

# CHAPTER 3

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## Normal Force, Shear Force, Bending Moment and Torsion

The purpose of a structure is to support the loads for which it has been designed. To accomplish this it must be able to transmit a load from one point to another, i.e. from the loading point to the supports. In Fig. 2.21, for example, the beam transmits the effects of the loads at B and C to the built-in end A. This it achieves by developing an *internal force* system and it is the distribution of these internal forces which must be determined before corresponding stress distributions and displacements can be found.

A knowledge of stress is essential in structural design where the cross-sectional area of a member must be such that stresses do not exceed values that would cause breakdown in the crystalline structure of the material of the member; in other words, a structural failure. In addition to stresses, strains, and thereby displacements, must be calculated to ensure that as well as strength a structural member possesses sufficient stiffness to prevent excessive distortions damaging surrounding portions of a complete structure.

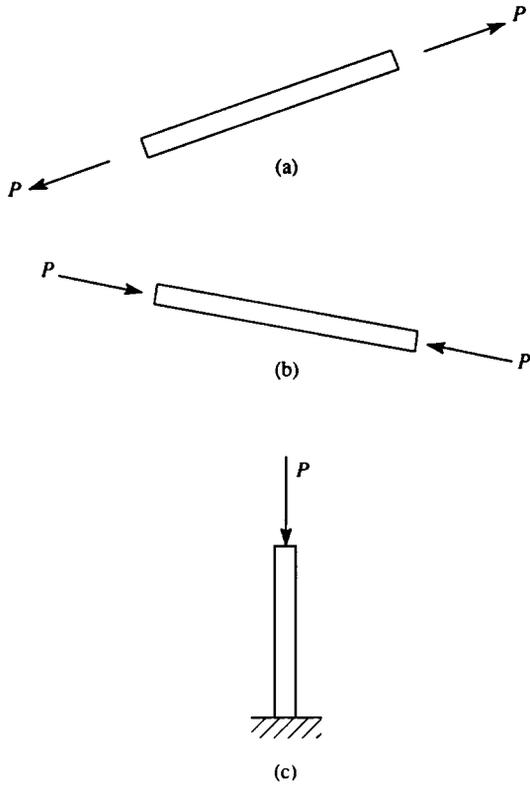
In this chapter we shall examine the different types of load to which a structural member may be subjected and then determine corresponding internal force distributions.

### 3.1 Types of load

Structural members may be subjected to complex loading systems apparently comprised of several different types of load. However, no matter how complex such systems appear to be they consist of a maximum of four basic load types: axial loads, shear loads, bending moments and torsion.

#### Axial load

Axial loads are applied along the longitudinal or centroidal axis of a structural member. If the action of the load is to increase the length of the member, the member is said to be in *tension* (Fig. 3.1(a)) and the applied load is *tensile*. A load that tends to shorten a member places the member in *compression* and is known as a *compressive* load (Fig. 3.1(b)). Members such as those shown in Figs 3.1(a) and (b) are commonly found in pin-jointed frameworks where a member in tension is called a *tie* and one in compression a *strut* or *column*. More frequently, however, the name



**Fig. 3.1** Axially loaded members

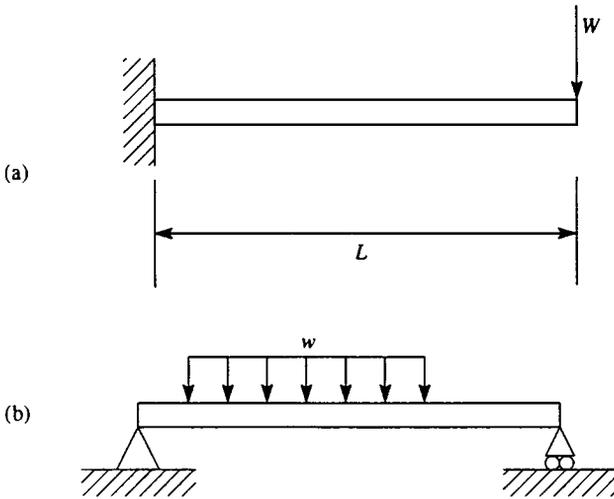
‘column’ is associated with a vertical member carrying a compressive load, as illustrated in Fig. 3.1(c).

**Shear load**

Shear loads act perpendicularly to the axis of a structural member and have one of the forms shown in Fig. 3.2; in this case the members are *beams*. Fig. 3.2(a) shows a *concentrated* shear load,  $W$ , applied to a cantilever beam. The shear load in Fig. 3.2(b) is *distributed* over a length of the beam and is of *intensity*  $w$  (force units) per unit length.

A concentrated load on a beam is one which, theoretically, can be regarded as acting wholly at one point. In reality such a situation could not arise since this would imply that the bearing pressure between the load and the beam was infinitely large. Thus, in practice, all loads must be distributed over a finite length of beam. It is when this length of beam is small that we can consider, for the purposes of calculation, the load to be concentrated at one point.

Practical examples of loads that may be regarded as concentrated arise when a beam supports other transverse beams. Distributed loads occur in situations where a girder, for example, supports a wall or floor slab; other distributed loads result from wind forces. All beams in fact support a distributed load, their self-weight.



**Fig. 3.2** Shear loads applied to beams

### Bending moment

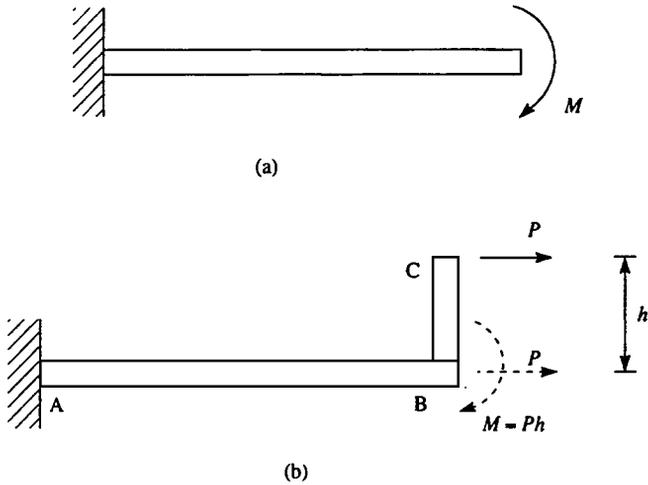
In practice it is difficult to apply a pure bending moment such as that shown in Fig. 3.3(a) to a beam. Generally, pure bending moments arise through the application of other types of load to adjacent structural members. For example, in Fig. 3.3(b), a vertical member BC is attached to the cantilever AB and carries a horizontal shear load,  $P$  (as far as BC is concerned). AB is therefore subjected to a pure moment,  $M = Ph$ , at B together with an axial load,  $P$ .

### Torsion

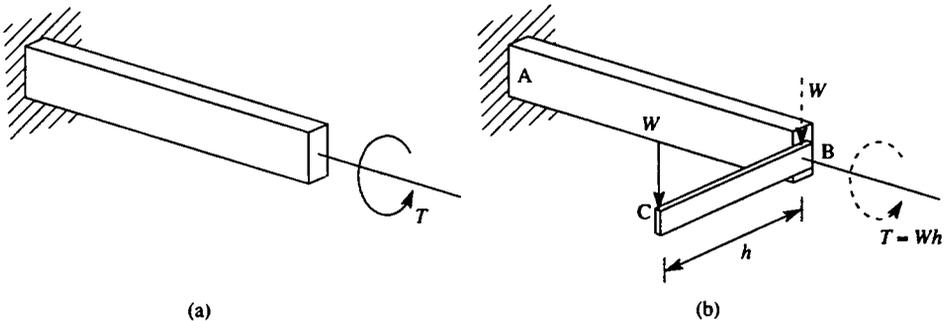
A similar situation arises in the application of a pure torque,  $T$  (Fig. 3.4(a)), to a beam. A practical example of a torque applied to a cantilever beam is given in Fig. 3.4(b) where the horizontal member BC supports a vertical shear load at C. The cantilever AB is then subjected to a pure torque,  $T = Wh$ , plus a shear load,  $W$ .

All the loads illustrated in Figs 3.1–3.4 are applied to the various members by some external agency and are therefore *externally applied loads*. Each of these loads induces reactions in the support systems of the different beams; examples of the calculation of support reactions are given in Section 2.5. Since structures are in equilibrium under a force system of externally applied loads and support reactions, it follows that the support reactions are themselves externally applied loads.

Now consider the cantilever beam of Fig. 3.2(a). If we were to physically cut through the beam at some section mm (Fig. 3.5(a)) the portion BC would no longer be able to support the load,  $W$ . The portion AB of the beam therefore performs the same function for the portion BC as does the wall for the complete beam. Thus at the section mm the portion AB applies a force  $W$  and a moment  $M$  to the portion BC at B, thereby maintaining its equilibrium (Fig. 3.5(b)); by the law of action and reaction, BC exerts an equal force system on AB, but opposite in direction. The complete force systems acting on the two faces of the section mm are shown in Fig. 3.5(b).



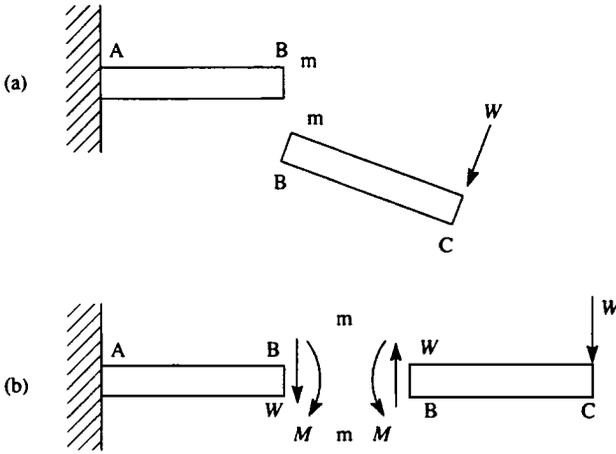
**Fig. 3.3** Moments applied to beams



**Fig. 3.4** Torques applied to a beam

Systems of forces such as those at the section mm are known as *internal forces*. Generally, they vary throughout the length of a structural member as can be seen from Fig. 3.5(b) where the internal moment,  $M$ , increases in magnitude as the built-in end is approached due to the increasing rotational effect of  $W$ . We note that applied loads of one type can induce internal forces of another. Thus in Fig. 3.5(b) the external shear load,  $W$ , produces both shear and bending at the section mm.

Internal forces are distributed throughout beam sections in the form of stresses. It follows that the resultant of each individual stress distribution must be the corresponding internal force; internal forces are therefore often known as *stress resultants*. However, before an individual stress distribution can be found it is necessary to determine the corresponding internal force. Also, in design problems, it is necessary to determine the position and value of maximum stress and displacement. Thus, usually, the first step in the analysis of a structure is to calculate the distribution of each of the four basic internal force types throughout the component structural members. We shall therefore determine the distributions of the four internal force systems in a variety of structural members. First, however, we shall establish a notation and sign convention for each type of force.



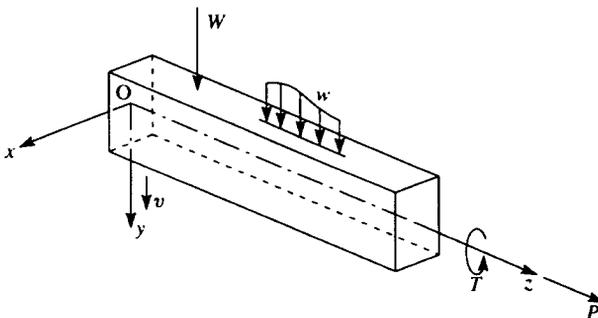
**Fig. 3.5** Internal force system generated by an external shear load

### 3.2 Notation and sign convention

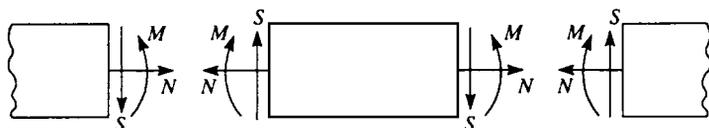
We shall be concerned initially with structural members having at least one longitudinal plane of symmetry. Normally this will be a vertical plane and will contain the externally applied loads. Later, however, we shall investigate the bending and shear of beams having unsymmetrical sections so that as far as possible the notation and sign convention we adopt now will be consistent with that required later. Some modifications will, however, be necessary.

The coordinates of all points in a structural member will be referred to a right-handed system of axes  $Oxyz$  as shown in Fig. 3.6.  $Oz$  forms the longitudinal axis of the beam and  $Oy$  is vertically downwards. Vertical externally applied loads  $W$  (concentrated) and  $w$  (distributed) are positive in the positive direction of the  $y$  axis as are vertical displacements of the member, which are given the symbol  $v$ . An axial load  $P$  is positive when tensile while a torque  $T$  is positive if applied in an anticlockwise sense when viewed in the direction  $zO$ .

Positive internal force systems are shown acting on a length of a structural member in Fig. 3.7. For equilibrium of the length of beam the internal forces must



**Fig. 3.6** Notation and sign conventions for displacements and externally applied loads



**Fig. 3.7** Positive internal force systems

act in opposite directions on the opposite faces of the length of beam. Thus when deciding the sign of an internal force, attention must be paid to the face of the section on which it acts. For example, a positive shear force,  $S$ , acts downwards on the right-hand face of the element but upwards on the left-hand face.

A positive bending moment,  $M$ , will cause the upper surface of a beam to become concave and is therefore known as a *sagging* bending moment. Negative bending moments produce convex upper surfaces on beams and are termed *hogging* bending moments.

The axial force,  $N$ , acts normally to the cross-section of a member and is therefore referred to as a normal force;  $N$  is positive when tensile.

A positive internal torque is anticlockwise when viewed in the direction  $zO$ .

We see from the above that positive externally applied loads are associated with positive corresponding internal forces.

Generally it is advantageous to represent the distribution of internal forces, moments and torques by *internal force diagrams*; the methods of construction of these diagrams will be illustrated by examples.

### 3.3 Normal force

**Example 3.1** Construct a normal force diagram for the beam AB shown in Fig. 3.8(a).

The first step is to calculate the support reactions using the methods described in Section 2.5. In this case, since the beam is on a roller support at B, the horizontal load at B is reacted at A; clearly  $R_{A,H} = 10$  kN acting to the left.

Generally the distribution of an internal force will change at a loading discontinuity. In this case there is no loading discontinuity at any section of the beam so that we can determine the complete distribution of the normal force by calculating the normal force at any section Z, a distance  $z$  from A.

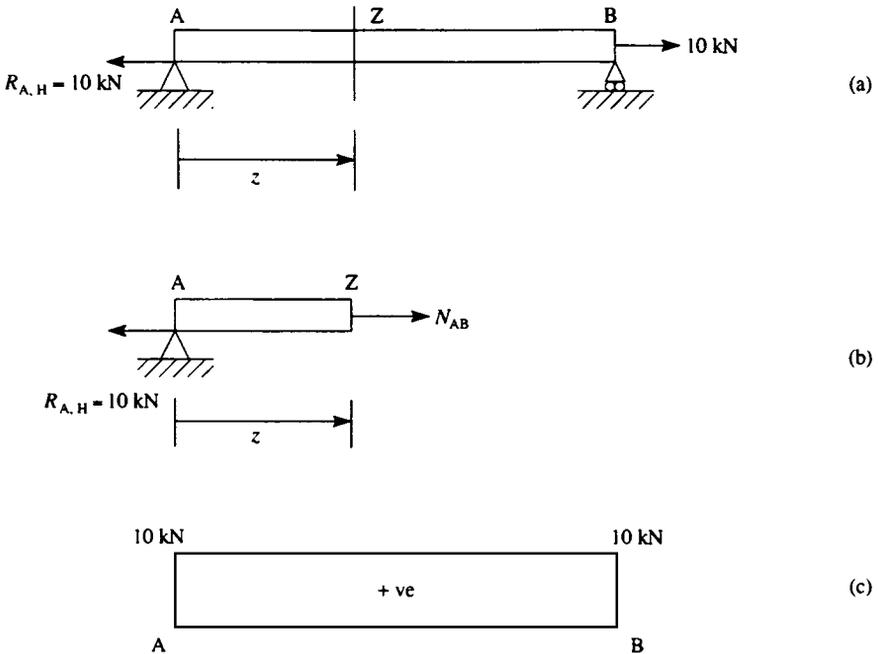
Consider the length AZ of the beam as shown in Fig. 3.8(b) (equally we could consider the length ZB). The internal normal force acting at Z is  $N_{AB}$  which is shown acting in a positive (tensile) direction. The length AZ of the beam is in equilibrium under the action of  $R_{A,H}$  ( $= 10$  kN) and  $N_{AB}$ . Thus, from Section 2.4, for equilibrium in the  $z$  direction,

$$N_{AB} - R_{A,H} = N_{AB} - 10 = 0$$

whence

$$N_{AB} = +10 \text{ kN}$$

$N_{AB}$  is positive and therefore acts in the assumed positive direction; the normal force diagram for the complete beam is then as shown in Fig. 3.8(c).



**Fig. 3.8** Normal force diagram for the beam of Ex. 3.1

When the equilibrium of a portion of a structure is considered as in Fig. 3.8(b) we are using what is termed a *free body diagram*.

**Example 3.2** Draw a normal force diagram for the beam ABC shown in Fig. 3.9(a).

Again by considering the overall equilibrium of the beam we see that  $R_{A,H} = 10$  kN acting to the left (C is the roller support).

In this example there is a loading discontinuity at B so that the distribution of the normal force in AB will be different to that in BC. We must therefore determine the normal force at an arbitrary section  $Z_1$  between A and B and then at an arbitrary section  $Z_2$  between B and C.

The free body diagram for the portion of the beam  $AZ_1$  is shown in Fig. 3.9(b). (Alternatively we could consider the portion  $Z_1C$ .) As before, we draw in a positive normal force,  $N_{AB}$ . Then, for equilibrium of  $AZ_1$  in the  $z$  direction.

$$N_{AB} - 10 = 0$$

so that

$$N_{AB} = +10 \text{ kN (tension)}$$

Now consider the length  $ABZ_2$  of the beam; again we draw in a positive normal force,  $N_{BC}$ . Then for equilibrium of  $ABZ_2$  in the  $z$  direction.

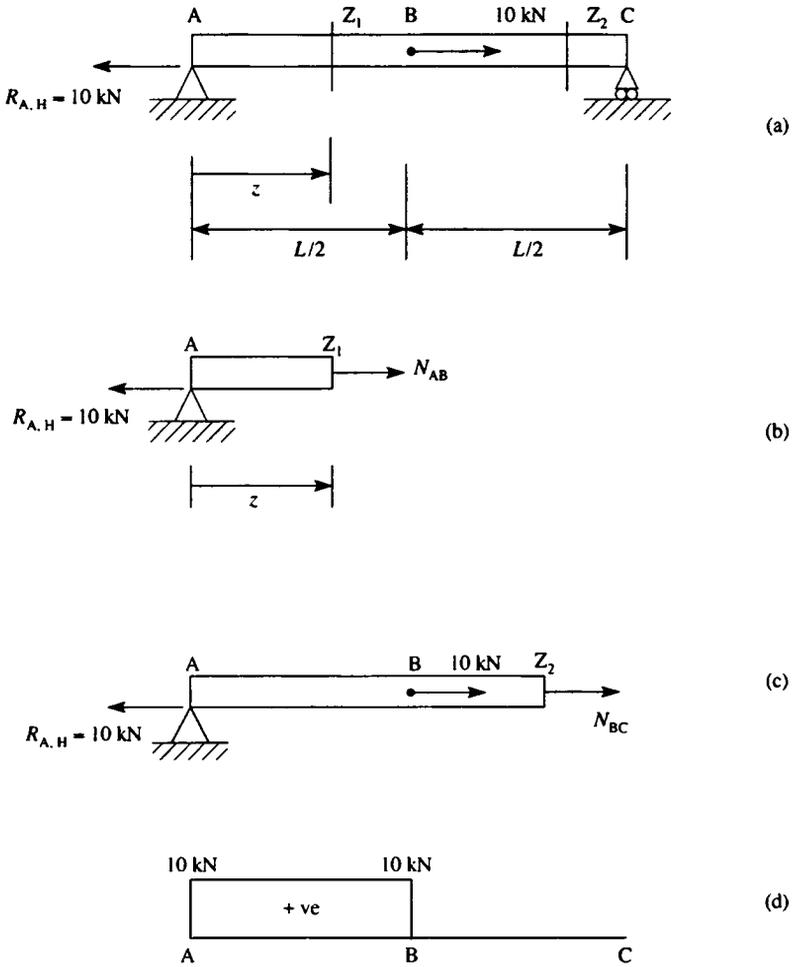
$$N_{BC} + 10 - 10 = 0$$

which gives

$$N_{BC} = 0$$

Note that we would have obtained the same result by considering the portion  $Z_2C$  of the beam.

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**Fig. 3.9** Normal force diagram for the beam of Ex. 3.2

Finally the complete normal force diagram for the beam is drawn as shown in Fig. 3.9(d).

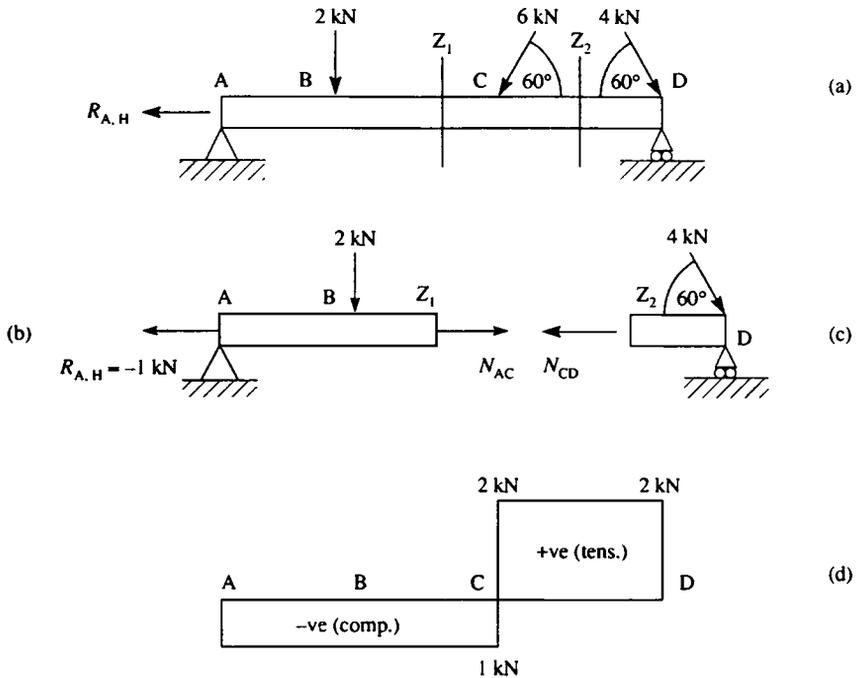
**Example 3.3** Fig. 3.10(a) shows a beam ABCD supporting three concentrated loads, two of which are inclined to the longitudinal axis of the beam. Construct the normal force diagram for the beam and determine the maximum value.

In this example we are only concerned with determining the normal force distribution in the beam, so that it is unnecessary to calculate the vertical reactions at the supports. Further, the horizontal components of the inclined loads can only be resisted at A since D is a roller support. Thus, considering the horizontal equilibrium of the beam,

$$R_{A,H} + 6 \cos 60^\circ - 4 \cos 60^\circ = 0$$

whence

$$R_{A,H} = -1 \text{ kN}$$



**Fig. 3.10** Normal force diagram for the beam of Ex. 3.3

The negative sign of  $R_{A,H}$  indicates that the reaction acts to the right and not to the left as originally assumed. However, rather than change the direction of  $R_{A,H}$  in the diagram, it is simpler to retain the assumed direction and then insert the negative value as required.

Although there is an apparent loading discontinuity at B, the 2 kN load acts perpendicularly to the longitudinal axis of the beam and will therefore not affect the normal force. We may therefore consider the normal force at any section  $Z_1$  between A and C. The free body diagram for the portion  $AZ_1$  of the beam is shown in Fig. 3.10(b); again we draw in a positive normal force  $N_{AC}$ . For equilibrium of  $AZ_1$

$$N_{AC} - R_{A,H} = 0$$

so that

$$N_{AC} = R_{A,H} = -1 \text{ kN (compression)}$$

The horizontal component of the inclined load at C produces a loading discontinuity so that we now consider the normal force at any section  $Z_2$  between C and D. Here it is slightly simpler to consider the equilibrium of the length  $Z_2D$  of the beam rather than the length  $AZ_2$ . Thus, from Fig. 3.10(c)

$$N_{CD} - 4 \cos 60^\circ = 0$$

which gives

$$N_{CD} = +2 \text{ kN (tension)}$$

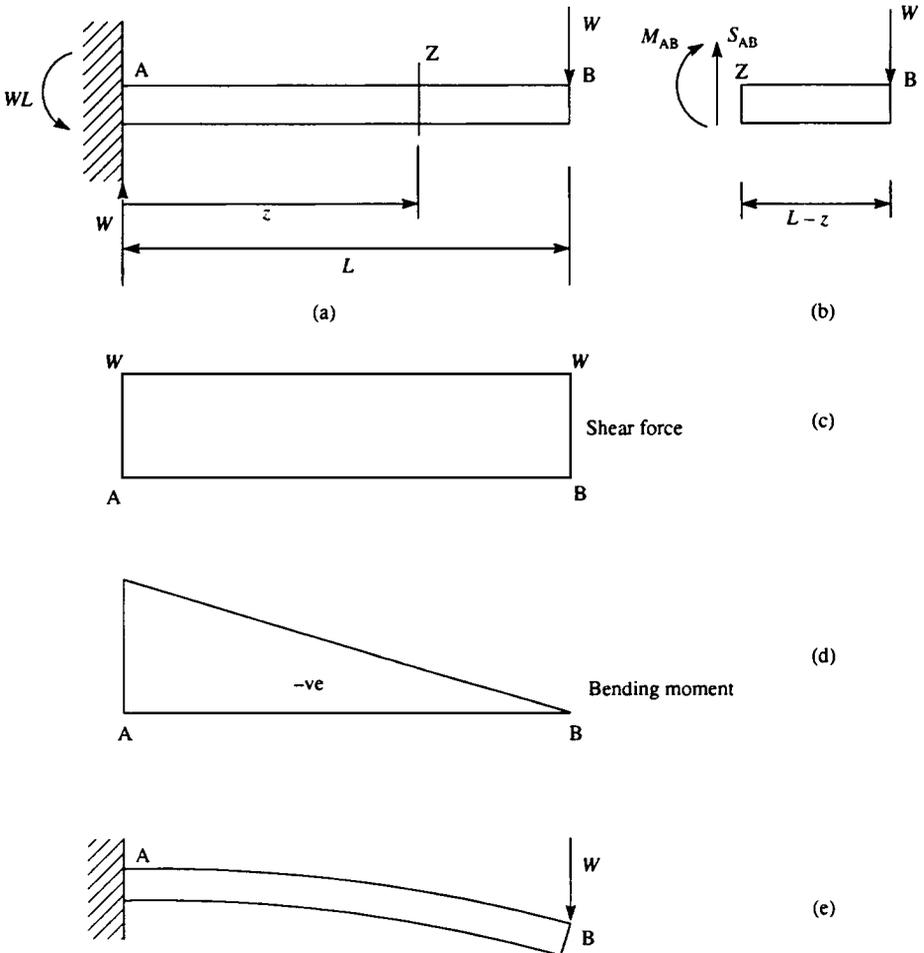
From the completed normal force diagram in Fig. 3.10(d) we see that the maximum normal force in the beam is 2 kN (tension) acting at all sections between C and D.

### 3.4 Shear force and bending moment

It is convenient to consider shear force and bending moment distributions in beams simultaneously since, as we shall see in Section 3.5, they are directly related. Again the method of construction of shear force and bending moment diagrams will be illustrated by examples.

**Example 3.4** Cantilever beam with a concentrated load at the free end (Fig. 3.11).

Generally, as in the case of normal force distributions, we require the variation in shear force and bending moment along the length of a beam. Again, loading discontinuities, such as concentrated loads and/or a sudden change in the intensity of a distributed load, cause discontinuities in the distribution of shear force and bending moment so that it is necessary to consider a series of sections, one between



**Fig. 3.11** Shear force and bending moment diagrams for the beam of Ex. 3.4

each loading discontinuity. In this example, however, there are no loading discontinuities between the built-in end A and the free end B so that we may consider a section Z at any point between A and B.

For many beams the value of each support reaction must be calculated before the shear force and bending moment distributions can be obtained. In Fig. 3.11(a) a consideration of the overall equilibrium of the beam (see Section 2.5) gives a vertical reaction,  $W$ , and a moment reaction,  $WL$ , at the built-in end. However, if we consider the equilibrium of the length ZB of the beam as shown in the free body diagram in Fig. 3.11(b), this calculation is unnecessary.

As in the case of normal force distributions we assign positive directions to the shear force,  $S_{AB}$ , and bending moment,  $M_{AB}$ , at the section Z. Then, for vertical equilibrium of the length ZB of the beam we have

$$S_{AB} - W = 0$$

which gives

$$S_{AB} = +W$$

Thus the shear force is constant along the length of the beam and the shear force diagram is rectangular in shape, as shown in Fig. 3.11(c).

The bending moment,  $M_{AB}$ , is now found by considering the moment equilibrium of the length ZB of the beam about the section Z. Alternatively we could take moments about B, but this would involve the moment of the shear force,  $S_{AB}$ , about B. This approach, although valid, is not good practice since it includes a previously calculated quantity; in some cases, however, this is unavoidable. Thus, taking moments about the section Z we have

$$M_{AB} + W(L - z) = 0$$

whence

$$M_{AB} = -W(L - z) \quad (i)$$

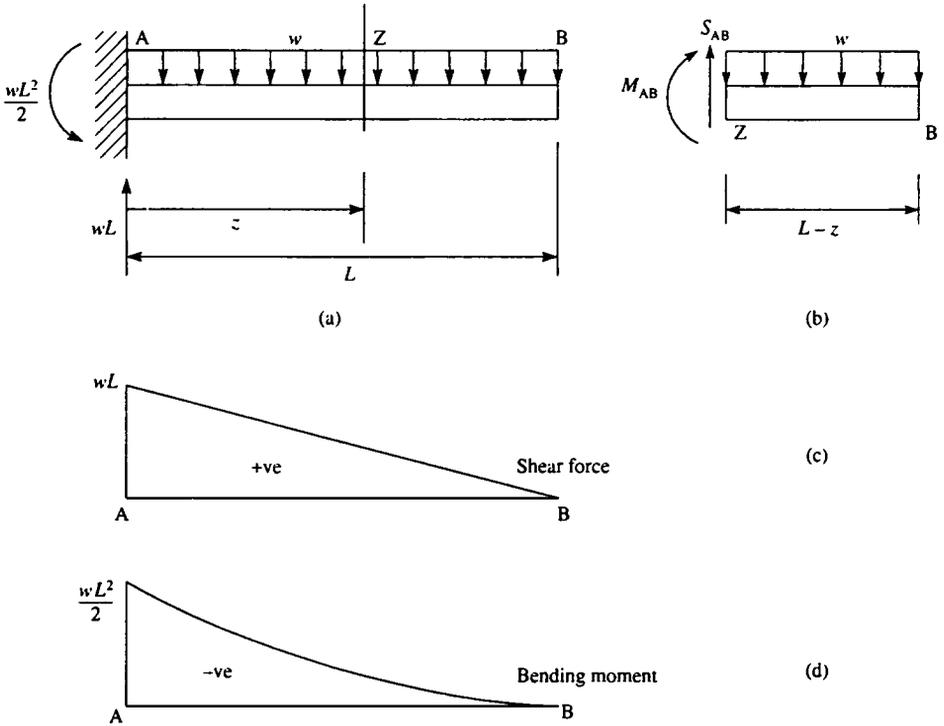
Eq. (i) shows that  $M_{AB}$  varies linearly along the length of the beam, is negative, i.e. hogging, at all sections and increases from zero at the free end ( $z = L$ ) to  $-WL$  at the built-in end where  $z = 0$ .

It is usual to draw the bending moment diagram on the tension side of a beam. This procedure is particularly useful in the design of reinforced concrete beams since it shows directly the surface of the beam near which the major steel reinforcement should be provided. Also, drawing the bending moment diagram on the tension side of a beam can give an indication of the deflected shape as illustrated in Exs 3.4–3.7. This is not always the case, however, as we shall see in Exs 3.8 and 3.9.

In this case the beam will bend as shown in Fig. 3.11(e) so that the upper surface of the beam is in tension and the lower one in compression; the bending moment diagram is therefore drawn on the upper surface as shown in Fig. 3.11(d). Note that negative (hogging) bending moments applied in a vertical plane will always result in the upper surface of a beam being in tension.

**Example 3.5** Cantilever beam carrying a uniformly distributed load of intensity  $w$ .

Again it is unnecessary to calculate the reactions at the built-in end of the cantilever; their values are, however, shown in Fig. 3.12(a). Note that for the purpose of calculating the moment reaction the uniformly distributed load may be replaced by a concentrated load ( $=wL$ ) acting at a distance  $L/2$  from A.



**Fig. 3.12** Shear force and bending moment diagrams for the beam of Ex. 3.5

There is no loading discontinuity between A and B so that we may consider the shear force and bending moment at any section Z between A and B. As before, we insert positive directions for the shear force,  $S_{AB}$ , and bending moment,  $M_{AB}$ , in the free body diagram of Fig. 3.12(b). Then, for vertical equilibrium of the length ZB of the beam,

$$S_{AB} - w(L - z) = 0$$

so that

$$S_{AB} = +w(L - z) \tag{i}$$

Therefore  $S_{AB}$  varies linearly with  $z$  and increases from zero at B to  $+wL$  at A (Fig. 3.12(c)).

Now consider the moment equilibrium of the length AB of the beam and take moments about Z:

$$M_{AB} + \frac{w}{2} (L - z)^2 = 0$$

which gives

$$M_{AB} = -\frac{w}{2} (L - z)^2 \tag{ii}$$

Note that the total load on the length ZB of the beam is  $w(L - z)$ , which we may consider acting as a concentrated load at a distance  $(L - z)/2$  from Z. From Eq. (ii)

we see that the bending moment,  $M_{AB}$ , is negative at all sections of the beam and varies parabolically as shown in Fig. 3.12(d) where the bending moment diagram is again drawn on the tension side of the beam. The actual shape of the bending moment diagram may be found by plotting values or, more conveniently, by examining Eq. (ii). Differentiating with respect to  $z$  we obtain

$$\frac{dM_{AB}}{dz} = w(L - z) \quad (\text{iii})$$

so that when  $z = L$ ,  $dM_{AB}/dz = 0$  and the bending moment diagram is tangential to the datum line AB at B. Furthermore it can be seen from Eq. (iii) that the gradient ( $dM_{AB}/dz$ ) of the bending moment diagram decreases as  $z$  increases, so that its shape is as shown in Fig. 3.12(d).

**Example 3.6** Simply supported beam carrying a central concentrated load.

In this example it is necessary to calculate the value of the support reactions, both of which are seen, from symmetry, to be  $W/2$  (Fig. 3.13(a)). Also, there is a loading discontinuity at B, so that we must consider the shear force and bending moment first at an arbitrary section  $Z_1$ , say, between A and B and then at an arbitrary section  $Z_2$  between B and C.

From the free body diagram in Fig. 3.13(b) in which both  $S_{AB}$  and  $M_{AB}$  are in positive directions we see, by considering the vertical equilibrium of the length  $AZ_1$  of the beam, that

$$S_{AB} - \frac{W}{2} = 0$$

whence 
$$S_{AB} = +\frac{W}{2}$$

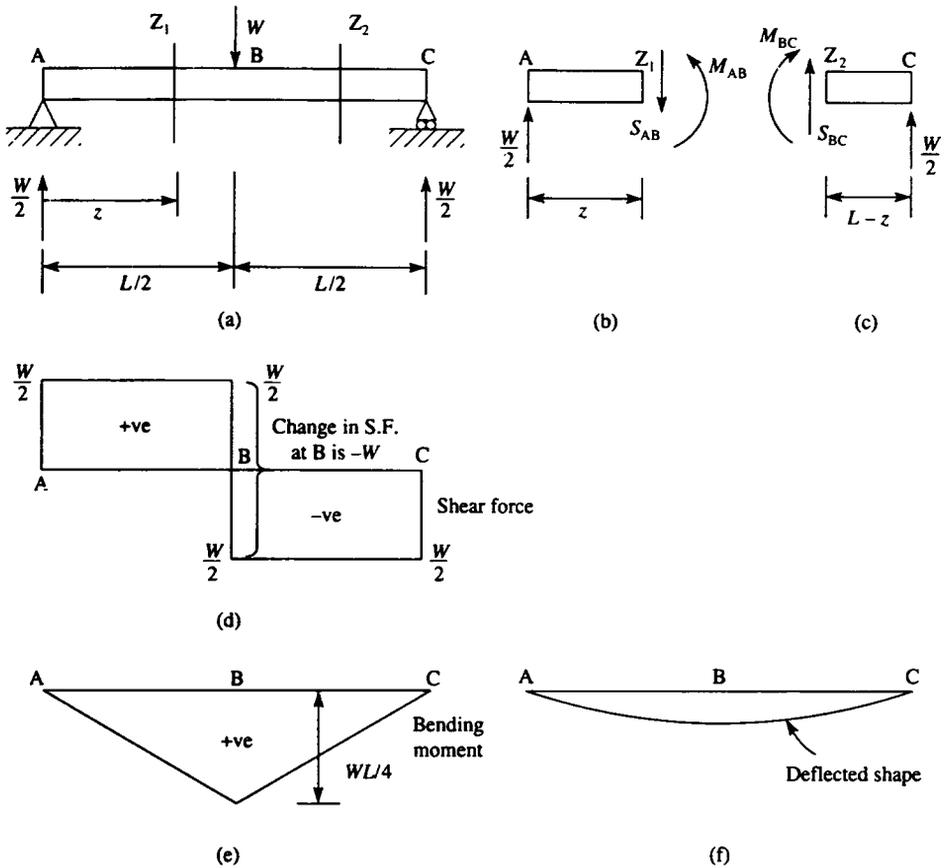
$S_{AB}$  is therefore constant at all sections of the beam between A and B, in other words, from a section immediately to the right of A to a section immediately to the left of B.

Now consider the free body diagram of the length  $Z_2C$  of the beam in Fig. 3.13(c). Note that, equally, we could have considered the length  $ABZ_2$ , but this would have been slightly more complicated in terms of the number of loads acting. For vertical equilibrium of  $Z_2C$

$$S_{AB} + \frac{W}{2} = 0$$

from which 
$$S_{AB} = -\frac{W}{2}$$

and we see that  $S_{BC}$  is constant at all sections of the beam between B and C so that the complete shear force diagram has the form shown in Fig. 3.13(d). Note that the *change* in shear force from that at a section immediately to the left of B to that at a section immediately to the right of B is  $-W$ . We shall consider the implications of this later in the chapter.



**Fig. 3.13** Shear force and bending moment diagrams for the beam of Ex 3.6

It would also appear from Fig. 3.13(d) that there are two different values of shear force at the same section B of the beam. This results from the assumption that  $W$  is concentrated at a point which, practically, is impossible since there would then be an infinite bearing pressure on the surface of the beam. In practice, the load  $W$  and the support reactions would be distributed over a small length of beam (Fig. 3.14(a)) so that the actual shear force distribution would be that shown in Fig. 3.14(b).

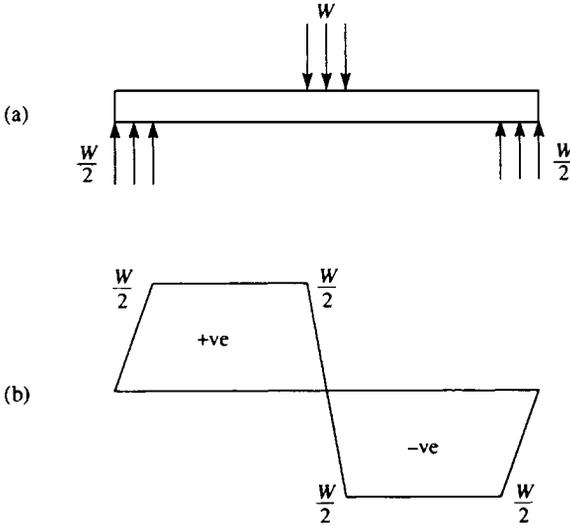
The distribution of the bending moment in AB is now found by considering the moment equilibrium about  $Z_1$  of the length  $AZ_1$  of the beam in Fig. 3.13(b). Thus

$$M_{AB} - \frac{W}{2} z = 0$$

or 
$$M_{AB} = \frac{W}{2} z \tag{i}$$

Therefore  $M_{AB}$  varies linearly from zero at A ( $z=0$ ) to  $+WL/4$  at B ( $z=L/2$ ).

Now considering the length  $Z_2C$  of the beam in Fig 3.13(c) and taking moments about  $Z_2$



**Fig. 3.14** Shear force diagram in a practical situation

$$M_{BC} - \frac{W}{2} (L - z) = 0$$

which gives

$$M_{BC} = +\frac{W}{2} (L - z) \tag{ii}$$

From Eq. (ii) we see that  $M_{BC}$  varies linearly from  $+WL/4$  at B ( $z = L/2$ ) to zero at C ( $z = L$ ).

The complete bending moment diagram is shown in Fig. 3.13(e). Note that the bending moment is positive (sagging) at all sections of the beam so that the lower surface of the beam is in tension. In this example the deflected shape of the beam would be that shown in Fig. 3.13(f).

**Example 3.7** Simply supported beam carrying a uniformly distributed load.

The symmetry of the beam and its load may again be used to determine the support reactions which are each  $wL/2$ . Furthermore, there is no loading discontinuity between the ends A and B of the beam so that it is sufficient to consider the shear force and bending moment at just one section Z, a distance  $z$ , say, from A; again we draw in positive directions for the shear force and bending moment at the section Z in the free body diagram shown in Fig. 3.15(b).

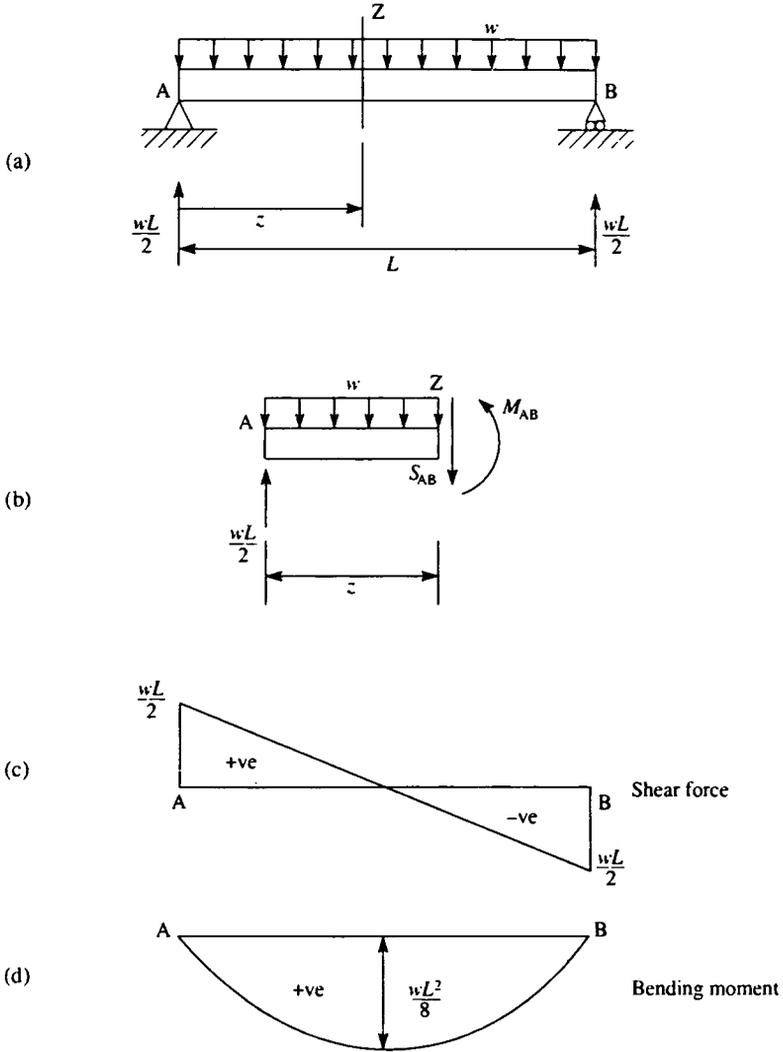
Considering the vertical equilibrium of the length AZ of the beam gives

$$S_{AB} + wz - w \frac{L}{2} = 0$$

whence

$$S_{AB} = +w \left( \frac{L}{2} - z \right) \tag{i}$$

$S_{AB}$  therefore varies linearly along the length of the beam from  $+wL/2$  at A ( $z = 0$ ) to  $-wL/2$  at B ( $z = L$ ). Note that  $S_{AB} = 0$  at mid-span ( $z = L/2$ ).



**Fig. 3.15** Shear force and bending moment diagrams for the beam of Ex. 3.7

Now taking moments about  $Z$  for the length  $AZ$  of the beam in Fig. 3.15(b) we have

$$M_{AB} + \frac{wz^2}{2} - \frac{wL}{2} z = 0$$

from which

$$M_{AB} = +\frac{wz}{2} (L - z) \tag{ii}$$

Thus  $M_{AB}$  varies parabolically along the length of the beam and is positive (sagging) at all sections of the beam except at the supports ( $z = 0$  and  $z = L$ ) where it is zero.

Also, differentiating Eq. (ii) with respect to  $z$  gives

$$\frac{dM_{AB}}{dz} = w\left(\frac{L}{2} - z\right) \quad (\text{iii})$$

From Eq. (iii) we see that  $dM_{AB}/dz = 0$  at mid-span where  $z = L/2$ , so that the bending moment diagram has a turning value or mathematical maximum at this section. In this case this mathematical maximum is the maximum value of the bending moment in the beam and is, from Eq. (ii),  $+wL^2/8$ .

The bending moment diagram for the beam is shown in Fig. 3.15(d) where it is again drawn on the tension side of the beam; the deflected shape of the beam will be identical in form to the bending moment diagram.

Examples 3.4–3.7 may be regarded as ‘standard’ cases and it is useful to memorize the form that the shear force and bending moment diagrams take including the principal values.

**Example 3.8** Simply supported beam with cantilever overhang (Fig. 3.16(a)).

The support reactions are calculated using the methods described in Section 2.5. Thus, taking moments about B in Fig. 3.16(a) we have

$$R_A \times 2 - 2 \times 3 \times 0.5 + 1 \times 1 = 0$$

which gives

$$R_A = 1 \text{ kN}$$

From vertical equilibrium

$$R_B + R_A - 2 \times 3 - 1 = 0$$

so that

$$R_B = 6 \text{ kN}$$

The support reaction at B produces a loading discontinuity at B so that we must consider the shear force and bending moment at two arbitrary sections of the beam,  $Z_1$  in AB and  $Z_2$  in BC. Free body diagrams are therefore drawn for the lengths  $AZ_1$  and  $Z_2C$  of the beam and positive directions for the shear force and bending moment drawn in as shown in Figs 3.16(b) and (c). Alternatively, we could have considered the lengths  $Z_1BC$  and  $ABZ_2$ , but this approach would have involved slightly more complicated solutions in terms of the number of loads applied.

Now from the vertical equilibrium of the length  $AZ_1$  of the beam in Fig. 3.16(b) we have

$$S_{AB} + 2z - 1 = 0$$

or

$$S_{AB} = 1 - 2z \quad (\text{i})$$

The shear force therefore varies linearly in AB from +1 kN at A ( $z = 0$ ) to -3 kN at B ( $z = 2$  m). Note that  $S_{AB} = 0$  at  $z = 0.5$  m.

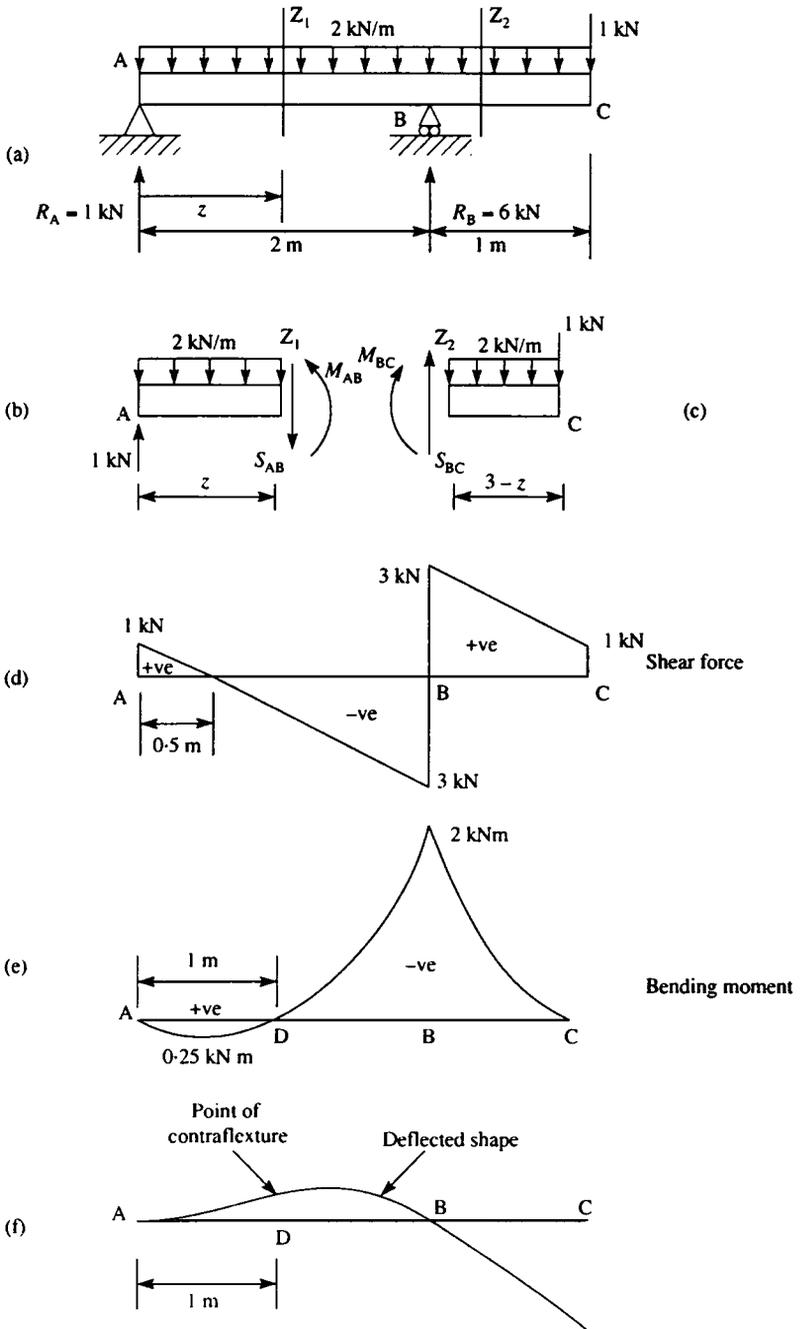
Consideration of the vertical equilibrium of the length  $Z_2C$  of the beam in Fig. 3.16(c) gives

$$S_{BC} - 2(3 - z) - 1 = 0$$

from which

$$S_{BC} = 7 - 2z \quad (\text{ii})$$

Eq. (ii) shows that  $S_{BC}$  varies linearly in BC from +3 kN at B ( $z = 2$  m) to +1 kN at C ( $z = 3$  m).



**Fig. 3.16** Shear force and bending moment diagrams for the beam of Ex. 3.8

The complete shear force diagram for the beam is shown in Fig. 3.16(d).

The bending moment,  $M_{AB}$ , is now obtained by considering the moment equilibrium of the length  $AZ_1$  of the beam about  $Z_1$  in Fig. 3.16(b). Hence

$$M_{AB} + 2z \frac{z}{2} - 1z = 0$$

so that

$$M_{AB} = z - z^2 \tag{iii}$$

which is a parabolic function of  $z$ . The distribution may be plotted by selecting a series of values of  $z$  and calculating the corresponding values of  $M_{AB}$ . However, this would not necessarily produce accurate estimates of either the magnitudes and positions of the maximum values of  $M_{AB}$  or, say, the positions of the zero values of  $M_{AB}$  which, as we shall see later, are important in beam design. A better approach is to examine Eq. (iii) as follows. Clearly when  $z = 0$ ,  $M_{AB} = 0$  as would be expected at the simple support at A. Also at B, where  $z = 2$  m,  $M_{AB} = -2$  kN so that although the support at B is a simple support and allows rotation of the beam, there is a moment at B; this is produced by the loads on the cantilever overhang BC. Rewriting Eq. (iii) in the form

$$M_{AB} = z(1 - z) \tag{iv}$$

we see immediately that  $M_{AB} = 0$  at  $z = 0$  (as demonstrated above) and that  $M_{AB} = 0$  at  $z = 1$  m, the point D in Fig. 3.16(e). We shall see later in Chapter 9 that at the point in the beam where the bending moment changes sign the curvature of the beam is zero; this point is known as a *point of contraflexure* or *point of inflection*. Now differentiating Eq. (iii) with respect to  $z$  we obtain

$$\frac{dM_{AB}}{dz} = 1 - 2z \tag{v}$$

and we see that  $dM_{AB}/dz = 0$  at  $z = 0.5$  m. In other words  $M_{AB}$  has a turning value or mathematical maximum at  $z = 0.5$  m at which point  $M_{AB} = 0.25$  kN m. Note that this is not the greatest value of bending moment in the span AB. Also it can be seen that for  $0 < z < 0.5$  m,  $dM_{AB}/dz$  decreases with  $z$  while for  $0.5 \text{ m} < z < 2$  m,  $dM_{AB}/dz$  increases negatively with  $z$ .

Now we consider the moment equilibrium of the length  $Z_2C$  of the beam in Fig. 3.16(c) about  $Z_2$ .

$$M_{BC} + \frac{2}{2}(3 - z)^2 + 1(3 - z) = 0$$

so that

$$M_{BC} = -12 + 7z - z^2 \tag{vi}$$

from which we see that  $dM_{BC}/dz$  is not zero at any point in BC and that as  $z$  increases  $dM_{BC}/dz$  decreases.

The complete bending moment diagram is therefore as shown in Fig. 3.16(e). Note that the value of zero shear force in AB coincides with the turning value of the bending moment.

In this particular example it is not possible to deduce the displaced shape of the beam from the bending moment diagram. Only three facts relating to the displaced

shape can be stated with certainty; these are, the deflections at A and B are zero and there is a point of contraflexure at D, 1 m from A. However, using the method described in Section 13.2 gives the displaced shape shown in Fig. 3.16(f). Note that, although the beam is subjected to a sagging bending moment over the length AD, the actual deflection is upwards; clearly this could not have been deduced from the bending moment diagram.

**Example 3.9** Simply supported beam carrying a point moment.

From a consideration of the overall equilibrium of the beam (Fig. 3.17(a)) the support reactions are  $R_A = M_0/L$  acting vertically upward and  $R_C = M_0/L$  acting vertically downward. Note that  $R_A$  and  $R_C$  are independent of the point of application of  $M_0$ .

Although there is a loading discontinuity at B it is a point moment and will not affect the distribution of shear force. Thus, by considering the vertical equilibrium of either  $AZ_1$  in Fig. 3.17(b) or  $Z_2C$  in Fig. 3.17(c) we see that

$$S_{AB} = S_{BC} = +\frac{M_0}{L} \quad (\text{i})$$

The shear force is therefore constant along the length of the beam as shown in Fig. 3.17(d).

Now considering the moment equilibrium about  $Z_1$  of the length  $AZ_1$  of the beam in Fig. 3.17(b),

$$M_{AB} - \frac{M_0}{L} z = 0$$

or 
$$M_{AB} = \frac{M_0}{L} z \quad (\text{ii})$$

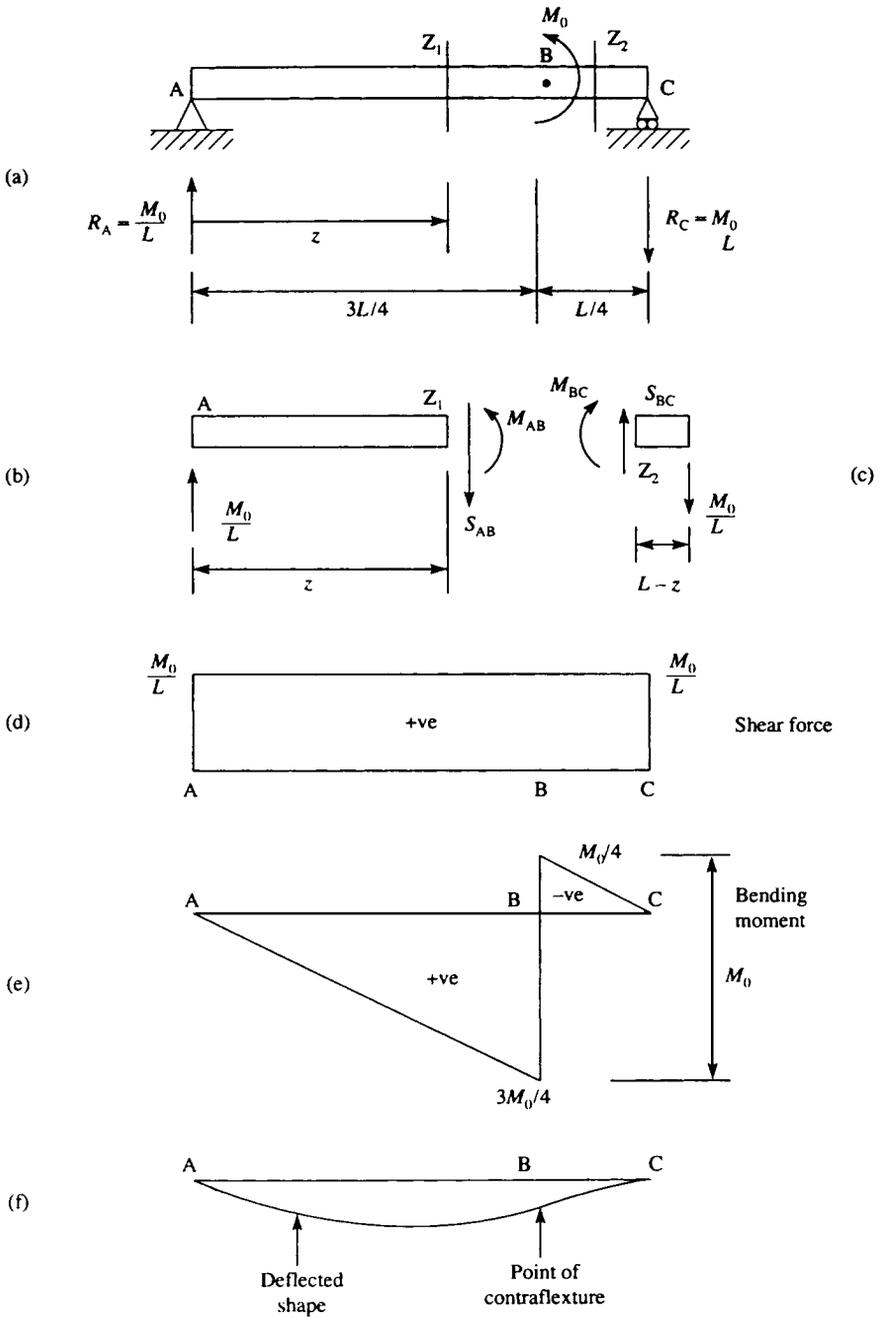
$M_{AB}$  therefore increases linearly from zero at A ( $z = 0$ ) to  $+3M_0/4$  at B ( $z = 3L/4$ ). From Fig. 3.17(c) and taking moments about  $Z_2$  we have

$$M_{BC} + \frac{M_0}{L} (L - z) = 0$$

or 
$$M_{BC} = \frac{M_0}{L} (z - L) \quad (\text{iii})$$

$M_{BC}$  therefore decreases linearly from  $-M_0/4$  at B ( $z = 3L/4$ ) to zero at C ( $z = L$ ); the complete distribution of bending moment is shown in Fig. 3.17(e). The deflected form of the beam is shown in Fig. 3.17(f) where a point of contraflexure occurs at B, the section at which the bending moment changes sign.

In this example, as in Ex. 3.8, the exact form of the deflected shape cannot be deduced from the bending moment diagram without analysis. However, using the method of singularities described in Section 13.2, it may be shown that the deflection at B is positive and that the slope of the beam at C is negative, giving the displaced shape shown in Fig. 3.17(f).



**Fig. 3.17** Shear force and bending moment diagrams for the beam of Ex. 3.9

### 3.5 Load, shear force and bending moment relationships

It is clear from Exs 3.4–3.9 that load, shear force and bending moment are related. Thus, for example, uniformly distributed loads produce linearly varying shear forces and maximum values of bending moment coincide with zero shear force. We shall now examine these relationships mathematically.

The length of beam shown in Fig. 3.18(a) carries a general system of loading comprising concentrated loads and a distributed load  $w(z)$ . An elemental length  $\delta z$  of the beam is subjected to the force and moment system shown in Fig. 3.18(b); since  $\delta z$  is very small the distributed load may be regarded as constant over the length  $\delta z$ . For vertical equilibrium of the element

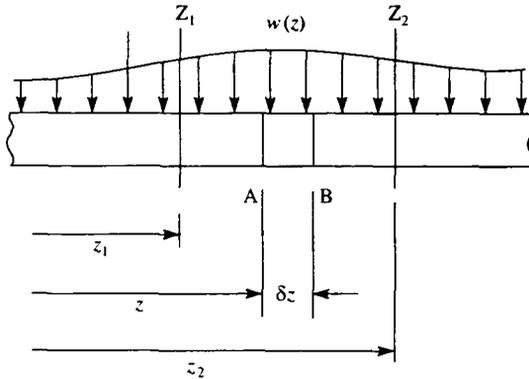
$$S - w(z)\delta z - (S + \delta S) = 0$$

so that

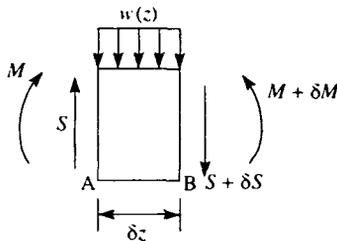
$$-w(z)\delta z - \delta S = 0$$

Thus, in the limit as  $\delta z \rightarrow 0$

$$\frac{dS}{dz} = -w(z) \tag{3.1}$$



(a)



(b)

**Fig. 3.18** Load, shear force and bending moment relationships

From Eq. (3.1) we see that the rate of change of shear force at a section of a beam, in other words the gradient of the shear force diagram, is equal to minus the value of the load intensity at that section. In Fig. 3.12(c), for example, the shear force changes linearly from  $wL$  at A to zero at B so that the gradient of the shear force diagram at any section of the beam is  $-wL/L = -w$  where  $w$  is the load intensity. Equation (3.1) also applies at beam sections subjected to concentrated loads. In Fig. 3.13(a) the load intensity at B, theoretically, is infinite, as is the gradient of the shear force diagram at B (Fig. 3.13(d)). In practice the shear force diagram would have a finite gradient at this section as illustrated in Fig. 3.14.

Now integrating Eq. (3.1) with respect to  $z$  we obtain

$$S = - \int w(z) dz + C_1 \tag{3.2}$$

in which  $C_1$  is a constant of integration which may be determined in a particular case from the loading boundary conditions.

If, for example,  $w(z)$  is a uniformly distributed load of intensity  $w$ , that is it is not a function of  $z$ , Eq. (3.2) becomes

$$S = -wz + C_1$$

which is the equation of a straight line of gradient  $-w$  as demonstrated for the cantilever beam of Fig. 3.12 in the previous paragraph. Furthermore, for this particular example,  $S = 0$  at  $z = L$  so that  $C_1 = wL$  and  $S = w(L - z)$  as before.

In the case of a beam carrying only concentrated loads then, in the bays between the loads,  $w(z) = 0$  and Eq. (3.2) reduces to

$$S = C_1$$

so that the shear force is constant over the unloaded length of beam (see Figs 3.11 and 3.13).

Suppose now that Eq. (3.1) is integrated over the length of beam between the sections  $Z_1$  and  $Z_2$ . Then

$$\int_{z_1}^{z_2} \frac{dS}{dz} dz = - \int_{z_1}^{z_2} w(z) dz$$

which gives

$$S_2 - S_1 = \int_{z_1}^{z_2} w(z) dz \tag{3.3}$$

where  $S_1$  and  $S_2$  are the shear forces at the sections  $Z_1$  and  $Z_2$  respectively. Equation (3.3) shows that the *change* in shear force between two sections of a beam is equal to minus the area under the load distribution curve over that length of beam.

The argument may be applied to the case of a concentrated load  $W$  which may be regarded as a uniformly distributed load acting over an extremely small elemental length of beam, say  $\delta z$ . The area under the load distribution curve would then be  $w\delta z (=W)$  and the change in shear force from the section  $z$  to the section  $z + \delta z$  would be  $-W$ . In other words, the change in shear force from a section immediately to the left of a concentrated load to a section immediately to the right is equal to minus the value of the load, as noted in Ex. 3.6.

## 56 Normal Force, Shear Force, Bending Moment and Torsion

Now consider the rotational equilibrium of the element  $\delta z$  in Fig. 3.18(b) about B. Thus

$$M + S \delta z - w(z) \delta z \frac{\delta z}{2} - (M + \delta M) = 0$$

The term involving the square of  $\delta z$  is a second-order term and may be neglected. Hence

$$S \delta z - \delta M = 0$$

or, in the limit as  $\delta z \rightarrow 0$

$$\frac{dM}{dz} = S \quad (3.4)$$

Equation (3.4) establishes for the general case what may be observed in particular in the shear force and bending moment diagrams of Exs 3.4–3.9, i.e. the gradient of the bending moment diagram at a beam section is equal to the value of the shear force at that section. For example, in Fig. 3.16(e) the bending moment in AB is a mathematical maximum at the section where the shear force is zero.

Integrating Eq. (3.4) with respect to  $z$  we have

$$M = \int S dz + C_2 \quad (3.5)$$

in which  $C_2$  is a constant of integration. Substituting for  $S$  in Eq. (3.5) from Eq. (3.2) gives

$$M = \int \left[ - \int w(z) dz + C_1 \right] dz + C_2$$

or

$$M = - \iint w(z) dz + C_1 z + C_2 \quad (3.6)$$

If  $w(z)$  is a uniformly distributed load of intensity  $w$ , Eq. (3.6) becomes

$$M = -w \frac{z^2}{2} + C_1 z + C_2$$

which shows that the equation of the bending moment diagram on a length of beam carrying a uniformly distributed load is parabolic.

In the case of a beam carrying concentrated loads only then, between the loads,  $w(z) = 0$  and Eq. (3.6) reduces to

$$M = C_1 z + C_2$$

which shows that the bending moment varies linearly between the loads and has a gradient  $C_1$ .

The constants  $C_1$  and  $C_2$  in Eq. (3.6) may be found, for a given beam, from the loading boundary conditions. Thus, for the cantilever beam of Fig. 3.12, we have already shown that  $C_1 = wL$  so that  $M = -wz^2/2 + wLz + C_2$ . Also, when  $z = L$ ,  $M = 0$  which gives  $C_2 = -wL^2/2$  and hence  $M = -wz^2/2 + wLz - wL^2/2$  as before.

Now integrating Eq. (3.4) over the length of beam between the sections  $Z_1$  and  $Z_2$  (Fig. 3.18(a)),

$$\int_{z_1}^{z_2} \frac{dM}{dz} dz = \int_{z_1}^{z_2} S dz$$

which gives 
$$M_2 - M_1 = \int_{z_1}^{z_2} S dz \quad (3.7)$$

where  $M_1$  and  $M_2$  are the bending moments at the sections  $Z_1$  and  $Z_2$ , respectively. Equation (3.7) shows that the *change* in bending moment between two sections of a beam is equal to the area of the shear force diagram between those sections. Again, using the cantilever beam of Fig. 3.12 as an example, we see that the change in bending moment from A to B is  $wL^2/2$  and that the area of the shear force diagram between A and B is  $wL^2/2$ .

Finally, from Eqs (3.1) and (3.4)

$$\frac{d^2M}{dz^2} = \frac{dS}{dz} = -w(z) \quad (3.8)$$

The relationships established above may be used to construct shear force and bending moment diagrams for some beams more readily than when the methods illustrated in Exs 3.4–3.9 are employed. In addition they may be used to provide simpler solutions in some beam problems.

**Example 3.10** Construct shear force and bending moment diagrams for the beam shown in Fig. 3.19(a).

Initially the support reactions are calculated using the methods described in Section 2.5. Hence, for moment equilibrium of the beam about E

$$R_A \times 4 - 2 \times 3 - 5 \times 2 - 4 \times 1 \times 0.5 = 0$$

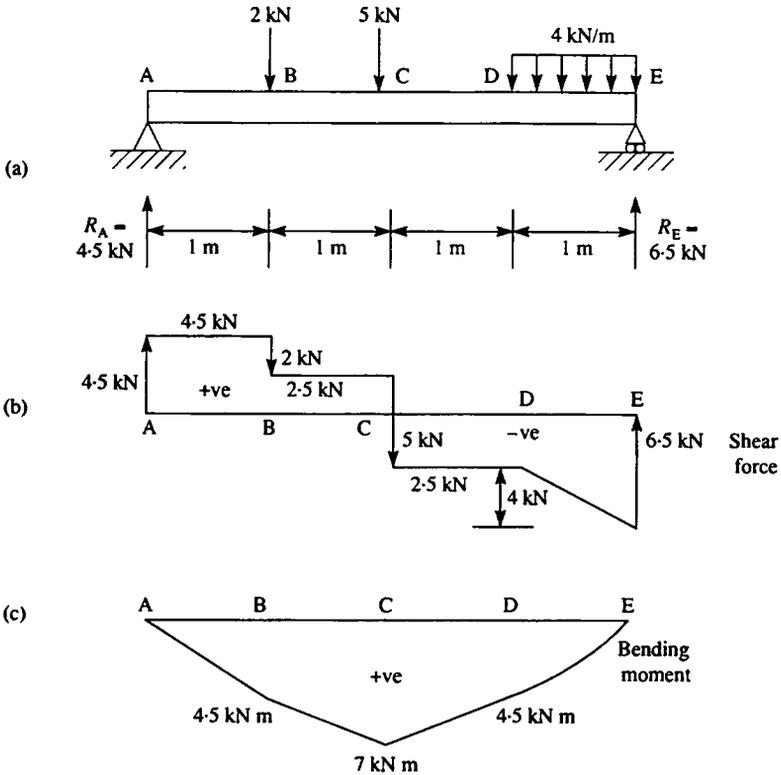
from which 
$$R_A = 4.5 \text{ kN}$$

Now considering the vertical equilibrium of the beam

$$R_E + R_A - 2 - 5 - 4 \times 1 = 0$$

so that 
$$R_E = 6.5 \text{ kN}$$

In constructing the shear force diagram we can make use of the facts that, as established above, the shear force is constant over unloaded bays of the beam, varies linearly when the loading is uniformly distributed and changes negatively as a vertically downward concentrated load is crossed in the positive  $z$  direction by the value of the load. Thus in Fig. 3.19(b) the shear force increases positively by 4.5 kN as we move from the left of A to the right of A, is constant between A and B, changes negatively by 2 kN as we move from the left of B to the right of B, and so on. In effect the shear force diagram is constructed by following the loading pattern. Note that between D and E the shear force changes linearly from  $-2.5$  kN at D to  $-6.5$  kN at a section immediately to the left of E, in other words it changes by  $-4$  kN, the total value of the downward-acting uniformly distributed load.



**Fig. 3.19** Shear force and bending moment diagrams for the beam of Ex. 3.10

The bending moment diagram may also be constructed using the above relationships, namely, the bending moment varies linearly over unloaded lengths of beam and parabolically over lengths of beam carrying a uniformly distributed load. Also, the change in bending moment between two sections of a beam is equal to the area of the shear force diagram between those sections. Thus in Fig. 3.19(a) we know that the bending moment at the pinned support at A is zero and that it varies linearly in the bay AB. The bending moment at B is then equal to the area of the shear force diagram between A and B, which is +4.5 kN·m. This represents, in fact, the change in bending moment from the zero value at A to the value at B. At C the area of the shear force diagram to the right or left of C is 7 kN·m (note that the bending moment at E is also zero), and so on. In the bay DE the shape of the parabolic curve representing the distribution of bending moment over the length of the uniformly distributed load may be found using part of Eq. (3.8), i.e.

$$\frac{d^2M}{dz^2} = -w(z)$$

For a vertically downward uniformly distributed load this expression becomes

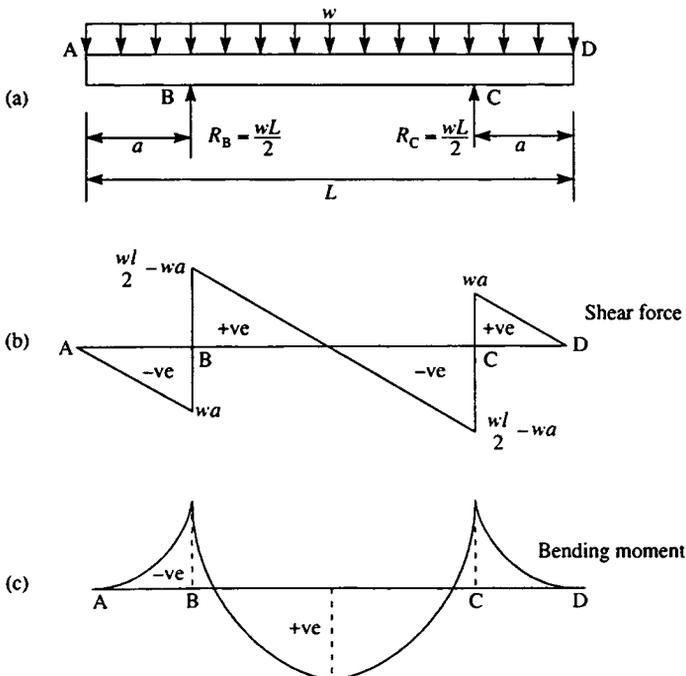
$$\frac{d^2M}{dz^2} = -w$$

which from mathematical theory shows that the curve representing the variation in bending moment is convex in the positive direction of bending moment. This may be observed in the bending moment diagrams in Figs 3.12(d), 3.15(d) and 3.16(e). In this example the bending moment diagram for the complete beam is shown in Fig. 3.19(c) and is again drawn on the tension side of the beam.

**Example 3.11** A precast concrete beam of length  $L$  is to be lifted from the casting bed and transported so that the maximum bending moment is as small as possible. If the beam is lifted by two slings placed symmetrically, show that each sling should be  $0.21L$  from the adjacent end.

The external load on the beam is comprised solely of its own weight, which is uniformly distributed along its length. The problem is therefore resolved into that of a simply supported beam carrying a uniformly distributed load in which the supports are positioned at some distance  $a$  from each end (Fig. 3.20(a)).

The shear force and bending moment diagrams may be constructed in terms of  $a$  using the methods described above and would take the forms shown in Figs 3.20(b) and (c). Examination of the bending moment diagram shows that there are two possible positions for the maximum bending moment. First at B and C where the bending moment is hogging and has equal values from symmetry; second at the mid-span point where the bending moment has a turning value and is sagging if the supports at B and C are spaced a sufficient distance apart. Suppose that B and C are positioned such that the value of the hogging bending moment at B and C is



**Fig. 3.20** Determination of optimum position for supports in a precast concrete beam (Ex. 3.11)

numerically equal to the sagging bending moment at the mid-span point. If now B and C are moved further apart the mid-span moment will increase while the moment at B and C decreases. Conversely, if B and C are brought closer together, the hogging moment at B and C increases while the mid-span moment decreases. It follows that the maximum bending moment will be as small as possible when the hogging moment at B and C is numerically equal to the sagging moment at mid-span.

The solution will be simplified if use is made of the relationship in Eq. (3.7). Thus, when the supports are in the optimum position, the change in bending moment from A to B (negative) is equal to minus half the change in the bending moment from B to the mid-span point (positive). It follows that the area of the shear force diagram between A and B is equal to minus half of that between B and the mid-span point. Then

$$-\frac{1}{2} awa = -\frac{1}{2} \left[ \frac{1}{2} \left( \frac{L}{2} - a \right) w \left( \frac{L}{2} - a \right) \right]$$

which reduces to

$$a^2 + La - L^2/4 = 0$$

the solution of which gives

$$a = 0.21L \text{ (the negative solution has no practical significance)}$$

### 3.6 Torsion

The distribution of torque along a structural member may be obtained by considering the equilibrium in free body diagrams of lengths of member in a similar manner to that used for the determination of shear force distributions in Exs 3.4–3.9.

**Example 3.12** Construct a torsion diagram for the beam shown in Fig. 3.21(a).

There is a loading discontinuity at B so that we must consider the torque at separate sections  $Z_1$  and  $Z_2$  in AB and BC, respectively. Thus, in the free body diagrams shown in Figs 3.21(b) and (c) we insert positive internal torques.

From Fig. 3.21(b)

$$T_{AB} - 10 + 8 = 0$$

so that

$$T_{AB} = +2 \text{ kNm}$$

From Fig. 3.21(c)

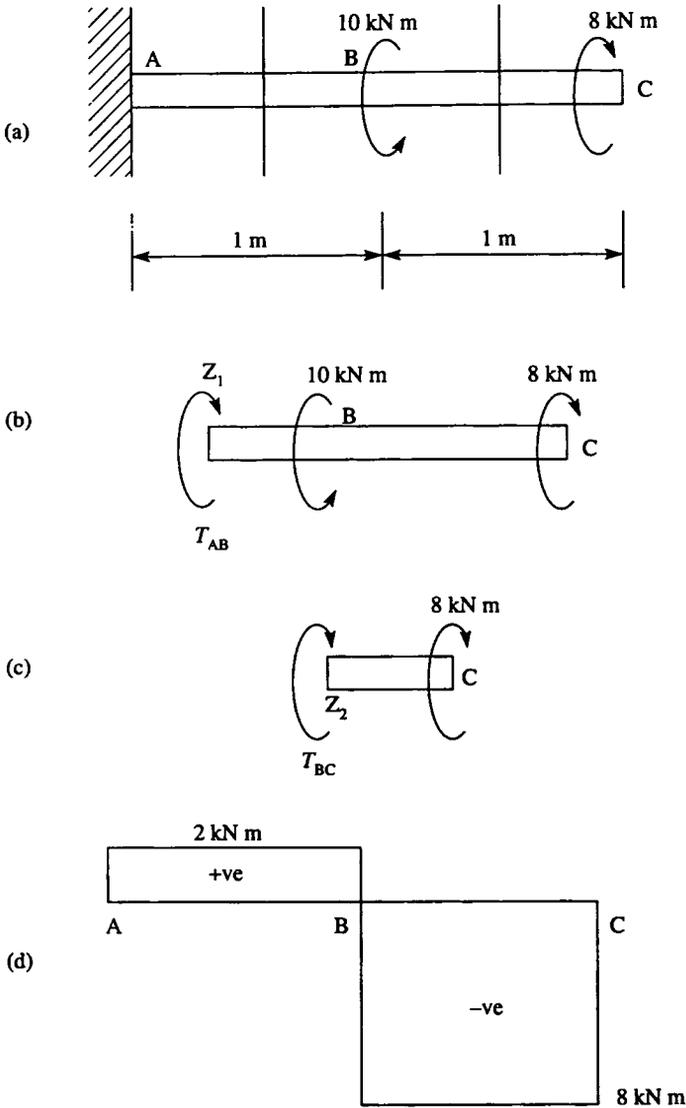
$$T_{BC} + 8 = 0$$

from which

$$T_{BC} = -8 \text{ kNm}$$

The complete torsion diagram is shown in Fig. 3.21(d).

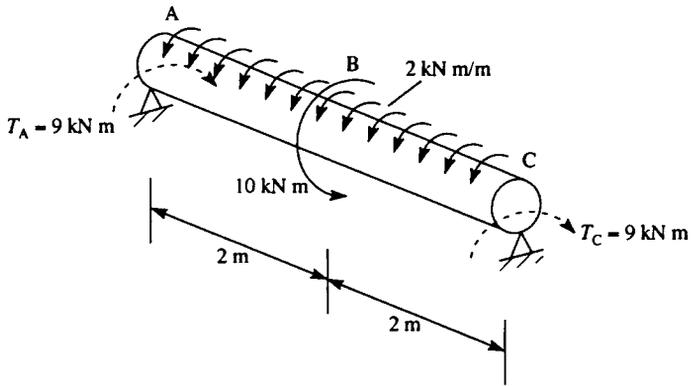
**Example 3.13** The structural member ABC shown in Fig. 3.22 carries a distributed torque of 2 kNm/m together with a concentrated torque of 10 kNm at mid-span. The supports at A and C prevent rotation of the member in planes perpendicular to its axis. Construct a torsion diagram for the member and determine the maximum value of torque.



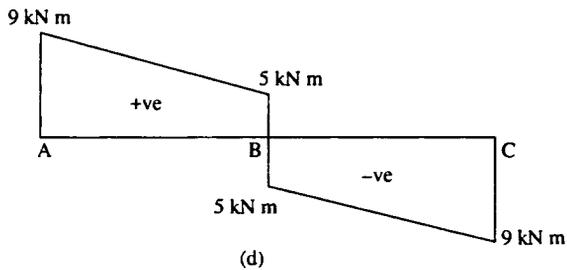
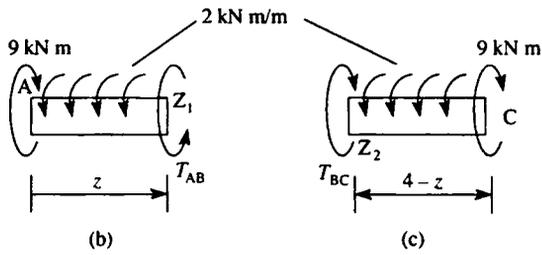
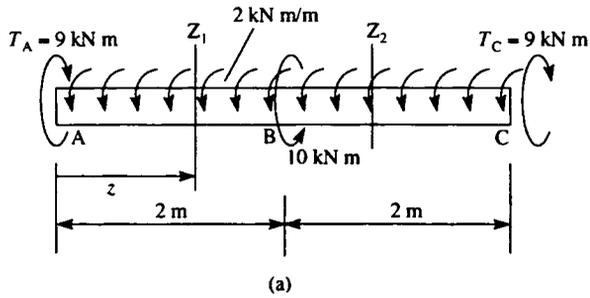
**Fig. 3.21** Torsion diagram for a cantilever beam

From the rotational equilibrium of the member about its longitudinal axis and its symmetry about the mid-span section at B, we see that the reactive torques  $T_A$  and  $T_C$  are each  $-9$  kN m, i.e. clockwise when viewed in the direction CBA. In general, as we shall see in Chapter 11, reaction torques at supports form a statically indeterminate system.

In this particular problem there is a loading discontinuity at B so that we must consider the internal torques at two arbitrary sections  $Z_1$  and  $Z_2$  as shown in Fig. 3.23(a).



**Fig. 3.22** Beam of Ex. 3.13



**Fig. 3.23** Torsion diagram for the beam of Ex. 3.13

From the free body diagram in Fig. 3.23(b)

$$T_{AB} + 2z - 9 = 0$$

which gives

$$T_{AB} = 9 - 2z \tag{i}$$

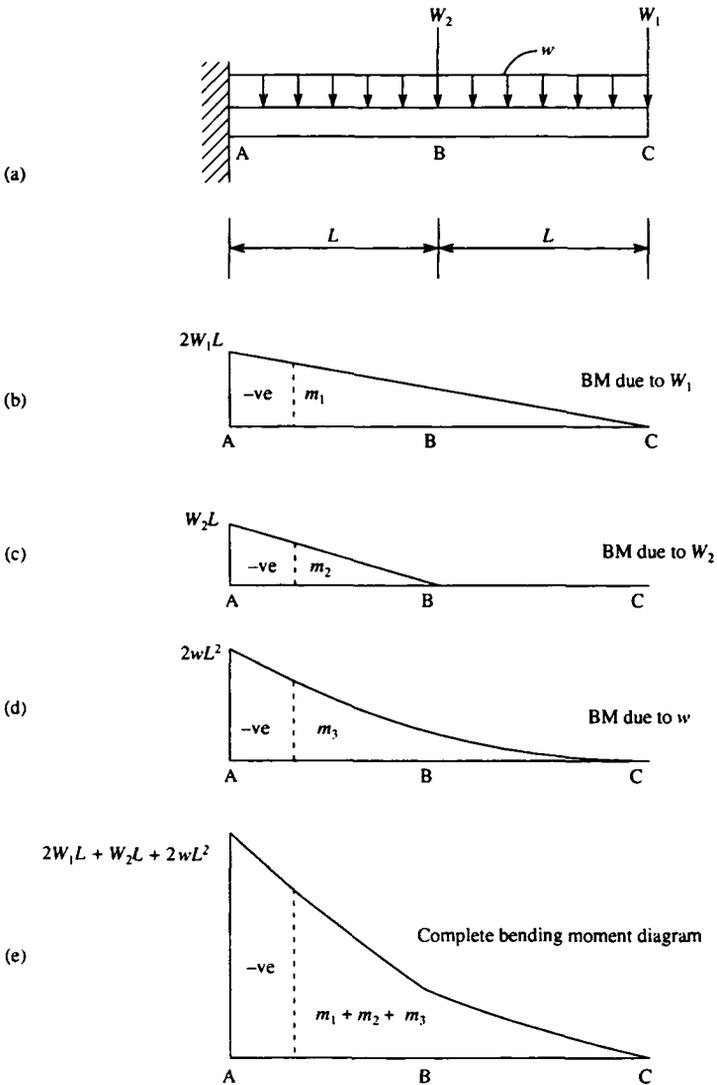
From Eq. (i) we see that  $T_{AB}$  varies linearly from +9 kN m at A ( $z = 0$ ) to +5 kN m at a section immediately to the left of B ( $z = 2$  m). Furthermore, from Fig. 3.23(c)

$$T_{BC} - 2(4 - z) + 9 = 0$$

so that

$$T_{BC} = -2z - 1 \tag{ii}$$

from which we see that  $T_{BC}$  varies linearly from -5 kN m at a section immediately



**Fig. 3.24** Bending moment (BM) diagram using the principle of superposition

to the right of B ( $z = 2$  m) to  $-9$  kN m at C ( $z = 4$  m). The resulting torsion diagram is shown in Fig. 3.23(d).

### 3.7 Principle of superposition

An extremely useful principle in the analysis of linearly elastic structures (see Chapter 8) is that of superposition. The principle states that if the displacements at all points in an elastic body are proportional to the forces producing them, that is the body is linearly elastic, the effect (i.e. stresses and displacements) on such a body of a number of forces acting simultaneously is the sum of the effects of the forces applied separately.

This principle can sometimes simplify the construction of shear force and bending moment diagrams.

**Example 3.14** Construct the bending moment diagram for the beam shown in Fig. 3.24(a).

Figures 3.24(b), (c) and (d) show the bending moment diagrams for the cantilever when each of the three loading systems acts separately. The bending moment diagram for the beam when the loads act simultaneously is obtained by adding the ordinates of the separate diagrams and is shown in Fig. 3.24(e).

### Problems

**P.3.1** A transmitting mast of height 40 m and weight 4.5 kN/m length is stayed by three groups of four cables attached to the mast at heights of 15, 25 and 35 m. If each cable is anchored to the ground at a distance of 20 m from the base of the mast and tensioned to a force of 15 kN, draw a diagram of the compressive force in the mast.

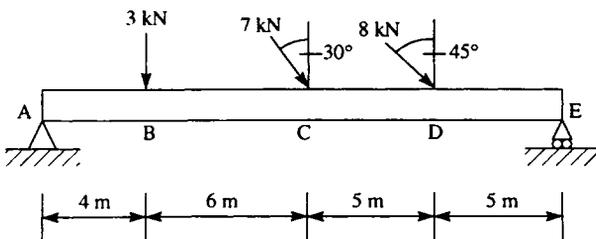
*Ans.* Max. force = 315 kN.

**P.3.2** Construct the normal force, shear force and bending moment diagrams for the beam shown in Fig. P.3.2.

*Ans.*  $N_{AB} = 9.2$  kN,  $N_{BC} = 9.2$  kN,  $N_{CD} = 5.7$  kN,  $N_{DE} = 0$ .

$S_{AB} = 6.8$  kN,  $S_{BC} = 3.8$  kN,  $S_{CD} = -2.3$  kN,  $S_{DE} = -7.9$  kN.

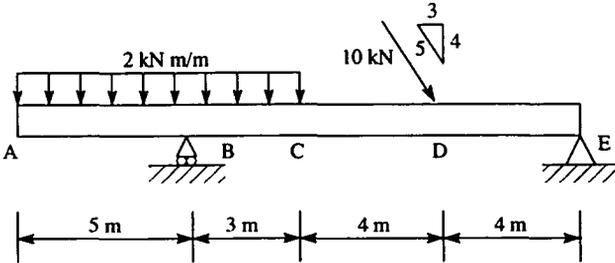
$M_B = 27.2$  kN m,  $M_C = 50$  kN m,  $M_D = 39.5$  kN m.



**Fig. P.3.2**

**P.3.3** Draw dimensioned sketches of the diagrams of normal force, shear force and bending moment for the beam shown in Fig. P.3.3.

Ans.  $N_{AB} = N_{BC} = N_{CD} = 0$ ,  $N_{DE} = -6$  kN.  
 $S_A = 0$ ,  $S_B$  (in AB) =  $-10$  kN,  $S_B$  (in BC) =  $10$  kN,  
 $S_C = 4$  kN,  $S_D$  (in CD) =  $4$  kN,  $S_{DE} = -4$  kN.  
 $M_B = -25$  kN m,  $M_C = -4$  kN m,  $M_D = 12$  kN m.

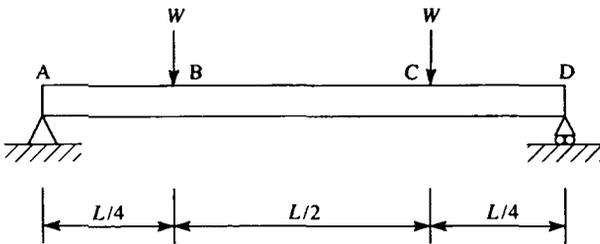


**Fig. P.3.3**

**P.3.4** Draw shear force and bending moment diagrams for the beam shown in Fig. P.3.4.

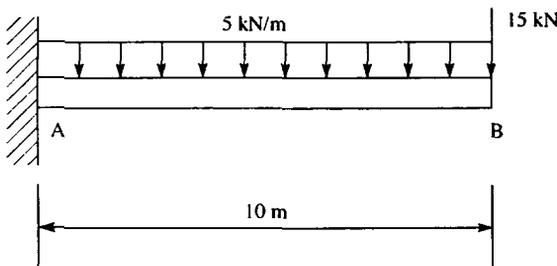
Ans.  $S_{AB} = W$ ,  $S_{BC} = 0$ ,  $S_{CD} = -W$ .  
 $M_B = M_C = WL/4$ .

Note zero shear and constant bending moment in central span.



**Fig. P.3.4**

**P.3.5** The cantilever AB shown in Fig. P.3.5 carries a uniformly distributed load of  $5$  kN/m and a concentrated load of  $15$  kN at its free end. Construct the shear force and bending moment diagrams for the beam.



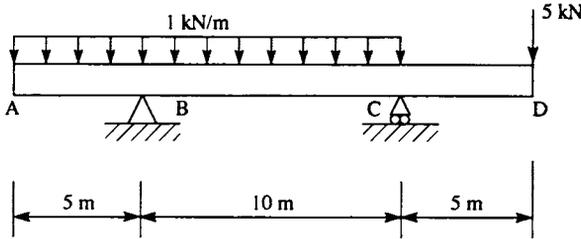
**Fig. P.3.5**

Ans.  $S_B = 15 \text{ kN}$ ,  $S_C = 65 \text{ kN}$ .  
 $M_B = 0$ ,  $M_A = -400 \text{ kN m}$ .

**P.3.6** Sketch the bending moment and shear force diagrams for the simply supported beam shown in Fig. P.3.6 and insert the principal values.

Ans.  $S_B$  (in AB) =  $-5 \text{ kN}$ ,  $S_B$  (in BC) =  $3.75 \text{ kN}$ ,  $S_C$  (in BC) =  $-6.25 \text{ kN}$ ,  
 $S_{CD} = 5 \text{ kN}$ ,  $M_B = -12.5 \text{ kN m}$ ,  $M_C = -25 \text{ kN m}$ .

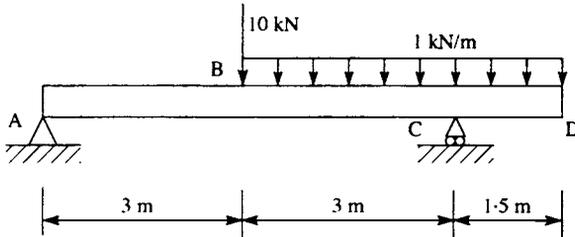
Turning value of bending moment of  $-5.5 \text{ kN m}$  in BC,  $3.75 \text{ m}$  from B.



**Fig. P.3.6**

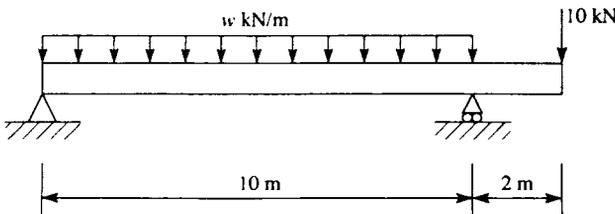
**P.3.7** Draw the shear force and bending moment diagrams for the beam shown in Fig. P.3.7 indicating the principal values.

Ans.  $S_{AB} = 5.6 \text{ kN}$ ,  $S_B$  (in BC) =  $-4.4 \text{ kN}$ ,  $S_C$  (in BC) =  $-7.4 \text{ kN}$ ,  $S_C$  (in CD) =  $1.5 \text{ kN}$ .  
 $M_B = 16.69 \text{ kN m}$ ,  $M_C = -1.13 \text{ kN m}$ .



**Fig. P.3.7**

**P.3.8** Find the value of  $w$  in the beam shown in Fig. P.3.8 for which the maximum sagging bending moment occurs at a point  $10/3 \text{ m}$  from the left-hand support and determine the value of this moment.

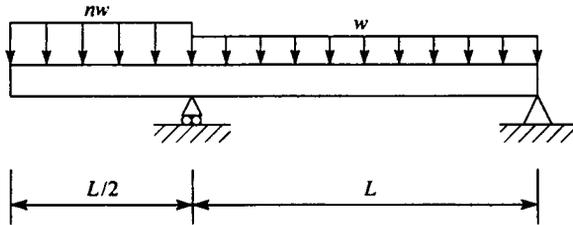


**Fig. P.3.8**

Ans.  $w = 1.2 \text{ kN/m}$ ,  $6.7 \text{ kNm}$ .

**P.3.9** Find the value of  $n$  for the beam shown in Fig. P.3.9 such that the maximum sagging bending moment occurs at  $L/3$  from the right-hand support. Using this value of  $n$  determine the position of the point of contraflexure in the beam.

Ans.  $n = 4/3$ ,  $L/3$  from left-hand support.



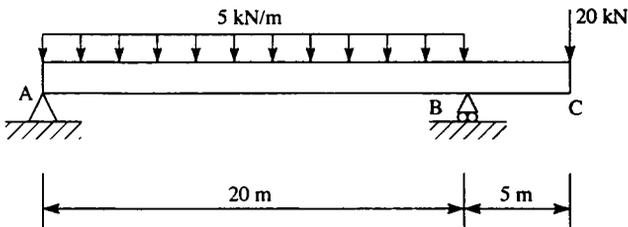
**Fig. P.3.9**

**P.3.10** Sketch the shear force and bending moment diagrams for the simply supported beam shown in Fig. P.3.10 and determine the positions of maximum bending moment and point of contraflexure. Calculate the value of the maximum moment.

Ans.  $S_A = 45 \text{ kN}$ ,  $S_B$  (in AB)  $= -55 \text{ kN}$ ,  $S_{BC} = 20 \text{ kN}$ .

$M_{\max} = 202.5 \text{ kNm}$  at  $9 \text{ m}$  from A,  $M_B = -100 \text{ kNm}$ .

Point of contraflexure is  $18 \text{ m}$  from A.



**Fig. P.3.10**

**P.3.11** Determine the position of maximum bending moment in a simply supported beam,  $8 \text{ m}$  span, which carries a load of  $100 \text{ kN}$  uniformly distributed over its complete length and, in addition, a load of  $120 \text{ kN}$  uniformly distributed over  $2.5 \text{ m}$  to the right from a point  $2 \text{ m}$  from the left support. Calculate the value of maximum bending moment and the value of bending moment at mid-span.

Ans.  $M_{\max} = 294 \text{ kNm}$  at  $3.6 \text{ m}$  from left-hand support.

$M$  (mid-span)  $= 289 \text{ kNm}$ .

**P.3.12** A simply supported beam AB has a span of  $6 \text{ m}$  and carries a distributed load which varies linearly in intensity from zero at A to  $2 \text{ kN/m}$  at B. Sketch the shear force and bending moment diagrams for the beam and insert the principal values.

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Ans.  $S_{AB} = 2 - z^2/6$ ,  $S_A = 2$  kN,  $S_B = -4$  kN.

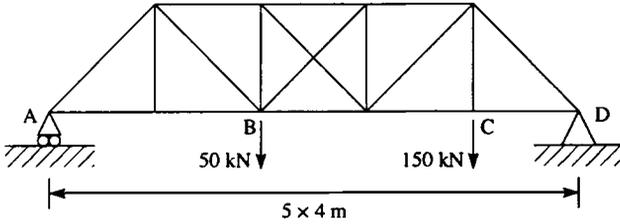
$M_{AB} = 2z - z^3/18$ ,  $M_{max} = 4.62$  kNm at 3.46 m from A.

**P.3.13** A precast concrete beam of length  $L$  is to be lifted by a single sling and has one end resting on the ground. Show that the optimum position for the sling is 0.29 m from the nearest end.

**P.3.14** Construct shear force and bending moment diagrams for the framework shown in Fig. P.3.14.

Ans.  $S_{AB} = 60$  kN,  $S_{BC} = 10$  kN,  $S_{CD} = -140$  kN.

$M_B = 480$  kNm,  $M_C = 560$  kNm.



**Fig. P.3.14**

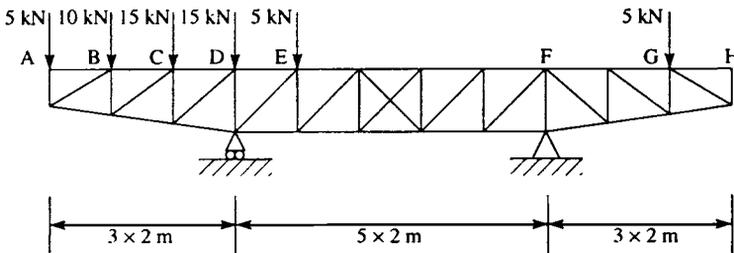
**P.3.15** Draw shear force and bending moment diagrams for the framework shown in Fig. P.3.15.

Ans.  $S_{AB} = -5$  kN,  $S_{BC} = -15$  kN,  $S_{CD} = -30$  kN,  $S_{DE} = 12$  kN,  $S_{EF} = 7$  kN,

$S_{FG} = 5$  kN,  $S_{GH} = 0$ .

$M_B = -10$  kNm,  $M_C = -40$  kNm,  $M_D = -100$  kNm,  $M_E = -76$  kNm,

$M_F = -20$  kNm,  $M_G = M_H = 0$ .



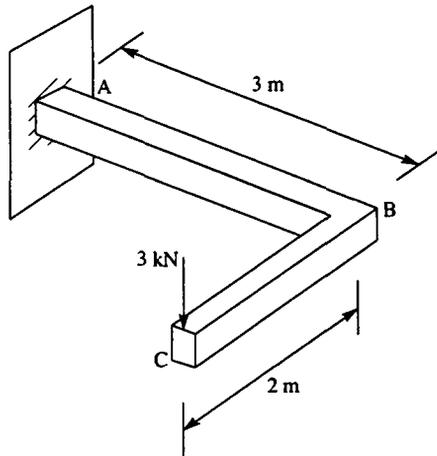
**Fig. P.3.15**

**P.3.16** The cranked cantilever ABC shown in Fig. P.3.16 carries a load of 3 kN at its free end. Draw shear force, bending moment and torsion diagrams for the complete beam.

Ans.  $S_{CB} = 3$  kN,  $S_{BA} = 3$  kN.

$M_C = 0$ ,  $M_B$  (in CB) =  $-6$  kNm,  $M_B$  (in BA) =  $0$ ,  $M_A = -9$  kNm.

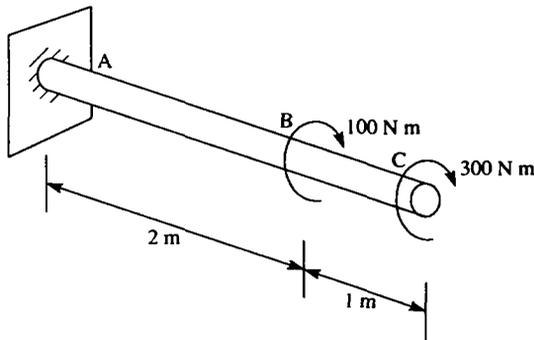
$T_{CB} = 0$ ,  $T_{BA} = 6$  kNm.



**Fig. P.3.16**

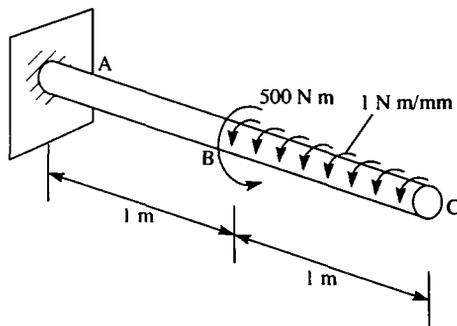
**P.3.17** Construct a torsion diagram for the beam shown in Fig. P.3.17.

*Ans.*  $T_{CB} = -300 \text{ N m}$ ,  $T_{BA} = -400 \text{ N m}$ .



**Fig. P.3.17**

**P.3.18** The beam ABC shown in Fig. P.3.18 carries a distributed torque of  $1 \text{ N m/mm}$  over its outer half BC and a concentrated torque of  $500 \text{ N m}$  at B. Sketch the torsion diagram for the beam inserting the principal values.



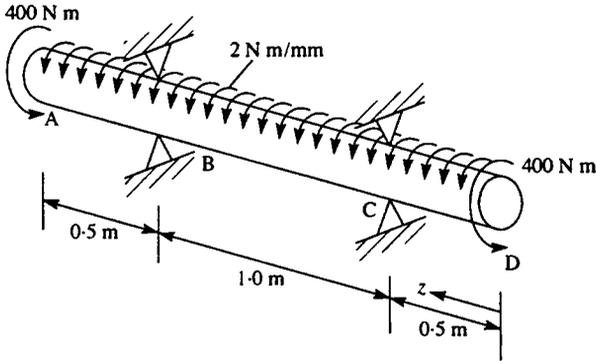
**Fig. P.3.18**

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Ans.  $T_C = 0$ ,  $T_B$  (in BC) = 1000 N m,  $T_B$  (in AB) = 1500 N m.

**P.3.19** The cylindrical bar ABCD shown in Fig. P.3.19 is supported symmetrically at B and C by supports that prevent rotation of the bar about its longitudinal axis. The bar carries a uniformly distributed torque of 2 N m/mm together with concentrated torques of 400 N m at each end. Draw the torsion diagram for the bar and determine the maximum value of torque.

Ans.  $T_{DC} = 400 + 2z$ ,  $T_{CB} = 2z - 2000$ ,  $T_{BA} = 2z - 4400$  ( $T$  in N m when  $z$  is in mm).  
 $T_{\max} = 1400$  N m at C and B.



**Fig. P.3.19**