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APPENDIX A
UNITS OF MEASUREMENT

The following units, abbreviations and prefixes are from the

Système International d'Unitès (designated SI in all Languages.)

Prefixes.

Abreviations		
Prefix	Multiplication factor	Symbol
tera	10^{12}	T
giga	10^{9}	G
mega	10^{6}	M
kilo	10^{3}	K
hecto	10^{2}	h
deka	10	da
deci	10^{-1}	d
centi	10^{-2}	c
milli	10^{-3}	m
micro	10^{-6}	μ
nano	10^{-9}	n
pico	10^{-12}	p

Basic Units.

Basic units of measurement		
Unit	Name	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	degree Kelvin	$^\circ$K
Luminous intensity	candela	cd

Supplementary units		
Unit	Name	Symbol
Plane angle	radian	rad
Solid angle	steradian	sr

DERIVED UNITS		
Name	Units	Symbol
Area	square meter	m^2
Volume	cubic meter	m^3
Frequency	hertz	Hz (s^{-1})
Density	kilogram per cubic meter	kg/m^3
Velocity	meter per second	m/s
Angular velocity	radian per second	rad/s
Acceleration	meter per second squared	m/s^2
Angular acceleration	radian per second squared	rad/s^2
Force	newton	N $(kg \cdot m/s^2)$
Pressure	newton per square meter	N/m^2
Kinematic viscosity	square meter per second	m^2/s
Dynamic viscosity	newton second per square meter	$N \cdot s/m^2$
Work, energy, quantity of heat	joule	J $(N \cdot m)$
Power	watt	W (J/s)
Electric charge	coulomb	C $(A \cdot s)$
Voltage, Potential difference	volt	V (W/A)
Electromotive force	volt	V (W/A)
Electric force field	volt per meter	V/m
Electric resistance	ohm	Ω (V/A)
Electric capacitance	farad	F $(A \cdot s/V)$
Magnetic flux	weber	Wb $(V \cdot s)$
Inductance	henry	H $(V \cdot s/A)$
Magnetic flux density	tesla	T (Wb/m^2)
Magnetic field strength	ampere per meter	A/m
Magnetomotive force	ampere	A

Physical constants.

$$4 \arctan 1 = \pi = 3.14159\,26535\,89793\,23846\,2643 \ldots$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.71828\,18284\,59045\,23536\,0287 \ldots$$

$$\text{Euler's constant} \quad \gamma = 0.57721\,56649\,01532\,86060\,6512 \ldots$$

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$

$$\text{speed of light in vacuum} = 2.997925(10)^8\, m\ s^{-1}$$

$$\text{electron charge} = 1.60210(10)^{-19}\, C$$

$$\text{Avogadro's constant} = 6.02252(10)^{23}\, mol^{-1}$$

$$\text{Plank's constant} = 6.6256(10)^{-34}\, J\, s$$

$$\text{Universal gas constant} = 8.3143\, J\, K^{-1}\, mol^{-1}$$

$$\text{Boltzmann constant} = 1.38054(10)^{-23}\, J\, K^{-1}$$

$$\text{Stefan–Boltzmann constant} = 5.6697(10)^{-8}\, W\, m^{-2}\, K^{-4}$$

$$\text{Gravitational constant} = 6.67(10)^{-11}\, N\, m^2 kg^{-2}$$

APPENDIX B
CHRISTOFFEL SYMBOLS OF SECOND KIND

1. **Cylindrical coordinates** $(r, \theta, z) = (x^1, x^2, x^3)$

$$
\begin{array}{lll}
x = r\cos\theta & r \geq 0 & h_1 = 1 \\
y = r\sin\theta & 0 \leq \theta \leq 2\pi & h_2 = r \\
z = z & -\infty < z < \infty & h_3 = 1
\end{array}
$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$
\begin{array}{ll}
x^2 + y^2 = r^2, & \text{Cylinders} \\
y/x = \tan\theta & \text{Planes} \\
z = Constant & \text{Planes.}
\end{array}
$$

$$
\left\{ {1 \atop 2\,2} \right\} = -r \qquad \left\{ {2 \atop 1\,2} \right\} = \left\{ {2 \atop 2\,1} \right\} = \frac{1}{r}
$$

2. **Spherical coordinates** $(\rho, \theta, \phi) = (x^1, x^2, x^3)$

$$
\begin{array}{lll}
x = \rho\sin\theta\cos\phi & \rho \geq 0 & h_1 = 1 \\
y = \rho\sin\theta\sin\phi & 0 \leq \theta \leq \pi & h_2 = \rho \\
z = \rho\cos\theta & 0 \leq \phi \leq 2\pi & h_3 = \rho\sin\theta
\end{array}
$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$
\begin{array}{ll}
x^2 + y^2 + z^2 = \rho^2 & \text{Spheres} \\
x^2 + y^2 = \tan^2\theta\, z & \text{Cones} \\
y = x\,\tan\phi & \text{Planes.}
\end{array}
$$

$$
\begin{array}{ll}
\left\{ {1 \atop 2\,2} \right\} = -\rho & \left\{ {2 \atop 1\,2} \right\} = \left\{ {2 \atop 2\,1} \right\} = \dfrac{1}{\rho} \\
\left\{ {1 \atop 3\,3} \right\} = -\rho\sin^2\theta & \left\{ {3 \atop 1\,3} \right\} = \left\{ {3 \atop 3\,1} \right\} = \dfrac{1}{\rho} \\
\left\{ {2 \atop 3\,3} \right\} = -\sin\theta\cos\theta & \left\{ {3 \atop 3\,2} \right\} = \left\{ {3 \atop 2\,3} \right\} = \cot\theta
\end{array}
$$

3. **Parabolic cylindrical coordinates** $(\xi, \eta, z) = (x^1, x^2, x^3)$

$$
\begin{array}{lll}
x = \xi\eta & -\infty < \xi < \infty & h_1 = \sqrt{\xi^2 + \eta^2} \\
y = \frac{1}{2}(\xi^2 - \eta^2) & -\infty < z < \infty & h_2 = \sqrt{\xi^2 + \eta^2} \\
z = z & \eta \ge 0 & h_3 = 1
\end{array}
$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$
\begin{array}{ll}
x^2 = -2\xi^2(y - \frac{\xi^2}{2}) & \text{Parabolic cylinders} \\
x^2 = 2\eta^2(y + \frac{\eta^2}{2}) & \text{Parabolic cylinders} \\
z = Constant & \text{Planes.}
\end{array}
$$

$$
\begin{array}{ll}
\left\{ {1 \atop 1\,1} \right\} = \frac{\xi}{\xi^2 + \eta^2} & \left\{ {1 \atop 2\,2} \right\} = \frac{-\xi}{\xi^2 + \eta^2} \\
\left\{ {2 \atop 2\,2} \right\} = \frac{\eta}{\xi^2 + \eta^2} & \left\{ {1 \atop 1\,2} \right\} = \left\{ {1 \atop 2\,1} \right\} = \frac{\eta}{\xi^2 + \eta^2} \\
\left\{ {2 \atop 1\,1} \right\} = \frac{-\eta}{\xi^2 + \eta^2} & \left\{ {2 \atop 2\,1} \right\} = \left\{ {2 \atop 1\,2} \right\} = \frac{\xi}{\xi^2 + \eta^2}
\end{array}
$$

4. **Parabolic coordinates** $(\xi, \eta, \phi) = (x^1, x^2, x^3)$

$$
\begin{array}{lll}
x = \xi\eta\cos\phi & \xi \ge 0 & h_1 = \sqrt{\xi^2 + \eta^2} \\
y = \xi\eta\sin\phi & \eta \ge 0 & h_2 = \sqrt{\xi^2 + \eta^2} \\
z = \frac{1}{2}(\xi^2 - \eta^2) & 0 < \phi < 2\pi & h_3 = \xi\eta
\end{array}
$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$
\begin{array}{ll}
x^2 + y^2 = -2\xi^2(z - \frac{\xi^2}{2}) & \text{Paraboloids} \\
x^2 + y^2 = 2\eta^2(z + \frac{\eta^2}{2}) & \text{Paraboloids} \\
y = x\,\tan\phi & \text{Planes.}
\end{array}
$$

$$
\begin{array}{ll}
\left\{ {1 \atop 1\,1} \right\} = \frac{\xi}{\xi^2 + \eta^2} & \left\{ {1 \atop 3\,3} \right\} = \frac{-\xi\eta^2}{\xi^2 + \eta^2} \\
\left\{ {2 \atop 2\,2} \right\} = \frac{\eta}{\xi^2 + \eta^2} & \left\{ {1 \atop 2\,1} \right\} = \left\{ {1 \atop 2\,1} \right\} = \frac{\eta}{\xi^2 + \eta^2} \\
\left\{ {1 \atop 2\,2} \right\} = \frac{-\xi}{\xi^2 + \eta^2} & \left\{ {2 \atop 2\,1} \right\} = \left\{ {2 \atop 1\,2} \right\} = \frac{\xi}{\xi^2 + \eta^2} \\
\left\{ {2 \atop 1\,1} \right\} = \frac{-\eta}{\xi^2 + \eta^2} & \left\{ {3 \atop 3\,2} \right\} = \left\{ {3 \atop 2\,3} \right\} = \frac{1}{\eta} \\
\left\{ {2 \atop 3\,3} \right\} = \frac{-\eta\xi^2}{\xi^2 + \eta^2} & \left\{ {3 \atop 1\,3} \right\} = \left\{ {3 \atop 3\,1} \right\} = \frac{1}{\xi}
\end{array}
$$

5. **Elliptic cylindrical coordinates** $(\xi,\eta,z)=(x^1,x^2,x^3)$

$$x=\cosh\xi\cos\eta \qquad \xi\ge 0 \qquad h_1=\sqrt{\sinh^2\xi+\sin^2\eta}$$
$$y=\sinh\xi\sin\eta \qquad 0\le\eta\le 2\pi \qquad h_2=\sqrt{\sinh^2\xi+\sin^2\eta}$$
$$z=z \qquad -\infty<z<\infty \qquad h_3=1$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\frac{x^2}{\cosh^2\xi}+\frac{y^2}{\sinh^2\xi}=1 \qquad \text{Elliptic cylinders}$$
$$\frac{x^2}{\cos^2\eta}-\frac{y^2}{\sin^2\eta}=1 \qquad \text{Hyperbolic cylinders}$$
$$z=Constant \qquad \text{Planes.}$$

$$\left\{ {1 \atop 1\,1} \right\}=\frac{\sinh\xi\cosh\xi}{\sinh^2\xi+\sin^2\eta} \qquad \left\{ {2 \atop 2\,2} \right\}=\frac{\sin\eta\cos\eta}{\sinh^2\xi+\sin^2\eta}$$
$$\left\{ {1 \atop 2\,2} \right\}=\frac{-\sinh\xi\cosh\xi}{\sinh^2\xi+\sin^2\eta} \qquad \left\{ {2 \atop 1\,1} \right\}=\frac{-\sin\eta\cos\eta}{\sinh^2\xi+\sin^2\eta}$$
$$\left\{ {1 \atop 1\,2} \right\}=\left\{ {1 \atop 2\,1} \right\}=\frac{\sin\eta\cos\eta}{\sinh^2\xi+\sin^2\eta} \qquad \left\{ {2 \atop 1\,2} \right\}=\left\{ {2 \atop 2\,1} \right\}=\frac{\sinh\xi\cosh\xi}{\sinh^2\xi+\sin^2\eta}$$

6. **Elliptic coordinates** $(\xi,\eta,\phi)=(x^1,x^2,x^3)$

$$x=\sqrt{(1-\eta^2)(\xi^2-1)}\cos\phi \qquad 1\le\xi<\infty \qquad h_1=\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}$$
$$y=\sqrt{(1-\eta^2)(\xi^2-1)}\sin\phi \qquad -1\le\eta\le 1 \qquad h_2=\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}$$
$$z=\xi\eta \qquad 0\le\phi<2\pi \qquad h_3=\sqrt{(1-\eta^2)(\xi^2-1)}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\frac{x^2}{\xi^2-1}+\frac{y^2}{\xi^2-1}+\frac{z^2}{\xi^2}=1 \qquad \text{Prolate ellipsoid}$$
$$\frac{z^2}{\eta^2}-\frac{x^2}{1-\eta^2}-\frac{y^2}{1-\eta^2}=1 \qquad \text{Two-sheeted hyperboloid}$$
$$y=x\,\tan\phi \qquad \text{Planes}$$

$$\left\{ {1 \atop 1\,1} \right\}=-\frac{\xi}{-1+\xi^2}+\frac{\xi}{\xi^2-\eta^2} \qquad \left\{ {2 \atop 3\,3} \right\}=\frac{\left(-1+\xi^2\right)\eta\left(1-\eta^2\right)}{\xi^2-\eta^2}$$
$$\left\{ {2 \atop 2\,2} \right\}=\frac{\eta}{1-\eta^2}-\frac{\eta}{\xi^2-\eta^2} \qquad \left\{ {1 \atop 1\,2} \right\}=-\frac{\eta}{\xi^2-\eta^2}$$
$$\left\{ {1 \atop 2\,2} \right\}=-\frac{\xi\left(-1+\xi^2\right)}{\left(1-\eta^2\right)\left(\xi^2-\eta^2\right)} \qquad \left\{ {2 \atop 2\,1} \right\}=\frac{\xi}{\xi^2-\eta^2}$$
$$\left\{ {1 \atop 3\,3} \right\}=-\frac{\xi\left(-1+\xi^2\right)\left(1-\eta^2\right)}{\xi^2-\eta^2} \qquad \left\{ {3 \atop 3\,1} \right\}=\frac{\xi}{-1+\xi^2}$$
$$\left\{ {2 \atop 1\,1} \right\}=\frac{\eta\left(1-\eta^2\right)}{\left(-1+\xi^2\right)\left(\xi^2-\eta^2\right)} \qquad \left\{ {3 \atop 3\,2} \right\}=-\frac{\eta}{1-\eta^2}$$

7. **Bipolar coordinates** $(u, v, z) = (x^1, x^2, x^3)$

$$x = \frac{a \sinh v}{\cosh v - \cos u}, \qquad 0 \le u < 2\pi \qquad h_1^2 = h_2^2$$

$$y = \frac{a \sin u}{\cosh v - \cos u}, \qquad -\infty < v < \infty \qquad h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}$$

$$z = z \qquad -\infty < z < \infty \qquad h_3^2 = 1$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$(x - a \coth v)^2 + y^2 = \frac{a^2}{\sinh^2 v} \qquad \text{Cylinders}$$

$$x^2 + (y - a \cot u)^2 = \frac{a^2}{\sin^2 u} \qquad \text{Cylinders}$$

$$z = Constant \qquad \text{Planes.}$$

$$\begin{Bmatrix} 1 \\ 1\,1 \end{Bmatrix} = \frac{\sin u}{\cos u - \cosh v} \qquad \begin{Bmatrix} 2 \\ 1\,1 \end{Bmatrix} = \frac{\sinh v}{-\cos u + \cosh v}$$

$$\begin{Bmatrix} 2 \\ 2\,2 \end{Bmatrix} = \frac{\sinh v}{\cos u - \cosh v} \qquad \begin{Bmatrix} 1 \\ 1\,2 \end{Bmatrix} = \frac{\sinh v}{\cos u - \cosh v}$$

$$\begin{Bmatrix} 1 \\ 2\,2 \end{Bmatrix} = \frac{\sin u}{-\cos u + \cosh v} \qquad \begin{Bmatrix} 2 \\ 2\,1 \end{Bmatrix} = \frac{\sin u}{\cos u - \cosh v}$$

8. **Conical coordinates** $(u, v, w) = (x^1, x^2, x^3)$

$$x = \frac{uvw}{ab}, \qquad b^2 > v^2 > a^2 > w^2, \quad u \ge 0 \qquad h_1^2 = 1$$

$$y = \frac{u}{a}\sqrt{\frac{(v^2 - a^2)(w^2 - a^2)}{a^2 - b^2}} \qquad h_2^2 = \frac{u^2(v^2 - w^2)}{(v^2 - a^2)(b^2 - v^2)}$$

$$z = \frac{v}{b}\sqrt{\frac{(v^2 - b^2)(w^2 - b^2)}{b^2 - a^2}} \qquad h_3^2 = \frac{u^2(v^2 - w^2)}{(w^2 - a^2)(w^2 - b^2)}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$x^2 + y^2 + z^2 = u^2 \qquad \text{Spheres}$$

$$\frac{x^2}{v^2} + \frac{y^2}{v^2 - a^2} + \frac{z^2}{v^2 - b^2} = 0, \qquad \text{Cones}$$

$$\frac{x^2}{w^2} + \frac{y^2}{w^2 - a^2} + \frac{z^2}{w^2 - b^2} = 0, \qquad \text{Cones.}$$

$$\begin{Bmatrix} 2 \\ 2\,2 \end{Bmatrix} = \frac{v}{b^2 - v^2} - \frac{v}{-a^2 + v^2} + \frac{v}{v^2 - w^2} \qquad \begin{Bmatrix} 3 \\ 2\,2 \end{Bmatrix} = \frac{w\left(-a^2 + w^2\right)\left(-b^2 + w^2\right)}{\left(b^2 - v^2\right)\left(-a^2 + v^2\right)\left(v^2 - w^2\right)}$$

$$\begin{Bmatrix} 3 \\ 3\,3 \end{Bmatrix} = -\frac{w}{v^2 - w^2} - \frac{w}{-a^2 + w^2} - \frac{w}{-b^2 + w^2} \qquad \begin{Bmatrix} 2 \\ 2\,1 \end{Bmatrix} = \frac{1}{u}$$

$$\begin{Bmatrix} 1 \\ 2\,2 \end{Bmatrix} = -\frac{u\left(v^2 - w^2\right)}{\left(b^2 - v^2\right)\left(-a^2 + v^2\right)} \qquad \begin{Bmatrix} 2 \\ 2\,3 \end{Bmatrix} = -\frac{w}{v^2 - w^2}$$

$$\begin{Bmatrix} 1 \\ 3\,3 \end{Bmatrix} = -\frac{u\left(v^2 - w^2\right)}{\left(-a^2 + w^2\right)\left(-b^2 + w^2\right)} \qquad \begin{Bmatrix} 3 \\ 3\,1 \end{Bmatrix} = \frac{1}{u}$$

$$\begin{Bmatrix} 2 \\ 3\,3 \end{Bmatrix} = -\frac{v\left(b^2 - v^2\right)\left(-a^2 + v^2\right)}{\left(v^2 - w^2\right)\left(-a^2 + w^2\right)\left(-b^2 + w^2\right)} \qquad \begin{Bmatrix} 3 \\ 3\,2 \end{Bmatrix} = \frac{v}{v^2 - w^2}$$

9. **Prolate spheroidal coordinates** $(u, v, \phi) = (x^1, x^2, x^3)$

$$x = a \sinh u \sin v \cos\phi, \quad u \geq 0 \qquad h_1^2 = h_2^2$$
$$y = a \sinh u \sin v \sin\phi, \quad 0 \leq v \leq \pi \qquad h_2^2 = a^2(\sinh^2 u + \sin^2 v)$$
$$z = a \cosh u \cos v, \quad 0 \leq \phi < 2\pi \qquad h_3^2 = a^2 \sinh^2 u \sin^2 v$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\frac{x^2}{(a\sinh u)^2} + \frac{y^2}{a\sinh u)^2} + \frac{z^2}{a\cosh u)^2} = 1, \qquad \text{Prolate ellipsoids}$$
$$\frac{x^2}{(a\cos v)^2} - \frac{y^2}{(a\sin v)^2} - \frac{z^2}{(a\cos v)^2} = 1, \qquad \text{Two-sheeted hyperpoloid}$$
$$y = x\ \tan\phi, \qquad \text{Planes.}$$

$$\left\{ \begin{matrix} 1 \\ 1\,1 \end{matrix} \right\} = \frac{\cosh u \sinh u}{\sin^2 v + \sinh^2 u} \qquad \left\{ \begin{matrix} 2 \\ 3\,3 \end{matrix} \right\} = -\frac{\cos v \sin v \sinh^2 u}{\sin^2 v + \sinh^2 u}$$
$$\left\{ \begin{matrix} 2 \\ 2\,2 \end{matrix} \right\} = \frac{\cos v \sin v}{\sin^2 v + \sinh^2 u} \qquad \left\{ \begin{matrix} 1 \\ 1\,2 \end{matrix} \right\} = \frac{\cos v \sin v}{\sin^2 v + \sinh^2 u}$$
$$\left\{ \begin{matrix} 1 \\ 2\,2 \end{matrix} \right\} = -\frac{\cosh u \sinh u}{\sin^2 v + \sinh^2 u} \qquad \left\{ \begin{matrix} 2 \\ 2\,1 \end{matrix} \right\} = \frac{\cosh u \sinh u}{\sin^2 v + \sinh^2 u}$$
$$\left\{ \begin{matrix} 1 \\ 3\,3 \end{matrix} \right\} = -\frac{\sin^2 v \cosh u \sinh u}{\sin^2 v + \sinh^2 u} \qquad \left\{ \begin{matrix} 3 \\ 3\,1 \end{matrix} \right\} = \frac{\cosh u}{\sinh u}$$
$$\left\{ \begin{matrix} 2 \\ 1\,1 \end{matrix} \right\} = -\frac{\cos v \sin v}{\sin^2 v + \sinh^2 u} \qquad \left\{ \begin{matrix} 3 \\ 3\,2 \end{matrix} \right\} = \frac{\cos v}{\sin v}$$

10. **Oblate spheroidal coordinates** $(\xi, \eta, \phi) = (x^1, x^2, x^3)$

$$x = a \cosh\xi \cos\eta \cos\phi, \quad \xi \geq 0 \qquad h_1^2 = h_2^2$$
$$y = a \cosh\xi \cos\eta \sin\phi, \quad -\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2} \qquad h_2^2 = a^2(\sinh^2\xi + \sin^2\eta)$$
$$z = a \sinh\xi \sin\eta, \quad 0 \leq \phi \leq 2\pi \qquad h_3^2 = a^2 \cosh^2\xi \cos^2\eta$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$\frac{x^2}{(a\cosh\xi)^2} + \frac{y^2}{(a\cosh\xi)^2} + \frac{z^2}{(a\sinh\xi)^2} = 1, \qquad \text{Oblate ellipsoids}$$
$$\frac{x^2}{(a\cos\eta)^2} + \frac{y^2}{(a\cos\eta)^2} - \frac{z^2}{(a\sin\eta)^2} = 1, \qquad \text{One-sheet hyperboloids}$$
$$y = x\ \tan\phi, \qquad \text{Planes.}$$

$$\left\{ \begin{matrix} 1 \\ 1\,1 \end{matrix} \right\} = \frac{\cosh\xi \sinh\xi}{\sin^2\eta + \sinh^2\xi} \qquad \left\{ \begin{matrix} 2 \\ 3\,3 \end{matrix} \right\} = \frac{\cos\eta \sin\eta \cosh^2\xi}{\sin^2\eta + \sinh^2\xi}$$
$$\left\{ \begin{matrix} 2 \\ 2\,2 \end{matrix} \right\} = \frac{\cos\eta \sin\eta}{\sin^2\eta + \sinh^2\xi} \qquad \left\{ \begin{matrix} 1 \\ 1\,2 \end{matrix} \right\} = \frac{\cos\eta \sin\eta}{\sin^2\eta + \sinh^2\xi}$$
$$\left\{ \begin{matrix} 1 \\ 2\,2 \end{matrix} \right\} = -\frac{\cosh\xi \sinh\xi}{\sin^2\eta + \sinh^2\xi} \qquad \left\{ \begin{matrix} 2 \\ 2\,1 \end{matrix} \right\} = \frac{\cosh\xi \sinh\xi}{\sin^2\eta + \sinh^2\xi}$$
$$\left\{ \begin{matrix} 1 \\ 3\,3 \end{matrix} \right\} = -\frac{\cos^2\eta \cosh\xi \sinh\xi}{\sin^2\eta + \sinh^2\xi} \qquad \left\{ \begin{matrix} 3 \\ 3\,1 \end{matrix} \right\} = \frac{\sinh\xi}{\cosh\xi}$$
$$\left\{ \begin{matrix} 2 \\ 1\,1 \end{matrix} \right\} = -\frac{\cos\eta \sin\eta}{\sin^2\eta + \sinh^2\xi} \qquad \left\{ \begin{matrix} 3 \\ 3\,2 \end{matrix} \right\} = -\frac{\sin\eta}{\cos\eta}$$

11. <u>**Toroidal coordinates**</u> $(u, v, \phi) = (x^1, x^2, x^3)$

$$x = \frac{a \sinh v \cos \phi}{\cosh v - \cos u}, \quad 0 \le u < 2\pi \qquad h_1^2 = h_2^2$$

$$y = \frac{a \sinh v \sin \phi}{\cosh v - \cos u}, \quad -\infty < v < \infty \qquad h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}$$

$$z = \frac{a \sin u}{\cosh v - \cos u}, \quad 0 \le \phi < 2\pi \qquad h_3^2 = \frac{a^2 \sinh^2 v}{(\cosh v - \cos u)^2}$$

The coordinate curves are formed by the intersection of the coordinate surfaces

$$x^2 + y^2 + \left(z - \frac{a \cos u}{\sin u}\right)^2 = \frac{a^2}{\sin^2 u}, \qquad \text{Spheres}$$

$$\left(\sqrt{x^2 + y^2} - a\frac{\cosh v}{\sinh v}\right)^2 + z^2 = \frac{a^2}{\sinh^2 v}, \qquad \text{Tores}$$

$$y = x \ \tan \phi, \qquad \text{planes}$$

$$\left\{ \begin{matrix} 1 \\ 1\,1 \end{matrix} \right\} = \frac{\sin u}{\cos u - \cosh v} \qquad \left\{ \begin{matrix} 2 \\ 3\,3 \end{matrix} \right\} = -\frac{\sinh v\,(\cos u \cosh v - 1)}{\cos u - \cosh v}$$

$$\left\{ \begin{matrix} 2 \\ 2\,2 \end{matrix} \right\} = \frac{\sinh v}{\cos u - \cosh v} \qquad \left\{ \begin{matrix} 1 \\ 1\,2 \end{matrix} \right\} = \frac{\sinh v}{\cos u - \cosh v}$$

$$\left\{ \begin{matrix} 1 \\ 2\,2 \end{matrix} \right\} = \frac{\sin u}{-\cos u + \cosh v} \qquad \left\{ \begin{matrix} 2 \\ 2\,1 \end{matrix} \right\} = \frac{\sin u}{\cos u - \cosh v}$$

$$\left\{ \begin{matrix} 1 \\ 3\,3 \end{matrix} \right\} = \frac{\sin u \sinh v^2}{-\cos u + \cosh v} \qquad \left\{ \begin{matrix} 3 \\ 3\,1 \end{matrix} \right\} = \frac{\sin u}{\cos u - \cosh v}$$

$$\left\{ \begin{matrix} 2 \\ 1\,1 \end{matrix} \right\} = \frac{\sinh v}{-\cos u + \cosh v} \qquad \left\{ \begin{matrix} 3 \\ 3\,2 \end{matrix} \right\} = \frac{\cos u \cosh v - 1}{\cos u \sinh v - \cosh v \sinh v}$$

12. **Confocal ellipsoidal coordinates** $(u, v, w) = (x^1, x^2, x^3)$

$$x^2 = \frac{(a^2-u)(a^2-v)(a^2-w)}{(a^2-b^2)(a^2-c^2)}, \qquad u < c^2 < b^2 < a^2$$
$$y^2 = \frac{(b^2-u)(b^2-v)(b^2-w)}{(b^2-a^2)(b^2-c^2)}, \qquad c^2 < v < b^2 < a^2$$
$$z^2 = \frac{(c^2-u)(c^2-v)(c^2-w)}{(c^2-a^2)(c^2-b^2)}, \qquad c^2 < b^2 < v < a^2$$

$$h_1^2 = \frac{(u-v)(u-w)}{4(a^2-u)(b^2-u)(c^2-u)}$$
$$h_2^2 = \frac{(v-u)(v-w)}{4(a^2-v)(b^2-v)(c^2-v)}$$
$$h_3^2 = \frac{(w-u)(w-v)}{4(a^2-w)(b^2-w)(c^2-w)}$$

$$\left\{ \begin{matrix} 1 \\ 1\,1 \end{matrix} \right\} = \frac{1}{2\,(a^2-u)} + \frac{1}{2\,(b^2-u)} + \frac{1}{2\,(c^2-u)} + \frac{1}{2\,(u-v)} + \frac{1}{2\,(u-w)}$$
$$\left\{ \begin{matrix} 2 \\ 2\,2 \end{matrix} \right\} = \frac{1}{2\,(a^2-v)} + \frac{1}{2\,(b^2-v)} + \frac{1}{2\,(c^2-v)} + \frac{1}{2\,(-u+v)} + \frac{1}{2\,(v-w)}$$
$$\left\{ \begin{matrix} 3 \\ 3\,3 \end{matrix} \right\} = \frac{1}{2\,(a^2-w)} + \frac{1}{2\,(b^2-w)} + \frac{1}{2\,(c^2-w)} + \frac{1}{2\,(-u+w)} + \frac{1}{2\,(-v+w)}$$

$$\left\{ \begin{matrix} 1 \\ 2\,2 \end{matrix} \right\} = \frac{\left(a^2-u\right)\left(b^2-u\right)\left(c^2-u\right)(v-w)}{2\,(a^2-v)\,(b^2-v)\,(c^2-v)\,(u-v)\,(u-w)} \qquad \left\{ \begin{matrix} 1 \\ 1\,2 \end{matrix} \right\} = \frac{-1}{2\,(u-v)}$$
$$\left\{ \begin{matrix} 1 \\ 3\,3 \end{matrix} \right\} = \frac{\left(a^2-u\right)\left(b^2-u\right)\left(c^2-u\right)(-v+w)}{2\,(u-v)\,(a^2-w)\,(b^2-w)\,(c^2-w)\,(u-w)} \qquad \left\{ \begin{matrix} 1 \\ 1\,3 \end{matrix} \right\} = \frac{-1}{2\,(u-w)}$$
$$\left\{ \begin{matrix} 2 \\ 1\,1 \end{matrix} \right\} = \frac{\left(a^2-v\right)\left(b^2-v\right)\left(c^2-v\right)(u-w)}{2\,(a^2-u)\,(b^2-u)\,(c^2-u)\,(-u+v)\,(v-w)} \qquad \left\{ \begin{matrix} 2 \\ 2\,1 \end{matrix} \right\} = \frac{-1}{2\,(-u+v)}$$
$$\left\{ \begin{matrix} 2 \\ 3\,3 \end{matrix} \right\} = \frac{\left(a^2-v\right)\left(b^2-v\right)\left(c^2-v\right)(-u+w)}{2\,(-u+v)\,(a^2-w)\,(b^2-w)\,(c^2-w)\,(v-w)} \qquad \left\{ \begin{matrix} 2 \\ 2\,3 \end{matrix} \right\} = \frac{-1}{2\,(v-w)}$$
$$\left\{ \begin{matrix} 3 \\ 1\,1 \end{matrix} \right\} = \frac{(u-v)\left(a^2-w\right)\left(b^2-w\right)\left(c^2-w\right)}{2\,(a^2-u)\,(b^2-u)\,(c^2-u)\,(-u+w)\,(-v+w)} \qquad \left\{ \begin{matrix} 3 \\ 3\,1 \end{matrix} \right\} = \frac{-1}{2\,(-u+w)}$$
$$\left\{ \begin{matrix} 3 \\ 2\,2 \end{matrix} \right\} = \frac{(-u+v)\left(a^2-w\right)\left(b^2-w\right)\left(c^2-w\right)}{2\,(a^2-v)\,(b^2-v)\,(c^2-v)\,(-u+w)\,(-v+w)} \qquad \left\{ \begin{matrix} 3 \\ 3\,2 \end{matrix} \right\} = \frac{-1}{2\,(-v+w)}$$

APPENDIX C
VECTOR IDENTITIES

The following identities assume that $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ are differentiable vector functions of position while f, f_1, f_2 are differentiable scalar functions of position.

1.	$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$
2.	$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$
3.	$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$
4.	$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \vec{0}$
5.	$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = \vec{B}(\vec{A} \cdot \vec{C} \times \vec{D}) - \vec{A}(\vec{B} \cdot \vec{C} \times \vec{D})$ $= \vec{C}(\vec{A} \cdot \vec{B} \times \vec{C}) - \vec{D}(\vec{A} \cdot \vec{B} \times \vec{C})$
6.	$(\vec{A} \times \vec{B}) \cdot (\vec{B} \times \vec{C}) \times (\vec{C} \times \vec{A}) = (\vec{A} \cdot \vec{B} \times \vec{C})^2$
7.	$\nabla(f_1 + f_2) = \nabla f_1 + \nabla f_2$
8.	$\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$
9.	$\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$
10.	$\nabla(f\vec{A}) = (\nabla f) \cdot \vec{A} + f\nabla \cdot \vec{A}$
11.	$\nabla(f_1 f_2) = f_1 \nabla f_2 + f_2 \nabla f_1$
12.	$\nabla \times (f\vec{A}) =)\nabla f) \times \vec{A} + f(\nabla \times \vec{A})$
13.	$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
14.	$(\vec{A} \cdot \nabla)\vec{A} = \nabla \left(\frac{\lvert\vec{A}\rvert^2}{2} \right) - \vec{A} \times (\nabla \times \vec{A})$
15.	$\nabla(\vec{A} \cdot \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} + (\vec{A} \cdot \nabla)\vec{B} + \vec{B} \times (\nabla \times \vec{A}) + \vec{A} \times (\nabla \times \vec{B})$
16.	$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{B} + \vec{A}(\nabla \cdot \vec{B})$
17.	$\nabla \cdot (\nabla f) = \nabla^2 f$
18.	$\nabla \times (\nabla f) = \vec{0}$
19.	$\nabla \cdot (\nabla \times \vec{A}) = 0$
20.	$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$