6.1 THE VIBRATION OF STRUCTURES WITH ONE DEGREE OF FREEDOM

1. A structure is modelled by a rigid horizontal member of mass 3000 kg, supported at each end by a light elastic vertical member of flexural stiffness 2 MN/m.
   Find the frequency of small-amplitude horizontal vibrations of the rigid member.

2. Part of a structure is modelled by a thin rigid rod of mass \( m \) pivoted at the lower end, and held in the vertical position by two springs, each of stiffness \( k \), as shown.
   Find the frequency of small-amplitude oscillation of the rod about the pivot.

3. A uniform beam of length 8 m, simply supported at the ends, carries a uniformly distributed mass of 300 kg/m and three bodies, one of mass 1000 kg at mid-span and
two of mass 1500 kg each, at 2 m from each end. The second moment of area of the beam is $10^{-4}$ m$^4$ and the modulus of elasticity of the material is 200 GN/m$^2$.

Estimate the lowest natural frequency of flexural vibration of the beam assuming that the deflection $y_x$ at a distance $x$ from one end is given by:

$$y_x = y_c \sin \pi(x/l),$$

where $y_c$ is the deflection at mid-span and $l$ is the length of the beam.

4. A section of steel pipe in a distillation plant is 80 mm in diameter, 5 mm thick and 4 m long. The pipe may be assumed to be built-in at each end, so that the deflection $y$, at a distance $x$ from one end of a pipe of length $l$ is

$$y = \frac{mg}{24EI} x^2 (l - x)^2,$$

$m$ being the mass per unit length.

Calculate the lowest natural frequency of transverse vibration of the pipe when full of water. Take $\rho_{\text{steel}} = 7750$ kg/m$^3$, $\rho_{\text{water}} = 930$ kg/m$^3$ and $E_{\text{steel}} = 200$ GN/m$^2$.

5. A uniform horizontal steel beam is built in to a rigid structure at one end and pinned at the other end; the pinned end cannot move vertically but is otherwise unconstrained. The beam is 8 m long, the relevant flexural second moment of area of the cross section is $4.3 \times 10^6$ mm$^4$, and the beam's own mass together with the mass attached to the beam is equivalent to a uniformly distributed mass of 600 kg/m.

Using a combination of sinusoidal functions for the deflected shape of the beam, estimate the lowest natural frequency of flexural vibrations in the vertical plane.

6. Estimate the lowest frequency of natural transverse vibration of a chimney 100 m high, which can be represented by a series of lumped masses $M$ at distances $y$ from its base as follows:

<table>
<thead>
<tr>
<th>$y$ (m)</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$ (10$^3$ kg)</td>
<td>700</td>
<td>540</td>
<td>400</td>
<td>280</td>
<td>180</td>
</tr>
</tbody>
</table>

With the chimney considered as a cantilever on its side the static deflection in bending, $x$ along the chimney is calculated to be

$$x = X \left(1 - \cos \pi \frac{y}{2l}\right),$$

where $l = 100$ m, and $X = 0.2$ m.

How would you expect the actual frequency to compare with the frequency that you have calculated?
7. Estimate the lowest natural frequency of a light beam 7 m long carrying six concentrated masses equally spaced along its length. The measured static deflections under each mass are:

<table>
<thead>
<tr>
<th>Mass (kg)</th>
<th>1070</th>
<th>970</th>
<th>370</th>
<th>370</th>
<th>670</th>
<th>670</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deflection (mm)</td>
<td>2.5</td>
<td>2.8</td>
<td>5.5</td>
<td>5.0</td>
<td>2.5</td>
<td>1.0</td>
</tr>
</tbody>
</table>

8. An elastic part of a structure has a dynamic force-deflection loop at a frequency of 10 rad/s as shown below.

![Dynamic force-deflection loop](image)

Find
(a) the stiffness $k$,
(b) assuming viscous damping find $c$ and $\zeta$, and
(c) assuming hysteretic damping find $\eta$.

9. A solid steel shaft, 25 mm in diameter and 0.45 m long, is mounted in long bearings in a rigid frame at one end and has at its other end, which is unsupported, a steel flywheel. The flywheel can be treated as a rim 0.6 m in outer diameter and 20 mm square cross section, with rigid spokes of negligible mass lying in the mid-plane of the rim.

Find the frequency of free flexural vibrations.

10. A uniform rigid building, height 30 m and cross section 10 m $\times$ 10 m, rests on an elastic soil of stiffness $0.6 \times 10^6$ N/m$^3$. (Stiffness is defined as the force per unit area to produce unit deflection.)

If the mass of the building is $2 \times 10^6$ kg and its inertia about its axis of rocking at the base is $500 \times 10^6$ kg m$^2$, calculate the period of the rocking motion (small amplitudes).

What wind speed would excite this motion if the Strouhal number is 0.22? Calculate also the maximum height the building could be before becoming unstable.

11. A single degree of freedom system with a body of mass 10 kg, a spring of stiffness 1 kN/m and negligible damping is subjected to an input force $F$ which varies with time as shown overleaf.
Determine the amplitude of free vibration of the body after the force is removed.

12. A uniform rigid tower of height 30 m and cross section $3 \times 3$ m, is symmetrically mounted on a rigid foundation of depth 2 m and section $5 \times 5$ m. The mass of the tower is calculated to be $1.5 \times 10^5$ kg, and of the foundation, $10^3$ kg. The foundation rests on an elastic soil which has a uniform stiffness of $2 \times 10^6$ N/m$^3$. (Stiffness is defined as the force per unit area to produce unit deflection.)

If the mass moment of inertia of the tower and foundation about its axis of rocking at the base of the foundation is $6 \times 10^7$ kg m$^2$, find the period of small-amplitude rocking motion. The axis of rocking is parallel to a side of the foundation.

What is the greatest height the tower could have and still be stable on this foundation?

13. The foundation of a rigid tower is a circular concrete block of diameter $D$, set into an elastic soil. The effective stiffness of the soil, $k$, is defined as the force per unit area to produce a unit displacement and is constant for small deflections. The tower is uniform with a total mass $M$. The centre of mass is situated on the centre line at a height $h$ above the base. The moment of inertia of the tower about an axis of rocking at the base is $I$.

Show that the natural frequency of rocking is given by:

$$\omega = \sqrt{\frac{\pi k D^4}{64 I} - \frac{M g h}{I}} \text{ Hz.}$$

14. A body supported by an elastic structure performs a damped oscillation of period 1 s, in a medium whose resistance is proportional to the velocity. At a given instant the amplitude was observed to be 100 mm, and in 10 s this had reduced to 1 mm.

What would be the period of the free vibration if the resistance of the medium were negligible?
15. To determine the amount of damping in a bridge it was set into vibration in the fundamental mode by dropping a weight on it at centre span. The observed frequency was 1.5 Hz, and the amplitude was found to have decreased to 0.9 of the initial maximum after 2 s. The equivalent mass of the bridge (estimated by the Rayleigh Energy method) was $10^5$ kg.

Assuming viscous damping and simple harmonic motion, calculate the damping coefficient, the logarithmic decrement and the damping ratio.

16. A new concert hall is to be protected from the ground vibrations from an adjacent highway by mounting the hall on rubber blocks. The predominant frequency of the sinusoidal ground vibrations is 40 Hz, and a motion transmissibility of 0.1 is to be attained at that frequency.

Calculate the static deflection required in the rubber blocks, assuming that these act as linear, undamped springs.

17. When considering the vibrations of a structure, what is meant by the $Q$ factor? Derive a simple relationship between the $Q$ factor and the damping ratio for a single degree of freedom system with light viscous damping.

Measurements of the vibration of a bridge section resulting from impact tests show that the period of each cycle is 0.6 s, and that the amplitude of the third cycle is twice the amplitude of the ninth cycle. Assuming the damping to be viscous estimate the $Q$ factor of the section.

When a vehicle of mass 4000 kg is positioned at the centre of the section the period of each cycle increases to 0.62 s; no change is recorded in the rate of decay of the vibration. What is the effective mass of the section?

18. The vibration on the floor of a laboratory is found to be simple harmonic motion at a frequency in the range 15–60 Hz, (depending on the speed of some nearby reciprocating plant). It is desired to install in the laboratory a sensitive instrument which requires insulating from the floor vibration. The instrument is to be mounted on a small platform which is supported by three similar springs resting on the floor, arranged to carry equal loads; the motion is restrained to occur in a vertical direction only. The combined mass of the instrument and the platform is 40 kg: the mass of the springs can be neglected and the equivalent viscous damping ratio of the suspension is 0.2.

Calculate the maximum value for the spring stiffness, if the amplitude of the transmitted vibration is to be less than 10% of that of the floor vibration over the given frequency range.

19. A machine of mass 520 kg produces a vertical disturbing force which oscillates sinusoidally at a frequency of 25 Hz. The force transmitted to the floor is to have an amplitude, at this frequency, not more than 0.4 times that of the disturbing force in the machine, and the static deflection of the machine on its mountings is to be as small as possible consistent with this.

For this purpose, rubber mountings are to be used, which are available as units, each of which has a stiffness of 359 kN/m and a damping coefficient of 2410 N s/m. How many of these units are needed?
20. The basic element of many vibration-measuring devices is the seismic unit, which consists of a mass \( m \) supported from a frame by a spring of stiffness \( k \) in parallel with a damper of viscous damping coefficient \( c \). The frame of the unit is attached to the structure whose vibration is to be determined, the quantity measured being \( z \), the relative motion between the seismic mass and the frame. The motion of both the structure and the seismic mass is translation in the vertical direction only.

Derive the equation of motion of the seismic mass, assuming that the structure has simple harmonic motion of circular frequency \( \nu \), and deduce the steady state amplitude of \( z \).

Given that the undamped natural circular frequency \( \omega \) of the unit is much greater than \( \nu \), show why the unit may be used to measure the acceleration of the structure.

Explain why in practice some damping is desirable.

If the sensitivity of the unit (that is, the amplitude of \( z \) as a multiple of the amplitude of the acceleration of the frame) is to have the same value when \( \nu = 0.2 \omega \) as when \( \nu \ll \omega \), find the necessary value of the damping ratio.

21. A two-wheeled trailer of sprung mass 700 kg is towed at 60 km/h, along an undulating straight road whose surface may be considered sinusoidal. The distance from peak to peak of the road surface is 30 m, and the height from hollow to crest is 0.1 m. The effective stiffness of the trailer suspension is 60 kN/m, and the shock absorbers, which provide linear viscous damping, are set to give a damping ratio of 0.67.

Assuming that only vertical motion of the trailer is excited, find the absolute amplitude of this motion and its phase angle relative to the undulations.

22. Find the Fourier series representation of the following triangular wave.

23. A wooden floor, 6 m by 3 m, is simply supported along the two shorter edges. The mass is 300 kg and the static deflection at the centre under the self-weight is 7 mm.
It is proposed to determine the dynamic properties of the floor by dropping a sand bag of 50 kg mass on it at the centre, and to measure the response at that position with an accelerometer and a recorder.

In order to select the instruments required, estimate:
(i) the frequency of the fundamental mode of vibration that would be recorded,
(ii) the number of oscillations at the fundamental frequency for the signal amplitude on the recorder to be reduced to half, assuming a loss factor of 0.05, and
(iii) the height of drop of the sand bag so that the dynamic deflection shall not exceed 10 mm, and the corresponding maximum acceleration.

24. The figure shows a diagrammatic end view of one half of a swing-axle suspension of a motor vehicle which consists of a horizontal half-axle OA pivoted to the chassis at O, a wheel rotating about the centre line of the axle and a spring of stiffness \( k \) and a viscous damper with a damping coefficient \( c \) both located vertically between the axle and the chassis. The mass of the half-axle is \( m_1 \) and its radius of gyration about \( O \) is \( h \). The mass of the wheel is \( m_2 \) and it may be regarded as a thin uniform disc having an external radius \( r \) and located at a horizontal distance \( s \) from the pivot \( O \). The spring and damper are located at horizontal distances \( q \) and \( p \) from the pivot \( O \), as shown.

Derive the equation for angular displacement of the axle-wheel assembly about the pivot \( O \), and obtain from it an expression for the frequency of damped free oscillations of the assembly. Express this frequency in terms of the given parameters and the undamped natural frequency of the assembly.

25. The T-shaped body shown overleaf pivots about a point \( O \) on a horizontal ground surface and is held upright, so that its mass centre \( G \) is a distance \( h \) vertically above \( O \), by two springs pinned to it and to the ground. Each spring has a stiffness \( k \) and its vertical
centre line is at a distance $c$ from the pivot $O$. The T-shaped body has a mass $m$ and a radius of gyration $r$ about its mass centre.

The body is acted on by a force $F$ whose line of action is horizontal and at a height $d$ above the ground, where $d > h$. Derive an expression for the rotation of the body if the force rises suddenly from zero to $F$, assuming that the angular displacement of the body is small.

If the suddenly applied force $F$ drops equally suddenly to zero after a time $t_0$ from its original application, derive the equation of the rotational motion of the body for times greater than $t_0$.

26. The figure shows a vibration exciter which consists of two contra-rotating wheels, each carrying an eccentric body of mass 0.1 kg at an effective radius of 10 mm from its axis of rotation. The vertical positions of the eccentric bodies occur at the same instant. The total mass of the exciter is 2 kg and damping is negligible.

Find a value for the stiffness of the spring mounting so that a force of amplitude 100 N, due to rotor unbalance, is transmitted to the fixed support when the wheels rotate at 150 rad/s.
6.2 THE VIBRATION OF STRUCTURES WITH MORE THAN ONE DEGREE OF FREEDOM

27. A two-storey building is represented by the two degree of freedom lumped mass system shown below.

![Diagram of two-storey building](image)

Obtain the frequency equation for swaying motion in the X–Y plane; hence calculate the natural frequencies and sketch the corresponding mode shapes.

28. A vehicle has a mass of 2000 kg and a 3 m wheelbase. The mass moment of inertia about the centre of mass is 500 kg m², and the centre of mass is located 1 m from the front axle. Considering the vehicle as a two degree of freedom system, find the natural frequencies and the corresponding modes of vibration, if the front and rear springs have stiffnesses of 50 kN/m and 80 kN/m, respectively.

The expansion joints of a concrete road are 5 m apart. These joints cause a series of impulses at equal intervals to vehicles travelling at a constant speed. Determine the speeds at which pitching motion and up and down motion are most likely to arise for the above vehicle.

29. To analyse the vibrations of a two-storey building it is represented by the lumped mass system shown overleaf, where \( m_1 = \frac{1}{3} m_2 \), and \( k_1 = \frac{1}{2} k_2 \) (\( k_1 \) and \( k_2 \) represent the shear stiffnesses of the parts of the building shown).

Calculate the natural frequencies of free vibrations, and sketch the corresponding mode shapes of the building, showing the amplitude ratios.

If a horizontal harmonic force \( F_1 \sin \omega t \) is applied to the top floor, determine expressions for the amplitudes of the steady state vibration of each floor.
30. The rigid beam, shown in its position of static equilibrium in the figure, has a mass $m$ and a mass moment of inertia $2ma^2$ about an axis perpendicular to the plane of the diagram, and through its centre of mass $G$.

Assuming no horizontal motion of $G$, find the frequencies of small oscillations in the plane of the diagram, and the corresponding positions of the nodes.

31. A small electronic package is supported by springs as shown opposite. The mass of the package is $m$, each spring has a constant axial stiffness $k$, and damping is negligible.
Sec. 6.21 The vibration of structures with more than one degree of freedom

Considering motion in the plane of the figure only, and assuming that the amplitude of vibration of the package is small enough for the lateral spring forces to be negligible, write down the equations of motion and hence obtain the frequencies of free vibration of the package.

Explain how the vibration mode shapes can be found.

32. Explain, in one sentence each, what is meant by a natural frequency and mode shape of a dynamic system.

33. Part of a machine can be modelled by the system shown. Two uniform discs A and B, which are free to rotate about fixed parallel axes through their centres, are coupled by a spring. Similar springs connect the discs to the fixed frame as shown in the figure. Each of the springs has a stiffness of 2.5 kN/m which is the same in tension or compression. Disc A has a mass moment of inertia about its axis of rotation of 0.05 kg m², and a radius of 0.1 m, whilst for the disc B the corresponding figures are 0.3 kg m² and 0.2 m. Damping is negligible.

Determine the natural frequencies of small amplitude oscillation of the system and the corresponding mode shapes.

34. Part of a building structure is modelled by the triple pendulum shown overleaf. Obtain the equations of motion of small-amplitude oscillation in the plane of the figure by using the Lagrange equation.

Hence determine the natural frequencies of the structure and the corresponding mode shapes.
35. A simplified model for studying the dynamics of a motor vehicle is shown. The body has a mass $M$, and a moment of inertia about an axis through its mass centre of $I_g$. It is considered to be free to move in two directions – vertical translation and rotation in the vertical plane. Each of the unsprung wheel masses, $m$, are free to move in vertical translation only. Damping effects are ignored.

(i) Derive equations of motion for this system. Define carefully the coordinates used.

(ii) Is it possible to determine the natural frequency of a ‘wheel hop’ mode without solving all the equations of motion? If not, suggest an approximation that might be made, in order to obtain an estimate of the wheel hop frequency, and calculate such an estimate given the following data:

$$k = 20 \text{ kN/m}; \quad K = 70 \text{ kN/m}; \quad m = 22.5 \text{ kg}.$$
36. If the building in problem 27 were enlarged by adding a further floor of mass \( m \) and shear stiffness \( k \) on top of the existing building, obtain the frequency equation for the three degree of freedom system formed. Given \( m \) and \( k \) contemplate how this equation may be solved. What if the building were 20 storeys high?

37. To analyse the vibration of a two-coach rail unit, it is modelled as the system shown. Each coach is represented by a rigid uniform beam of length \( l \) and mass \( m \); the coupling is a simple ball-joint. The suspension is considered to be three similar springs, each of stiffness \( k \), positioned as shown. Damping can be neglected.

Considering motion in the plane of the figure only, obtain the equations of motion for small-amplitude free vibrations, and hence obtain the natural frequencies of the system.

Explain how the mode shapes may be found.

38. A bridge structure is modelled by a simply supported beam of length \( l \), with three equal bodies each of mass \( m \) attached to it at equal distances as shown. Show that the influence coefficients are (where \( \Delta = l^3/256 EI \)):

\[
\begin{align*}
\alpha_{11} &= 3\Delta, & \alpha_{12} &= 3.67\Delta, & \alpha_{13} &= 2.33\Delta, \\
\alpha_{21} &= 3.67\Delta, & \alpha_{22} &= 5.33\Delta, & \alpha_{23} &= 3.67\Delta, \\
\alpha_{31} &= 2.33\Delta, & \alpha_{32} &= 3.67\Delta, & \alpha_{33} &= 3\Delta.
\end{align*}
\]

Proceed to find the flexibility matrix and, by iteration, deduce the lowest natural frequency and associated mode shape.

39. A solid cylinder, has a mass \( M \) and radius \( R \). Pinned to the axis of the cylinder is an arm of length \( l \) which carries a bob of mass \( m \) as shown overleaf. Obtain the natural frequency of free vibration of the bob.
40. An aeroplane has a fuselage mass of 4000 kg. Each wing has an engine of mass 500 kg, and a fuel tank of mass 200 kg at its tip, as shown. Neglecting the mass of each wing, calculate the frequencies of flexural vibrations in a vertical plane. Take the stiffness of the wing sections to be $3k$ and $k$ as shown, where $k = 100$ kN/m.

41. A marine propulsion installation is shown in the figure opposite. For the analysis of torsional vibration, the installation can be modelled as the system shown, where the mass moments of inertia for the engine, gearbox and propeller taken about the axis of rotation are $I_e$, $I_G$ and $I_p$ respectively, and the stiffnesses of the gearbox and propeller shafts are $k_G$ and $k_p$ respectively. The numerical values are
The vibration of structures with more than one degree of freedom

\[ I_E = 0.8 \text{ kg m}^2, \]
\[ I_G = 0.3 \text{ kg m}^2, \]
\[ I_P = 2.0 \text{ kg m}^2, \]
\[ k_G = 400 \text{ kNm/rad}, \]
\[ k_P = 120 \text{ kNm/rad}, \]

and damping can be neglected.

Calculate the natural frequencies of free torsional vibration and give the positions of the node for each frequency.

42. A machine is modelled by the system shown. The masses of the main elements are \( m_1 \) and \( m_2 \), and the spring stiffnesses are as shown. Each roller has a mass \( m \), diameter \( d \), and mass moment of inertia \( J \) about its axis, and rolls without slipping.

Considering motion in the longitudinal direction only, use Lagrange’s equation to obtain the equations of motion for small free oscillations of the system. If \( m_1 = 4 \text{ m} \), \( m_2 = 2 \text{ m} \) and \( J = md^2/8 \), deduce the natural frequencies of the system and the corresponding mode shapes.

43. Vibrations of a particular structure can be analysed by considering the equivalent system shown overleaf.
The bodies are mounted on small frictionless rollers whose mass is negligible, and motion occurs in a horizontal direction only.

Write down the equations of motion of the system and determine the frequency equation in determinant form. Indicate how you would

(i) solve the frequency equation, and
(ii) determine the mode shapes associated with each natural frequency.

Briefly describe how the Lagrange equation could be used to obtain the natural frequencies of free vibration of the given system.

44. A simply supported beam of negligible mass and length \( l \), has three bodies each of mass \( m \) attached as shown. The influence coefficients are, using standard notation,

\[
\begin{align*}
\alpha_{11} &= 3l^3/256 \, EI, \\
\alpha_{21} &= 2.33l^3/256 \, EI, \\
\alpha_{22} &= 5.33l^3/256 \, EI.
\end{align*}
\]

Write down the flexibility matrix, and determine by iteration the frequency of the first mode of vibration, correct to 2 significant figures, if \( EI = 10 \, \text{Nm}^2 \), \( m = 2 \, \text{kg} \) and \( l = 1 \, \text{m} \).

Comment on the physical meaning of the eigenvector you have obtained, and use the orthogonality principle to obtain the frequencies of the higher modes.

45. A structure is modelled by three identical long beams and rigid bodies, connected by two springs as shown opposite. The rigid bodies are each of mass \( M \) and the mass of the beams is negligible. Each beam has a transverse stiffness \( K \) at its unsupported end; and the springs have stiffnesses \( k \) and \( 2k \) as shown.

Determine the frequencies and corresponding mode shapes of small-amplitude oscillation of the bodies in the plane of the figure. Damping can be neglected.
46. Find the dynamic matrix of the system shown.
If \( k = 20 \text{kN/m} \) and \( m = 5 \text{ kg} \), find the lowest natural frequency of the system and the associated mode shape.

47. A structure is modelled by the three degree of freedom system shown overleaf. Only translational motion in a vertical direction can occur.
Show that the influence coefficients are
\[ \alpha_{11} = \alpha_{22} = \alpha_{33} = \frac{1}{3}k \]
and
\[ \alpha_{21} = \alpha_{31} = \alpha_{32} = \frac{1}{4}k, \]
and proceed to find the flexibility matrix. Hence obtain the lowest natural frequency of the system and the corresponding mode shape.

48. A delicate instrument is to be mounted on an antivibration installation so as to minimize the risk of interference caused by groundborne vibration. An elevation of the installation is shown opposite, and the point A indicates the location of the most sensitive part of the instrument. The installation is free to move in the vertical plane, but horizontal translation is not to be considered.
It is decided to use as a design criterion the transmissibility $T_{AB}$, being the sinusoidal vibration amplitude at A for a unit amplitude of vibration on the ground at B. One of the major sources of groundborne vibration is a nearby workshop where there are several machines which run at 3000 rev/min. Accordingly, it is proposed that the installation should have a transmissibility $|T_{AB}|$ of 1% at 50 Hz.

Given the following data:

$$M = 3175 \text{ kg; } l = 0.75 \text{ m; } R = 0.43 \text{ m (where } I_G = MR^2)$$

determine the maximum value of stiffness $K$ that the mounts may possess in order to meet the requirement, and find the two natural frequencies of the installation.

Repeat the analysis using a simpler model of the system having just one degree of freedom – vertical translation of the whole installation – and establish whether this simpler approach provides an acceptable means of designing such a vibration isolation system.

For the purpose of these basic isolation design calculations, damping may be ignored.

49. In a vibration isolation system, a group of machines are firmly mounted together onto a rigid concrete raft which is then isolated from the foundation by four antivibration pads. For the purposes of analysis, the system may be modelled as a symmetrical body of mass 1150 kg and moments of inertia about rolling and pitching axes through the mass centre of 175 kgm$^2$ and 250 kgm$^2$, respectively, supported at each corner by a spring of stiffness $7.5 \times 10^5$ N/m.

The model is shown overleaf.
The major disturbing force is generated by a machine at one corner of the raft and may be represented by a harmonically varying vertical force with a frequency of 50 Hz, acting directly through the axis of one of the mounts.

(i) Considering vertical vibration only, show that the force transmitted to the foundation by each mount will be different, and calculate the magnitude of the largest, expressed as a percentage of the excitation force.

(ii) Identify the mode of vibration that is responsible for the largest component of this transmitted force and suggest ways of improving the isolation performance using the same mounts but without modifying the raft.

(iii) Show that a considerable improvement in isolation would be obtained by moving the disturbing machine to the centre of the raft, and calculate the transmitted force for this case, again expressed as a percentage of the exciting force.

50. Find the driving point impedance of the system shown. The bodies move on frictionless rollers in a horizontal direction only.

\[ k = 100 \text{ N/m} \]

\[ F = F_0 e^{i\omega t} \]

\[ m_1 = 0.2 \text{ kg} \]

\[ m_2 = 0.25 \text{ kg} \]

\[ c = 20 \text{ N/m/s} \]

\[ k = 100 \text{ N/m} \]

Hence show that the amplitude of vibration of body 1 is

\[ \frac{\sqrt{[72000 + 2620v^2 + 0.2v^2]^2 + (20v^2)^2]}}{v^2 (0.04v^4 + 1224v^2 + 32400)} \]
51. Find the driving point mobility of the system shown; only motion in the vertical direction occurs and damping is negligible.

\[ m_1 = 0.5 \text{ kg} \]
\[ k_1 = 10^6 \text{ N/m} \]
\[ m_2 = 1 \text{ kg} \]
\[ k_2 = 4 \times 10^6 \text{ N/m} \]
\[ m_3 = 2 \text{ kg} \]
\[ k_3 = 5 \times 10^6 \text{ N/m} \]
\[ m_4 = 1 \text{ kg} \]
\[ k_4 = 2 \times 10^6 \text{ N/m} \]

Hence obtain the frequency equation; check your result by using a different method of analysis.

6.3 THE VIBRATION OF CONTINUOUS STRUCTURES

52. A uniform beam of length \( l \) is built-in at one end, and rests on a spring of stiffness \( k \) at the other, as shown.
Determine the frequency equation for small-amplitude transverse vibration, and show how the first natural frequency changes as $k$ increases from zero at the free end, to infinity, at the simply supported end.

Comment on the effect of the value of $k$ on the frequency of the 10th mode.

53. A structure is modelled as a uniform beam of length $l$, hinged at one end, and resting on a spring of stiffness $k$ at the other, as shown.

Determine the first three natural frequencies of the beam, and sketch the corresponding mode shapes.

54. Part of a structure is modelled as a uniform cross-section beam having a pinned attachment at one end and a sliding constraint at the other (where it is free to translate, but not to rotate) as shown.

(i) Derive the frequency equation for this beam and find expressions for the $n$th natural frequency and the corresponding mode shape. Sketch the shapes of the first three modes.

(ii) The beam is to be stiffened by adding a spring of stiffness $k$ to the sliding end. Derive the frequency equation for this case and use the result to deduce the frequency equation for a pinned clamped beam.

(iii) Estimate how much the fundamental frequency of the original beam is raised by adding a very stiff spring to its sliding end.
55. A portal frame consists of three uniform beams, each of length \( l \), mass \( m \), and flexural rigidity \( EI \), attached as shown. There is no relative rotation between beams at their joints.

56. A uniform cantilever of length \( l \) and flexural rigidity \( EI \), is subjected to a transverse harmonic exciting force \( F \sin vt \) at the free end. Show that the displacement at the free end is

\[
\frac{\sin \lambda l \cdot \cosh \lambda l - \cos \lambda l \cdot \sinh \lambda l}{EI\lambda^3 (1 + \cos \lambda l \cdot \cosh \lambda l)} F \sin vt,
\]

where \( \lambda = (\rho A v^2 / EI)^{1/4} \).

57. A thin rectangular plate has its long sides simply supported, and both its short sides unsupported. Find the first three natural frequencies of flexural vibration, and sketch the corresponding mode shapes.

58. Derive the frequency equation for flexural vibration of a uniform beam that is pinned at one end and free at the other.

Show that the fundamental mode of vibration has a natural frequency of zero, and explain the physical significance of this mode.

Obtain an approximate value for the natural frequency of the first bending mode of vibration, and compare this with the corresponding value for a beam that is rigidly clamped at one end and free at the other.

6.4 DAMPING IN STRUCTURES

59. The ‘half-power’ method of determining the damping in a particular mode of vibration from a receptance plot can be extended to a more general form in which the two points used – one below resonance and one above – need not be at an amplitude exactly 0.707 times the peak value.
(i) A typical Nyquist plot of the receptance for a single degree of freedom system with structural damping is shown, with two points corresponding to frequencies \( \nu_1 \) and \( \nu_2 \). The natural frequency, \( \alpha \), is also indicated. Prove that the damping loss factor, \( \eta \), is given exactly by:

\[
\eta = \left[ \frac{(\nu_2^2 - \nu_1^2)}{\omega^2} \right] \left[ \frac{1}{\tan \frac{1}{2} \phi_1 + \tan \frac{1}{2} \phi_2} \right]^{-1},
\]

where \( \phi_1 \) and \( \phi_2 \) are the angles subtended by points 1 and 2 with the resonance point and the centre of the circle. Show how this expression relates to the half-power points formula.

(ii) Some receptance data from measurements on a practical structure are listed in the table opposite. By application of the formula above to the data given, obtain a best estimate for the damping of the mode under investigation.
60. A large symmetric machine tool structure is supported by four suspension units, one at each corner, intended to provide isolation against vibration. Each unit consists of a primary spring (which can be considered massless and undamped) of stiffness 250 kN/m, in parallel with a viscous dashpot of rate 20 kN s/m. The 'bouncing' natural frequency of the installation is 2.8 Hz while the two rocking modes are 1.9 Hz and 2.2 Hz.

It is found that excessive high-frequency vibration forces are transmitted from the machine to the floor, particularly above 20 Hz. Some modifications are required to improve the isolation performance, but a constraint is imposed by the pipes and other service connections to the machine which cannot withstand significantly larger displacements than are currently encountered.

It is suggested that a rubber bush be inserted either at one end of the dashpot (for example, between the dashpot and the machine structure), or between the entire suspension unit and the machine. The same bush would be used in either configuration and it may be modelled as an undamped spring with a stiffness of 700 kN/m.

Show analytically which of the two proposed modifications provides the greatest improvement in high-frequency isolation, and calculate the increased attenuation (in dB) for both cases at 25 Hz and at 50 Hz. Consider motion in the vertical direction only.

Comment on the suitability of the two proposed modifications, and indicate what additional calculations should be made to define completely the dynamic behaviour of the modified installation.

61. A machine having a mass of 1250 kg is isolated from floor vibration by a resilient mount whose stiffness is 0.2 x 10^6 N/m, and which has negligible damping. The machine generates a strong excitation which can be considered as an externally applied harmonic force at its running speed of 480 revolutions per minute, and the vibration isolation required is specified as a force transmissibility of –35 dB at this frequency.
(i) Show that the single-stage system described above will not provide the necessary attenuation.

It is decided to improve the effectiveness of the installation by introducing a second mass-spring stage between the resilient mount and the floor. The maximum deadweight that can be supported by the floor is 2500 kg, and so the second-stage mass is taken as 1250 kg.

(ii) Calculate the stiffness of the second-stage spring in order to attain the required force transmissibility.

(iii) Determine the frequency at which this two-stage system has the same transmissibility as the simpler single-stage one, and sketch the transmissibility curve for each case, indicating the frequency above which the isolation system gives a definite attenuation.

62. (a) The traditional 'half-power points' formula for estimating damping, that is, loss factor $\zeta = \Delta f / f_0$ (where $\Delta f$ is the frequency bandwidth at the half-power points and $f_0$ is the frequency of maximum response), is an approximation that becomes unreliable when applied to modes with relatively high damping.

Sketch a graph indicating the error incurred in using this formula instead of the exact one, as a function of damping loss factor in the range 0.1 to 1.0.

(b) The measured receptance data given in the table were taken in the frequency region near a mode of vibration of interest on a scale model of a chemical reactor.

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Receptance</th>
<th>Frequency (Hz)</th>
<th>Receptance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Modulus</td>
<td></td>
<td>Phase</td>
</tr>
<tr>
<td></td>
<td>($\times 10^{-6}$ m/N)</td>
<td></td>
<td>(degrees)</td>
</tr>
<tr>
<td>380.0</td>
<td>41.6</td>
<td>31</td>
<td></td>
</tr>
<tr>
<td>390.0</td>
<td>49.9</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>400.0</td>
<td>66.5</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>410.0</td>
<td>86.1</td>
<td>41</td>
<td></td>
</tr>
<tr>
<td>420.0</td>
<td>70.7</td>
<td>67</td>
<td></td>
</tr>
<tr>
<td>430.0</td>
<td>64.0</td>
<td>65</td>
<td></td>
</tr>
<tr>
<td>440.0</td>
<td>67.0</td>
<td>60</td>
<td></td>
</tr>
</tbody>
</table>

Obtain estimates for the damping in this mode of vibration, using (a) a modulus-frequency plot and (b) a polar (or Nyquist) plot of the receptance data. Present your answers in terms of $Q$ factors. State which of the two estimates obtained you consider to be the more reliable, and justify your choice.

63. The results given opposite are of an incomplete resonance test on a structure. The response at different frequencies was measured at the point of application of a sinusoidal driving force and is given as the receptance, being the ratio of the amplitude of vibration to the maximum value of the force. The phase angle between the amplitude and force was also measured.
Estimate the effective mass, dynamic stiffness and loss factor, assuming material type damping.

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Receptance ($\times 10^{-6}$ m/N)</th>
<th>Phase angle (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>70</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>82</td>
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<td>24</td>
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<td>88</td>
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<td>94</td>
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<tr>
<td>100</td>
<td>32</td>
<td>85</td>
</tr>
<tr>
<td>109</td>
<td>10</td>
<td>135</td>
</tr>
<tr>
<td>115</td>
<td>9</td>
<td>180</td>
</tr>
<tr>
<td>130</td>
<td>7</td>
<td>180</td>
</tr>
</tbody>
</table>

64. A test is conducted in order to measure the dynamic properties of an antivibration mount. A mass of 900 kg is supported on the mount to form a single degree of freedom system, and measurements are made of the receptance of this system in the region of its major resonance.

In addition to the hysteretic damping provided by the mount (which is to be measured), some additional damping is introduced by friction in the apparatus, and so tests are made at two different amplitudes of vibration ($x_o$) in order to determine the magnitude of each of the two sources of damping.

It may be assumed that the loss factor of the mount is a constant, valid for all vibration amplitudes, but the dynamic stiffness is not a constant and so the two tests have slightly different natural frequencies.

Details of some receptance measurements are given overleaf. Assuming that the additional damping has the characteristic of Coulomb friction damping, estimate the hysteretic damping loss factor of the mount.
### Problems

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Test (a)</th>
<th>Test (b)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_o = 0.1$ mm</td>
<td>$x_o = 0.02$ mm</td>
</tr>
<tr>
<td>Re($\alpha$) ($\times 10^{-7}$ m/N)</td>
<td>Im($\alpha$)</td>
<td>Re($\alpha$) ($\times 10^{-7}$ m/N)</td>
</tr>
<tr>
<td>13.25</td>
<td>7.6</td>
<td>-6.9</td>
</tr>
<tr>
<td>13.50</td>
<td>7.6</td>
<td>-8.6</td>
</tr>
<tr>
<td>13.75</td>
<td>6.6</td>
<td>-10.7</td>
</tr>
<tr>
<td>14.00</td>
<td>4.4</td>
<td>-12.4</td>
</tr>
<tr>
<td>14.25</td>
<td>1.6</td>
<td>-12.8</td>
</tr>
<tr>
<td>14.50</td>
<td>-1.0</td>
<td>-11.9</td>
</tr>
<tr>
<td>14.75</td>
<td>-2.5</td>
<td>-10.1</td>
</tr>
<tr>
<td>15.00</td>
<td>-3.1</td>
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</tr>
<tr>
<td>15.25</td>
<td>-3.2</td>
<td>-7.0</td>
</tr>
<tr>
<td>15.50</td>
<td>-3.0</td>
<td>-6.0</td>
</tr>
</tbody>
</table>

65. A resonance test on a flexible structure at a constant energy level has revealed a prominent mode at 120 Hz with a half-power frequency bandwidth of 4.8 Hz and a peak acceleration of 480 m/s². The effective mass of the structure for this mode has been estimated at 20 kg.

It is proposed to introduce Coulomb type friction at the point of measurement of the response so as to reduce the motion by $1/5$ for the same energy input as before. Estimate the friction force required, assuming this to be independent of amplitude and frequency of vibration.

66. (i) Derive a relationship between the logarithmic decrement of a system with velocity type damping performing free vibrations, and the loss factor for structural damping. Define clearly any assumptions made.

(ii) A concrete floor slab is to be supported on four columns, spaced in a square grid of sides 7 m. The detail of one column is shown opposite. The slab is to be isolated from vibrations being transmitted up the column by rubber pads, installed as shown. The slab thickness is 150 mm and the density of concrete is 2250 kg/m³. The first resonance frequency of the slab is estimated to be 20 Hz and the logarithmic decrement for concrete is about 0.2. Measurements of the vibration in a column have shown a strong peak at 20 Hz. The rubber pads have a loss factor of 0.1.

(a) Estimate the resonance frequency the pad system should have to provide a reduction of $4/5$ in the vibration being transmitted at 20 Hz to the centre of the floor slab. Comment upon the result.

(b) Estimate the additional attenuation in dB the isolation will provide at a frequency of 160 Hz.

Some of the information given may be superfluous.
67. (i) Often it is required to introduce into a structure some additional form of damping in order to keep resonance vibrations down to an acceptable level. One method is to use a damped dynamic absorber where a (relatively) small mass is suspended from the primary mass (of the vibrating structure) by a spring and a dashpot. If the absorber spring–mass system is tuned to the natural frequency of the effective mass–spring model of the structure, then the absorber dashpot may introduce some damping to the structural resonance.

Without performing analysis, but using physical reasoning only, sketch a family of curves for the point receptance on the primary mass for a range of different magnitudes for the absorber dashpot between 0 and ∞, and hence show that there will be an optimum value for that dashpot rate.

(ii) In one application of this type of damper-absorber, it is required to increase the damping in a new suspension bridge.

In moderate to high winds the airflow over the bridge generates an effectively steady-state excitation force at the bridge's fundamental natural frequency. The airflow also provides some damping. The amplitude of steady vibration under this excitation is found to be 20 mm and this is twice the maximum amplitude considered to be 'safe'. Accordingly, it is proposed to introduce extra damping to reduce the resonance amplitude to 10 mm.

Tests on the bridge show that it possesses some structural damping and this is estimated from measurement of free decay curves. The amplitude of vibration is
found to halve after 40 cycles. A more significant source of damping is the airflow over the bridge and this is most readily described in terms of energy dissipation, which is estimated to be $\beta x_0^3 \text{ N m per cycle}$ where $\beta = 2.5 \times 10^7 \text{ N/m}^2$ and $x_0$ = vibration amplitude. The effective mass and stiffness of the bridge (for its fundamental mode) are 500 000 kg and $5 \times 10^7 \text{ N/m}$, respectively. Determine the equivalent viscous dashpot rate which must be added in order to reduce the resonance vibration amplitude to 10 mm. Assume the excitation force remains the same.

(iii) If, due to miscalculation, the actual dashpot used has a rate of only 70% of that specified, what then will be the vibration amplitude?

68. A partition is made from several layers of metal and a plastic material. Experiments with the metal layers alone have shown that the energy dissipated per cycle of vibration at the lowest natural frequency is $3 \times 10^4 x_0^2$ joule/cycle, where $x_0$ is the amplitude in metres at the centre of the partition. The stiffness of the plastic layers themselves when measured at the centre is $4 \times 10^7 \text{ N/m}$ with a loss factor of 0.3. The acoustic energy loss from one face alone of the partition when vibrating at the lowest natural frequency of 70 Hz is $1.5 \dot{x}^2$ joules/cycle, where $\dot{x}$ is the maximum velocity in m/s at the centre.

Calculate the amplitude of vibration at the centre of the partition when one face receives an acoustic energy input of 50 watts at 70 Hz. Explain carefully any assumptions that have to be made.

69. A sketch is given below of the essential parts of the front suspension of a motor car, showing the unsprung mass consisting of the tyre, the wheel, and the stub axle, connected at point A by a rubber bush to a hydraulic shock absorber and the main coil spring. The other end of the shock absorber is connected at point B by another rubber bush to a subframe of the car body. A set of wishbone link arms with rubber bushes at each end serve to stabilize the unit.
(a) Devise a representative model for this suspension system comprising lumped masses, springs and dampers. Indicate how the equations of motion can be obtained, but do not solve these. Define the symbols introduced carefully.

(b) A massless model of the main spring, the shock absorber and the bushes at points A and B is shown, assuming that the car body represents an infinite impedance. The rubber bushes A and B are identical and have a complex stiffness

\[ k^* = k(1 + j\eta), \]

where the elastic stiffness \( k = 600 \text{ kN/m} \) and the loss factor \( \eta = 0.25 \).

The main spring stiffness \( K = 25 \text{ kN/m} \), and the shock absorber behaves as a viscous damper with a coefficient \( c = 3 \text{ kN s/m} \).

Estimate the percentage contribution by the two bushes to the total energy being dissipated per cycle, for an input motion \( y = y_0 \sin \omega t \) with \( y_0 = 25 \text{ mm} \) and \( \omega = 30 \text{ rad/s} \), assuming that the maximum possible displacement of 5 mm across each bush is being taken up.

70. A machine produces a vertical harmonic force and is to be isolated from the foundations by a suspension system consisting of metal springs in series with blocks of a viscoelastic material.

(i) Show analytically whether it will be better to place the blocks of viscoelastic material above or below the springs from the point of view of:

(a) the force transmitted to the foundation,

(b) damping out high-frequency resonances in the metal spring for the type of installation where the attachment points to the machine are slender metal brackets.

(ii) Compare the system described above with one in which the blocks of viscoelastic material are placed in parallel with the springs.

Define carefully all symbols introduced.

71. Consider the simple joint shown, in which metal to metal contact occurs.
A harmonic exciting force $F \sin \nu t$ is applied to one joint member which has a mass $m$, and is supported by an element of stiffness $k$. The other member is rigidly fixed, so that it is infinitely stiff in the direction of this exciting force. A constant force $N$ is applied normal to the joint interfaces by a clamping arrangement not shown. It is to be assumed that the coefficient of friction $\mu$ existing at the joint interfaces is constant.

Assuming the motion $y$ to be sinusoidal, show that

$$y = \frac{F \sin \nu t - \mu N}{k \left(1 - \left(\frac{\nu}{\omega}\right)^2\right)},$$

and hence obtain an expression for the energy dissipated per cycle by slipping. Show that the maximum energy is dissipated when $\mu N/F = 0.5$, and that $y$ then has an amplitude 50% of the amplitude when $N = 0$. Furthermore, by drawing force–slip hysteresis loops and plotting energy dissipation as a function of $\mu N/F$, show that at least 50% of the maximum energy dissipation can be achieved by maintaining $\mu N/F$ between 0.15 and 0.85.

Comment on the practical significance of this.

72. A beam on elastic supports with dry friction damped joints is modelled by the system shown.

By considering equivalent viscous damping for the friction damper, show that

$$|\delta|^2 = Y^2 - (4F_d/nk)^2,$$

where $\delta = y - y_1$, $Y$ is the amplitude of the body motion and $F_d$ is the tangential friction force in the damper. Hence deduce that
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Consider the response when $F_a = 0$ and $F_a = \infty$, and show that the amplitude of the body for all values of $F_a$ is $2F/nK$ when $\nu/\omega = \sqrt{1 + (n/2)}$, and assess the significance of this.

*Hint:* Write equations of motion for a system with equivalent viscous damping $c = 4F_d/k\nu$, and put $y_1 = Y_1 e^{i\omega t}$, etc. From the equations of motion,

$$Y = \frac{F}{k} \left[ \frac{1 + (cv/nk)^2}{1 - (v/\omega)^2} \right]^{1/2} \left[ 1 - \left( \frac{v}{\omega} \right)^2 \right]^{1/2} \left[ 1 + n - \left( \frac{v}{\omega} \right)^2 \right].$$

Substituting for $c$ and $I\delta l$ gives the required expression for $Y$. Note that as $F_d \to \infty$, $I\delta l \to 0$.

73. Part of a structure is modelled by a cantilever with a friction joint at the free end, as shown. The cantilever has a harmonic exciting force $F \sin \nu t$ applied at a distance $a$ from the root. The tangential friction force generated in the joint by the clamping force $N$ can be represented by a series of linear periodic functions, $F_d(t)$.

$$F \sin \nu t$$

Show that $y_c(t) = \alpha_{yc} F \sin \nu t + \alpha_{yc} F_d(t)$, where $c$ is an arbitrary position along the cantilever and $\alpha$ is a receptance.

By assuming that the friction force is harmonic and always opposes the exciting force, find the energy dissipated per cycle, and hence show that $F_a = 2\mu N$. Is this assumption reasonable for all modes of vibration?

Thus, this linearization of the damping replaces the actual friction force during slipping, $\mu N$, by a sinusoidally varying force of amplitude $2\mu N$. Compare this representation with a Fourier series for the friction damping force.

74. A fabricated steel mast is observed to oscillate violently under certain wind conditions. In order to increase the damping some relative motion is to be allowed in a number of the bolted joints by inserting spring washers under the nuts and by opening
the holes to give a definite clearance. Rubber blocks are to be provided to keep the joint central.

In 15 joints metal to metal sliding friction is to be introduced with a coefficient of friction of 0.2 for a clamping force of $2 \times 10^4$ N. The clearance in each bolt hole is 2.5 mm on diameter. To keep the joint nominally at its central position two rubber blocks are fitted as shown below. The blocks are pressed in position to provide a centring force in excess of the static friction force. Each block is square in cross section, 60 mm by 60 mm, and nominally 18 mm thick. The rubber has a loss factor of 0.12.

The maximum energy input per cycle of oscillation of the mast by the wind is estimated as 1500 joules.

(i) Calculate the modulus of elasticity for the rubber material so that the full clearance in the bolt holes of all the joints is just taken up during an oscillation under the maximum wind excitation, neglecting any structural damping in the mast itself.

(ii) Estimate the $Q$ factor for the mast with the damping in the joints, given that the stored energy in the structure is 2500 joules for a deflection which takes up the total clearance in each joint.
75. The receptance at a point in a structure is measured over a frequency range, and it is found that a resonance occurs in the excitation range. It is therefore decided to add an undamped vibration absorber to the structure.

Sketch a typical receptance–frequency plot for the structure, and by adding the receptance plot of an undamped vibration absorber, predict the new natural frequencies. Show the effect of changes in the absorber mass and stiffness on the natural frequencies, by drawing new receptance–frequency curves for the absorber.

76. Briefly derive the equations that describe the operation of an undamped dynamic vibration absorber.

A milling machine of mass 2700 kg demonstrates a large resonant vibration in the vertical direction at a cutter speed of 300 rev/min when fitted with a cutter having 20 teeth. To overcome this effect it is proposed to add an undamped vibration absorber.

Calculate the minimum absorber mass and the relevant spring stiffness required if the resonance frequency is to lie outside the range corresponding to a cutter speed of 250 to 350 rev/min.

77. In a pumping station, a section of pipe resonated at a pump speed of 120 rev/min. To eliminate this vibration, it was proposed to clamp a spring–mass system to the pipe to act as an absorber. In the first test, an absorber mass of 2 kg tuned to 120 cycle/min resulted in the system having a natural frequency of 96 cycle/min.

If the absorber system is to be designed so that the natural frequencies lie outside the range 85–160 cycle/min, what are the limiting values of the absorber mass and spring stiffness?

78. A certain machine of mass 300 kg produces a harmonic disturbing force $F \cos 15\tau$. Because the frequency of this force coincides with the natural frequency of the machine on its spring mounting an undamped vibration absorber is to be fitted.

If no resonance is to be within 10% of the exciting frequency, find the minimum mass and corresponding stiffness of a suitable absorber. Derive your analysis from the equations of motion, treating the problem as one-dimensional.

79. A machine tool of mass 3000 kg has a large resonance vibration in the vertical direction at 120 Hz. To control this resonance, an undamped vibration absorber of mass 600 kg is fitted, tuned to 120 Hz.

Find the frequency range in which the amplitude of the machine vibration is less with the absorber fitted than without.

80. The figure overleaf shows a body of mass $m_1$ which is supported by a spring of stiffness $k_1$ and which is excited by a harmonic force $P \sin \nu t$. An undamped dynamic vibration absorber consisting of a mass $m_2$ and a spring of stiffness $k_2$ is attached to the body as shown.
Derive an expression for the amplitude of the vibrations of the body.
The body shows a violent resonance at 152 Hz. As a trial remedy a vibration absorber is attached which results in a resonance frequency at 140 Hz. How many such absorbers are required if no resonance is to occur between 120 and 180 Hz?

81. (i) Show that the frequency at which an undamped vibration absorber is most effective ($\omega_a$) is given by the expression

$$\omega_a = \frac{k_a}{m_a}$$

(where $m_a$ and $k_a$ are the mass and stiffness of the added absorber system) and is therefore independent of the properties of the system to which the absorber has been added. Also, derive an expression for the steady-state amplitude of the absorber mass when the system is being driven at its natural frequency $\omega_n$.

(ii) In order to suppress vibration, a vibration absorber system is to be attached to a machine tool which operates over a range of speeds. The design of the absorber is chosen to be a light beam, which is rigidly fixed at one end to the machine tool, and a mass, which may be clamped at various positions along the length of the beam so as to tune the absorber to a required frequency.

Given that the beam is made of aluminium which has a Young’s Modulus of 70 GN/m$^2$ and is of square section, 60 mm $\times$ 60 mm, and the absorber mass is 25 kg, calculate the minimum length of the beam required for the absorber to function over the frequency range 40–50 Hz. Ignore the mass of the beam itself.

Also, calculate how far from the fixed end of the beam the mass would have to be clamped in order to tune the absorber to the maximum frequency of its range of operation.