathbf{F} = \sum_{i=1}^{3} \lambda_i \, \mathbf{n}_i \otimes \mathbf{N}_i \tag{2.8.21}$$

is

$$\dot{\mathbf{F}} = \sum_{i=1}^{3} \left[\dot{\lambda}_i \, \mathbf{n}_i \otimes \mathbf{N}_i + \lambda_i \left(\dot{\mathbf{n}}_i \otimes \mathbf{N}_i + \mathbf{n}_i \otimes \dot{\mathbf{N}}_i \right) \right]. \tag{2.8.22}$$

Since $\dot{\mathbf{n}}_i = \boldsymbol{\Omega} \cdot \mathbf{n}_i$ and $\dot{\mathbf{N}}_i = \boldsymbol{\Omega}_0 \cdot \mathbf{N}_i$, it follows that

$$\dot{\mathbf{F}} = \sum_{i=1}^{3} \dot{\lambda}_i \, \mathbf{n}_i \otimes \mathbf{N}_i + \boldsymbol{\Omega} \cdot \mathbf{F} - \mathbf{F} \cdot \boldsymbol{\Omega}_0, \tag{2.8.23}$$

and

$$\dot{\mathbf{F}} = \sum_{i=1}^{3} \dot{\lambda}_i \, \mathbf{n}_i \otimes \mathbf{N}_i + \sum_{i \neq j} \left(\lambda_j \Omega_{ij} - \lambda_i \Omega_{ij}^0 \right) \mathbf{n}_i \otimes \mathbf{N}_j. \tag{2.8.24}$$

2.8.1. Spins of Lagrangian and Eulerian Triads

The inverse of the deformation gradient can be written in terms of the principal stretches as

$$\mathbf{F}^{-1} = \sum_{i=1}^{3} \frac{1}{\lambda_i} \, \mathbf{N}_i \otimes \mathbf{n}_i. \tag{2.8.25}$$

Using this and Eq. (2.8.24) we obtain an expression for the velocity gradient

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \sum_{i=1}^{3} \frac{\dot{\lambda}_i}{\lambda_i} \, \mathbf{n}_i \otimes \mathbf{n}_i + \sum_{i \neq j} \left(\Omega_{ij} - \frac{\lambda_i}{\lambda_j} \, \Omega_{ij}^0 \right) \mathbf{n}_i \otimes \mathbf{n}_j. \tag{2.8.26}$$

The symmetric part of this is the rate of deformation tensor,

$$\mathbf{D} = \sum_{i=1}^{3} \frac{\dot{\lambda}_i}{\lambda_i} \, \mathbf{n}_i \otimes \mathbf{n}_i + \sum_{i \neq j} \frac{\lambda_j^2 - \lambda_i^2}{2\lambda_i \lambda_j} \, \Omega_{ij}^0 \, \mathbf{n}_i \otimes \mathbf{n}_j, \tag{2.8.27}$$

while the antisymmetric part is the spin tensor

$$\mathbf{W} = \sum_{i \neq j} \left(\Omega_{ij} - \frac{\lambda_i^2 + \lambda_j^2}{2\lambda_i \lambda_j} \, \Omega_{ij}^0 \right) \mathbf{n}_i \otimes \mathbf{n}_j. \tag{2.8.28}$$

Evidently, for $i \neq j$ from Eq. (2.8.27) we have

$$\Omega_{ij}^0 = \frac{2\lambda_i \lambda_j}{\lambda_j^2 - \lambda_i^2} \, D_{ij}, \quad \lambda_i \neq \lambda_j, \tag{2.8.29}$$

which is an expression for the components of the Lagrangian spin $\boldsymbol{\Omega}_0$ in terms of the stretch ratios and the components of the rate of deformation tensor. Substituting (2.8.29) into (2.8.28) we obtain an expression for the components of the Eulerian spin $\boldsymbol{\Omega}$ in terms of the stretch ratios and the components of the rate of deformation and spin tensors, i.e.,

$$\Omega_{ij} = W_{ij} + \frac{\lambda_i^2 + \lambda_j^2}{\lambda_j^2 - \lambda_i^2} \, D_{ij}, \quad \lambda_i \neq \lambda_j. \tag{2.8.30}$$

Lastly, we note that the inverse of the rotation tensor $\mathbf{R}$ is

$$\mathbf{R}^{-1} = \sum_{i=1}^{3} \mathbf{N}_i \otimes \mathbf{n}_i, \tag{2.8.31}$$

so that, by virtue of Eq. (2.8.20), the spin $\boldsymbol{\omega}$ can be expressed as

$$\boldsymbol{\omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1} = \sum_{i \neq j} \left(\Omega_{ij} - \Omega_{ij}^0\right) \mathbf{n}_i \otimes \mathbf{n}_j. \tag{2.8.32}$$

Thus,

$$\omega_{ij} = \Omega_{ij} - \Omega_{ij}^0, \tag{2.8.33}$$

where Ω_{ij}^0 are the components of $\boldsymbol{\Omega}_0$ on the Lagrangian triad $\{\mathbf{N}_i\}$, while Ω_{ij} are the components of $\boldsymbol{\Omega}$ on the Eulerian triad $\{\mathbf{n}_i\}$. When Eqs. (2.8.29) and (2.8.30) are substituted into Eq. (2.8.33), we obtain an expression for the spin components ω_{ij} in terms of the stretch ratios and the components of the rate of deformation and spin tensors, which is

$$\omega_{ij} = W_{ij} + \frac{\lambda_j - \lambda_i}{\lambda_i + \lambda_j} D_{ij}. \tag{2.8.34}$$

This complements the previously derived expression for the spin $\boldsymbol{\omega}$ in terms of $\mathbf{V}$, $\mathbf{D}$, and $\mathbf{W}$, given by Eq. (2.7.14). Further analysis can be found in Biot (1965) and Hill (1970, 1978).

2.9. Behavior under Superimposed Rotation

If a time-dependent rotation $\mathbf{Q}$ is superimposed to the deformed configuration at time t, an infinitesimal material line element $\mathrm{d}\mathbf{x}$ becomes (Fig. 2.6)

$$\mathrm{d}\mathbf{x}^* = \mathbf{Q} \cdot \mathrm{d}\mathbf{x}, \tag{2.9.1}$$

while in the undeformed configuration

$$\mathrm{d}\mathbf{X}^* = \mathrm{d}\mathbf{X}. \tag{2.9.2}$$

Consequently, since $\mathrm{d}\mathbf{x} = \mathbf{F} \cdot \mathrm{d}\mathbf{X}$, we have

$$\mathbf{F}^* = \mathbf{Q} \cdot \mathbf{F}. \tag{2.9.3}$$

This implies that

$$\mathbf{U}^* = \mathbf{U}, \quad \mathbf{C}^* = \mathbf{C}, \quad \mathbf{E}_{(n)}^* = \mathbf{E}_{(n)}, \tag{2.9.4}$$

and

$$\mathbf{V}^* = \mathbf{Q} \cdot \mathbf{V} \cdot \mathbf{Q}^T, \quad \mathbf{B}^* = \mathbf{Q} \cdot \mathbf{B} \cdot \mathbf{Q}^T, \quad \boldsymbol{\mathcal{E}}_{(n)}^* = \mathbf{Q} \cdot \boldsymbol{\mathcal{E}}_{(n)} \cdot \mathbf{Q}^T. \tag{2.9.5}$$

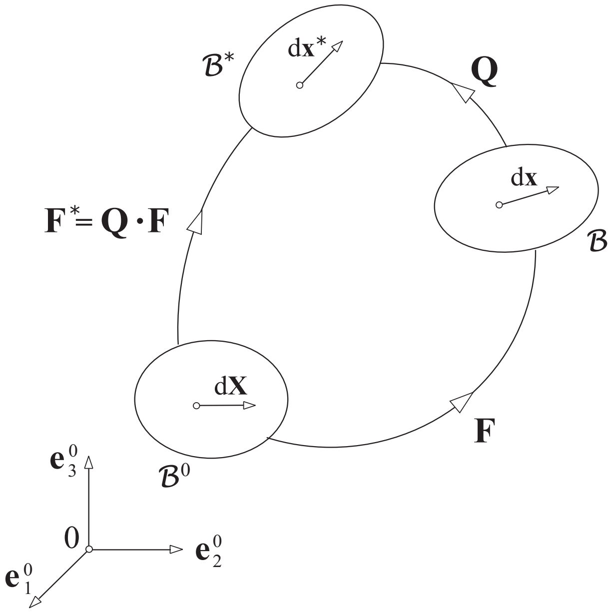

FIGURE 2.6. The material element $\mathrm{d}\mathbf{X}$ from the undeformed configuration $\mathcal{B}^0$ becomes $\mathrm{d}\mathbf{x} = \mathbf{F} \cdot \mathrm{d}\mathbf{X}$ in the deformed configuration $\mathcal{B}$, and $\mathrm{d}\mathbf{x}^* = \mathbf{Q} \cdot \mathrm{d}\mathbf{x}$ in the rotated deformed configuration $\mathcal{B}^*$.

The objective rates of the spatial vector $\mathbf{a}$ transform according to

$$\overset{\triangle}{\mathbf{a}}{}^* = \mathbf{Q} \cdot \overset{\triangle}{\mathbf{a}}, \quad \overset{\triangledown}{\mathbf{a}}{}^* = \mathbf{Q} \cdot \overset{\triangledown}{\mathbf{a}}, \quad \overset{\circ}{\mathbf{a}}{}^* = \mathbf{Q} \cdot \overset{\circ}{\mathbf{a}}, \tag{2.9.6}$$

as do the objective rates of the deformation gradient $\mathbf{F}$. The rotation $\mathbf{R}$ becomes

$$\mathbf{R}^* = \mathbf{Q} \cdot \mathbf{R}. \tag{2.9.7}$$

The spin $\boldsymbol{\omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{-1}$ changes to

$$\boldsymbol{\omega}^* = \boldsymbol{\Omega} + \mathbf{Q} \cdot \boldsymbol{\omega} \cdot \mathbf{Q}^T, \quad \boldsymbol{\Omega} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1}. \tag{2.9.8}$$

The velocity gradient transforms as

$$\mathbf{L}^* = \boldsymbol{\Omega} + \mathbf{Q} \cdot \mathbf{L} \cdot \mathbf{Q}^T, \tag{2.9.9}$$

while the velocity strain and the spin tensors become

$$\mathbf{D}^* = \mathbf{Q} \cdot \mathbf{D} \cdot \mathbf{Q}^T, \tag{2.9.10}$$

$$\mathbf{W}^* = \boldsymbol{\Omega} + \mathbf{Q} \cdot \mathbf{W} \cdot \mathbf{Q}^T. \tag{2.9.11}$$

The rates of the material and spatial strain tensors change according to

$$\dot{\mathbf{E}}^*_{(n)} = \dot{\mathbf{E}}_{(n)}, \tag{2.9.12}$$

$$\dot{\boldsymbol{\mathcal{E}}}^*_{(n)} = \mathbf{Q} \cdot \left(\dot{\boldsymbol{\mathcal{E}}}_{(n)} + \hat{\boldsymbol{\Omega}} \cdot \boldsymbol{\mathcal{E}}_{(n)} - \boldsymbol{\mathcal{E}}_{(n)} \cdot \hat{\boldsymbol{\Omega}} \right) \cdot \mathbf{Q}^T, \tag{2.9.13}$$

where

$$\hat{\boldsymbol{\Omega}} = \mathbf{Q}^T \cdot \boldsymbol{\Omega} \cdot \mathbf{Q}, \quad \boldsymbol{\Omega} = \dot{\mathbf{Q}} \cdot \mathbf{Q}^{-1}. \tag{2.9.14}$$

The transformation formulas for the convected rates of spatial strain tensors are

$$\overset{\triangle}{\boldsymbol{\mathcal{E}}}{}^*_{(1)} = \mathbf{Q} \cdot \overset{\triangle}{\boldsymbol{\mathcal{E}}}_{(1)} \cdot \mathbf{Q}^T, \quad \overset{\triangledown}{\boldsymbol{\mathcal{E}}}{}^*_{(-1)} = \mathbf{Q} \cdot \overset{\triangledown}{\boldsymbol{\mathcal{E}}}_{(-1)} \cdot \mathbf{Q}^T. \tag{2.9.15}$$

Since $\overset{\triangle}{\boldsymbol{\mathcal{E}}}_{(1)} = \overset{\triangledown}{\boldsymbol{\mathcal{E}}}_{(-1)}$ by Eqs. (2.6.11) and (2.6.12), it follows that

$$\overset{\triangle}{\boldsymbol{\mathcal{E}}}{}^*_{(1)} = \overset{\triangledown}{\boldsymbol{\mathcal{E}}}{}^*_{(-1)}, \tag{2.9.16}$$

as expected. The same transformation, as in Eq. (2.9.15), applies to other objective rates of spatial tensors, such as $\overset{\triangledown}{\boldsymbol{\mathcal{E}}}_{(1)}$ and $\overset{\triangle}{\boldsymbol{\mathcal{E}}}_{(-1)}$, or $\overset{\circ}{\mathbf{B}}$ and $\overset{\bullet}{\mathbf{B}}$. Furthermore,

$$\overset{\bullet}{\boldsymbol{\mathcal{E}}}{}^*_{(n)} = \mathbf{Q} \cdot \overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)} \cdot \mathbf{Q}^T, \tag{2.9.17}$$

where $\overset{\bullet}{\boldsymbol{\mathcal{E}}}_{(n)}$ is defined in Eq. (2.6.19).

In summary, while objective material tensors remain unchanged by the rotation of the deformed configuration, e.g., Eqs. (2.9.4) and (2.9.12), the objective spatial tensors change according to transformation rules specified by equations such as (2.9.5) and (2.9.10).

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