Appendix A

Non-conservative form of Navier–Stokes equations

To derive the Navier–Stokes equations in their non-conservative form, we start with the conservative form.

Conservation of mass:
\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial x_i} + u_i \frac{\partial \rho}{\partial x_i} = 0
\] (A.1)

Conservation of momentum:
\[
\frac{\partial (\rho u_i)}{\partial t} + \frac{\partial (u_j \rho u_i)}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0
\] (A.2)

Conservation of energy:
\[
\frac{\partial (\rho E)}{\partial t} + \frac{\partial (u_j \rho E)}{\partial x_j} - \frac{\partial}{\partial x_i} \left( k \frac{\partial T}{\partial x_i} \right) + \frac{\partial (u_j p)}{\partial x_j} - \frac{\partial (\tau_{ij} u_j)}{\partial x_j} = 0
\] (A.3)

Rewriting the momentum equation with terms differentiated as
\[
\rho \frac{\partial u_i}{\partial t} + u_i \left( \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \rho}{\partial x_j} \right) + \rho u_j \frac{\partial u_i}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0
\] (A.4)

and substituting the equation of mass conservation (Eq. A.1) into the above equation gives the reduced momentum equation
\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} - \frac{1}{\rho} \frac{\partial \tau_{ij}}{\partial x_j} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} = 0
\] (A.5)

Similarly as above, the energy equation (Eq. A.3) can be written with differentiated terms as
\[
E \left( \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \rho}{\partial x_j} \right) + \rho \frac{\partial E}{\partial t} + \rho u_j \frac{\partial E}{\partial x_j} - \frac{\partial}{\partial x_i} \left( k \frac{\partial T}{\partial x_i} \right) + \frac{\partial (u_j p)}{\partial x_j} - \frac{\partial (\tau_{ij} u_j)}{\partial x_j} = 0
\] (A.6)
Again substituting the continuity equation into the above equation, we have the reduced form of the energy equation

\[
\frac{\partial E}{\partial t} + u_j \frac{\partial E}{\partial x_j} - \frac{1}{\rho} \frac{\partial}{\partial x_i} \left( k \frac{\partial T}{\partial x_i} \right) + \frac{1}{\rho} \frac{\partial (u_i p)}{\partial x_i} - \frac{1}{\rho} \frac{\partial (\tau_i u_j)}{\partial x_i} \tag{A.7}
\]

Some authors use Eqs. (A.1), (A.5) and (A.7) to study compressible flow problems. However, these non-conservative equations can result in multiple or incorrect solutions in certain cases. This is true especially for high-speed compressible flow problems with shocks. The reader should note that such non-conservative equations are not suitable for simulation of compressible flow problems.