where $\xi$ is a one-dimensional natural coordinate for the surface side, $e_3$ is the unit vector normal to the plane of deformation (which is constant), and $\bar{p}(\xi)$ is now the force per unit length of surface side. For this case the nodal forces for the follower pressure load are given explicitly by

$$\mathbf{f}_\alpha = \int_{-1}^{1} N_\alpha \bar{p}(\xi) \left\{ \frac{-x_{2,\xi}}{x_{1,\xi}} \right\} \, d\xi$$  \hspace{1cm} (10.146)

where $x_{i,\xi}$ are derivatives computed from the one-dimensional finite element interpolation used to approximate the element side. The case for axisymmetry involves additional terms and the reader is referred to reference 21 for details.

### 10.7 Material constitution for finite deformation

In order to complete any finite element development it is necessary to describe how the material behaves when subjected to deformation or deformation histories. In the discussion above we considered elastic behaviour without introducing details on how to model specific material behaviour. Clearly, restriction to elastic behaviour is inadequate to model the behaviour of many engineering materials as we have already shown in many previous applications. The modelling of engineering materials at finite strain is a subject of much research and any complete summary on the state of the art is clearly outside the scope of what can be presented here. In this chapter we present only some classical methods which may be used to model elastic and elasto-plastic type behaviours. The reader is directed to literature for details on other constitutive models (e.g. see references 3 and 24).

We first consider some methods which may be used to describe the behaviour of isotropic elastic materials which undergo finite deformation. In this section we restrict attention to those materials in which a stored energy function is used. Later we will extend this to permit the use of plasticity models and show that much of the material presented in Chapter 3 is here again useful. Finally, to permit the modelling of materials which are not isotropic or cannot be expressed as an extension to elastic behaviour (e.g. generalized plasticity models of Chapter 3) we introduce a rate form – here again many options are possible.

#### 10.7.1 Isotropic elasticity – formulation in invariants

We consider a finite deformation form for hyperelasticity in which a stored energy density function, $W$, is used to compute stresses. For a stored energy density expressed in terms of right Cauchy–Green deformation tensor, $C_{ij}$, the second Piola–Kirchhoff stress is computed by using Eq. (10.38). Through standard transformation we can also obtain the Kirchhoff stress as

$$\tau_{ij} = 2b_{ik} \frac{\partial W}{\partial b_{kj}}$$  \hspace{1cm} (10.147)

and thus, by using Eq. (10.19), also obtain directly the Cauchy stress.
For an isotropic material the stored energy density depends only on three invariants of the deformation. Here we consider the three invariants (noting they also are equal to those for $b_{ij}$) expressed as

$$I = C_{kk} = b_{kk} \quad (10.148)$$

$$II = \frac{1}{2} (I^2 - C_{KL} C_{LK}) = \frac{1}{2} (I^2 - b_{kl} b_{lk}) \quad (10.149)$$

and

$$III = \det C_{KL} = \det b_{kl} = J^2 \quad \text{where} \quad J = \det F_{KL} \quad (10.150)$$

and write the strain energy density as

$$W(C_{KL}) = W(b_{kl}) \equiv W(I, II, J) \quad (10.151)$$

where we select $J$ instead of $III$ as the measure of the volume change. Thus, the second Piola–Kirchhoff stress is computed as

$$S_{ij} = 2 \left[ \frac{\partial W}{\partial I} \frac{\partial I}{\partial C_{ij}} + \frac{\partial W}{\partial II} \frac{\partial II}{\partial C_{ij}} + \frac{\partial W}{\partial J} \frac{\partial J}{\partial C_{ij}} \right] \quad (10.152)$$

The derivatives of the invariants may be evaluated as (see Appendix A)

$$\frac{\partial I}{\partial C_{ij}} = \delta_{ij}, \quad \frac{\partial II}{\partial C_{ij}} = I \delta_{ij} - C_{ij}, \quad \frac{\partial J}{\partial C_{ij}} = \frac{1}{2} J C_{ij}^{-1} \quad (10.153)$$

Thus, the stress is given by

$$S_{ij} = 2 \left[ \delta_{ij} \left( I \delta_{ij} - C_{ij} \right) \frac{1}{2} J C_{ij}^{-1} \right] \left\{ \frac{\partial W}{\partial I} \frac{\partial W}{\partial II} \frac{\partial W}{\partial J} \right\} \quad (10.154)$$

The second Piola–Kirchhoff stress may be transformed to the Cauchy stress by using Eq. (10.20), and gives

$$\sigma_{ij} = \frac{2}{J} \left[ b_{ij} \left( I b_{ij} - b_{im} b_{mj} \right) \frac{1}{2} J \delta_{ij} \right] \left\{ \frac{\partial W}{\partial I} \frac{\partial W}{\partial II} \frac{\partial W}{\partial J} \right\} \quad (10.155)$$

Use of a Newton–Raphson type solution process requires computation of the elastic moduli for the finite elasticity model. The elastic moduli with respect to the reference configuration are deduced from

$$D_{ijkl} = 4 \frac{\partial^2 W}{\partial C_{ij} \partial C_{kl}} = 2 \frac{\partial S_{ij}}{\partial C_{kl}} \quad (10.156)$$
Using Eq. (10.154) the general form for the elastic moduli of an isotropic material is obtained from

\[ D_{IJKL} = 4 \left[ \delta_{IJ}, (I \delta_{IJ} - C_{IJ}), \frac{1}{2} J C_{IJ}^{-1} \right] \left[ \begin{array}{ccc} \frac{\partial^2 W}{\partial I^2} & \frac{\partial^2 W}{\partial I \partial II} & \frac{\partial^2 W}{\partial I \partial J} \\ \frac{\partial^2 W}{\partial II \partial I} & \frac{\partial^2 W}{\partial II^2} & \frac{\partial^2 W}{\partial II \partial J} \\ \frac{\partial^2 W}{\partial J \partial I} & \frac{\partial^2 W}{\partial J \partial II} & \frac{\partial^2 W}{\partial J^2} \end{array} \right] \left\{ \begin{array}{c} \delta_{KL} \\ (I \delta_{KL} - C_{KL}) \\ \frac{1}{2} J C_{KL}^{-1} \end{array} \right\} \]

\[ + \left[ \delta_{IJ} \delta_{KL} - \frac{1}{2} (\delta_{IK} \delta_{JL} + \delta_{IL} \delta_{JK}) \right], \quad J \left[ C_{IJ} C_{KL}^{-1} - 2 C_{IKLJ}^{-1} \right] \left\{ \begin{array}{c} 4 \frac{\partial W}{\partial I} \\ \frac{\partial W}{\partial II} \\ \frac{\partial W}{\partial J} \end{array} \right\} \] (10.157)

where

\[ C_{IKLJ}^{-1} = \frac{1}{2} \left[ C_{IK}^{-1} C_{JL}^{-1} + C_{IL}^{-1} C_{JK}^{-1} \right] \] (10.158)

The spatial elasticities related to the Cauchy stress are obtained by the push forward transformation

\[ J d_{ijkl} = F_{il} F_{jk} F_{ki} F_{lj} D_{IKL} \] (10.159)

which, applied to Eq. (10.157), gives

\[ J d_{ijkl} = 4 \left[ b_{ij}, (I b_{ij} - b_{im} b_{mj}), \frac{1}{2} J \delta_{ij} \right] \left[ \begin{array}{ccc} \frac{\partial^2 W}{\partial I^2} & \frac{\partial^2 W}{\partial I \partial II} & \frac{\partial^2 W}{\partial I \partial J} \\ \frac{\partial^2 W}{\partial II \partial I} & \frac{\partial^2 W}{\partial II^2} & \frac{\partial^2 W}{\partial II \partial J} \\ \frac{\partial^2 W}{\partial J \partial I} & \frac{\partial^2 W}{\partial J \partial II} & \frac{\partial^2 W}{\partial J^2} \end{array} \right] \left\{ \begin{array}{c} b_{kl} \\ (I b_{kl} - b_{km} b_{ml}) \\ \frac{1}{2} J \delta_{kl} \end{array} \right\} \]

\[ + \left[ b_{ij} b_{kl} - \frac{1}{2} (b_{ik} b_{jl} + b_{il} b_{jk}) \right], \quad J \left[ \delta_{kl} \delta_{ij} - 2 I_{ijkl} \right] \left\{ \begin{array}{c} 4 \frac{\partial W}{\partial II} \\ \frac{\partial W}{\partial J} \end{array} \right\} \] (10.160)

where

\[ I_{ijkl} = \frac{1}{2} \left[ \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right] \] (10.161)

The above expressions describe completely the necessary equations to construct a finite element model for any isotropic hyperelastic material. All that remains is to select a specific form for the stored energy function \( W \). Here, many options exist and we include below only a very simple model. For others the reader is referred to literature on the subject.
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Example: compressible neo-Hookean material

As an example, we consider the case of a neo-Hookean material that includes a compressibility effect. The stored energy density is expressed as

$$ W(I, J) = \frac{1}{2} \mu (I - 3 - 2 \ln J) + \frac{1}{2} \lambda (J - 1)^2 $$

(10.162)

where the material constants \( \lambda \) and \( \mu \) are selected to give the same response in small deformations as a linear elastic material using Lamé parameters. Substitution into Eq. (10.154) gives

$$ S_{IJ} = \mu (\delta_{IJ} - C_{IJ}^{-1}) + \lambda J (J - 1) C_{IJ}^{-1} $$

(10.163)

which may be transformed to give the Cauchy stress

$$ \delta_y = \frac{\mu}{J} (b_{ij} - \delta_{ij}) + \lambda (J - 1) \delta_y $$

(10.164)

For the neo-Hookean model the material moduli with respect to the reference configuration are given as

$$ D_{ijkl} = \lambda (2J - 1) C_{ij}^{-1} C_{kl}^{-1} + 2[\mu - \lambda J (J - 1)] C_{ijkl}^{-1} $$

(10.165)

Transformation to spatial configuration moduli gives

$$ d_{ijkl} = \lambda (2J - 1) \delta_{ij} \delta_{kl} + 2 \left[ \frac{\mu}{J} - \lambda (J - 1) \right] I_{ijkl} $$

(10.166)

We note that when \( J \approx 1 \) the small deformation result

$$ d_{ijkl} = \lambda \delta_{ij} \delta_{kl} + 2 \mu I_{ijkl} $$

(10.167)

is obtained and thus matches the usual linear elastic relations. This permits the finite deformation formulation to be used directly for analyses in which the small strain assumptions hold as well as for situations in which deformations are large. The above model may also be used with the mixed forms described above for situations where the ratio \( \lambda/\mu \) is large (i.e. nearly incompressible behaviour). Indeed this was an early use of the model.

10.7.2 Isotropic elasticity – formulation in principal stretches

Other forms of elastic constitutive equations may be introduced by using appropriate expansions of the stored energy density function. As an alternative, an elastic formulation expressed in terms of principal stretches (which are the square root of the eigenvalues of \( C_{ij} \) or \( b_{ij} \)) may be introduced. This approach has been presented by Ogden and by Simo and Taylor.

We first consider a change of coordinates given by (see Appendix B, Volume 1)

$$ x_i = \Lambda_{m'i} x_{m'} $$

(10.168)

where \( \Lambda_{m'i} \) are direction cosines between the two Cartesian systems. The transformation equations for a second-rank tensor, say \( b_{ij} \), may then be written in the form

$$ b_{ij} = \Lambda_{m'i} b_{m'n'} \Lambda_{nj} $$

(10.169)
To compute specific relations for the transformation array we consider the solution of the eigenproblem

\[ b_i q_j^{(n)} = q_j^{(n)} b_i; \quad n = 1, 2, 3 \quad \text{with} \quad q_k^{(m)} q_k^{(n)} = \delta_{mn} \]  

(10.170)

where \( b_i \) are the principal values of \( b_{ij} \), and \( q_i^{(n)} \) are direction cosines for the principal directions. The principal values of \( b_{ij} \) are equal to the square of the principal stretches, \( \lambda_n \), that is,

\[ b_i = \lambda_n^2 \]  

(10.171)

If we assign the direction cosines in the transformation equation (10.169) as

\[ \Lambda_{ij} = q_i^{(n)} \]  

(10.172)

the spectral representation of the deformation tensor results and may be expressed as

\[ b_{ij} = \sum_m \lambda_m^2 q_i^{(m)} q_j^{(m)} \]  

(10.173)

An advantage of a spectral form is that other forms of the tensor may easily be represented. For example,

\[ b_{ik} b_{kj} = \sum_m \lambda_m^4 q_i^{(m)} q_j^{(m)} \quad \text{and} \quad b_{ik}^{-1} = \sum_m \lambda_m^{-2} q_i^{(m)} q_j^{(m)} \]  

(10.174)

Also, we note that an identity tensor may be represented as

\[ \delta_{ij} = \sum_m q_i^{(m)} q_j^{(m)} \]  

(10.175)

From Eq. (10.155) we can immediately observe that Cauchy and Kirchhoff stresses have the same principal directions as the left Cauchy–Green tensor. Thus, for example, the Kirchhoff stress has the representation

\[ \tau_{ij} = \sum_m \tau_m q_i^{(m)} q_j^{(m)} \]  

(10.176)

where \( \tau_m \) denote principal values.

If we now represent the stored energy function in terms of principal stretch values as \( \hat{w}(\lambda_1, \lambda_2, \lambda_3) \) the principal values of the Kirchhoff stress may be deduced from

\[ \tau_m = \lambda_m \frac{\partial \hat{w}}{\partial \lambda_m} \]  

(10.177)

The reader is referred to the literature for a more general discussion on formulations in principal stretches for use in general elasticity problems. Here we wish to consider one form which is useful to develop solution algorithms for finite elasto-plastic behaviour of isotropic materials in which elastic strains are quite small. Such a form is useful, for example, in modelling metal plasticity.

**Logarithmic principal stretch form**

A particularly simple result is obtained by writing the stored energy function in terms of logarithmic principal stretches. Accordingly, we take

\[ \hat{w}(\lambda_1, \lambda_2, \lambda_3) = w(\varepsilon_1, \varepsilon_2, \varepsilon_3) \quad \text{where} \quad \varepsilon_m = \log(\lambda_m) \]  

(10.178)
From Eq. (10.177) it follows that

$$\tau_m = \frac{\partial W}{\partial \varepsilon_m}$$  \hspace{1cm} (10.179)

which is now identical to the form from linear elasticity, but expressed in principal directions. It also follows that the elastic moduli may be written as\textsuperscript{24,26} (summation convention is not used to write this expression)

$$J d_{ijkl} = \sum_{m=1}^{3} \sum_{n=1}^{3} \left[ c_{mn} - 2 \tau_m \delta_{mn} \right] q_i^{(m)} q_j^{(m)} q_k^{(n)} q_l^{(n)}$$

$$+ \frac{1}{2} \sum_{m=1}^{3} \sum_{n \neq m}^{3} g_{mn} [q_i^{(m)} q_j^{(n)} q_k^{(n)} q_l^{(n)} + q_i^{(m)} q_j^{(n)} q_k^{(n)} q_l^{(m)}]$$  \hspace{1cm} (10.180)

where

$$c_{mn} = \frac{\partial^2 W}{\partial \varepsilon_m \partial \varepsilon_n} \quad \text{and} \quad g_{mn} = \begin{cases} \frac{\tau_m \lambda_n^2 - \tau_n \lambda_m^2}{\lambda_m^2 - \lambda_n^2} ; & \lambda_m \neq \lambda_n \\ \frac{\partial (\tau_m - \tau_n)}{\partial \varepsilon_m} ; & \lambda_m = \lambda_n \end{cases}$$  \hspace{1cm} (10.181)

In practice the equal root form is used whenever differences are less than a small tolerance (say $10^{-8}$).

Use of a quadratic form for $w$ given by

$$w = \frac{1}{2} (K - \frac{2}{3} G) [\varepsilon_1 + \varepsilon_2 + \varepsilon_3]^2 + G \left[ \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 \right]$$  \hspace{1cm} (10.182)

yields principal Kirchhoff stresses given by

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} K + \frac{4}{3} G & K - \frac{2}{3} G & K - \frac{2}{3} G \\ K - \frac{2}{3} G & K + \frac{4}{3} G & K - \frac{2}{3} G \\ K - \frac{2}{3} G & K - \frac{2}{3} G & K + \frac{4}{3} G \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}$$  \hspace{1cm} (10.183)

in which the $3 \times 3$ elasticity matrix is given by a constant coefficient matrix which is identical to the usual linear elastic expression in terms of bulk and shear moduli. We also note that when roots are equal

$$\frac{\partial (\tau_m - \tau_n)}{\partial \varepsilon_m} (K + \frac{4}{3} G) - (K - \frac{2}{3} G) = 2G$$  \hspace{1cm} (10.184)

which defines the usual shear modulus form in isotropic linear elasticity.

### 10.7.3 Plasticity models

For isotropic materials, the modelling of elasto-plastic behaviour in which the total deformations are large may be performed by an extension of a hyperelastic
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In this case the deformation gradient is decomposed in a product form (instead of the additive form assumed in Chapter 3) written as

$$F_{ij} = F^e_{ij} F^p_{ij}$$  \hspace{1cm} (10.185)

where $F^e_{ij}$ is the elastic part and $F^p_{ij}$ the plastic part. The deformation picture is often shown as three parts, a reference state, a deformed state, and an intermediate state. The intermediate state is assumed to be the state of a point in a stress-free condition.*

From this decomposition deformation tensors may be defined as

$$b^e_{ij} = F^e_{ik} F^e_{kj} \quad \text{and} \quad C^p_{ij} = F^p_{ki} F^p_{kj}$$  \hspace{1cm} (10.186)

which when combined with Eq. (10.185) give the alternate representation

$$b^e_{ij} = F_{ij} \left( C^p_{ij} \right)^{-1} F^e_{ij}$$  \hspace{1cm} (10.187)

An incremental setting may now be established that obtains a solution for a time $t_{n+1}$ given the state at time $t_n$. The steps to establish the algorithm are too lengthy to include here and the interested reader is referred to literature for details.\(^5,24,30,33\)

The components $(b^e_{ij})_n$ denote values of the converged elastic deformation tensor at time $t_n$. We assume at the start of a new load step a trial value of the elastic tensor is determined from

$$f_{ik} = (F^e_{ik})_n (F^e_{kj})_n$$  \hspace{1cm} (10.188)

where an incremental deformation gradient is computed as

$$f_{ij} = (F_{ik})_n (F^e_{kj})_n$$  \hspace{1cm} (10.189)

A spectral representation of the trial tensor is then determined by using Eq. (10.173) giving

$$b^e_{ij} = \sum_m (\lambda_m^{(m)})_n + 1 q_{ij}^{(m),tr} q_{ij}^{(m),tr}$$  \hspace{1cm} (10.190)

Owing to isotropy $q_{ij}^{(m),tr}$ can be shown to equal the final directions $q_{ij}^{(m)}$.\(^24\)

Trial logarithmic strains are computed as

$$\varepsilon^{tr}_{m,n+1} = \log (\lambda^e_m)_{n+1}$$  \hspace{1cm} (10.191)

and used with the stored energy function $W(b^e_{ij})$ to compute trial values of the principal Kirchhoff stress $(\tau^r_m)_{n+1}$. This may be used in conjunction with the return map algorithm (see Section 3.4.2) and a yield function written in principal stresses $\tau_m$ to compute a final stress state and any internal hardening variables. This part of the algorithm is identical to the small strain form and needs no additional description except to emphasize that only the normal stress is included in the calculation of yield and flow directions. We note in particular that any of the yield functions for isotropic materials which we discussed in Chapter 3 may be used. The use of the return map algorithm also yields the consistent elasto-plastic tangent in principal space which can be transformed by means of Eq. (10.180) for subsequent use in the finite element matrix form.

* The intermediate state is not a configuration, as it is generally discontinuous across interfaces between elastic and inelastic response.
The last step in the algorithm is to compute the final elastic deformation tensor. This is accomplished from the spectral form and final elastic logarithmic strains resulting from the return map solution as

\[
(b_{ij}^e)_{n+1} = \sum_{m=1}^{3} \exp[2(\varepsilon^e_m)_{n+1}] q_i^{(m)} q_j^{(m)}
\]  

(10.192)

The advantages of the above algorithm are numerous. The form again permits a consistent linearization of the algorithm resulting in optimal performance when used with the Newton–Raphson solution scheme. Most important, all the steps previously developed for the small deformation case are here used. For example, although not discussed here, extension to viscoplastic and generalized plastic forms for isotropic materials is again given by results contained in Secs 3.6.2 and 3.9. The primary difficulty is an inability to treat materials which are anisotropic. Here recourse to a rate form of the constitutive equation is possible, as discussed next.

### 10.7.4 Rate constitutive models

The construction of a rate form for elastic constitutive equations deduced from a stored energy function is easily performed in the reference configuration by taking a time derivative of Eq. (10.38), which gives

\[
\dot{S}_{IJ} = D_{IJKL} \dot{E}_{KL}
\]

(10.193)

where, as before, \(D_{IJKL}\) are moduli given by Eq. (10.156). The above result follows naturally from the notion of a derivative since

\[
\dot{S}_{IJ} = \lim_{\eta \to 0} \frac{S_{IJ}(t + \eta) - S_{IJ}(t)}{\eta}
\]

(10.194)

Such a definition is clearly not appropriate for the Cauchy or Kirchhoff stress since they are related to different configurations at time \(t + \eta\) and \(t\) and thus would not satisfy the requirements of objectivity. A definition of an objective time derivative may be computed for the Kirchhoff stress by using Eq. (10.20) and is sometimes referred to as the Truesdell rate or equivalently a Lie derivative. Accordingly, we note that the objective time derivative is given by

\[
\dot{\tau}_{ij} = F_{il} \dot{S}_{lj} F_{jl} + \dot{F}_{il} S_{lj} F_{jl} + F_{il} S_{lj} \dot{F}_{jl}
\]

(10.195)

Introducing the rate of deformation tensor \(l_{ij}\) defined as

\[
\dot{F}_{il} = \dot{x}_{i,l} = x_{i,l} x_{j,l} = l_{ij} F_{jl}
\]

(10.196)

the stress rate may now be written as

\[
\dot{\tau}_{ij} = F_{il} \dot{S}_{lj} F_{jl} + l_{ik} \tau_{kj} + \tau_{ik} l_{kj}
\]

(10.197)

The rate of the second Piola–Kirchhoff stress may be transformed by noting

\[
\dot{E}_{KL} = \frac{1}{2} (F_{KL} \dot{F}_{KK} + F_{KK} \dot{F}_{KL}) = \frac{1}{2} (F_{IK} F_{KL} l_{kl} + F_{KK} F_{IL} l_{kl}) = F_{KL} F_{IL} \dot{\varepsilon}_{kl}
\]

(10.198)
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where

\[ \dot{\varepsilon}_{kl} = \frac{1}{2} (l_{kl} + l_{lk}) = \frac{1}{2} \left( \frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \right) \]  

(10.199)

in which \( v_k \) = \( \dot{x}_k \) = \( \dot{u}_k \) is the velocity vector. The form \( \dot{\varepsilon}_{kl} \) is identical to the rate of small strain form. Furthermore we have upon grouping terms the rate of stress expression

\[ \dot{\tau}_{ij} = J d_{ijkl} \dot{\varepsilon}_{kl} + l_{ik} \tau_{kj} + \tau_{ik} l_{kj} \]  

(10.200)

in which \( d_{ijkl} \) is computed now by means of Eq. (10.79). Incremental forms may be deduced for a rate equation which involve objective approximations for the Lie derivative. For example an approximation to the ‘strain rate’ may be computed from

\[ (\dot{\varepsilon}_{ij})_{n+1/2} \approx \frac{1}{\Delta t} \left( f^{-1}_{ik} \right)_{n+1/2} \Delta E_{kl} \left( f^{-1}_{jl} \right)_{n+1/2} \]  

(10.201)

\[ \Delta E_{kl} = \frac{1}{2} \left[ (f_{km})_{n+1} (f_{ln})_{n+1} - \delta_{kl} \right] \]  

(10.202)

where

\[ (f_{ij})_{n+\alpha} = \delta_{ij} + \alpha \frac{\partial \Delta (u_i)_{n+1}}{\partial (x_j)_n} \]  

(10.203)

with \( \Delta (u_i)_{n+1} = (u_i)_{n+1} - (u_i)_n \). Similarly, an approximation to the Lie derivative of Kirchhoff stress may be taken as

\[ (\dot{\tau}_{ij})_{n+1/2} \approx \frac{1}{\Delta t} \left( f_{ij} \right)_{n+1/2} \left[ (f^{-1}_{km})_{n+1} (\tau_{mn})_{n+1} (f^{-1}_{ln})_{n+1} - (\tau_{kl})_n \right] \left( f_{jl} \right)_{n+1/2} \]  

(10.204)

Other approximations may be used; however, the above are quite convenient. In the approximation a modulus array \( d_{ijkl} \) must also be obtained. Here there is no simple form which is always consistent with the tangent needed for a full Newton–Raphson solution scheme and, often, a constant array is used based on results from linear elasticity.

Extension of the above to include general material constitution may be performed by replacing the strain rate by an additive form given as

\[ \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \]  

(10.205)

Once again we can use all the constitutive equations discussed in Chapter 3 (including those which are not isotropic) to construct a finite element model for the large strain problem. Here, since approximations not consistent with a Newton–Raphson scheme are generally used for the moduli, convergence generally does not achieve an asymptotic quadratic rate. Use of quasi-Newton schemes and line search, as described in Chapter 2, can improve the convergence properties and leads to excellent performance in most situations.

Many other stress rates may be substituted for the Lie derivative. For example, the Jaumann–Zaremba stress rate form given as

\[ \dot{\tau}_{ij} = J d_{ijkl} \dot{\varepsilon}_{kl} + \dot{\omega}_{ik} \tau_{kj} - \dot{\omega}_{jk} \tau_{ik} \]  

(10.206)
may be used. This form is deduced by noting that the rate of deformation tensor may be split into a symmetric and skew-symmetric form as

\[ \dot{I}_{ij} = \dot{e}_{ij} + \omega_{ij} \]  

(10.207)

where \( \omega_{ij} \) is the rate of spin or vorticity. This form was often used in early developments of finite element solutions to large strain problems and enjoys considerable popularity even today.

**10.8 Contact problems**

In many problems situations are encountered where the points on a boundary of one body come into contact with points on the boundary of the same or another object. Such problems are commonly referred to as contact problems. Finite element methods have been used for many years to solve contact problems. Contact problems are inherently non-linear since, prior to contact, boundary conditions are given by traction conditions (often the traction being simply zero) whereas during 'contact' kinematic constraints must be imposed which prevent penetration of one boundary through the other, called the impenetrability condition.

The solution of a contact problem involves first identifying which points on a boundary interact and second the insertion of appropriate conditions to prevent the penetration. Figure 10.2 shows a typical situation in which one body is being pressed into a second body. In Fig. 10.2(a) the two objects are not in contact and the boundary conditions are specified by zero traction conditions for both bodies. In Fig. 10.2(b) the two objects are in contact along a part of the boundary segment.

![Figure 10.2](image_url)
and here conditions must be inserted to ensure that penetration does not occur. Along this boundary different types of contact can be modelled, the simplest being a frictionless condition in which the only non-zero contact traction is normal to the contact surface. A more complex condition occurs in which traction tangential to the surface can be generated by frictional conditions. The simplest model for a frictional condition is Coulomb friction where
\[ |t_s| \leq \mu |t_n| \] (10.208)
in which \( \mu \) is a positive frictional parameter, \( t_n \) is the magnitude of the normal traction, and \( t_s \) is the tangential traction. If the magnitude of \( t_s \) is less than the limit condition the points on the surface are assumed to stick; whereas if the magnitude is at the limit condition slip occurs with an imposed tangential traction on each surface opposite to the direction of slip and equal to \( \mu |t_n| \).

In modelling contact problems by finite element methods immediate difficulties arise. First, it is not possible to model contact at every point along a boundary. This is primarily because of the fact that the finite element representation of the boundary is not smooth. For example in the two-dimensional case in which boundaries of individual elements are straightline segments as shown in Fig. 10.3 nodes \( A \) and \( B \) are in contact with the lower body but the segment between the nodes is not in contact. Second, finite element modelling results in non-unique representation of a normal between the two bodies and, again because of finite element discretization, the normals are not continuous between elements. This is illustrated also in Fig. 10.3 where it is evident that the normal to the segment between nodes \( A \) and \( B \) is not the same as the negative normal of the facets around node \( C \) (which indeed are not unique at node \( C \)).

### 10.8.1 Geometric modelling

**Node–node contact — Hertzian contact**

For applications in which displacements on the contact boundary are small it is sometimes possible to model the contact by means of nodes. For this to be possible, the finite element mesh must be constructed such that boundary nodes on one body, here referred to as slave nodes, match the location of the boundary nodes for the other body, referred to as master nodes, to within conditions acceptable for
small deformation analysis. Such conditions may also be extended for cases where the boundary of one body is treated as flat and rigid (unilateral contact). A problem in which such conditions may be used is the interaction between two half discs (or hemispheres) which are pressed together along the line of action between their centres. A simple finite element model for such a problem is shown in Fig. 10.4(a) where it is observed that the horizontal alignment of potential contact nodes on the boundary of each disc are identical. The solution after pressing the bodies together is indicated in Fig. 10.4(b) and contours for the vertical normal stress are shown in Fig. 10.4(c). It is evident that the contours do not match perfectly along the vertical axis owing to lack of alignment of the nodes in the deformed position. However, the mismatch is not severe, and useful engineering results are possible. Later we will consider methods which give a more accurate representation; however, before doing so we consider the methods available to prevent penetration.

The determination of which nodes are in contact for such a problem can be monitored simply by comparing the vertical position of each node pair, which may be treated as a simple two-node element. Thus denoting the upper disc as slave body 's' and the lower one as master body 'm' we can monitor the vertical gap

\[ g = x_2^{(s)} - x_2^{(m)} = [X_2^{(s)} + u_2^{(s)}] - [X_2^{(m)} + u_2^{(m)}] \]  

(10.209)

If \( g > 0 \) no contact exists, whereas if \( g \leq 0 \) contact or penetration has occurred. (We note that penetration can exist for any iteration in which no modification of the
formulation has been inserted.) Thus, the next step is to insert a constraint condition for any nodal pair (element) in which the gap $g$ is negative or zero (here some tolerance usually is necessary to define 'zero'). There are many approaches which may be used to insert the constraint. Here we discuss use of a Lagrange multiplier form, a penalty approach, and an augmented lagrangian approach.  

**Lagrange multiplier form**

A Lagrange multiplier approach is given simply by multiplying the gap condition given in Eq. (10.209) by the multiplier. Accordingly, we can write for each nodal pair for which contact has been assigned a variational term

$$\Pi_c = \lambda g$$  \hspace{1cm} (10.210)

and add its first variation to the variational equations being used to solve the problem. The first variation to Eq. (10.210) is given as

$$\delta \Pi_c = \delta \lambda g + [\delta u_2^{(s)} - \delta u_2^{(m)}] \lambda$$  \hspace{1cm} (10.211)

and thus we identify $\lambda$ as a 'force' applied to each node to prevent penetration. Linearization of Eq. (10.211) produces a tangent matrix term for use in a Newton–Raphson solution process. The final tangent and residual for the nodal contact element may be written as

$$\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & -1 \\
1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
\delta u_2^{(s)} \\
\delta u_2^{(m)} \\
d\lambda
\end{bmatrix} =
\begin{bmatrix}
-\lambda \\
\lambda \\
-g
\end{bmatrix}$$  \hspace{1cm} (10.212)

and is added into the equations in a manner identical to any element assembly process. It is evident that the equations in this form introduce a new unknown for each contact pair. Also, as for any Lagrange multiplier approach, the equations are not positive definite and indeed have a zero diagonal for each multiplier term, thus, special care is needed in the solution process to avoid divisions by the zero diagonal.

**Penalty function form**

An approach which avoids equation solution difficulties of a Lagrange multiplier method is the penalty method, as described many times in Volume 1. In this the contact term is given by

$$\Pi = \kappa g^2$$  \hspace{1cm} (10.213)

where $\kappa$ is a penalty parameter. The matrix equation for a nodal pair is now given by

$$\begin{bmatrix}
\kappa & -\kappa \\
-\kappa & \kappa
\end{bmatrix}
\begin{bmatrix}
\delta u_2^{(s)} \\
\delta u_2^{(m)}
\end{bmatrix} =
\begin{bmatrix}
-\kappa g \\
\kappa g
\end{bmatrix}$$  \hspace{1cm} (10.214)

In a penalty approach the final gap will not be zero but becomes a small number depending on the value of the parameter $\kappa$ selected. Thus, the advantage of the
penalty method is somewhat offset by a need to identify the value of the parameter that gives an acceptable answer. Indeed, in a complex problem this is not a trivial task, especially for problems involving contact between beam, plate, or shell elements and solid elements.

**Augmented lagrangian form**

A compromise between the penalty method and the Lagrange multiplier method may be achieved by using an iterative update for the multiplier combined with a penalty-like form. We discussed this for incompressibility problems in Sec. 12.6 of Volume 1 and here indicate briefly how it applies equally to the contact problem. Based on results from Volume 1 we may write the augmented form as

\[
\begin{bmatrix}
\kappa & -\kappa \\
-\kappa & \kappa 
\end{bmatrix}
\begin{bmatrix}
du^{(s)}_2 \\
du^{(m)}_2 
\end{bmatrix}
= \begin{cases}
-\lambda_k - \kappa g \\
\lambda_k + \kappa g 
\end{cases}
\]

(10.215)

where an update to the Lagrange multiplier is computed by using

\[
\lambda_{k+1} = \lambda_k + \kappa g
\]

(10.216)

Such an update may be computed after each Newton–Raphson iteration or in an added iteration loop after convergence of the Newton–Raphson iteration. In either case a loss of quadratic convergence results for the simple augmented strategy shown. Improvements to superlinear convergence are possible as shown by Zavarise and Wriggers, and a more complex approach which restores the quadratic convergence rate may be introduced at the expense of retaining an added variable. In general, however, use of a fairly large value of the penalty parameter in the simple scheme shown above is sufficient to achieve good solutions with few added iterations.

**Node–surface contact**

The simplest form for contact between bodies in which nodes on surfaces of one body do not interact directly with nodes on a second body is defined by a node–surface treatment. A two-dimensional treatment for this case is shown in Fig. 10.5 where a node, called the slave node, with deformed position \(x\), can contact a segment, called the master surface, defined for simplicity in two dimensions by an interpolation

\[
x = N_\alpha(\xi)x_\alpha
\]

(10.217)

This interpolation may be treated either as the usual interpolation along the edge facets of elements describing the target body as shown in Fig. 10.5(a) or by an interpolation which smooths the slope discontinuity between adjacent element surface facets as shown in Fig. 10.5(b).

A contact between the two bodies occurs when the gap \(g\) shown in Fig. 10.5 becomes zero. The determination of a contact requires a search to find which target facet is a potential contact surface and computation of the associated gap for each one. If the gap is positive no contact condition exists and, thus, no modification to the governing equations is required. If the gap is negative a 'penetration' of the two bodies has occurred and it is necessary to modify the equilibrium equations to reflect the contact forces which occur.
To determine the gap it is necessary to find the point on the target (master) facet which is closest to the slave node. This can be accomplished by expressing points on the facet by using Eq. (10.217) and finding the value of $\xi$ which minimizes the function

$$f(\xi) = \frac{1}{2} (x_s^T - x^T)(x_s - x) = \text{minimum}$$

(10.218)

Here again a Newton–Raphson solution method may be used to find a solution. Linearizing, we solve for iterates from

$$[x_x^T x_x - (x_s - x(\xi_i))^T x_x] d\xi_i = x_x^T (x_s - x(\xi_i)) = \frac{df}{d\xi} = R$$

(10.219)

with updates

$$\xi_{i+1} = \xi_i + d\xi_i$$

until $R = 0$ is satisfied to within a specified tolerance. For the two-dimensional problem in which linear interpolation is used to define $N_\alpha(\xi)$, the expression for $R$ is linear in $\xi$ and, thus, convergence is achieved in one iteration. Denoting the solution as $\xi_c$, the location of the closest point on the target facet becomes $x_c$ as shown in Fig. 10.5 and, using Eq. (10.217), is given by

$$x_c = N_\alpha(\xi_c)x_\alpha$$

(10.220)

For frictionless contact only normal tractions are involved on the surfaces between the two bodies; thus sliding can occur without generation of tangential forces and the traction is given by

$$t = \lambda_n n_c$$

(10.221)

where $\lambda_n$ is the magnitude of a normal traction applied to the contact target and $n_c$ is a unit normal to the master facet at the point $\xi_c$. This case can be included by appending
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the variation of a Lagrange multiplier term to the Galerkin (weak) form describing
equilibrium of the problem for each contact slave node. This term may be expressed as

\[ \Pi_c = \left( \lambda_n \mathbf{n}_c^T \right) \left( g_n \mathbf{n}_c \right) A_c \] (10.222)

where at the solution point \( \xi_c \)

\[ g_n \mathbf{n}_c = x_s - x(\xi_c) \] (10.223)

and \( A_c \) is a surface area associated with the slave node. In the development summarized here the surface area term is based on the reference configuration and kept constant during the analysis. Thus, the traction measure \( \lambda_n \) is a reference surface measure which must be scaled by the current surface area ratio to obtain the magnitude of the traction in the deformed state.

Use of the Lagrange multiplier form introduces an additional unknown \( \lambda_n \) for each master–slave contact pair. Since a contact traction interacts with both bodies it must be determined as part of the solution of the global equilibrium equations. Of course, we again can eliminate the contact tractions by using a penalty form for the constraint in a manner similar to that used for treating node–node contact [Eq. (10.213)]. However, even then the problem is more complex as we do not know \textit{a priori} which master facet a contact node will interact with. This implies that the non-zero structure of the global tangent matrix will change during the solution of any contact problem and continual updates are required to describe the profile or sparse structure.

The variation of the potential given in Eq. (10.222) may be expressed as

\[ \delta \Pi_c = \left[ \delta \left( \lambda_n \mathbf{n}_c^T \right) g_n \mathbf{n}_c + \left( \delta x_c^T - \delta x_c^T \right) \mathbf{n}_c \lambda_n \right] A_c \] (10.224)

where the variation of the coordinate \( x_c \) is given by

\[ \delta x_c = N_\alpha \delta x_\alpha + x_\xi \delta \xi_c \] (10.225)

with \( x_\xi \) computed by differentiation of the interpolation functions in Eq. (10.217) and then evaluated at \( \xi_c \). Noting now that

\[ \mathbf{n}_x^T \mathbf{n}_c = 1 \quad \text{and} \quad \mathbf{n}_c^T \delta \mathbf{n}_c = 0 \] (10.226)

the added contact term may be written in matrix form as

\[ \delta \Pi_c = \left[ \delta \lambda_n, \delta x_\alpha^T, \delta x_\alpha^T, \delta \xi_c \right] \left\{ \begin{array}{c} g_n A_c \\ \lambda_n A_c \mathbf{n}_c \\ -\lambda_n A_c N_\alpha \mathbf{n}_c \\ -\lambda_n A_c x_\alpha^T \mathbf{n}_c \end{array} \right\} \] (10.227)

where we note that the entry multiplying \( \delta \xi_c \) vanishes at \( \xi = \xi_c \). At a solution \( g_n \) should also be zero; however, in some iterations it may be non-zero as a result of penetrations occurring before the contact term is inserted (also for penalty type methods it will never be exactly zero).

* One is tempted to use \( \mathbf{n}_x^T \mathbf{n}_c = 1 \) to simplify Eq. (10.222); however, doing this the advantages of Eq. (10.223) are then lost and computation of the tangent becomes more complex.
Using a Newton–Raphson solution procedure generates a tangent matrix expressed formally by the term

\[
\frac{d}{dT} = \left\{ \left( \delta \mathbf{x}_s^T - \delta \mathbf{x}_c^T \right) d(\lambda_n \mathbf{n}_c) + \delta (\lambda_n \mathbf{n}_c^T) (dx_s - dx_c) \right\}
\]

The linearization is straightforward except for the terms involving \( \mathbf{n}_c \). It is necessary here to use a form which does not divide by \( g_n \) [which would result in using Eq. (10.223) directly]. This may be achieved by introducing a surface unit tangent vector

\[
\mathbf{t}_c = \frac{\mathbf{x}_c}{||\mathbf{x}_c||}
\]

and using

\[
\mathbf{n}_c = \mathbf{t}_c \times \mathbf{e}_3 \quad \text{and} \quad \delta \mathbf{n}_c = \delta \mathbf{t}_c \times \mathbf{e}_3
\]

where \( \mathbf{e}_3 \) is a unit vector normal to the plane of deformation. Now

\[
\delta \mathbf{t}_c = \frac{1}{||\mathbf{x}_c||} \left[ \mathbf{I} - \mathbf{t}_c \mathbf{t}_c^T \right] \delta \mathbf{x}_c = \frac{1}{||\mathbf{x}_c||} \mathbf{n}_c \mathbf{n}_c^T \delta \mathbf{x}_c
\]

which upon using Eq. (10.230) gives

\[
\delta \mathbf{n}_c = -\frac{\mathbf{x}_c \mathbf{n}_c^T \delta \mathbf{x}_c}{||\mathbf{x}_c||^2}
\]

Performing all linearizations results in the tangent term

\[
\left[ \begin{array}{cccc}
0 & A_c \mathbf{n}_c^T & -N_\beta A_c \mathbf{n}_c^T & 0 \\
A_c \mathbf{n}_c & 0 & G_{s\beta} & G_{s\xi} \\
-N_\alpha A_c \mathbf{n}_c & G_{\alpha s} & G_{\alpha\beta} & G_{\alpha\xi} \\
0 & G_{\xi s} & G_{\xi\beta} & G_{\xi\xi}
\end{array} \right] \left\{ \begin{array}{c}
d(\lambda_n) \\
dx_s \\
dx_\alpha \\
d\xi_c
\end{array} \right\}
\]

where

\[
G_{s\beta} = -\frac{\lambda_n A_c N_\beta \mathbf{x}_s \mathbf{n}_c}{||\mathbf{x}||^2} = G_{s\beta}^T
\]

\[
G_{s\xi} = -\frac{\lambda_n A_c \kappa_c \mathbf{x}_\xi \mathbf{n}_c}{||\mathbf{x}||^2} = G_{s\xi}^T
\]

\[
G_{\alpha\beta} = \frac{\lambda_n A_c}{||\mathbf{x}||^2} \left[ N_\alpha N_\beta \mathbf{x}_s \mathbf{n}_c^T + N_\alpha \mathbf{x}_\xi \mathbf{n}_c^T \mathbf{x}_\xi \mathbf{n}_c + g_n N_\alpha \mathbf{x}_\xi \mathbf{n}_c \mathbf{n}_c^T \right]
\]

\[
G_{\alpha\xi} = \frac{\lambda_n A_c \kappa_c}{||\mathbf{x}||^2} \left[ N_\alpha \mathbf{x}_\xi - g_n N_\alpha \mathbf{x}_\xi \mathbf{n}_c \right]
\]

\[
G_{\xi\xi} = \lambda_n A_c \left[ 1 - \frac{g_n \kappa_c}{||\mathbf{x}_\xi||^2} \right]
\]
in which $\kappa_c = n_c^T \mathbf{x}_{\xi \xi}$ is related to the target facet curvature. Clearly, for a straight contact facet with linear interpolation defining $\mathbf{x}(\xi)$ many of the terms in Eq. (10.234) vanish and the tangent is greatly simplified. In such a case, however, special attention must be devoted to the case where a slave node is very near a master node and oscillations on which facet to use occur during subsequent Newton–Raphson iterations. Here an expedient solution is to use concepts from multi-surface plasticity to define a ‘continuous’ approximation for the normal. This leads to additional considerations which are not given here and are left for the reader to develop (see reference 55).

The tangent matrix may be reduced by eliminating the dependence on $d\xi_c$. The reduced tangent then depends only on the Lagrange multiplier $\lambda_n$ and geometry terms computed from current values of nodal parameters.

Extension to three-dimensional problems is straightforward and involves addition of a second natural coordinate $\eta$ and replacing $\mathbf{e}_3$ by a second surface tangent vector deduced from $\mathbf{x}_{\eta \eta}$. Extension to include frictional effects may also be performed and the reader is referred to the literature for additional details.\textsuperscript{67–81}

### 10.9 Numerical examples

#### 10.9.1 Node–surface contact between discs

As a first example here we consider the contact problem previously solved using a node–node approach. In that case we observed a small but significant discontinuity between the contours of vertical stress between the bodies, indicating that traction is not correctly transmitted across the section. Here we use the node–surface method given above in which the contact area of each body is taken as the boundary of elements. The solution is achieved by using a penalty method and a two-pass solution procedure where on the first pass one body is the slave and the other the master and on the second pass the designation is reversed. This approach has been shown to be necessary in order to satisfy the mixed patch test for contact.\textsuperscript{65} The results using this approach are shown in Fig. 10.6. For the solution, the two-dimensional plane

Fig. 10.6 Contact between semicircular disks: vertical contours for node-to-surface solution.
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strain finite deformation displacement element described in Sec. 10.3.2 is used with material behaviour given by the neo-Hookean hyperelastic model described in Sec. 10.7.1. Properties are: $E = 100\,000$ for the upper body and $E = 1000$ for the lower body. A Poisson ratio of $\nu = 0.25$ is used to compute Lamé parameters $\lambda$ and $\mu$. As can readily be seen in the figure, the results obtained are significantly better than those from the node-to-node analysis.

10.9.2 Upsetting of a cylindrical billet

To illustrate performance in highly strained regimes, we consider large compression of a three-dimensional cylindrical billet. The initial configuration is a cylinder

Fig. 10.7 Initial and final configurations for a billet.
Numerical examples

with radius, \( r = 10 \), and height, \( h = 15 \). The mesh consists of 459 eight-noded hexahedral elements based on the mixed-enhanced formulation presented in Sec. 10.5. The billet is loaded via displacement control on the upper surface, while the lower edge is fully restrained. A full Newton–Raphson solution process is used in which the upper displacement is increased by increments of displacement equal to 0.25 units.

To prevent penetration with the rigid base during large deformations a simple node-on-node penalty formulation with a penalty parameter \( k = 10^6 \) is defined for nodes on the lower part of the cylindrical boundary. A neo-Hookean material model with \( \lambda = 10^4 \) and \( \mu = 10 \) is used for the simulation. Figure 10.7 depicts the initial mesh and progression of deformation.

**10.9.3 Necking of circular bar**

The last example we include is a large deformation plasticity problem. Here we consider the three-dimensional behaviour of a cylindrical bar subjected to tension. In the presence of plastic deformation an unstable plastic necking will occur at some location along a bar of mild steel, or similar elasto-plastic behaving material. This is easily observed from the tension test of a cylindrical specimen which tapers by a small amount to a central location to ensure that the location of necking will occur in a specified location. A finite element model is constructed having the same taper, and here only one-eighth of the bar need be modelled as shown in Fig. 10.8(a). In Fig. 10.8(b) we show the half-bar model which is projected by symmetry and reflection and on which the behaviour will be illustrated.

Fig. 10.8 Necking of a cylindrical bar: eight-noded elements. (a) Finite element model; (b) half-bar by symmetry.
This problem has been studied by several authors and here the properties are taken as described by Simo and co-workers.\textsuperscript{5,9,24} The one-eighth quadrant model consists of 960 eight-noded hexahedra of the mixed type discussed in Sec. 10.5. The radius at the loading end is taken as $R = 6.413$ and a uniform taper to a central radius of $R_c = 0.982 \times R$ is used. The total length of the bar is $L = 53.334$ (giving a half length of 25.667). The mesh along the length is uniform between the centre (0) and a distance of 10, and again from 10 to the end. A blending function mesh generation is used (see Sec. 9.12, Volume 1) to ensure that exterior nodes lie exactly on the circular radius. This ensures that, as much as possible for the discretization employed, the response will be axisymmetric.

The finite deformation plasticity model based on the logarithmic stretch elastic behaviour from Sec. 10.7.2 and the finite plasticity as described in Sec. 10.7.3 is used for the analysis. The material properties used are as follows: elastic properties are $K = 164.21$ and $G = 80.1938$; a $J_2$ plasticity model in terms of principal Kirchhoff stresses $\tau$ with an initial yield in tension of $\tau_y = 0.45$ is used. Only isotropic hardening is included and a saturation type model defined by

$$\kappa = H_i e^p + \left[\tau^\infty_y - \tau_y\right] \left(1 - \exp\left[-\delta e^p\right]\right)$$

with the parameters

$$H_i = 0.12924, \quad \tau^\infty_y = 0.715, \quad \text{and} \quad \delta = 16.93$$

is employed. An alternative to this is a piecewise linear behavior as suggested by some authors; however, the above model is very easy to implement and gives a smooth behaviour with increase in the accumulated plastic strain $e^p$ as the hardening parameter $\kappa$.

In Fig. 10.9 we show the deformed configuration of the bar at an elongation of 22.5 per cent (elongation = 6 units). Figure 10.9(a) has the contours of the first invariant ($I_1$); (b) second invariant ($I_2$).

Fig. 10.9 Deformed configuration and contours for necking of bar: (a) first invariant ($I_1$); (b) second invariant ($I_2$).
10.10 Concluding remarks

This chapter presents a unified approach for all finite deformation problems. The various procedures for solution of the resulting non-linear algebraic system have followed those presented in Chapter 2. Although not discussed extensively in the chapter, the extension to consider transient (dynamic) situations is easily accomplished. The long-term integration of dynamic problems occasionally presents difficulties using the time integration procedures discussed in Volume 1. Here schemes which conserve momentum and energy for hyperelastic materials can be considered as alternatives, and the reader is referred to literature on the subject for additional details. 56, 82–87

We have also presented some mixed forms for developing elements which perform well at finite strains and with materials which can exhibit nearly incompressible behaviour. These elements are developed in a form which allow the introduction of finite elastic and inelastic material models without difficulty. Indeed, we have
shown that there is no need to decouple the constitutive behaviour between volumetric and deviatoric response as often assumed in many presentations. We usually find that transformation to a current configuration form in which either the Kirchhoff stress or the Cauchy stress is used directly will lead to a form which admits a simple extension of existing small deformation finite element procedures for developing the necessary residual (force) and stiffness matrices. An exception here is the presentation of the mixed-enhanced form in which all basic development is shown using the deformation gradient and first Piola–Kirchhoff stress. Here we could express final answers in a current configuration form also, but we leave these steps for the reader to perform.

In this chapter we have concentrated on developments for continuum problems in which the full two- or three-dimensional equations are modelled by finite elements. In the next chapter we address problems which can be represented using beam (rod), plate, or shell models and thus permit a reduction of the discretization space to one or two dimensions.

References


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