Part 1

ELEMENTS OF CONTINUUM MECHANICS

CHAPTER 1

TENSOR PRELIMINARIES

1.1. Vectors

An orthonormal basis for the three-dimensional Euclidean vector space is a set of three orthogonal unit vectors. The scalar product of any two of these vectors is

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{1.1.1}$$

δ_{ij} being the Kronecker delta symbol. An arbitrary vector $\mathbf{a}$ can be decomposed in the introduced basis as

$$\mathbf{a} = a_i \mathbf{e}_i, \quad a_i = \mathbf{a} \cdot \mathbf{e}_i. \tag{1.1.2}$$

The summation convention is assumed over the repeated indices. The scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i. \tag{1.1.3}$$

The vector product of two base vectors is defined by

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k, \tag{1.1.4}$$

where ϵ_{ijk} is the permutation symbol

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } ijk \text{ is an even permutation of 123}, \\ -1, & \text{if } ijk \text{ is an odd permutation of 123}, \\ 0, & \text{otherwise}. \end{cases} \tag{1.1.5}$$

The vector product of the vectors $\mathbf{a}$ and $\mathbf{b}$ can consequently be written as

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_i b_j \mathbf{e}_k. \tag{1.1.6}$$

The triple scalar product of the base vectors is

$$(\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k = \epsilon_{ijk}, \tag{1.1.7}$$

so that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \epsilon_{ijk} a_i b_j c_k = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{1.1.8}$$

In view of the vector relationship

$$(\mathbf{e}_i \times \mathbf{e}_j) \cdot (\mathbf{e}_k \times \mathbf{e}_l) = (\mathbf{e}_i \cdot \mathbf{e}_k)(\mathbf{e}_j \cdot \mathbf{e}_l) - (\mathbf{e}_i \cdot \mathbf{e}_l)(\mathbf{e}_j \cdot \mathbf{e}_k), \tag{1.1.9}$$

there is an $\epsilon - \delta$ identity

$$\epsilon_{ijm}\epsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{1.1.10}$$

In particular,

$$\epsilon_{ikl}\epsilon_{jkl} = 2\delta_{ij}, \quad \epsilon_{ijk}\epsilon_{ijk} = 6. \tag{1.1.11}$$

The triple vector product of the base vectors is

$$(\mathbf{e}_i \times \mathbf{e}_j) \times \mathbf{e}_k = \epsilon_{ijm}\epsilon_{klm}\mathbf{e}_l = \delta_{ik}\mathbf{e}_j - \delta_{jk}\mathbf{e}_i. \tag{1.1.12}$$

Thus,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = a_i b_j (c_i \mathbf{e}_j - c_j \mathbf{e}_i), \tag{1.1.13}$$

which confirms the vector identity

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}. \tag{1.1.14}$$

1.2. Second-Order Tensors

A dyadic product of two base vectors is the second-order tensor $\mathbf{e}_i \otimes \mathbf{e}_j$, such that

$$(\mathbf{e}_i \otimes \mathbf{e}_j) \cdot \mathbf{e}_k = \mathbf{e}_k \cdot (\mathbf{e}_j \otimes \mathbf{e}_i) = \delta_{jk}\mathbf{e}_i. \tag{1.2.1}$$

For arbitrary vectors **a**, **b** and **c**, it follows that

$$(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{e}_k = b_k \mathbf{a}, \quad (\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}. \tag{1.2.2}$$

The tensors $\mathbf{e}_i \otimes \mathbf{e}_j$ serve as base tensors for the representation of an arbitrary second-order tensor,

$$\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad A_{ij} = \mathbf{e}_i \cdot \mathbf{A} \cdot \mathbf{e}_j. \tag{1.2.3}$$

A dot product of the second-order tensor **A** and the vector **a** is the vector

$$\mathbf{b} = \mathbf{A} \cdot \mathbf{a} = b_i \mathbf{e}_i, \quad b_i = A_{ij} a_j. \tag{1.2.4}$$

Similarly, a dot product of two second-order tensors **A** and **B** is the second-order tensor

$$\mathbf{C} = \mathbf{A} \cdot \mathbf{B} = C_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad C_{ij} = A_{ik}B_{kj}. \tag{1.2.5}$$

The unit (identity) second-order tensor is

$$\mathbf{I} = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \tag{1.2.6}$$

which satisfies

$$\mathbf{A} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{A} = \mathbf{A}, \quad \mathbf{I} \cdot \mathbf{a} = \mathbf{a}. \tag{1.2.7}$$

The transpose of the tensor **A** is the tensor $\mathbf{A}^T$, which, for any vectors **a** and **b**, meets

$$\mathbf{A} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{A}^T, \quad \mathbf{b} \cdot \mathbf{A} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{A}^T \cdot \mathbf{b}. \tag{1.2.8}$$

Thus, if $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, then

$$\mathbf{A}^T = A_{ji}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.2.9}$$

The tensor $\mathbf{A}$ is symmetric if $\mathbf{A}^T = \mathbf{A}$; it is antisymmetric (or skew-symmetric) if $\mathbf{A}^T = -\mathbf{A}$. If $\mathbf{A}$ is nonsingular ($\det \mathbf{A} \neq 0$), there is a unique inverse tensor $\mathbf{A}^{-1}$ such that

$$\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}. \tag{1.2.10}$$

In this case, $\mathbf{b} = \mathbf{A} \cdot \mathbf{a}$ implies $\mathbf{a} = \mathbf{A}^{-1} \cdot \mathbf{b}$. For an orthogonal tensor $\mathbf{A}^T = \mathbf{A}^{-1}$, so that $\det \mathbf{A} = \pm 1$. The plus sign corresponds to proper and minus to improper orthogonal tensors.

The trace of the tensor $\mathbf{A}$ is a scalar obtained by the contraction $(i = j)$ operation

$$\operatorname{tr} \mathbf{A} = A_{ii}. \tag{1.2.11}$$

For a three-dimensional identity tensor, $\operatorname{tr} \mathbf{I} = 3$. Two inner (scalar or double-dot) products of two second-order tensors are defined by

$$\mathbf{A} \cdot\cdot\, \mathbf{B} = \operatorname{tr} (\mathbf{A} \cdot \mathbf{B}) = A_{ij} B_{ji}, \tag{1.2.12}$$

$$\mathbf{A} : \mathbf{B} = \operatorname{tr} \left(\mathbf{A} \cdot \mathbf{B}^T \right) = \operatorname{tr} \left(\mathbf{A}^T \cdot \mathbf{B} \right) = A_{ij} B_{ij}. \tag{1.2.13}$$

The connections are

$$\mathbf{A} \cdot\cdot\, \mathbf{B} = \mathbf{A}^T : \mathbf{B} = \mathbf{A} : \mathbf{B}^T. \tag{1.2.14}$$

If either $\mathbf{A}$ or $\mathbf{B}$ is symmetric, $\mathbf{A} \cdot\cdot\, \mathbf{B} = \mathbf{A} : \mathbf{B}$. Also,

$$\operatorname{tr} \mathbf{A} = \mathbf{A} : \mathbf{I}, \quad \operatorname{tr} (\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}. \tag{1.2.15}$$

Since the trace product is unaltered by any cyclic rearrangement of the factors, we have

$$\mathbf{A} \cdot\cdot\, (\mathbf{B} \cdot \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B}) \cdot\cdot\, \mathbf{C} = (\mathbf{C} \cdot \mathbf{A}) \cdot\cdot\, \mathbf{B}, \tag{1.2.16}$$

$$\mathbf{A} : (\mathbf{B} \cdot \mathbf{C}) = \left(\mathbf{B}^T \cdot \mathbf{A} \right) : \mathbf{C} = \left(\mathbf{A} \cdot \mathbf{C}^T \right) : \mathbf{B}. \tag{1.2.17}$$

A deviatoric part of $\mathbf{A}$ is defined by

$$\mathbf{A}' = \mathbf{A} - \frac{1}{3} (\operatorname{tr} \mathbf{A}) \mathbf{I}, \tag{1.2.18}$$

with the property $\operatorname{tr} \mathbf{A}' = 0$. It is easily verified that $\mathbf{A}' : \mathbf{A} = \mathbf{A}' : \mathbf{A}'$ and $\mathbf{A}' \cdot\cdot\, \mathbf{A} = \mathbf{A}' \cdot\cdot\, \mathbf{A}'$. A nonsymmetric tensor $\mathbf{A}$ can be decomposed into its symmetric and antisymmetric parts, $\mathbf{A} = \mathbf{A}_{\mathrm{s}} + \mathbf{A}_a$, such that

$$\mathbf{A}_{\mathrm{s}} = \frac{1}{2} \left(\mathbf{A} + \mathbf{A}^T \right), \quad \mathbf{A}_a = \frac{1}{2} \left(\mathbf{A} - \mathbf{A}^T \right). \tag{1.2.19}$$

If $\mathbf{A}$ is symmetric and $\mathbf{W}$ is antisymmetric, the trace of their dot product is equal to zero, $\operatorname{tr} (\mathbf{A} \cdot \mathbf{W}) = 0$. The axial vector $\boldsymbol{\omega}$ of an antisymmetric tensor $\mathbf{W}$ is defined by

$$\mathbf{W} \cdot \mathbf{a} = \boldsymbol{\omega} \times \mathbf{a}, \tag{1.2.20}$$

for every vector $\mathbf{a}$. This gives the component relationships

$$W_{ij} = -\epsilon_{ijk} \omega_k, \quad \omega_i = -\frac{1}{2} \epsilon_{ijk} W_{jk}. \tag{1.2.21}$$

Since $\mathbf{A} \cdot \mathbf{e}_i = A_{ji}\mathbf{e}_j$, the determinant of $\mathbf{A}$ can be calculated from Eq. (1.1.8) as

$$\det \mathbf{A} = [(\mathbf{A} \cdot \mathbf{e}_1) \times (\mathbf{A} \cdot \mathbf{e}_2)] \cdot (\mathbf{A} \cdot \mathbf{e}_3) = \epsilon_{ijk} A_{i1} A_{j2} A_{k3}. \tag{1.2.22}$$

Thus,

$$\epsilon_{\alpha\beta\gamma}(\det \mathbf{A}) = \epsilon_{ijk} A_{i\alpha} A_{j\beta} A_{k\gamma}, \tag{1.2.23}$$

and by second of Eq. (1.1.11)

$$\det \mathbf{A} = \frac{1}{6}\, \epsilon_{ijk}\epsilon_{\alpha\beta\gamma} A_{i\alpha} A_{j\beta} A_{k\gamma}. \tag{1.2.24}$$

For further details, standard texts such as Brillouin (1964) can be consulted.

1.3. Eigenvalues and Eigenvectors

The vector $\mathbf{n}$ is an eigenvector of the second-order tensor $\mathbf{A}$ if there is a scalar λ such that $\mathbf{A} \cdot \mathbf{n} = \lambda \mathbf{n}$, i.e.,

$$(\mathbf{A} - \lambda \mathbf{I}) \cdot \mathbf{n} = \mathbf{0}. \tag{1.3.1}$$

A scalar λ is called an eigenvalue of $\mathbf{A}$ corresponding to the eigenvector $\mathbf{n}$. Nontrivial solutions for $\mathbf{n}$ exist if $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$, which gives the characteristic equation for $\mathbf{A}$,

$$\lambda^3 - J_1 \lambda^2 - J_2 \lambda - J_3 = 0. \tag{1.3.2}$$

The scalars J_1, J_2 and J_3 are the principal invariants of $\mathbf{A}$, which remain unchanged under any orthogonal transformation of the orthonormal basis of $\mathbf{A}$. These are

$$J_1 = \operatorname{tr} \mathbf{A}, \tag{1.3.3}$$

$$J_2 = \frac{1}{2}\left[\operatorname{tr}\left(\mathbf{A}^2\right) - (\operatorname{tr} \mathbf{A})^2\right], \tag{1.3.4}$$

$$J_3 = \det \mathbf{A} = \frac{1}{6}\left[2 \operatorname{tr}\left(\mathbf{A}^3\right) - 3 (\operatorname{tr} \mathbf{A}) \operatorname{tr}\left(\mathbf{A}^2\right) + (\operatorname{tr} \mathbf{A})^3\right]. \tag{1.3.5}$$

If $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$, there are three mutually orthogonal eigenvectors $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$, so that $\mathbf{A}$ has a spectral representation

$$\mathbf{A} = \sum_{i=1}^{3} \lambda_i \mathbf{n}_i \otimes \mathbf{n}_i. \tag{1.3.6}$$

If $\lambda_1 \neq \lambda_2 = \lambda_3$,

$$\mathbf{A} = (\lambda_1 - \lambda_2)\mathbf{n}_1 \otimes \mathbf{n}_1 + \lambda_2 \mathbf{I}, \tag{1.3.7}$$

while $\mathbf{A} = \lambda \mathbf{I}$, if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$.

A symmetric real tensor has all real eigenvalues. An antisymmetric tensor has only one real eigenvalue, which is equal to zero. The corresponding eigendirection is parallel to the axial vector of the antisymmetric tensor. A proper orthogonal (rotation) tensor has also one real eigenvalue, which is equal to one. The corresponding eigendirection is parallel to the axis of rotation.

1.4. Cayley–Hamilton Theorem

A second-order tensor satisfies its own characteristic equation

$$\mathbf{A}^3 - J_1\mathbf{A}^2 - J_2\mathbf{A} - J_3\mathbf{I} = \mathbf{0}. \tag{1.4.1}$$

This is a Cayley–Hamilton theorem. Thus, if $\mathbf{A}^{-1}$ exists, it can be expressed as

$$J_3\mathbf{A}^{-1} = \mathbf{A}^2 - J_1\mathbf{A} - J_2\mathbf{I}, \tag{1.4.2}$$

which shows that eigendirections of $\mathbf{A}^{-1}$ are parallel to those of $\mathbf{A}$. A number of useful results can be extracted from the Cayley–Hamilton theorem. An expression for $(\det \mathbf{F})$ in terms of traces of $\mathbf{A}$, $\mathbf{A}^2$, $\mathbf{A}^3$, given in Eq. (1.3.5), is obtained by taking the trace of Eq. (1.4.1). Similarly,

$$\det(\mathbf{I} + \mathbf{A}) - \det \mathbf{A} = 1 + J_1 - J_2. \tag{1.4.3}$$

If $\mathbf{X}^2 = \mathbf{A}$, an application of Eq. (1.4.1) to $\mathbf{X}$ gives

$$\mathbf{A} \cdot \mathbf{X} - I_1\mathbf{A} - I_2\mathbf{X} - I_3\mathbf{I} = \mathbf{0}, \tag{1.4.4}$$

where I_i are the principal invariants of $\mathbf{X}$. Multiplying this with I_1 and $\mathbf{X}$, and summing up the resulting two equations yields

$$\mathbf{X} = \frac{1}{I_1 I_2 + I_3}\left[\mathbf{A}^2 - \left(I_1^2 + I_2\right)\mathbf{A} - I_1 I_3 \mathbf{I}\right]. \tag{1.4.5}$$

The invariants I_i can be calculated from the principal invariants of $\mathbf{A}$, or from the eigenvalues of $\mathbf{A}$. Alternative route to solve $\mathbf{X}^2 = \mathbf{A}$ is via eigendirections and spectral representation (diagonalization) of $\mathbf{A}$.

1.5. Change of Basis

Under a rotational change of basis, the new base vectors are $\mathbf{e}_i^* = \mathbf{Q}\cdot\mathbf{e}_i$, where $\mathbf{Q}$ is a proper orthogonal tensor. An arbitrary vector $\mathbf{a}$ can be decomposed in the two bases as

$$\mathbf{a} = a_i\mathbf{e}_i = a_i^*\mathbf{e}_i^*, \quad a_i^* = Q_{ji}a_j. \tag{1.5.1}$$

If the vector $\mathbf{a}^*$ is introduced, with components a_i^* in the original basis ($\mathbf{a}^* = a_i^*\mathbf{e}_i$), then $\mathbf{a}^* = \mathbf{Q}^T \cdot \mathbf{a}$.

Under an arbitrary orthogonal transformation $\mathbf{Q}$ ($\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{I}$, $\det \mathbf{Q} = \pm 1$), the components of so-called axial vectors transform according to $\omega_i^* = (\det \mathbf{Q})Q_{ji}\omega_j$. On the other hand, the components of absolute vectors transform as $a_i^* = Q_{ji}a_j$. If attention is confined to proper orthogonal transformations, i.e., the rotations of the basis only ($\det \mathbf{Q} = 1$), no distinction is made between axial and absolute vectors.

An invariant of $\mathbf{a}$ is $\mathbf{a} \cdot \mathbf{a}$. A scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is an even invariant of vectors $\mathbf{a}$ and $\mathbf{b}$, since it remains unchanged under both proper and improper orthogonal transformation of the basis (rotation and reflection). A triple scalar product of three vectors is an odd invariant of those vectors, since it remains unchanged under all proper orthogonal transformations ($\det \mathbf{Q} = 1$), but changes the sign under improper orthogonal transformations ($\det \mathbf{Q} = -1$).

A second-order tensor $\mathbf{A}$ can be decomposed in the considered bases as

$$\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = A^*_{ij}\mathbf{e}^*_i \otimes \mathbf{e}^*_j, \quad A^*_{ij} = Q_{ki}A_{kl}Q_{lj}. \tag{1.5.2}$$

If the tensor $\mathbf{A}^* = A^*_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ is introduced, it is related to $\mathbf{A}$ by $\mathbf{A}^* = \mathbf{Q}^T \cdot \mathbf{A} \cdot \mathbf{Q}$. The two tensors share the same eigenvalues, which are thus invariants of $\mathbf{A}$ under rotation of the basis. Invariants are also symmetric functions of the eigenvalues, such as

$$\operatorname{tr}\mathbf{A} = \lambda_1 + \lambda_2 + \lambda_3, \quad \operatorname{tr}\left(\mathbf{A}^2\right) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad \operatorname{tr}\left(\mathbf{A}^3\right) = \lambda_1^3 + \lambda_2^3 + \lambda_3^3, \tag{1.5.3}$$

or the principal invariants of Eqs. (1.3.3)–(1.3.5),

$$J_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad J_2 = -\left(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1\right), \quad J_3 = \lambda_1\lambda_2\lambda_3. \tag{1.5.4}$$

All invariants of the second-order tensors under orthogonal transformations are even invariants.

1.6. Higher-Order Tensors

Triadic and tetradic products of the base vectors are

$$\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \quad \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \tag{1.6.1}$$

with obvious extension to higher-order polyadic products. These tensors serve as base tensors for the representation of higher-order tensors. For example, the permutation tensor is

$$\boldsymbol{\epsilon} = \epsilon_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \tag{1.6.2}$$

where ϵ_{ijk} is defined by Eq. (1.1.5). If $\mathbf{A}$ is a symmetric second-order tensor,

$$\boldsymbol{\epsilon} : \mathbf{A} = \epsilon_{ijk}A_{jk}\mathbf{e}_i = \mathbf{0}. \tag{1.6.3}$$

The fourth-order tensor $\boldsymbol{\mathcal{L}}$ can be expressed as

$$\boldsymbol{\mathcal{L}} = \mathcal{L}_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l. \tag{1.6.4}$$

A dot product of $\boldsymbol{\mathcal{L}}$ with a vector $\mathbf{a}$ is

$$\boldsymbol{\mathcal{L}} \cdot \mathbf{a} = \mathcal{L}_{ijkl}a_l\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k. \tag{1.6.5}$$

Two inner products of the fourth- and second-order tensors can be defined by

$$\boldsymbol{\mathcal{L}} \cdot\cdot\, \mathbf{A} = \mathcal{L}_{ijkl}A_{lk}\mathbf{e}_i \otimes \mathbf{e}_j, \quad \boldsymbol{\mathcal{L}} : \mathbf{A} = \mathcal{L}_{ijkl}A_{kl}\mathbf{e}_i \otimes \mathbf{e}_j. \tag{1.6.6}$$

If $\mathbf{W}$ is antisymmetric and $\boldsymbol{\mathcal{L}}$ has the symmetry in its last two indices,

$$\boldsymbol{\mathcal{L}} : \mathbf{W} = \mathbf{0}. \tag{1.6.7}$$

The symmetries of the form $\mathcal{L}_{ijkl} = \mathcal{L}_{jikl} = \mathcal{L}_{ijlk}$ will frequently, but not always, hold for the fourth-order tensors considered in this book. We also introduce the scalar products

$$\boldsymbol{\mathcal{L}} :: (\mathbf{A} \otimes \mathbf{B}) = \mathbf{B} : \boldsymbol{\mathcal{L}} : \mathbf{A} = B_{ij}\mathcal{L}_{ijkl}A_{kl}, \tag{1.6.8}$$

and

$$\boldsymbol{\mathcal{L}} \cdot\cdot\cdot\cdot\, (\mathbf{A} \otimes \mathbf{B}) = \mathbf{B} \cdot\cdot\, \boldsymbol{\mathcal{L}} \cdot\cdot\, \mathbf{A} = B_{ji}\mathcal{L}_{ijkl}A_{lk}. \tag{1.6.9}$$

The transpose of $\boldsymbol{\mathcal{L}}$ satisfies

$$\boldsymbol{\mathcal{L}} : \mathbf{A} = \mathbf{A} : \boldsymbol{\mathcal{L}}^T, \quad \mathbf{B} : \boldsymbol{\mathcal{L}} : \mathbf{A} = \mathbf{A} : \boldsymbol{\mathcal{L}}^T : \mathbf{B}, \tag{1.6.10}$$

hence, $\mathcal{L}^T_{ijkl} = \mathcal{L}_{klij}$. The tensor $\boldsymbol{\mathcal{L}}$ is symmetric if $\boldsymbol{\mathcal{L}}^T = \boldsymbol{\mathcal{L}}$, i.e., $\mathcal{L}_{ijkl} = \mathcal{L}_{klij}$ (reciprocal symmetry).

The symmetric fourth-order unit tensor $\boldsymbol{I}$ is

$$\boldsymbol{I} = I_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \quad I_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{1.6.11}$$

If $\boldsymbol{\mathcal{L}}$ possesses the symmetry in its leading and terminal pair of indices ($\mathcal{L}_{ijkl} = \mathcal{L}_{jikl}$ and $\mathcal{L}_{ijkl} = \mathcal{L}_{ijlk}$) and if $\mathbf{A}$ is symmetric ($A_{ij} = A_{ji}$), then

$$\boldsymbol{\mathcal{L}} : \boldsymbol{I} = \boldsymbol{I} : \boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{L}}, \quad \boldsymbol{I} : \mathbf{A} = \mathbf{A} : \boldsymbol{I} = \mathbf{A}. \tag{1.6.12}$$

For an arbitrary nonsymmetric second-order tensor $\mathbf{A}$,

$$\boldsymbol{I} : \mathbf{A} = \mathbf{A}_{\mathrm{s}} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T). \tag{1.6.13}$$

The fourth-order tensor with rectangular components

$$\hat{I}_{ijkl} = \frac{1}{2}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) \tag{1.6.14}$$

can also be introduced, such that

$$\hat{\boldsymbol{I}} : \mathbf{A} = \mathbf{A}_{\mathrm{a}} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T). \tag{1.6.15}$$

Note the symmetry properties

$$\hat{I}_{ijkl} = \hat{I}_{klij}, \quad \hat{I}_{jikl} = \hat{I}_{ijlk} = -\hat{I}_{ijkl}. \tag{1.6.16}$$

A fourth-order tensor $\boldsymbol{\mathcal{L}}$ is invertible if there exists another such tensor $\boldsymbol{\mathcal{L}}^{-1}$ which obeys

$$\boldsymbol{\mathcal{L}} : \boldsymbol{\mathcal{L}}^{-1} = \boldsymbol{\mathcal{L}}^{-1} : \boldsymbol{\mathcal{L}} = \boldsymbol{I}. \tag{1.6.17}$$

In this case, $\mathbf{B} = \boldsymbol{\mathcal{L}} : \mathbf{A}$ implies $\mathbf{A} = \boldsymbol{\mathcal{L}}^{-1} : \mathbf{B}$, and *vice versa*. The inner product of two fourth-order tensors $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{M}}$ is defined by

$$\boldsymbol{\mathcal{L}} : \boldsymbol{\mathcal{M}} = \mathcal{L}_{ijmn}\mathcal{M}_{mnkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l. \tag{1.6.18}$$

The trace of the fourth-order tensor $\boldsymbol{\mathcal{L}}$ is

$$\operatorname{tr}\boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{L}} :: \boldsymbol{I} = \mathcal{L}_{ijij}. \tag{1.6.19}$$

In particular, $\operatorname{tr}\boldsymbol{I} = 6$. A fourth-order tensor defined by

$$\boldsymbol{\mathcal{L}}^{\mathrm{d}} = \boldsymbol{\mathcal{L}} - \frac{1}{6}(\operatorname{tr}\boldsymbol{\mathcal{L}})\boldsymbol{I}, \tag{1.6.20}$$

satisfies

$$\operatorname{tr}\boldsymbol{\mathcal{L}}^{\,\mathrm{d}} = 0, \quad \boldsymbol{\mathcal{L}}^{\,\mathrm{d}} :: \boldsymbol{\mathcal{L}} = \boldsymbol{\mathcal{L}}^{\,\mathrm{d}} :: \boldsymbol{\mathcal{L}}^{\,\mathrm{d}}. \tag{1.6.21}$$

The tensor

$$\hat{\boldsymbol{\mathcal{L}}}^{\,\mathrm{d}} = \boldsymbol{\mathcal{L}} - \frac{1}{3}(\operatorname{tr}\boldsymbol{\mathcal{L}})\mathbf{I} \otimes \mathbf{I} \tag{1.6.22}$$

also has the property $\operatorname{tr}\hat{\boldsymbol{\mathcal{L}}}^{\,\mathrm{d}} = 0$.

Under rotational change of the basis specified by a proper orthogonal tensor $\mathbf{Q}$, the components of the fourth-order tensor change according to

$$\mathcal{L}^*_{ijkl} = Q_{\alpha i} Q_{\beta j} \mathcal{L}_{\alpha\beta\gamma\delta} Q_{\gamma k} Q_{\delta l}. \tag{1.6.23}$$

The trace of the fourth-order tensor is one of its invariants under rotational change of basis. Other invariants are discussed in the paper by Betten (1987).

1.6.1. Traceless Tensors

A traceless part of the symmetric second-order tensor $\mathbf{A}$ has the rectangular components

$$A'_{ij} = A_{ij} - \frac{1}{3} A_{kk} \delta_{ij}, \tag{1.6.24}$$

such that $A'_{ii} = 0$. For a symmetric third-order tensor $\mathbf{Z}$ ($Z_{ijk} = Z_{jik} = Z_{jki}$), the traceless part is

$$Z'_{ijk} = Z_{ijk} - \frac{1}{5} \left(Z_{mmi} \delta_{jk} + Z_{mmj} \delta_{ki} + Z_{mmk} \delta_{ij} \right), \tag{1.6.25}$$

which is defined so that the contraction of any two of its indices gives a zero vector, e.g.,

$$Z'_{iij} = Z'_{jii} = Z'_{iji} = 0. \tag{1.6.26}$$

A traceless part of the symmetric fourth-order tensor ($\mathcal{L}_{ijkl} = \mathcal{L}_{jikl} = \mathcal{L}_{ijlk} = \mathcal{L}_{klij}$) is defined by

$$\begin{aligned} \mathcal{L}'_{ijkl} = \mathcal{L}_{ijkl} &- \frac{1}{7} \left(\mathcal{L}_{mmij} \delta_{kl} + \mathcal{L}_{mmkl} \delta_{ij} + \mathcal{L}_{mmjk} \delta_{il} + \mathcal{L}_{mmil} \delta_{jk} \right. \\ &\left. + \mathcal{L}_{mmik} \delta_{jl} + \mathcal{L}_{mmjl} \delta_{ik} \right) + \frac{1}{35} \mathcal{L}_{mmnn} \left(\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right). \end{aligned} \tag{1.6.27}$$

A contraction of any two of its indices also yields a zero tensor, e.g.,

$$\mathcal{L}'_{iikl} = \mathcal{L}'_{kiil} = \mathcal{L}'_{ikli} = 0. \tag{1.6.28}$$

For further details see the papers by Spencer (1970), Kanatani (1984), and Lubarda and Krajcinovic (1993).

1.7. Covariant and Contravariant Components

1.7.1. Vectors

A pair of vector bases, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, are said to be reciprocal if

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^{\ j}, \tag{1.7.1}$$

where $\delta_i^{\ j}$ is the Kronecker delta symbol (Fig. 1.1). The base vectors of each basis are neither unit nor mutually orthogonal vectors, so that

$$2D\,\mathbf{e}^i = \epsilon_{ijk} (\mathbf{e}_j \times \mathbf{e}_k), \quad D = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3). \tag{1.7.2}$$

Any vector $\mathbf{a}$ can be decomposed in the primary basis as

$$\mathbf{a} = a^i \mathbf{e}_i, \quad a^i = \mathbf{a} \cdot \mathbf{e}^i, \tag{1.7.3}$$

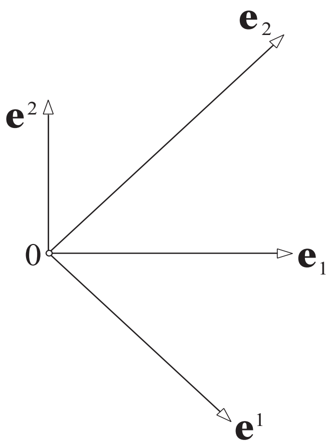

FIGURE 1.1. Primary and reciprocal bases in two dimensions ($\mathbf{e}_1 \cdot \mathbf{e}^2 = \mathbf{e}_2 \cdot \mathbf{e}^1 = 0$).

and in the reciprocal basis as

$$\mathbf{a} = a_i \mathbf{e}^i, \quad a_i = \mathbf{a} \cdot \mathbf{e}_i. \tag{1.7.4}$$

The components a^i are called contravariant, and a_i covariant components of the vector $\mathbf{a}$.

1.7.2. Second-Order Tensors

Denoting the scalar products of the base vectors by

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j = g^{ji}, \quad g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = g_{ji}, \tag{1.7.5}$$

there follows

$$a^i = g^{ij} a_j, \quad a_i = g_{ij} a^j, \tag{1.7.6}$$

$$\mathbf{e}^i = g^{ik} \mathbf{e}_k, \quad \mathbf{e}_i = g_{ik} \mathbf{e}^k. \tag{1.7.7}$$

This shows that the matrices of g^{ij} and g_{ij} are mutual inverses. The components g^{ij} and g_{ij} are contravariant and covariant components of the second-order unit (metric) tensor

$$\mathbf{I} = g^{ij} \mathbf{e}_i \otimes \mathbf{e}_j = g_{ij} \mathbf{e}^i \otimes \mathbf{e}^j = \mathbf{e}^j \otimes \mathbf{e}_j = \mathbf{e}_j \otimes \mathbf{e}^j. \tag{1.7.8}$$

Note that $g^i{}_j = \delta^i{}_j$ and $g_i{}^j = \delta_i{}^j$, both being the Kronecker delta. The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ can be calculated from

$$\mathbf{a} \cdot \mathbf{b} = g^{ij} a_i b_j = g_{ij} a^i b^j = a^i b_i = a_i b^i. \tag{1.7.9}$$

The second-order tensor has four types of decompositions

$$\mathbf{A} = A^{ij} \mathbf{e}_i \otimes \mathbf{e}_j = A_{ij} \mathbf{e}^i \otimes \mathbf{e}^j = A^i{}_j \mathbf{e}_i \otimes \mathbf{e}^j = A_i{}^j \mathbf{e}^i \otimes \mathbf{e}_j. \tag{1.7.10}$$

These are, respectively, contravariant, covariant, and two kinds of mixed components of $\mathbf{A}$, such that

$$A^{ij} = \mathbf{e}^i \cdot \mathbf{A} \cdot \mathbf{e}^j, \quad A_{ij} = \mathbf{e}_i \cdot \mathbf{A} \cdot \mathbf{e}_j, \quad A^i{}_j = \mathbf{e}^i \cdot \mathbf{A} \cdot \mathbf{e}_j, \quad A_i{}^j = \mathbf{e}_i \cdot \mathbf{A} \cdot \mathbf{e}^j. \tag{1.7.11}$$

The relationships between different components are easily established by using Eq. (1.7.7). For example,

$$A_{ij} = g_{ik} A^k{}_j = A_i{}^k g_{kj} = g_{ik} A^{kl} g_{lj}. \tag{1.7.12}$$

The transpose of $\mathbf{A}$ can be decomposed as

$$\mathbf{A}^T = A^{ji} \mathbf{e}_i \otimes \mathbf{e}_j = A_{ji} \mathbf{e}^i \otimes \mathbf{e}^j = A_j{}^i \mathbf{e}_i \otimes \mathbf{e}^j = A^j{}_i \mathbf{e}^i \otimes \mathbf{e}_j. \tag{1.7.13}$$

If $\mathbf{A}$ is symmetric ($\mathbf{A} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{A}$), one has

$$A^{ij} = A^{ji}, \quad A_{ij} = A_{ji}, \quad A^i{}_j = A_j{}^i, \tag{1.7.14}$$

although $A^i{}_j \neq A_i{}^j$.

A dot product of a second-order tensor $\mathbf{A}$ and a vector $\mathbf{a}$ is the vector

$$\mathbf{b} = \mathbf{A} \cdot \mathbf{a} = b^i \mathbf{e}_i = b_i \mathbf{e}^i. \tag{1.7.15}$$

The contravariant and covariant components of $\mathbf{b}$ are

$$b^i = A^{ij} a_j = A^i{}_j a^j, \quad b_i = A_{ij} a^j = A_i{}^j a_j. \tag{1.7.16}$$

A dot product of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ is the second-order tensor $\mathbf{C}$, such that

$$\mathbf{C} \cdot \mathbf{a} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{a}), \tag{1.7.17}$$

for any vector $\mathbf{a}$. Each type of components of $\mathbf{C}$ has two possible representations. For example,

$$C^{ij} = A^{ik} B_k{}^j = A^i{}_k B^{kj}, \quad C^i{}_j = A^{ik} B_{kj} = A^i{}_k B^k{}_j. \tag{1.7.18}$$

The trace of a tensor $\mathbf{A}$ is the scalar obtained by contraction of the subscript and superscript in the mixed component tensor representation. Thus,

$$\operatorname{tr} \mathbf{A} = A^i{}_i = A_i{}^i = g_{ij} A^{ij} = g^{ij} A_{ij}. \tag{1.7.19}$$

Two kinds of inner products are defined by

$$\mathbf{A} \cdot\cdot\, \mathbf{B} = \operatorname{tr}(\mathbf{A} \cdot \mathbf{B}) = A^{ij} B_{ji} = A_{ij} B^{ji} = A^i{}_j B^j{}_i = A_i{}^j B_j{}^i, \tag{1.7.20}$$

$$\mathbf{A} : \mathbf{B} = \operatorname{tr}\left(\mathbf{A} \cdot \mathbf{B}^T\right) = A^{ij} B_{ij} = A_{ij} B^{ij} = A^i{}_j B_i{}^j = A_i{}^j B^i{}_j. \tag{1.7.21}$$

If either $\mathbf{A}$ or $\mathbf{B}$ is symmetric, $\mathbf{A} \cdot\cdot\, \mathbf{B} = \mathbf{A} : \mathbf{B}$. The trace of $\mathbf{A}$ in Eq. (1.7.19) can be written as $\operatorname{tr} \mathbf{A} = \mathbf{A} : \mathbf{I}$, where $\mathbf{I}$ is defined by (1.7.8).

1.7.3. Higher-Order Tensors

An n-th order tensor has one completely contravariant, one completely covariant, and $(2^n - 2)$ kinds of mixed component representations. For a third-order tensor $\boldsymbol{\Gamma}$, for example, these are respectively Γ^{ijk}, Γ_{ijk}, and

$$\Gamma^{ij}{}_{k}, \quad \Gamma^{i}{}_{j}{}^{k}, \quad \Gamma_{i}{}^{jk}, \quad \Gamma^{i}{}_{jk}, \quad \Gamma_{i}{}^{j}{}_{k}, \quad \Gamma_{ij}{}^{k}. \tag{1.7.22}$$

As an illustration,

$$\boldsymbol{\Gamma} = \Gamma^{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k = \Gamma^{ij}{}_{k}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}^k. \tag{1.7.23}$$

The relationships between various components are analogous to those in Eq. (1.7.12), e.g.,

$$\Gamma^{i}{}_{jk} = \Gamma^{i}{}_{j}{}^{m}g_{mk} = \Gamma_{m}{}^{n}{}_{k}g^{mi}g_{nj} = \Gamma_{m}{}^{np}g^{mi}g_{nj}g_{pk}. \tag{1.7.24}$$

Four types of components of the inner product of the fourth- and second-order tensors, $\mathbf{C} = \boldsymbol{\mathcal{L}} : \mathbf{A}$, can all be expressed in terms of the components of $\boldsymbol{\mathcal{L}}$ and $\mathbf{A}$. For example, contravariant and mixed (right-covariant) components are

$$C^{ij} = \mathcal{L}^{ijkl}A_{kl} = \mathcal{L}^{ij}{}_{kl}A^{kl} = \mathcal{L}^{ijk}{}_{l}A_{k}{}^{l} = \mathcal{L}^{ij}{}_{k}{}^{l}A^{k}{}_{l}, \tag{1.7.25}$$

$$C^{i}{}_{j} = \mathcal{L}^{i}{}_{j}{}^{kl}A_{kl} = \mathcal{L}^{i}{}_{jkl}A^{kl} = \mathcal{L}^{i}{}_{j}{}^{k}{}_{l}A_{k}{}^{l} = \mathcal{L}^{i}{}_{jk}{}^{l}A^{k}{}_{l}. \tag{1.7.26}$$

1.8. Induced Tensors

Let $\{\mathbf{e}_i\}$ and $\{\mathbf{e}^i\}$ be a pair of reciprocal bases, and let $\mathbf{F}$ be a nonsingular mapping that transforms the base vectors $\mathbf{e}_i$ into

$$\hat{\mathbf{e}}_i = \mathbf{F}\cdot\mathbf{e}_i = F^{j}{}_{i}\mathbf{e}_j, \tag{1.8.1}$$

and the vectors $\mathbf{e}^i$ into

$$\hat{\mathbf{e}}^i = \mathbf{e}^i\cdot\mathbf{F}^{-1} = (F^{-1})^{i}{}_{j}\mathbf{e}^j, \tag{1.8.2}$$

such that $\hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}^j = \delta_i{}^j$ (Fig. 1.2). Then, in view of Eqs. (1.7.10) and (1.7.13) applied to $\mathbf{F}$ and $\mathbf{F}^T$, we have

$$\mathbf{F}^T\cdot\mathbf{F} = \hat{g}_{ij}\mathbf{e}^i\otimes\mathbf{e}^j, \quad \mathbf{F}^{-1}\cdot\mathbf{F}^{-T} = \hat{g}^{ij}\mathbf{e}_i\otimes\mathbf{e}_j, \tag{1.8.3}$$

where $\hat{g}_{ij} = \hat{\mathbf{e}}_i\cdot\hat{\mathbf{e}}_j$ and $\hat{g}^{ij} = \hat{\mathbf{e}}^i\cdot\hat{\mathbf{e}}^j$. Thus, covariant components of $\mathbf{F}^T\cdot\mathbf{F}$ and contravariant components of $\mathbf{F}^{-1}\cdot\mathbf{F}^{-T}$ in the original bases are equal to covariant and contravariant components of the metric tensor in the transformed bases ($\mathbf{I} = \hat{g}_{ij}\hat{\mathbf{e}}^i\otimes\hat{\mathbf{e}}^j = \hat{g}^{ij}\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j$).

An arbitrary vector $\mathbf{a}$ can be decomposed in the original and transformed bases as

$$\mathbf{a} = a^i\mathbf{e}_i = a_i\mathbf{e}^i = \hat{a}^i\hat{\mathbf{e}}_i = \hat{a}_i\hat{\mathbf{e}}^i. \tag{1.8.4}$$

Evidently,

$$\hat{a}^i = (F^{-1})^{i}{}_{j}a^j, \quad \hat{a}_i = F^{j}{}_{i}a_j. \tag{1.8.5}$$

Introducing the vectors

$$\mathbf{a}^* = \hat{a}^i\mathbf{e}_i, \quad \mathbf{a}_* = \hat{a}_i\mathbf{e}^i, \tag{1.8.6}$$

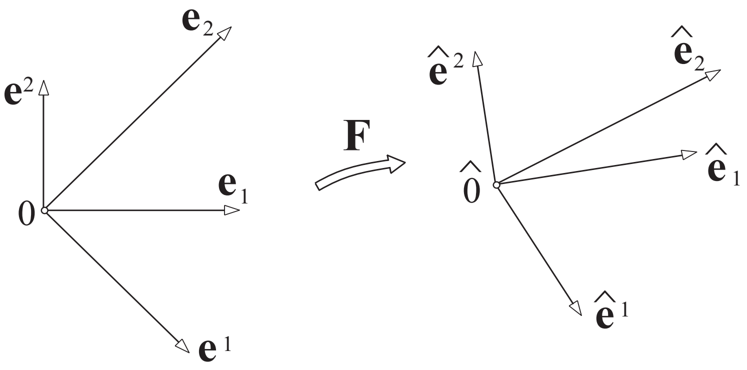

FIGURE 1.2. Upon mapping $\mathbf{F}$ the pair of reciprocal bases $\mathbf{e}_i$ and $\mathbf{e}^j$ transform into reciprocal bases $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}^j$.

it follows that

$$\hat{\mathbf{a}}^* = \mathbf{F}^{-1} \cdot \mathbf{a}, \quad \hat{\mathbf{a}}_* = \mathbf{F}^T \cdot \mathbf{a}. \tag{1.8.7}$$

The vectors $\mathbf{a}^*$ and $\mathbf{a}_*$ are induced from $\mathbf{a}$ by the transformation of bases. The contravariant components of $\mathbf{F}^{-1} \cdot \mathbf{a}$ in the original basis are numerically equal to contravariant components of $\mathbf{a}$ in the transformed basis. Analogous statement applies to covariant components.

Let $\mathbf{A}$ be a second-order tensor with components in the original basis given by Eq. (1.7.10), and in the transformed basis by

$$\mathbf{A} = \hat{A}^{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j = \hat{A}_{ij} \hat{\mathbf{e}}^i \otimes \hat{\mathbf{e}}^j = \hat{A}^i{}_j \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}^j = \hat{A}_i{}^j \hat{\mathbf{e}}^i \otimes \hat{\mathbf{e}}_j. \tag{1.8.8}$$

The components are related through

$$\hat{A}^{ij} = (F^{-1})^i{}_k A^{kl} (F^{-1})^j{}_l, \quad \hat{A}_{ij} = F^k{}_i A_{kl} F^l{}_j, \tag{1.8.9}$$

$$\hat{A}^i{}_j = (F^{-1})^i{}_k A^k{}_l F^l{}_j, \quad \hat{A}_i{}^j = F^k{}_i A_k{}^l (F^{-1})^j{}_l. \tag{1.8.10}$$

Introducing the tensors

$$\mathbf{A}^* = \hat{A}^{ij} \mathbf{e}_i \otimes \mathbf{e}_j, \quad \mathbf{A}_* = \hat{A}_{ij} \mathbf{e}^i \otimes \mathbf{e}^j, \tag{1.8.11}$$

$$\mathbf{A}^\star = \hat{A}^i{}_j \mathbf{e}_i \otimes \mathbf{e}^j, \quad \mathbf{A}_\star = \hat{A}_i{}^j \mathbf{e}^i \otimes \mathbf{e}_j, \tag{1.8.12}$$

we recognize from Eqs. (1.8.9) and (1.8.10) that

$$\mathbf{A}^* = \mathbf{F}^{-1} \cdot \mathbf{A} \cdot \mathbf{F}^{-T}, \quad \mathbf{A}_* = \mathbf{F}^T \cdot \mathbf{A} \cdot \mathbf{F}, \tag{1.8.13}$$

$$\mathbf{A}^\star = \mathbf{F}^{-1} \cdot \mathbf{A} \cdot \mathbf{F}, \quad \mathbf{A}_\star = \mathbf{F}^T \cdot \mathbf{A} \cdot \mathbf{F}^{-T}. \tag{1.8.14}$$

These four tensors are said to be induced from $\mathbf{A}$ by transformation of the bases (Hill, 1978). The contravariant components of the tensor $\mathbf{F}^{-1} \cdot \mathbf{A} \cdot \mathbf{F}^{-T}$ in the original basis are numerically equal to the contravariant components of the tensor $\mathbf{A}$ in the transformed basis. Analogous statements apply to covariant and mixed components.

1.9. Gradient of Tensor Functions

Let $f = f(\mathbf{A})$ be a scalar function of the second-order tensor argument $\mathbf{A}$. The change of f associated with an infinitesimal change of $\mathbf{A}$ can be determined from

$$\mathrm{d}f = \mathrm{tr}\left(\frac{\partial f}{\partial \mathbf{A}} \cdot \mathrm{d}\mathbf{A}\right). \tag{1.9.1}$$

If $\mathrm{d}\mathbf{A}$ is decomposed on the fixed primary and reciprocal bases as

$$\mathrm{d}\mathbf{A} = \mathrm{d}A^{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathrm{d}A_{ij}\mathbf{e}^i \otimes \mathbf{e}^j = \mathrm{d}A^i{}_j\mathbf{e}_i \otimes \mathbf{e}^j = \mathrm{d}A_i{}^j\mathbf{e}^i \otimes \mathbf{e}_j, \tag{1.9.2}$$

the gradient of f with respect to $\mathbf{A}$ is the second-order tensor with decompositions

$$\frac{\partial f}{\partial \mathbf{A}} = \frac{\partial f}{\partial A^{ji}}\,\mathbf{e}^i \otimes \mathbf{e}^j = \frac{\partial f}{\partial A_{ji}}\,\mathbf{e}_i \otimes \mathbf{e}_j = \frac{\partial f}{\partial A^j{}_i}\,\mathbf{e}_i \otimes \mathbf{e}^j = \frac{\partial f}{\partial A_j{}^i}\,\mathbf{e}^i \otimes \mathbf{e}_j, \tag{1.9.3}$$

since then (Ogden, 1984)

$$\mathrm{d}f = \frac{\partial f}{\partial A^{ij}}\,\mathrm{d}A^{ij} = \frac{\partial f}{\partial A_{ij}}\,\mathrm{d}A_{ij} = \frac{\partial f}{\partial A^i{}_j}\,\mathrm{d}A^i{}_j = \frac{\partial f}{\partial A_i{}^j}\,\mathrm{d}A_i{}^j. \tag{1.9.4}$$

Let $\mathbf{F} = \mathbf{F}(\mathbf{A})$ be a second-order tensor function of the second-order tensor argument $\mathbf{A}$. The change of $\mathbf{F}$ associated with an infinitesimal change of $\mathbf{A}$ can be determined from

$$\mathrm{d}\mathbf{F} = \frac{\partial \mathbf{F}}{\partial \mathbf{A}} \cdot\cdot\, \mathrm{d}\mathbf{A}. \tag{1.9.5}$$

If $\mathrm{d}\mathbf{A}$ is decomposed on the fixed primary and reciprocal bases as in Eq. (1.9.2), the gradient of $\mathbf{F}$ with respect to $\mathbf{A}$ is the fourth-order tensor, such that

$$\frac{\partial \mathbf{F}}{\partial \mathbf{A}} = \frac{\partial \mathbf{F}}{\partial A^{ji}}\,\mathbf{e}^i \otimes \mathbf{e}^j = \frac{\partial \mathbf{F}}{\partial A_{ji}}\,\mathbf{e}_i \otimes \mathbf{e}_j = \frac{\partial \mathbf{F}}{\partial A^j{}_i}\,\mathbf{e}_i \otimes \mathbf{e}^j = \frac{\partial \mathbf{F}}{\partial A_j{}^i}\,\mathbf{e}^i \otimes \mathbf{e}_j, \tag{1.9.6}$$

for then

$$\mathrm{d}\mathbf{F} = \frac{\partial \mathbf{F}}{\partial A^{ij}}\,\mathrm{d}A^{ij} = \frac{\partial \mathbf{F}}{\partial A_{ij}}\,\mathrm{d}A_{ij} = \frac{\partial \mathbf{F}}{\partial A^i{}_j}\,\mathrm{d}A^i{}_j = \frac{\partial \mathbf{F}}{\partial A_i{}^j}\,\mathrm{d}A_i{}^j. \tag{1.9.7}$$

For example,

$$\frac{\partial \mathbf{F}}{\partial \mathbf{A}} = \frac{\partial F_{ij}}{\partial A^{lk}}\,\mathbf{e}^i \otimes \mathbf{e}^j \otimes \mathbf{e}^k \otimes \mathbf{e}^l. \tag{1.9.8}$$

As an illustration, if $\mathbf{A}$ is symmetric and invertible second-order tensor, by taking a gradient of $\mathbf{A}\cdot\mathbf{A}^{-1} = \mathbf{I}$ with respect to $\mathbf{A}$, it readily follows that

$$\frac{\partial A^{-1}_{ij}}{\partial A_{kl}} = -\frac{1}{2}\left(A^{-1}_{ik}A^{-1}_{jl} + A^{-1}_{il}A^{-1}_{jk}\right). \tag{1.9.9}$$

The gradients of the three invariants of $\mathbf{A}$ in Eqs. (1.3.3)–(1.3.5) are

$$\frac{\partial J_1}{\partial \mathbf{A}} = \mathbf{I}, \quad \frac{\partial J_2}{\partial \mathbf{A}} = \mathbf{A} - J_1\mathbf{I}, \quad \frac{\partial J_3}{\partial \mathbf{A}} = \mathbf{A}^2 - J_1\mathbf{A} - J_2\mathbf{I}. \tag{1.9.10}$$

Since $\mathbf{A}^2$ has the same principal directions as $\mathbf{A}$, the gradients in Eq. (1.9.10) also have the same principal directions as $\mathbf{A}$. It is also noted that by the Cayley–Hamilton theorem (1.4.1), the last of Eq. (1.9.10) can be rewritten as

$$\frac{\partial J_3}{\partial \mathbf{A}} = J_3 \mathbf{A}^{-1}, \quad \text{i.e.,} \quad \frac{\partial(\det \mathbf{A})}{\partial \mathbf{A}} = (\det \mathbf{A})\, \mathbf{A}^{-1}. \tag{1.9.11}$$

Furthermore, if $\mathbf{F} = \mathbf{A} \cdot \mathbf{A}^T$, then with respect to an orthonormal basis

$$\frac{\partial A_{ij}}{\partial A_{kl}} = \delta_{ik}\delta_{jl}, \quad \frac{\partial F_{ij}}{\partial A_{kl}} = \delta_{ik} A_{jl} + \delta_{jk} A_{il}. \tag{1.9.12}$$

The gradients of the principal invariants $\bar{J}_i$ of $\mathbf{A} \cdot \mathbf{A}^T$ with respect to $\mathbf{A}$ are consequently

$$\frac{\partial \bar{J}_1}{\partial \mathbf{A}} = 2\mathbf{A}^T, \quad \frac{\partial \bar{J}_2}{\partial \mathbf{A}} = 2\left(\mathbf{A}^T \cdot \mathbf{A} \cdot \mathbf{A}^T - \bar{J}_1 \mathbf{A}^T\right), \quad \frac{\partial \bar{J}_3}{\partial \mathbf{A}} = 2\bar{J}_3 \mathbf{A}^{-1}. \tag{1.9.13}$$

1.10. Isotropic Tensors

An isotropic tensor is one whose components in an orthonormal basis remain unchanged by any proper orthogonal transformation (rotation) of the basis. All scalars are isotropic zero-order tensors. There are no isotropic first-order tensors (vectors), except the zero-vector. The only isotropic second-order tensors are scalar multiples of the second-order unit tensor δ_{ij}. The scalar multiples of the permutation tensor ϵ_{ijk} are the only isotropic third-order tensors. The most general isotropic fourth-order tensor has the components

$$\mathcal{L}_{ijkl} = a\, \delta_{ij}\delta_{kl} + b\, \delta_{ik}\delta_{jl} + c\, \delta_{il}\delta_{jk}, \tag{1.10.1}$$

where a, b, c are scalars. If $\boldsymbol{\mathcal{L}}$ is symmetric, $b = c$ and

$$\mathcal{L}_{ijkl} = a\, \delta_{ij}\delta_{kl} + 2b\, I_{ijkl}. \tag{1.10.2}$$

Isotropic tensors of even order can be expressed as a linear combination of outer products of the Kronecker deltas only; those of odd order can be expressed as a linear combination of outer products of the Kronecker deltas and permutation tensors. Since the outer product of two permutation tensors,

$$\epsilon_{ijk}\epsilon_{\alpha\beta\gamma} = \begin{vmatrix} \delta_{i\alpha} & \delta_{i\beta} & \delta_{i\gamma} \\ \delta_{j\alpha} & \delta_{j\beta} & \delta_{j\gamma} \\ \delta_{k\alpha} & \delta_{k\beta} & \delta_{k\gamma} \end{vmatrix}, \tag{1.10.3}$$

is expressed solely in terms of the Kronecker deltas, each term of an isotropic tensor of odd order contains at most one permutation tensor. Such tensors change sign under improper orthogonal transformation. Isotropic tensors of even order are unchanged under both proper and improper orthogonal transformations. For example, the components of an isotropic symmetric sixth-order tensor are

$$S_{ijklmn} = a\, \delta_{ij}\delta_{kl}\delta_{mn} + b\, \delta_{(ij} I_{klmn)} + c\, \delta_{(ik}\delta_{lm}\delta_{nj)}, \tag{1.10.4}$$

where the notation such as $\delta_{(ij}I_{klmn)}$ designates the symmetrization with respect to i and j, k and l, m and n, and ij, kl and mn (Eringen, 1971). Specifically,

$$\begin{aligned} \delta_{(ij}I_{klmn)} &= \frac{1}{3}\left(\delta_{ij}I_{klmn} + \delta_{kl}I_{mnij} + \delta_{mn}I_{ijkl}\right), \\ \delta_{(ik}\delta_{lm}\delta_{nj)} &= \frac{1}{4}\left(\delta_{ik}I_{jlmn} + \delta_{il}I_{jkmn} + \delta_{im}I_{klnj} + \delta_{in}I_{klmj}\right). \end{aligned} \tag{1.10.5}$$

In some applications it may be convenient to introduce the fourth-order base tensors (Hill, 1965; Walpole, 1981)

$$\boldsymbol{K} = \frac{1}{3}\,\mathbf{I}\otimes\mathbf{I}, \quad \boldsymbol{J} = \boldsymbol{I} - \boldsymbol{K}. \tag{1.10.6}$$

These tensors are such that $\operatorname{tr}\boldsymbol{K} = K_{ijij} = 1$, $\operatorname{tr}\boldsymbol{J} = J_{ijij} = 5$, and

$$\boldsymbol{J}:\boldsymbol{J} = \boldsymbol{J}, \quad \boldsymbol{K}:\boldsymbol{K} = \boldsymbol{K}, \quad \boldsymbol{J}:\boldsymbol{K} = \boldsymbol{K}:\boldsymbol{J} = \mathbf{0}. \tag{1.10.7}$$

Consequently,

$$(a_1\,\boldsymbol{J} + b_1\,\boldsymbol{K}) : (a_2\,\boldsymbol{J} + b_2\,\boldsymbol{K}) = a_1 a_2\,\boldsymbol{J} + b_1 b_2\,\boldsymbol{K}, \tag{1.10.8}$$

$$(a_1\,\boldsymbol{J} + b_1\,\boldsymbol{K})^{-1} = a_1^{-1}\,\boldsymbol{J} + b_1^{-1}\,\boldsymbol{K}. \tag{1.10.9}$$

An isotropic fourth-order tensor $\boldsymbol{\mathcal{L}}$ can be decomposed in this basis as

$$\boldsymbol{\mathcal{L}} = \mathcal{L}_J\,\boldsymbol{J} + \mathcal{L}_K\,\boldsymbol{K}, \tag{1.10.10}$$

where

$$\mathcal{L}_K = \operatorname{tr}(\boldsymbol{\mathcal{L}}:\boldsymbol{K}), \quad \mathcal{L}_K + 5\,\mathcal{L}_J = \operatorname{tr}\boldsymbol{\mathcal{L}}. \tag{1.10.11}$$

Product of any pair of isotropic fourth-order tensors is isotropic and commutative. The base tensors $\boldsymbol{K}$ and $\boldsymbol{J}$ partition the second-order tensor $\mathbf{A}$ into its spherical and deviatoric parts, such that

$$\mathbf{A}_{\text{sph}} = \boldsymbol{K}:\mathbf{A} = \frac{1}{3}(\operatorname{tr}\mathbf{A})\,\mathbf{I}, \quad \mathbf{A}_{\text{dev}} = \boldsymbol{J}:\mathbf{A} = \mathbf{A} - \mathbf{A}_{\text{sph}}. \tag{1.10.12}$$

1.11. Isotropic Functions

1.11.1. Isotropic Scalar Functions

A scalar function of the second-order symmetric tensor argument is said to be an isotropic function if

$$f\left(\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T\right) = f(\mathbf{A}), \tag{1.11.1}$$

where $\mathbf{Q}$ is an arbitrary proper orthogonal (rotation) tensor. Such a function depends on $\mathbf{A}$ only through its three invariants, $f = f(J_1, J_2, J_3)$. For isotropic $f(\mathbf{A})$, the principal directions of the gradient $\partial f/\partial\mathbf{A}$ are parallel to those of $\mathbf{A}$. This follows because the gradients $\partial J_i/\partial\mathbf{A}$ are all parallel to $\mathbf{A}$, by Eq. (1.9.10).

A scalar function of two symmetric second-order tensors $\mathbf{A}$ and $\mathbf{B}$ is said to be an isotropic function of both $\mathbf{A}$ and $\mathbf{B}$, if

$$f\left(\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T, \mathbf{Q}\cdot\mathbf{B}\cdot\mathbf{Q}^T\right) = f(\mathbf{A},\mathbf{B}). \tag{1.11.2}$$

Such a function can be represented as a polynomial of its irreducible integrity basis consisting of the individual and joint invariants of $\mathbf{A}$ and $\mathbf{B}$. The independent joint invariants are the traces of the following products

$$(\mathbf{A}\cdot\mathbf{B}),\quad \left(\mathbf{A}\cdot\mathbf{B}^2\right)^*,\quad \left(\mathbf{A}^2\cdot\mathbf{B}^2\right). \tag{1.11.3}$$

The joint invariants of three symmetric second-order tensors are the traces of

$$(\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{C}),\quad \left(\mathbf{A}^2\cdot\mathbf{B}\cdot\mathbf{C}\right)^*,\quad \left(\mathbf{A}^2\cdot\mathbf{B}^2\cdot\mathbf{C}\right)^*. \tag{1.11.4}$$

A superposed asterisk ($*$) indicates that the integrity basis also includes invariants formed by cyclic permutation of symmetric tensors involved. The integrity basis can be written for any finite set of second-order tensors. Spencer (1971) provides a list of invariants and integrity bases for a polynomial scalar function dependent on one up to six second-order symmetric tensors. An integrity basis for an arbitrary number of tensors is obtained by taking the bases for the tensors six at a time, in all possible combinations. For invariants of second-order tensors alone, it is not necessary to distinguish between the full and the proper orthogonal groups.

The trace of an antisymmetric tensor, or any power of it, is equal to zero, so that the integrity basis for the antisymmetric tensor $\mathbf{X}$ is $\operatorname{tr}(\mathbf{X}^2)$. A joint invariant of two antisymmetric tensors $\mathbf{X}$ and $\mathbf{Y}$ is $\operatorname{tr}(\mathbf{X}\cdot\mathbf{Y})$. The independent joint invariants of a symmetric tensor $\mathbf{A}$ and an antisymmetric tensor $\mathbf{X}$ are the traces of the products

$$\left(\mathbf{X}^2\cdot\mathbf{A}\right),\quad \left(\mathbf{X}^2\cdot\mathbf{A}^2\right),\quad \left(\mathbf{X}^2\cdot\mathbf{A}^2\cdot\mathbf{X}\cdot\mathbf{A}^2\right). \tag{1.11.5}$$

In the case of two symmetric and one antisymmetric tensor, the joint invariants include the traces of

$$\begin{array}{lll} (\mathbf{X}\cdot\mathbf{A}\cdot\mathbf{B}), & (\mathbf{X}\cdot\mathbf{A}^2\cdot\mathbf{B})^*, & (\mathbf{X}\cdot\mathbf{A}^2\cdot\mathbf{B}^2),\\ (\mathbf{X}\cdot\mathbf{A}^2\cdot\mathbf{B}\cdot\mathbf{A})^*, & (\mathbf{X}\cdot\mathbf{A}^2\cdot\mathbf{B}^2\cdot\mathbf{A})^*, & (\mathbf{X}^2\cdot\mathbf{A}\cdot\mathbf{B}),\\ (\mathbf{X}^2\cdot\mathbf{A}^2\cdot\mathbf{B})^*, & (\mathbf{X}^2\cdot\mathbf{A}\cdot\mathbf{X}\cdot\mathbf{B}), & (\mathbf{X}^2\cdot\mathbf{A}\cdot\mathbf{X}\cdot\mathbf{B}^2)^*. \end{array} \tag{1.11.6}$$

1.11.2. Isotropic Tensor Functions

A second-order tensor function is said to be an isotropic function of its second-order tensor argument if

$$\mathbf{F}\left(\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T\right)=\mathbf{Q}\cdot\mathbf{F}(\mathbf{A})\cdot\mathbf{Q}^T. \tag{1.11.7}$$

An isotropic symmetric function of a symmetric tensor $\mathbf{A}$ can be expressed as

$$\mathbf{F}(\mathbf{A})=a_0\mathbf{I}+a_1\mathbf{A}+a_2\mathbf{A}^2, \tag{1.11.8}$$

where a_i are scalar functions of the principal invariants of $\mathbf{A}$.

A second-order tensor function is said to be an isotropic function of its two second-order tensor arguments if

$$\mathbf{F}\left(\mathbf{Q}\cdot\mathbf{A}\cdot\mathbf{Q}^T,\mathbf{Q}\cdot\mathbf{B}\cdot\mathbf{Q}^T\right)=\mathbf{Q}\cdot\mathbf{F}(\mathbf{A},\mathbf{B})\cdot\mathbf{Q}^T. \tag{1.11.9}$$

An isotropic symmetric tensor function which is a polynomial of two symmetric tensors $\mathbf{A}$ and $\mathbf{B}$ can be expressed in terms of nine tensors, such that

$$\begin{aligned}\mathbf{F}(\mathbf{A},\mathbf{B}) = a_1\mathbf{I} + a_2\mathbf{A} + a_3\mathbf{A}^2 + a_4\mathbf{B} + a_5\mathbf{B}^2 \\ + a_6\left(\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A}\right) + a_7\left(\mathbf{A}^2\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A}^2\right) \\ + a_8\left(\mathbf{A}\cdot\mathbf{B}^2 + \mathbf{B}^2\cdot\mathbf{A}\right) + a_9\left(\mathbf{A}^2\cdot\mathbf{B}^2 + \mathbf{B}^2\cdot\mathbf{A}^2\right).\end{aligned} \tag{1.11.10}$$

The scalars a_i are scalar functions of ten individual and joint invariants of **A** and **B**. An antisymmetric tensor polynomial function of two symmetric tensors allows a representation

$$\begin{aligned}\mathbf{F}(\mathbf{A},\mathbf{B}) = a_1(\mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A}) + a_2\left(\mathbf{A}^2\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A}^2\right) \\ + a_3\left(\mathbf{B}^2\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B}^2\right) + a_4\left(\mathbf{A}^2\cdot\mathbf{B}^2 - \mathbf{B}^2\cdot\mathbf{A}^2\right) \\ + a_5\left(\mathbf{A}^2\cdot\mathbf{B}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{A}^2\right) + a_6\left(\mathbf{B}^2\cdot\mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A}\cdot\mathbf{B}^2\right) \\ + a_7\left(\mathbf{A}^2\cdot\mathbf{B}^2\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B}^2\cdot\mathbf{A}^2\right) + a_8\left(\mathbf{B}^2\cdot\mathbf{A}^2\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A}^2\cdot\mathbf{B}^2\right).\end{aligned} \tag{1.11.11}$$

A derivation of Eq. (1.11.11) is instructive. The most general scalar invariant of two symmetric and one antisymmetric tensor **X**, linear in **X**, can be written from Eq. (1.11.6) as

$$\begin{aligned}g(\mathbf{A},\mathbf{B},\mathbf{X}) = a_1\,\mathrm{tr}\left[(\mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A})\cdot\mathbf{X}\right] + a_2\,\mathrm{tr}\left[\left(\mathbf{A}^2\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A}^2\right)\cdot\mathbf{X}\right] \\ + a_3\,\mathrm{tr}\left[\left(\mathbf{B}^2\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B}^2\right)\cdot\mathbf{X}\right] + a_4\,\mathrm{tr}\left[\left(\mathbf{A}^2\cdot\mathbf{B}^2 - \mathbf{B}^2\cdot\mathbf{A}^2\right)\cdot\mathbf{X}\right] \\ + a_5\,\mathrm{tr}\left[\left(\mathbf{A}^2\cdot\mathbf{B}\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{A}^2\right)\cdot\mathbf{X}\right] + a_6\,\mathrm{tr}\left[\left(\mathbf{B}^2\cdot\mathbf{A}\cdot\mathbf{B}\right.\right. \\ \left.\left.-\ \mathbf{B}\cdot\mathbf{A}\cdot\mathbf{B}^2\right)\cdot\mathbf{X}\right] + a_7\,\mathrm{tr}\left[\left(\mathbf{A}^2\cdot\mathbf{B}^2\cdot\mathbf{A} - \mathbf{A}\cdot\mathbf{B}^2\cdot\mathbf{A}^2\right)\cdot\mathbf{X}\right] \\ + a_8\,\mathrm{tr}\left[\left(\mathbf{B}^2\cdot\mathbf{A}^2\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A}^2\cdot\mathbf{B}^2\right)\cdot\mathbf{X}\right].\end{aligned} \tag{1.11.12}$$

The coefficients a_i depend on the invariants of **A** and **B**. Recall that the trace of the product of symmetric and antisymmetric matrix, such as $(\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A})\cdot\mathbf{X}$, is equal to zero. The antisymmetric function $\mathbf{F}(\mathbf{A},\mathbf{B})$ is obtained from Eq. (1.11.12) as the gradient $\partial g/\partial\mathbf{X}$, which yields Eq. (1.11.11).

1.12. Rivlin's Identities

Applying the Cayley–Hamilton theorem to a second-order tensor $a\mathbf{A} + b\mathbf{B}$, where a and b are arbitrary scalars, and equating to zero the coefficient of a^2b, gives

$$\begin{aligned}\mathbf{A}^2\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A}^2 + \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{A} - I_A(\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A}) - I_B\mathbf{A}^2 - II_A\mathbf{B} \\ - \left[\mathrm{tr}\,(\mathbf{A}\cdot\mathbf{B}) - I_AI_B\right]\mathbf{A} - \left[III_A\,\mathrm{tr}\left(\mathbf{A}^{-1}\cdot\mathbf{B}\right)\right]\mathbf{I} = \mathbf{0}.\end{aligned} \tag{1.12.1}$$

The principal invariants of **A** and **B** are denoted by I_A, I_B, etc. Identity (1.12.1) is known as the Rivlin's identity (Rivlin, 1955). If $\mathbf{B} = \mathbf{A}$, the original Cayley–Hamilton theorem of Eq. (1.4.1) is recovered. In addition, from the Cayley–Hamilton theorem we have

$$III_A\,\mathrm{tr}\left(\mathbf{A}^{-1}\cdot\mathbf{B}\right) = \mathrm{tr}\left(\mathbf{A}^2\cdot\mathbf{B}\right) - I_A\,\mathrm{tr}\,(\mathbf{A}\cdot\mathbf{B}) - I_BII_A. \tag{1.12.2}$$

An identity among three tensors is obtained by applying the Cayley–Hamilton theorem to a second-order tensor $a\mathbf{A} + b\mathbf{B} + c\mathbf{C}$, and by equating to zero the coefficient of abc.

Suppose that $\mathbf{A}$ is symmetric, and $\mathbf{B}$ is antisymmetric. Equations (1.12.1) and (1.12.2) can then be rewritten as

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A}) + (\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A})\cdot\mathbf{A} - I_A(\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A})\\ - II_A\mathbf{B} - \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{A} = \mathbf{0}.\end{aligned} \tag{1.12.3}$$

Postmultiplying Eq. (1.12.3) with $\mathbf{A}$ and using the Cayley–Hamilton theorem yields another identity

$$\mathbf{A}\cdot(\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A})\cdot\mathbf{A} + III_A\mathbf{B} - \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{A} = \mathbf{0}. \tag{1.12.4}$$

If A is invertible, Eq. (1.12.4) is equivalent to

$$III_A\mathbf{A}^{-1}\cdot\mathbf{B}\cdot\mathbf{A}^{-1} = I_A\mathbf{B} - (\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A}). \tag{1.12.5}$$

1.12.1. Matrix Equation $\mathbf{A}\cdot\mathbf{X}+\mathbf{X}\cdot\mathbf{A}=\mathbf{B}$

The matrix equation

$$\mathbf{A}\cdot\mathbf{X}+\mathbf{X}\cdot\mathbf{A}=\mathbf{B} \tag{1.12.6}$$

can be solved by using Rivlin's identities. Suppose $\mathbf{A}$ is symmetric and $\mathbf{B}$ is antisymmetric. The solution $\mathbf{X}$ of Eq. (1.12.6) is then an antisymmetric matrix, and the Rivlin identities (1.12.3) and (1.12.4) become

$$\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A} - I_A\mathbf{B} - II_A\mathbf{X} - \mathbf{A}\cdot\mathbf{X}\cdot\mathbf{A} = \mathbf{0}, \tag{1.12.7}$$

$$\mathbf{A}\cdot\mathbf{B}\cdot\mathbf{A} + III_A\mathbf{X} - I_A\mathbf{A}\cdot\mathbf{X}\cdot\mathbf{A} = \mathbf{0}. \tag{1.12.8}$$

Upon eliminating $\mathbf{A}\cdot\mathbf{X}\cdot\mathbf{A}$, we obtain the solution for $\mathbf{X}$

$$(I_A II_A + III_A)\mathbf{X} = I_A(\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A}) - I_A^2\mathbf{B} - \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{A}, \tag{1.12.9}$$

which can be rewritten as

$$(I_A II_A + III_A)\mathbf{X} = -(I_A\mathbf{I}-\mathbf{A})\cdot\mathbf{B}\cdot(I_A\mathbf{I}-\mathbf{A}). \tag{1.12.10}$$

Since

$$I_A II_A + III_A = -\det(I_A\mathbf{I}-\mathbf{A}), \tag{1.12.11}$$

and having in mind Eq. (1.12.5), the solution for $\mathbf{X}$ in Eq. (1.12.10) can be expressed in an alternative form

$$\mathbf{X} = [\operatorname{tr}(I_A\mathbf{I}-\mathbf{A})^{-1}]\,\mathbf{B} - (I_A\mathbf{I}-\mathbf{A})^{-1}\cdot\mathbf{B} - \mathbf{B}\cdot(I_A\mathbf{I}-\mathbf{A})^{-1}, \tag{1.12.12}$$

provided that $I_A\mathbf{I}-\mathbf{A}$ is not a singular matrix.

Consider now the solution of Eq. (1.12.6) when both $\mathbf{A}$ and $\mathbf{B}$ are symmetric, and so is $\mathbf{X}$. If Eq. (1.12.6) is premultiplied by $\mathbf{A}$, it can be recast in the form

$$\mathbf{A}\cdot\left(\mathbf{A}\cdot\mathbf{X}-\frac{1}{2}\mathbf{B}\right) + \left(\mathbf{A}\cdot\mathbf{X}-\frac{1}{2}\mathbf{B}\right)\cdot\mathbf{A} = \frac{1}{2}\,(\mathbf{A}\cdot\mathbf{B}-\mathbf{B}\cdot\mathbf{A}). \tag{1.12.13}$$

Since the right-hand side of this equation is an antisymmetric matrix, it follows that

$$\frac{\mathbf{Y}}{2} = \mathbf{A}\cdot\mathbf{X} - \frac{1}{2}\,\mathbf{B} = \frac{1}{2}\,\mathbf{B} - \mathbf{X}\cdot\mathbf{A} \tag{1.12.14}$$

is also antisymmetric, and Eq. (1.12.13) has the solution for $\mathbf{Y}$ according to Eq. (1.12.10) or (1.12.12), e.g.,

$$(I_A II_A + III_A)\mathbf{Y} = -(I_A\mathbf{I} - \mathbf{A})\cdot(\mathbf{A}\cdot\mathbf{B} - \mathbf{B}\cdot\mathbf{A})\cdot(I_A\mathbf{I} - \mathbf{A}). \tag{1.12.15}$$

Thus, from Eq. (1.12.14), the solution for $\mathbf{X}$ is

$$\mathbf{X} = \frac{1}{4}\left[\mathbf{A}^{-1}(\mathbf{B}+\mathbf{Y}) + (\mathbf{B}-\mathbf{Y})\cdot\mathbf{A}^{-1}\right]. \tag{1.12.16}$$

For further analysis the papers by Sidoroff (1978), Guo (1984), and Scheidler (1994) can be consulted.

1.13. Tensor Fields

Tensors fields are comprised by tensors whose values depend on the position in space. For simplicity, consider the rectangular Cartesian coordinates. The position vector of an arbitrary point of three-dimensional space is $\mathbf{x} = x_i\mathbf{e}_i$, where $\mathbf{e}_i$ are the unit vectors in the coordinate directions. The tensor field is denoted by $\mathbf{T}(\mathbf{x})$. This can represent a scalar field $f(\mathbf{x})$, a vector field $\mathbf{a}(\mathbf{x})$, a second-order tensor field $\mathbf{A}(\mathbf{x})$, or any higher-order tensor field. It is assumed that $\mathbf{T}(\mathbf{x})$ is differentiable at a point $\mathbf{x}$ of the considered domain.

1.13.1. Differential Operators

The gradient of a scalar field $f = f(\mathbf{x})$ is the operator which gives a directional derivative of f, such that

$$\mathrm{d}f = \boldsymbol{\nabla} f\cdot\mathrm{d}\mathbf{x}. \tag{1.13.1}$$

Thus, with respect to rectangular Cartesian coordinates,

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial x_i}\,\mathbf{e}_i, \quad \boldsymbol{\nabla} = \frac{\partial}{\partial x_i}\,\mathbf{e}_i. \tag{1.13.2}$$

In particular, if $\mathrm{d}\mathbf{x}$ is taken to be parallel to the level surface $f(\mathbf{x}) = \text{const.}$, it follows that $\boldsymbol{\nabla} f$ is normal to the level surface at the considered point (Fig. 1.3).

The gradient of a vector field $\mathbf{a} = \mathbf{a}(\mathbf{x})$, and its transpose, are the second-order tensors

$$\boldsymbol{\nabla}\mathbf{a} = \boldsymbol{\nabla}\otimes\mathbf{a} = \frac{\partial a_j}{\partial x_i}\,\mathbf{e}_i\otimes\mathbf{e}_j, \quad \mathbf{a}\boldsymbol{\nabla} = \mathbf{a}\otimes\boldsymbol{\nabla} = \frac{\partial a_i}{\partial x_j}\,\mathbf{e}_i\otimes\mathbf{e}_j. \tag{1.13.3}$$

They are introduced such that

$$\mathrm{d}\mathbf{a} = (\mathbf{a}\boldsymbol{\nabla})\cdot\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{x}\cdot(\boldsymbol{\nabla}\mathbf{a}). \tag{1.13.4}$$

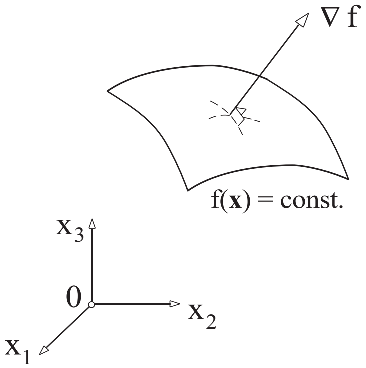

FIGURE 1.3. The gradient $\boldsymbol{\nabla} f$ is perpendicular to the level surface $f(\mathbf{x}) = \text{const.}$

The gradient of a second-order tensor field $\mathbf{A} = \mathbf{A}(\mathbf{x})$ is similarly

$$\boldsymbol{\nabla}\mathbf{A} = \boldsymbol{\nabla} \otimes \mathbf{A} = \frac{\partial A_{ij}}{\partial x_k}\, \mathbf{e}_k \otimes \mathbf{e}_i \otimes \mathbf{e}_j, \quad \mathbf{A}\boldsymbol{\nabla} = \mathbf{A} \otimes \boldsymbol{\nabla} = \frac{\partial A_{ij}}{\partial x_k}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \tag{1.13.5}$$

so that

$$\mathrm{d}\mathbf{A} = (\mathbf{A}\boldsymbol{\nabla}) \cdot \mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{x} \cdot (\boldsymbol{\nabla}\mathbf{A}). \tag{1.13.6}$$

The divergence of a vector field is the scalar

$$\boldsymbol{\nabla} \cdot \mathbf{a} = \mathrm{tr}\,(\boldsymbol{\nabla}\mathbf{a}) = \frac{\partial a_i}{\partial x_i}\,. \tag{1.13.7}$$

The divergence of the gradient of a scalar field is

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f) = \nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i}\,, \quad \nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i}\,. \tag{1.13.8}$$

The operator ∇^2 is the Laplacian operator. The divergence of the gradient of a vector field can be written as

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla}\mathbf{a}) = \nabla^2 \mathbf{a} = \frac{\partial^2 a_i}{\partial x_j \partial x_j}\, \mathbf{e}_i. \tag{1.13.9}$$

The divergence of a second-order tensor field is defined by

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \frac{\partial A_{ij}}{\partial x_i}\, \mathbf{e}_j, \quad \mathbf{A} \cdot \boldsymbol{\nabla} = \frac{\partial A_{ij}}{\partial x_j}\, \mathbf{e}_i. \tag{1.13.10}$$

The curl of a vector field is the vector

$$\boldsymbol{\nabla} \times \mathbf{a} = \epsilon_{ijk}\, \frac{\partial a_j}{\partial x_i}\, \mathbf{e}_k. \tag{1.13.11}$$

It can be shown that the vector field $\boldsymbol{\nabla} \times \mathbf{a}$ is an axial vector field of the antisymmetric tensor field $(\mathbf{a}\boldsymbol{\nabla} - \boldsymbol{\nabla}\mathbf{a})$. The curl of a second-order tensor

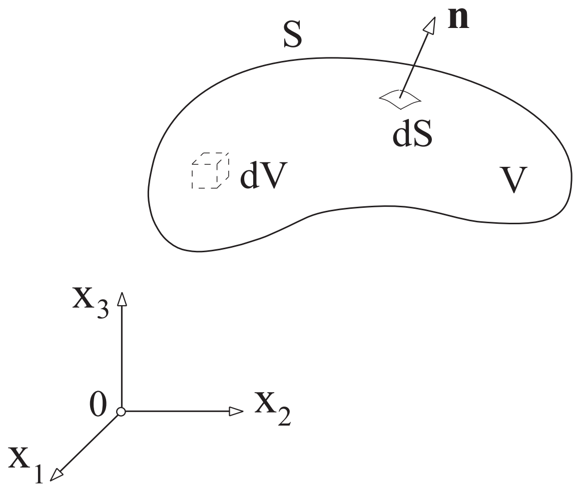

FIGURE 1.4. Three-dimensional domain V bounded by a closed surface S with unit outward normal $\mathbf{n}$.

field is similarly

$$\boldsymbol{\nabla} \times \mathbf{A} = \epsilon_{ijk} \frac{\partial A_{jl}}{\partial x_i} \mathbf{e}_k \otimes \mathbf{e}_l. \tag{1.13.12}$$

It is noted that $\mathbf{A} \times \boldsymbol{\nabla} = -\left(\boldsymbol{\nabla} \times \mathbf{A}^T\right)^T$, while $\mathbf{a} \times \boldsymbol{\nabla} = -\boldsymbol{\nabla} \times \mathbf{a}$.

We list bellow three formulas used later in the book. If $\mathbf{a}$ is an arbitrary vector, $\mathbf{x}$ is a position vector, and if $\mathbf{A}$ and $\mathbf{B}$ are two second-order tensors, then

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \cdot \mathbf{a}) = (\boldsymbol{\nabla} \cdot \mathbf{A}) \cdot \mathbf{a} + \mathbf{A} : (\boldsymbol{\nabla} \otimes \mathbf{a}), \tag{1.13.13}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \cdot \mathbf{B}) = (\boldsymbol{\nabla} \cdot \mathbf{A}) \cdot \mathbf{B} + \left(\mathbf{A}^T \cdot \boldsymbol{\nabla}\right) \cdot \mathbf{B}, \tag{1.13.14}$$

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{x}) = (\boldsymbol{\nabla} \cdot \mathbf{A}) \times \mathbf{x} - \boldsymbol{\epsilon} : \mathbf{A}. \tag{1.13.15}$$

The permutation tensor is $\boldsymbol{\epsilon}$, and : designates the inner product, defined by Eq. (1.2.13). The nabla operator in Eqs. (1.13.13)–(1.13.15) acts on the quantity to the right of it. The formulas can be easily proven by using the component tensor representations. A comprehensive treatment of tensor fields can be found in Truesdell and Toupin (1960), and Ericksen (1960).

1.13.2. Integral Transformation Theorems

Let V be a three dimensional domain bounded by a closed surface S with unit outward normal $\mathbf{n}$ (Fig. 1.4). For a tensor field $\mathbf{T} = \mathbf{T}(\mathbf{x})$, continuously

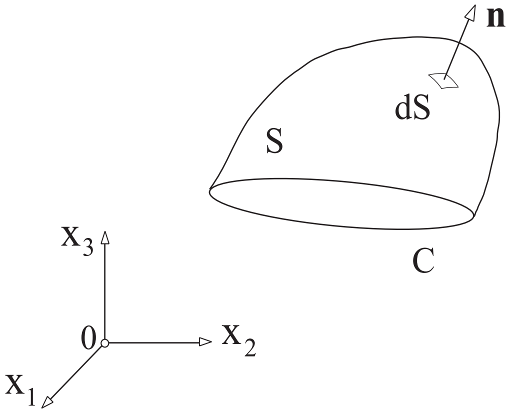

FIGURE 1.5. An open surface S with unit outward normal $\mathbf{n}$ and a bounding edge C.

differentiable in V and continuous on S, the generalized Gauss theorem asserts that

$$\int_V (\boldsymbol{\nabla} * \mathbf{T})\,\mathrm{d}V = \int_S \mathbf{n} * \mathbf{T}\,\mathrm{d}S. \tag{1.13.16}$$

The asterisk $(*)$ product can be either a dot $(\cdot)$ or cross $(\times)$ product, and $\mathbf{T}$ represents a scalar, vector, second- or higher-order tensor field (Malvern, 1969). For example, for a second-order tensor field $\mathbf{A}$, expressed in rectangular Cartesian coordinates,

$$\int_V \frac{\partial A_{ij}}{\partial x_i}\,\mathrm{d}V = \int_S n_i A_{ij}\,\mathrm{d}S. \tag{1.13.17}$$

Let S be a portion of an oriented surface with unit outward normal $\mathbf{n}$. The bounding edge of the surface is a closed curve C (Fig. 1.5). For tensors fields that are continuously differentiable in S and continuous on C, the generalized Stokes theorem asserts that

$$\int_S (\mathbf{n} \times \boldsymbol{\nabla}) * \mathbf{T}\,\mathrm{d}S = \int_C \mathrm{d}\mathbf{C} * \mathbf{T}. \tag{1.13.18}$$

For example, for a second-order tensor $\mathbf{A}$ this becomes, in the rectangular Cartesian coordinates,

$$\int_S \epsilon_{ijk} n_i \frac{\partial A_{kl}}{\partial x_j}\,\mathrm{d}S = \int_C A_{kl}\,\mathrm{d}C_k. \tag{1.13.19}$$

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