The comparison singles out only one displacement and each plot uses the number of mesh divisions in a quarter of the plate as abscissa. It is therefore difficult to deduce the convergence rate and the performance of elements with multiple nodes. A more convenient plot gives the energy norm $\|u\|$, versus the number of degrees of freedom $N$ on a logarithmic scale. We show such a comparison for some elements in Fig. 4.17 for a problem of a slightly skewed, simply supported plate. It is of interest to observe that, owing to the singularity, both high- and low-order elements converge at almost identical rates (though, of course, the former give better overall accuracy). Different rates of convergence would, of course, be obtained if no singularity existed (see Chapter 14 of Volume I).

Conforming shape functions with nodal singularities

4.9 General remarks

It has already been demonstrated in Sec. 4.3 that it is impossible to devise a simple polynomial function with only three nodal degrees of freedom that will be able to satisfy slope continuity requirements at all locations along element boundaries. The alternative of imposing curvature parameters at nodes has the disadvantage, however, of imposing excessive conditions of continuity (although we will investigate some of the elements that have been proposed from this class). Furthermore, it is desirable from many points of view to limit the nodal variables to three quantities only. These, with simple physical interpretation, allow the generalization of plate elements to shells to be easily interpreted also.

It is, however, possible to achieve $C_1$ continuity by provision of additional shape functions for which, in general, second-order derivatives have non-unique values at nodes. Providing the patch test conditions are satisfied, convergence is again assured.

Such shape functions will be discussed now in the context of triangular and quadrilateral elements. The simple rectangular shape will be omitted as it is a special case of the quadrilateral.

4.10 Singular shape functions for the simple triangular element

Consider for instance either of the following sets of functions:

$$\varepsilon_{jk} = \frac{L_i L_j^2 L_k^2 (L_k - L_I)}{(L_i + L_j)(L_j + L_k)} \quad (4.72)$$

or

$$\varepsilon_{jk} = \frac{L_i L_j^2 L_k^2 (1 + L_i)}{(L_i + L_j)(L_j + L_k)} \quad (4.73)$$

in which once again $i, j, k$ are a cyclic permutation of 1, 2, 3. Both have the property that along two sides ($i-j$ and $i-k$) of a triangle (Fig. 4.18) their values and the values
of their normal slope are zero. On the third side \((j-k)\) the function is zero but a normal slope exists. In both, its variation is parabolic. Now, all the functions used to define the non-conforming triangle [see Eq. (4.55)] were cubic and hence permit also a parabolic variation of the normal slope which is not uniquely defined by the two end nodal values (and hence resulted in non-conformity). However, if we specify as an additional variable the normal slope of \(w\) at a mid-point of each side then, by combining the new functions \(\epsilon_{jk}\) with the other functions previously given, a unique parabolic variation of normal slope along interelement faces is achieved and a compatible element results.

Apparently, this can be achieved by adding three such additional degrees of freedom to expression (4.55) and proceeding as described above. This will result in an element shown in Fig. 4.19(a), which has six nodes, three corner ones as before and three additional ones at which only normal slope is specified. Such an element requires the definition of a node (or an alternative) to define the normal slope and also involves assembly of nodes with differing numbers of degrees of freedom. It is necessary to define a unique normal slope for the parameter associated with the mid-point of adjacent elements. One simple solution is to use the direction of increasing node number of the adjacent vertices to define a unique normal.

Another alternative, which avoids the above difficulties, is to constrain the mid-side node degree of freedom. For instance, we can assume that the normal slope at the centre-point of a line is given as the average of the two slopes at the ends. This, after suitable transformation, results in a compatible element with exactly the same degrees of freedom as that described in previous sections [see Fig. 4.19(b)].

The algebra involved in the generation of suitable shape functions along the lines described here is quite extensive and will not be given fully.

First, the normal slopes at the mid-sides are calculated from the basic element shape functions [Eq. (4.57)] as

\[
\begin{bmatrix}
\frac{\partial w}{\partial n}_4 & \frac{\partial w}{\partial n}_5 & \frac{\partial w}{\partial n}_6
\end{bmatrix}^T = Z a_e
\]

(4.74)
Similarly, the average values of the nodal slopes in directions normal to the sides are calculated from these functions:

\[
\begin{bmatrix}
\frac{\partial w}{\partial n}_4 \\
\frac{\partial w}{\partial n}_5 \\
\frac{\partial w}{\partial n}_6
\end{bmatrix} = Z a_e^c
\]  

(4.75)

The contribution of the \( \varepsilon \) functions to these slopes is added in proportions of \( \varepsilon_{ik} - \gamma_i \) and is simply (as these give unit normal slope)

\[
\gamma = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}^T
\]  

(4.76)

On combining Eq. (4.57) and the last three relations we have

\[
\bar{Z} a_e^c = Z a_e^c + \gamma
\]  

(4.77)
from which it immediately follows on finding $\gamma$ that

$$w = N^0 a^e + [\varepsilon_{23}, \varepsilon_{31}, \varepsilon_{12}] (\bar{Z} - Z) a^e$$  \hspace{1cm} (4.78)

in which $N^0$ are the non-conforming shape functions defined in Eq. (4.57). Thus, new shape functions are now available from Eq. (4.78).

An alternative way of generating compatible triangles was developed by Clough and Tocher.$^{10}$ As shown in Fig. 4.19(a) each element triangle is first divided into three parts based on an internal point $p$. For each $ijk$ triangle a complete cubic expansion is written involving 10 terms which may be expressed in terms of the displacement and slopes at each vertex and the mid-side slope along the $ij$ edge.

Matching the values at the vertices for the three sub-triangles produces an element with 15 degrees of freedom: 12 conventional degrees of freedom at nodes 1, 2, 3 and $p$; and three normal slopes at nodes 4, 5, 6. Full $C_1$ continuity in the interior of the element is achieved by constraining the three parameters at the $p$ node to satisfy continuous normal slope at each internal mid-side. Thus, we achieve an element with 12 degrees of freedom similar to the one previously outlined using the singular shape functions. Constraining the normal slopes on the exterior mid-sides leads to an element with 9 degrees of freedom [see Fig. 4.19(b)].

These elements are achieved at the expense of providing non-unique values of second derivatives at the corners. We note, however, that strains are in general also non-unique in elements surrounding a node (e.g. constant strain triangles in elasticity have different strains in each element surrounding each node). In the previously developed shape functions $\varepsilon_{jk}$ an infinite number of values to the second derivatives are obtained at each node depending on the direction the corner is approached. Indeed, the derivation of the Clough and Tocher triangle can be obtained by defining an alternative set of $\varepsilon$ functions, as has been shown in reference 11.

As both types of elements lead to almost identical numerical results the preferable one is that leading to simplified computation. If numerical integration is used (as indeed is always strongly recommended for such elements) the form of functions continuously defined over the whole triangle as given by Eqs (4.57) and (4.78) is advantageous, although a fairly high order of numerical integration is necessary because of the singular nature of the functions.

### 4.11 An 18 degree-of-freedom triangular element with conforming shape functions

An element that presents a considerable improvement over the type illustrated in Fig. 4.19(a) is shown in Fig. 4.19(c). Here, the 12 degrees of freedom are increased to 18 by considering both the values of $w$ and its cross derivative $\partial^2 w / \partial s \partial n$, in addition to the normal slope $\partial w / \partial n$, at element mid-sides.*

Thus an equal number of degrees of freedom is presented at each node. Imposition of the continuity of cross derivatives at mid-sides does not involve any additional constraint as this indeed must be continuous in physical situations.

* This is, in fact, identical to specifying both $\partial w / \partial n$ and $\partial w / \partial s$ at the mid-side.
The derivation of this element is given by Irons\textsuperscript{14} and it will suffice here to say that in addition to the modes already discussed, fourth-order terms of the type illustrated in Fig. 4.10(d) and ‘twist’ functions of Fig. 4.18(b) are used. Indeed, it can be simply verified that the element contains all 15 terms of the quartic expansion in addition to the ‘singularity’ functions.

\subsection*{4.12 Compatible quadrilateral elements}

Any of the previous triangles can be combined to produce ‘composite’ compatible quadrilateral elements with or without internal degrees of freedom. Three such quadrilaterals are illustrated in Fig. 4.20 and, in all, no mid-side nodes exist on the external boundaries. This avoids the difficulties of defining a unique parameter and of assembly already mentioned.

In the first, no internal degrees of freedom are present and indeed no improvement on the comparable triangles is expected. In the following two, 3 and 7 internal degrees of freedom exist, respectively. Here, normal slope continuity imposed in the last one does not interfere with the assembly, as internal degrees of freedom are in all cases eliminated by static condensation.\textsuperscript{62} Much improved accuracy with these elements has been demonstrated by Clough and Felippa.\textsuperscript{15}

An alternative direct derivation of a quadrilateral element was proposed by Sander\textsuperscript{12} and Fraeijs de Veubeke.\textsuperscript{13,16} This is along the following lines. Within a quadrilateral of Fig. 4.21(a) a complete cubic with 10 constants is taken, giving the first component of the displacement which is defined by three functions. Thus,

\begin{equation}
\begin{aligned}
\mathbf{w} &= w^a + w^b + w^c \\
\mathbf{w}^a &= \alpha_1 + \alpha_2 x + \cdots + \alpha_{10} y^3
\end{aligned}
\end{equation}

The second function \( w^b \) is defined in a piecewise manner. In the lower triangle of Fig. 4.21(b) it is taken as zero; in the upper triangle a cubic expression with three constants merges with slope discontinuity into the field of the lower triangle. Thus, in \( jkm \),

\begin{equation}
\begin{aligned}
w^b &= \alpha_{11} y'^2 + \alpha_{12} y'^3 + \alpha_{13} x' y'^2
\end{aligned}
\end{equation}

in terms of the locally specified coordinates \( x' \) and \( y' \). Similarly, for the third function, Fig. 4.21(c), \( w^c = 0 \) in the lower triangle, and in \( imj \) we define

\begin{equation}
\begin{aligned}
w^c &= \alpha_{14} y''^2 + \alpha_{15} y''^3 + \alpha_{16} x'' y''^2
\end{aligned}
\end{equation}

\begin{figure}[h]
\centering
\begin{subfigure}{0.3\textwidth}
\centering
\includegraphics[width=\textwidth]{a.png}
\caption{(a)}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
\centering
\includegraphics[width=\textwidth]{b.png}
\caption{(b) Three internal degrees of freedom}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
\centering
\includegraphics[width=\textwidth]{c.png}
\caption{(c) Seven internal degrees of freedom}
\end{subfigure}
\caption{Some composite quadrilateral elements.}
\end{figure}
The 16 external degrees of freedom are provided by 12 usual corner variables and four normal mid-side slopes and allow the 16 constants $\alpha_1$ to $\alpha_{16}$ to be found by inversion. Compatibility is assured and once again non-unique second derivatives arise at corners.

Again it is possible to constrain the mid-side nodes if desired and thus obtain a 12 degree-of-freedom element. The expansion can be found explicitly, as shown by Fraeijs de Veubeke, and a useful element generated.\textsuperscript{16}

The element described above cannot be formulated if a corner of the quadrilateral is re-entrant. This is not a serious limitation but needs to be considered on occasion if such an element degenerates to a near triangular shape.

### 4.13 Quasi-conforming elements

The performance of some of the conforming elements discussed in Secs 4.10–4.12 is shown in the comparison graphs of Fig. 4.16. It should be noted that although monotonic convergence in energy norm is now guaranteed, by subdividing each mesh to obtain the next one, the conforming triangular elements of references 10 and 11 perform almost identically but are considerably stiffer and hence less accurate than many of the non-conforming elements previously cited.

To overcome this inaccuracy a quasi-conforming or smoothed element was derived by Razzaque and Irons.\textsuperscript{33,34} For the derivation of this element substitute shape functions are used.

The substitute functions are cubic functions (in area coordinates) so designed as to approximate in a least-square sense the singular functions $\varepsilon$ and their derivatives used to enforce continuity [see Eqs (4.72)–(4.78)], as shown in Fig. 4.22.

The algebra involved is complex but a full subprogram for stiffness computations is available in reference 33. It is noted that this element performs very similarly to the simper, non-conforming element previously derived for the triangle. It is interesting...
to observe that here the non-conforming element is developed by choice and not to avoid difficulties. Its validity, however, is established by patch tests.

**Conforming shape functions with additional degrees of freedom**

### 4.14 Hermitian rectangle shape function

With the rectangular element of Fig 4.7 the specification of $\frac{\partial^2 w}{\partial x \partial y}$ as a nodal parameter is always permissible as it does not involve ‘excessive continuity’. It is
easy to show that for such an element polynomial shape functions giving compatibility can be easily determined.

A polynomial expansion involving 16 constants [equal to the number of nodal parameters $w_i$, $(\partial w/\partial x)_i$, $(\partial w/\partial y)_i$, and $(\partial^2 w/\partial x \partial y)_i$] could, for instance, be written retaining terms that do not produce a higher-order variation of $w$ or its normal slope along the sides. Many alternatives will be present here and some may not produce invertible $C$ matrices [see Eq. (4.41)].

An alternative derivation uses Hermitian polynomials which permit the writing down of suitable functions directly. An Hermitian polynomial

$$H^n_{m}(x)$$

is a polynomial of order $2n + 1$ which gives, for $m = 0$ to $n$,

$$\frac{d^k H^n_{m}(x)}{dx^k} = \begin{cases} 1, & \text{when } k = m \text{ and } x = x_i \\ 0, & \text{when } k \neq m \text{ or when } x = x_j \end{cases}$$

A set of first-order Hermitian polynomials is thus a set of cubic terms giving shape functions for a line element $ij$ at the ends of which slopes and values of the function are used as variables. Figure 4.23 shows such a set of cubics, and it is easy to verify that the shape functions are given by

$$H^1_{01}(x) = 1 - 3 \frac{x^2}{L^2} + 2 \frac{x^3}{L^3}$$

$$H^1_{11}(x) = x - 2 \frac{x^2}{L} + \frac{x^3}{L^2}$$

$$H^1_{02}(x) = 3 \frac{x^2}{L^2} - 2 \frac{x^3}{L^3}$$

$$H^1_{12}(x) = -\frac{x^2}{L} + \frac{x^3}{L^2}$$

![Fig. 4.23 First-order Hermitian functions.](image)
The 21 and 18 degree-of-freedom triangle

where $L$ is the length of the side. These are precisely the ‘beam’ functions used in Chapter 2 of Volume 1.

It is easy to verify that the following shape functions

$$N_i = \begin{bmatrix} H_{00}(x)H_{00}(y), & H_{10}(x)H_{01}(y), & H_{01}(x)H_{10}(y), & H_1(x)H_1(y) \end{bmatrix}$$

(4.83)
correspond to the values of

$$w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x \partial y}$$
specified at the corner nodes, taking successively unit values at node $i$ and zero at other nodes.

An element based on these shape functions has been developed by Bogner et al. and used with success. Indeed it is the most accurate rectangular element available as indicated by results in Fig. 4.16. A development of this type of element to include continuity of higher derivatives is simple and outlined in reference 18. In their undistorted form the above elements are, as for all rectangles, of very limited applicability.

### 4.15 The 21 and 18 degree-of-freedom triangle

If continuity of higher derivatives than first is accepted at nodes (thus imposing a certain constraint on non-homogeneous material and discontinuous thickness situations as explained in Sec. 4.2.4), the generation of slope and deflection compatible elements presents less difficulty.

Considering as nodal degrees of freedom

$$w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}$$
a triangular element will involve at least 18 degrees of freedom. However, a complete fifth-order polynomial contains 21 terms. If, therefore, we add three normal slopes at the mid-side as additional degrees of freedom a sufficient number of equations appear to exist for which the shape functions can be found with a complete quintic polynomial.

Along any edge we have six quantities determining the variation of $w$ (displacement, slopes, and curvature at corner nodes), that is, specifying a fifth-order variation. Thus, this is uniquely defined and therefore $w$ is continuous between elements. Similarly, $\partial w/\partial n$ is prescribed by five quantities and varies as a fourth-order polynomial. Again this is as required by the slope continuity between elements.

If we write the complete quintic polynomial as

$$w = \alpha_1 + \alpha_2 x + \cdots + \alpha_{21} y^5$$

(4.84)

* For this derivation use of simple cartesian coordinates is recommended in preference to area coordinates. Symmetry is assured as the polynomial is complete.
Plate bending approximation

using the ordering in the Pascal triangle [see Fig. 8.12 of Volume 1] we can proceed along the lines of the argument used to develop the rectangle in Sec. 4.3 and write

\[ w_i = \alpha_1 + \alpha_2 x_i + \cdots + \alpha_{21} y_i^5 \]

\[ \left( \frac{\partial w}{\partial x} \right)_i = \alpha_2 + \cdots + \alpha_{21} y_i^4 \]

\[ \left( \frac{\partial w}{\partial y} \right)_i = \alpha_3 + \cdots + 5 \alpha_{21} y_i^4 \]

\[ \left( \frac{\partial^2 w}{\partial x^2} \right)_i = 2 \alpha_4 + \cdots + 2 \alpha_{19} y_i^3 \]

and so on, and finally obtain an expression

\[ a^e = C \alpha \]  \hspace{1cm} (4.85)

in which \( C \) is a \( 21 \times 21 \) matrix.

The only apparent difficulty in the process that the reader may experience in forming this is that of the definition of the normal slopes at the mid-side nodes. However, if one notes that

\[ \frac{\partial w}{\partial n} = \cos \phi \frac{\partial w}{\partial x} + \sin \phi \frac{\partial w}{\partial y} \]  \hspace{1cm} (4.86)

in which \( \phi \) is the angle of a particular side to the \( x \) axis, the manner of formulation becomes simple. It is not easy to determine an explicit inverse of \( C \), and the stiffness expressions, etc., are evaluated as in Eqs (4.30)–(4.33) by a numerical inversion.

The existence of the mid-side nodes with their single degree of freedom is an inconvenience. It is possible, however, to constrain these by allowing only a cubic variation of the normal slope along each triangle side. Now, explicitly, the matrix \( C \) and the degrees of freedom can be reduced to 18, giving an element illustrated in Fig. 4.19(e) with three corner nodes and 6 degrees of freedom at each node.

Both of these elements were described in several independently derived publications appearing during 1968 and 1969. The 21 degree-of-freedom element was described independently by Argyris et al.,\(^{23}\) Bell,\(^{19}\) Bosshard,\(^{22}\) and Visser,\(^{24}\) listing the authors alphabetically. The reduced 18 degree-of-freedom version was developed by Argyris et al.,\(^{23}\) Bell,\(^{19}\) Cowper et al.,\(^{21}\) and Irons.\(^{14}\) An essentially similar, but more complicated, formulation has been developed by Butlin and Ford\(^{20}\) and mention of the element shape functions was made earlier by Withum\(^{63}\) and Felippa.\(^{64}\)

It is clear that many more elements of this type could be developed and indeed some are suggested in the above references. A very inclusive study is found in the work of Zenisek,\(^{65}\) Peano,\(^{66}\) and others.\(^{67-69}\) However, it should always be borne in mind that all the elements discussed in this section involve an inconsistency when discontinuous variation of material properties occurs. Further, the existence of higher-order derivatives makes it more difficult to impose boundary conditions and indeed the simple interpretation of energy conjugates as 'nodal forces' is more complex. Thus, the engineer may still feel a justified preference for the more intuitive formulation involving displacements and slopes only, despite the fact that very good accuracy is demonstrated in the references cited for the quartic and quintic elements.
4.16 Mixed formulations – general remarks

Equations (4.13)–(4.18) of this chapter provide for many possibilities to approximate both thick and thin plates by using mixed (i.e. reducible) forms. In these, more than one set of variables is approximated directly, and generally continuity requirements for such approximations can be of either $C_1$ or $C_0$ type. The procedures used in mixed formulations generally have been described in Chapters 11–13 of Volume 1, and the reader is referred to these for the general principles involved. The options open are large and indeed so is the number of publications proposing various alternatives. We shall therefore limit the discussion to those that appear most useful.

To avoid constant reference to the beginning of this chapter, the four governing equations (4.13)–(4.18) are rewritten below in their abbreviated form with dependent variable sets $M$, $\theta$, $S$, and $w$:

\[ M - DL\theta = 0 \]  
(4.87)
\[ \frac{1}{\alpha} S + \theta - \nabla w = 0 \]  
(4.88)
\[ L^T M + S = 0 \]  
(4.89)
\[ \nabla^T S + q = 0 \]  
(4.90)

in which $\alpha = \kappa G t$. To these, of course, the appropriate boundary conditions can be added. For details of the operators, etc., the fuller forms previously quoted need to be consulted.

Mixed forms that utilize direct approximations to all the four variables are not common. The most obvious set arises from elimination of the moments $M$, that is

\[ L^T DL\theta + S = 0 \]  
(4.91)
\[ \frac{1}{\alpha} S + \theta - \nabla w = 0 \]  
(4.92)
\[ \nabla^T S + q = 0 \]  
(4.93)

and is the basis of a formulation directly related to the three-dimensional elasticity consideration. This is so important that we shall devote Chapter 5 entirely to it, and, of course, there it can be used for both thick and thin plates. We shall, however, return to one of its derivations in Sec. 4.18.

One of the earliest mixed approaches leaves the variables $M$ and $w$ to be approximated and eliminates $S$ and $\theta$. The form given is restricted to thin plates and thus $\alpha = \infty$ is taken.

We now can write for Eqs (4.87) and (4.88),

\[ D^{-1} M - L \nabla w = 0 \]  
(4.94)
Plate bending approximation

and for Eqs (4.89) and (4.90),

$$\nabla^T L^T M - q = 0$$

The approximation can now be made directly putting

$$M = N_M \bar{M} \quad \text{and} \quad w = N_w \bar{w}$$

where $\bar{M}$ and $\bar{w}$ list the nodal (or other) parameters of the expansions, and $N_M$ and $N_w$ are appropriate shape functions.

The approximation equations can, as is well known (see Chapter 3 of Volume I), be made either via a suitable variational principle or directly in a weighted residual, Galerkin form, both leading to identical results. We choose here the latter, although the first presentations of this approximation by Herrmann\textsuperscript{70} and others\textsuperscript{71-78} all use the Hellinger–Reissner principle.

A weak form from which the plate approximation may be deduced is given by

$$\delta \Pi = \int_\Omega \delta M (-D^{-1} M + L \nabla w) \, d\Omega + \int_\Omega \delta w (\nabla^T L^T M - q) \, d\Omega + \delta \Pi_{bt} = 0 \quad (4.97)$$

where $\delta \Pi_{bt}$ describes appropriate boundary condition terms. Using the Galerkin weighting approximations

$$\delta M = N_M \delta \bar{M} \quad \text{and} \quad \delta w = N_w \delta \bar{w}$$

(4.98)

gives on integration by parts the following equation set

$$\begin{bmatrix} A & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} \bar{M} \\ \bar{w} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

(4.99)

where

$$A = -\int_\Omega N_M^T D^{-1} N_M \, d\Omega \quad f_1 = \int_{\Gamma_1} (\nabla N_w)^T \left\{ \begin{bmatrix} \bar{M}_n \\ \bar{M}_{ns} \end{bmatrix} \right\} \, d\Gamma \quad (4.100)$$

$$C = -\int_\Omega (L N_M)^T \nabla N_w \, d\Omega \quad f_2 = \int_\Omega N_{wq}^T q \, d\Omega + \int_{\Gamma_0} N_{wq}^T S_n \, d\Gamma$$

where $\bar{M}_n$ and $\bar{M}_{ns}$ are the prescribed boundary moments, and $S_n$ is the prescribed boundary shear force.

Immediately, it is evident that only $C_0$ continuity is required for both $M$ and $w$ interpolation,\textsuperscript{*} and many forms of elements are therefore applicable. Of course, appropriate patch tests for the mixed formulation must be enforced\textsuperscript{43} and this requires a necessary condition that

$$n_m \geq n_w$$

(4.101)

where $n_m$ stands for the number of parameters describing the moment field and $n_w$ the number in the displacement field.

Many excellent elements have been developed by using this type of approximation, though their application is limited because of the difficulty of interconnection with

\textsuperscript{*} It should be observed that, if $C_0$ continuity to the whole $M$ field is taken, excessive continuity will arise and it is usual to ensure the continuity of $M_n$ and $M_{ns}$ at interfaces only.
other structures as well as the fact that the coefficient matrix in Eq. (4.99) is indefinite with many zero diagonal terms.

Indeed, a similar fate is encountered in numerous ‘equilibrium element’ forms in which the moment (stress) field is chosen a priori in a manner satisfying Eq. (4.95). Here the research of Fraeijs de Veubeke\textsuperscript{77} and others\textsuperscript{12,30} has to be noted. It must, however, be observed that the second of these elements\textsuperscript{30} is in fact identical to the mixed element developed by Herrmann\textsuperscript{71} and Hellan\textsuperscript{79} (see also reference 52).

4.17 Hybrid plate elements

Hybrid elements are essentially mixed elements in which the field inside the element is defined by one set of parameters and the one on the element frame by another, as shown in Fig. 4.24. The latter are generally chosen to be of a type identical to other displacement models and thus can be readily incorporated in a general program and indeed used in conjunction with the standard displacement types we have already discussed. The internal parameters can be readily eliminated (being confined to a single element) and thus the difference from displacement forms are confined to the element subprogram. The original concept is attributable to Pian\textsuperscript{80,81} who pioneered this approach, and today many variants of the procedures exist in the context of thin plate theory.\textsuperscript{82–91}

In the majority of approximations, an equilibrating stress field is assumed to be given by a number of suitable shape functions and unknown parameters. In others, a mixed stress field is taken in the interior. A more refined procedure, introduced by Jirousek,\textsuperscript{57,91} assumes in the interior a series solution exactly satisfying all the differential equations involved for a homogeneous field.

All procedures use a suitable linking of the interior parameters with those defined on the boundary by the ‘frame parameters’. The procedures for doing this are fully
Plate bending approximation

described in Chapter 13 of Volume 1 in the context of elasticity equations, and only a small change of variables is needed to adapt these to the present case. We leave this extension to the reader who can also consult appropriate references for details.

Some remarks need to be made in the context of hybrid elements.

**Remark 1.** The first is that the number of internal parameters, \( n_I \), must be at least as large as the number of frame parameters, \( n_F \), which describe the displacements, less the number of rigid-body modes if singularity of the final (stiffness) matrix is to be avoided. Thus, we require that

\[
2 \geq n_F - 3
\]

for plates.

**Remark 2.** The second remark is a simple statement that it is possible, but counterproductive, to introduce an excessive number of internal parameters that simply give a more exact solution to a ‘wrong’ problem in which the frame is constraining the interior of an element. Thus additional accuracy is not achieved overall.

**Remark 3.** Most of the formulations are available for non-homogeneous plates (and hence non-linear problems). However, this is not true for the Trefftz-hybrid elements\(^5^7\), where an exact solution to the differential equation needs to be available for the element interior. Such solutions are not known for arbitrary non-homogeneous interiors and hence the procedure fails. However, for homogeneous problems the elements can be made much more accurate than any of the others and indeed allow a general polygonal element with singularities and/or internal boundaries to be developed by the use of special functions (see Fig. 4.24). Obviously, this advantage needs to be borne in mind.

A number of elements matching (or duplicating) the displacement method have been developed and the performance of some of the simpler ones is shown in Fig. 4.16. Indeed, it can be shown that many hybrid-type elements duplicate precisely the various incompatible elements that pass the convergence requirement. Thus, it is interesting to note that the triangle of Allman\(^9^0\) gives precisely the same results as the ‘smoothed’ Razzaque element of references 33 and 34 or, indeed, the element of Sec. 4.5.

### 4.18 Discrete Kirchhoff constraints

Another procedure for achieving excellent element performance is achieved as a constrained (mixed) element. Here it is convenient (though by no means essential) to use a variational principle to describe Eqs (4.91) and (4.93). This can be written simply as the minimization of the functional

\[
\Pi = \frac{1}{2} \int_{\Omega} (L\theta)^T D(L\theta) \, d\Omega + \frac{1}{2} \int_{\Omega} S^T \frac{1}{\alpha} S \, d\Omega - \int_{\Omega} wq \, d\Omega + \Pi_{bt} = \text{minimum}
\]

(4.103)
subject to the constraint that Eq. (4.92) be satisfied, that is, that

\[
\frac{1}{\alpha} S + \Theta - \nabla w = 0
\] (4.104)

We shall use this form for general thick plates in Chapter 5, but in the case of thin plates with which this chapter is concerned, we can specialize by putting \( \alpha = \infty \) and rewrite the above as

\[
\Pi = \frac{1}{2} \int_\Omega (L\Theta)^T D(L\Theta) \, d\Omega - \int_\Omega w q \, d\Omega + \Pi_{bt} = \text{minimum} (4.105)
\]

subject to

\[
\Theta - \nabla w = 0
\] (4.106)

and we note that the explicit mention of shear forces \( S \) is no longer necessary.

To solve the problem posed by Eqs (4.105) and (4.106) we can

1. approximate \( w \) and \( \Theta \) by independent interpolations of \( C_0 \) continuity as

\[
w = N_w \hat{w} \quad \text{and} \quad \Theta = N_\Theta \hat{\Theta}
\] (4.107)

2. impose a discrete approximation to the constraint of Eq. (4.106) and solve the minimization problem resulting from substitution of Eq. (4.107) into Eq. (4.105) by either discrete elimination, use of suitable lagrangian multipliers, or penalty procedures.

In the application of the so-called discrete Kirchhoff constraints, Eq. (4.106) is approximated by point (or subdomain) collocation and direct elimination is used to reduce the number of nodal parameters. Of course, the other means of imposing the constraints could be used with identical effect and we shall return to these in the next chapter. However, direct elimination is advantageous in reducing the final total number of variables and can be used effectively.

### 4.18.1 One-dimensional beam example

We illustrate the process to impose discrete constraints on a simple, one-dimensional, example of a beam (or cylindrical bending of a plate) shown in Fig. 4.25. In this, initially the displacements and rotations are taken as determined by a quadratic interpolation of an identical kind and we write in place of Eq. (4.107),

\[
\begin{bmatrix} w \\ \theta \end{bmatrix} = \sum_{i=1}^3 N_i \begin{bmatrix} \hat{w}_i \\ \hat{\theta}_i \end{bmatrix}
\] (4.108)

where \( i \) are the three element nodes.

The constraint is now applied by point collocation at coordinates \( x_\alpha \) and \( x_\beta \) of the beam; that is, we require that at these points

\[
\Theta - \frac{\partial w}{\partial x} = 0
\] (4.109)
Plate bending approximation

![Constraint here]

\[ a_i = \begin{bmatrix} w \\ \theta \end{bmatrix} \]

\[ \begin{bmatrix} w \\ \theta \end{bmatrix} = \sum_{i=1}^{3} N_a_i \]

\[ \begin{bmatrix} w \\ \theta \end{bmatrix} = \sum_{i=1}^{2} N_\alpha_i \]

**Fig. 4.25** A beam element with independent, Lagrangian, interpolation of \( w \) and \( \theta \) with constraint \( \partial w/\partial x - \theta = 0 \) applied at points \( x \).

This can be written by using the interpolation of Eq. (4.108) as two simultaneous equations

\[
\sum_{i=1}^{3} N_i(x_\alpha) \tilde{\theta}_i - \sum_{i=1}^{3} N'_i(x_\alpha) \tilde{w}_i = 0 \\
(4.110)
\]

\[
\sum_{i=1}^{3} N_i(x_\beta) \tilde{\theta}_i - \sum_{i=1}^{3} N'_i(x_\beta) \tilde{w}_i = 0
\]

where

\[ N_i(x_\alpha) = N_i(x)|_{x=x_\alpha} \quad \text{and} \quad N'_i(x_\alpha) = \left( \frac{dN_i}{dx} \right)_{x=x_\alpha} \]

Equations (4.110) can be used to eliminate \( \tilde{w}_i \) and \( \tilde{\theta}_i \). Writing Eqs (4.110) explicitly we have

\[
A_3 \begin{bmatrix} \tilde{w}_3 \\ \tilde{\theta}_3 \end{bmatrix} = A_1 \begin{bmatrix} \tilde{w}_1 \\ \tilde{\theta}_1 \end{bmatrix} + A_2 \begin{bmatrix} \tilde{w}_2 \\ \tilde{\theta}_2 \end{bmatrix} \\
(4.111)
\]

where

\[ A_i = \begin{bmatrix} N_i(x_\alpha) - N'_i(x_\alpha) \\ N_i(x_\beta) - N'_i(x_\beta) \end{bmatrix} \]

Substitution of the above into Eq. (4.108) results directly in shape functions from which the centre node has been eliminated, that is,

\[
\begin{bmatrix} w \\ \theta \end{bmatrix} = \sum_{i=1}^{2} \tilde{N}_i \begin{bmatrix} \tilde{w}_i \\ \tilde{\theta}_i \end{bmatrix} \\
(4.112)
\]

with

\[ \tilde{N}_i = N_i I + A_3^{-1} A_i \]

where \( I \) is a \( 2 \times 2 \) identity matrix.

If these functions are used for a beam, we arrive at an element that is convergent. Indeed, in the particular case where \( x_\alpha \) and \( x_\beta \) are chosen to coincide with the two Gauss quadrature points the element stiffness coincides with that given by a
displacement formulation involving a cubic $w$ interpolation. In fact, the agreement is exact for a uniform beam.

For two-dimensional plate elements the situation is a little more complex, but if we imagine $x$ to coincide with the direction tangent to an element side, precisely identical elimination enforces complete compatibility along an element side when both gradients of $w$ are specified at the ends. However, with discrete imposition of the constraints it is

![Diagram of discrete Kirchhoff theory elements](image)

**Fig. 4.26** A series of discrete Kirchhoff theory (DKT)-type elements of quadrilateral type.
not clear \textit{a priori} that convergence will always occur — though, of course, one can argue heuristically that collocation applied in numerous directions should result in an acceptable element. Indeed, patch tests turn out to be satisfied by most elements in which the $w$ interpolation (and hence the $\partial w/\partial s$ interpolation) have $C_0$ continuity.

The constraints frequently applied in practice involve the use of line or subdomain collocation to increase their number (which must, of course, always be less than the number of remaining variables) and such additional constraint equations as

$$I_G = \int_{\Gamma_e} \left( \frac{\partial w}{\partial s} - \theta_s \right) ds = 0$$

$$I_{\Omega x} = \int_{\Omega_e} \left( \frac{\partial w}{\partial x} - \theta_x \right) d\Omega = 0 \quad (4.113)$$

$$I_{\Omega y} = \int_{\Omega_e} \left( \frac{\partial w}{\partial y} - \theta_y \right) d\Omega = 0$$

are frequently used. The algebra involved in the elimination is not always easy and the reader is referred to original references for details pertaining to each particular element.

The concept of discrete Kirchhoff constraints was first introduced by Wempner \textit{et al.},\textsuperscript{92} Stricklin \textit{et al.},\textsuperscript{59} and Dhatt\textsuperscript{60} in 1968–69, but it has been applied extensively since.\textsuperscript{93–103} In particular, the 9 degree-of-freedom triangle\textsuperscript{93,94} and the complex semi-loof element of Irons\textsuperscript{96} are elements which have been successfully used.

Figure 4.26 illustrates some of the possible types of quadrilateral elements achieved in these references.

### 4.19 Rotation-free elements

It is possible to construct elements for thin plates in terms of transverse displacement parameters alone. Nay and Utku used quadratic displacement approximation and minimum potential energy to construct a least-square fit for an element configuration shown in Fig. 4.27(a).\textsuperscript{104} The element is non-conforming but passes the patch test and therefore is an admissible form. An alternative, mixed field, construction is given by Oñate and Zárate for a composite element constructed from linear interpolation on each triangle.\textsuperscript{105} In this work a mixed variational principle is used together with a special approximation for the curvature. We summarize here the steps in the better approach.

A three-field mixed variational form for a thin plate problem based on the Hu–Washizu functional may be written as

$$\Pi = \frac{1}{2} \int_A \kappa^T D \kappa \, dA - \int_A \mathbf{M}^T [(\mathbf{L} \nabla)w - \kappa] \, dA - \int_A w q \, dA + \Pi_{bl} \quad (4.114)$$

where now $\kappa$ and $\mathbf{M}$ are mixed variables to be approximated, $(\mathbf{L} \nabla)w$ are again second derivatives of displacement $w$ given in Eq. (4.20) and integration is over the area of the plate middle surface. Variation of Eq. (4.114) with respect to $\kappa$ gives the discrete constitutive equation

$$\int_A \delta \kappa^T [D \kappa - \mathbf{M}] \, dA = 0 \quad (4.115)$$
Rotation-free elements

Fig. 4.27 Elements for rotation-free thin plates: (a) patch for Nay and Utku procedure\textsuperscript{104} BPT triangle; and (b) patch for BPN triangle.\textsuperscript{105}

where $A_e$ is the domain of the patch for the element. Two alternatives for $A_e$ are considered in reference 105 and named BPT and BPN as shown in Figs 4.27(a) and 4.27(b), respectively. For the BPT form the integration is taken over the area of the element ‘$ijk$’ with area $A_e$ and boundary $\Gamma_e$. For the type BPN integration is over the more complex area $A_i$ with boundary $\Gamma_i$. Each, however, are simple to construct. Similarly, variation of Eq. (4.114) with respect to moment gives the discrete curvature relation

\[
\int_{A_e} \delta M^T [(L\nabla)w - \kappa] \, dA = 0 \tag{4.116}
\]

Finally, the equilibrium equations are obtained from the variation with respect to the displacement, and are expressed as

\[
\int_{A_e} [(L\nabla)\delta w]^T M \, dA - \int_{A_e} \delta q \, dA + \delta b_l = 0 \tag{4.117}
\]

A finite element approximation may be constructed in the standard manner by writing

\[
M = N_i^m \tilde{M}_i, \quad \kappa = N_i^k \tilde{\kappa}_i, \quad \text{and} \quad w = N_i^w \tilde{w}_i \tag{4.118}
\]

The simplest approximations are for $N_i^m = N_i^k = 1$ and linear interpolation over each triangle for $N_i^w$. Equation (4.115) is easily evaluated; however, the other two integrals have apparent difficulty since a linear interpolation yields zero derivatives within each triangle. Indeed the curvature is now concentrated in the ‘kinks’ which occur between contiguous triangles. To obtain discrete approximations to the curvature changes an integration by parts is used (see Green’s theorem, Appendix G of Volume 1) to rewrite Eq. (4.116) as

\[
\int_{A_e} \delta M^T \kappa \, dA + \int_{\Gamma_e} \delta M^T g \nabla w \, d\Gamma = 0 \tag{4.119}
\]
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**Plate bending approximation**

where

\[
g = \begin{bmatrix}
n_x & 0 \\
0 & n_y \\
n_y & n_x
\end{bmatrix}
\tag{4.120}
\]

is a matrix of the direction cosines for an outward pointing normal vector \( \mathbf{n} \) to the boundary \( \Gamma_e \) and

\[
\nabla w = \begin{bmatrix}
w_{x} \\
w_{y}
\end{bmatrix}
\tag{4.121}
\]

In these expressions \( \Gamma_e \) is the part of the boundary within the area of integration \( A_e \). Thus, for the element type BPT it is just the contour \( \Gamma_p \) as shown in Fig. 4.27(a). For the element type BPN no slope discontinuity occurs on the boundary \( \Gamma_i \) shown in Fig. 4.27(b); however, it is necessary to integrate along the half sides of each triangle within the patch bounded by \( \Gamma_i \). The remainder of the derivation is now straightforward and the reader is referred to reference 105 for additional details and results.

In this paper results are also presented for thin shells. We note that the type of element discussed in this section is quite different from those presented previously in that nodes exist outside the boundary of the element. Thus, the definition of an element and the assembly process are somewhat different. In addition, boundary conditions need some special treatments to include in a general manner.\textsuperscript{105} Because of these differences we do not consider additional members in this family. We do note, however, that for explicit dynamic programs some advantages occur since no rotation parameters need be integrated. Results for thin shells subjected to impulsive loading are particularly noteworthy.\textsuperscript{105}

### 4.20 Inelastic material behaviour

The preceding discussion has assumed the plate to be a linear elastic material. In many situations it is necessary to consider a more general constitutive behaviour in order to represent the physical problem correctly. For thin plates, only the bending and twisting moment are associated with deformations and are related to the local stresses through

\[
M = \begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = - \int_{-t/2}^{t/2} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} z \, dz
\tag{4.122}
\]

Any of the material models discussed in Chapter 3 which have symmetric stress behaviour with respect to strains may be used in plate analysis provided an appropriate plane stress form is available. The symmetry is necessary to avoid the generation of in-plane force resultants – which are assumed to decouple from the bending behaviour. If such conditions do not exist it is necessary to use a shell formulation as described in Chapters 6–9.
In practice two approaches are considered – one dealing with the individual lamina using local stress components $\sigma_x, \sigma_y$ and $\tau_{xy}$ and the other using plate resultant forces $M_x, M_y$ and $M_{xy}$ directly.

### 4.20.1 Numerical integration through thickness

The most direct approach is to use a plane stress form of the stress–strain relation and perform the through-thickness integration numerically. In order to capture the maximum stresses at the top and bottom of the plate it is best to use Gauss–Lobatto-type quadrature formulae\textsuperscript{106} where integrals are approximated by

\[
\int_{-1}^{1} f(\xi) \, d\xi \approx f(-1) W_1 + \sum_{n=2}^{N-1} f(\xi_n) W_n + f(1) W_N \tag{4.123}
\]

These formula differ from the typical gaussian quadrature considered previously and use the end points on the interval directly. This allows computation of first yield to be more accurate. The values of the parameters $\xi_n$ and $W_n$ are given in Table 4.4 up to the six-point formula. Parameters for higher-order formulae may be found in reference 106.

Noting that the strain components in plates [see Eq. (4.10)] are asymmetric with respect to the middle surface of the plate and that the $z$ coordinate is also asymmetric we can compute the plate resultants by evaluating only half the integral. Accordingly, we may use

\[
\mathbf{M} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = -2 \int_{0}^{t/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z \, dz \tag{4.124}
\]

and here a six-point formula or less will generally be sufficient to compute integrals. Equation (4.29) is replaced by the non-linear equation given as

\[
\Psi(\mathbf{\tilde{w}}) = \mathbf{f} - \int_{\Omega} \mathbf{B}^T \mathbf{M} \, d\Omega = 0 \tag{4.125}
\]

<table>
<thead>
<tr>
<th>N</th>
<th>$\pm \xi_n$</th>
<th>$W_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.0</td>
<td>1/3</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>4/3</td>
</tr>
<tr>
<td>4</td>
<td>1.0</td>
<td>1/6</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{0.2}$</td>
<td>5/6</td>
</tr>
<tr>
<td>5</td>
<td>1.0</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{3/7}$</td>
<td>49/90</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>64/90</td>
</tr>
<tr>
<td>6</td>
<td>1.0</td>
<td>1/15</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{t_1}$</td>
<td>$0.6/[t_1(1-t_0)^2]$</td>
</tr>
<tr>
<td></td>
<td>$\sqrt{t_2}$</td>
<td>$0.6/[t_2(1-t_0)^2]$</td>
</tr>
<tr>
<td>$t_0 = \sqrt{7}$</td>
<td>$t_1 = (7 + 2t_0)/21$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$t_2 = (7 - 2t_0)/21$</td>
<td></td>
</tr>
</tbody>
</table>
The solution process (for a static case) may now proceed by using, for instance, a Newton–Raphson scheme in which the tangent moduli for the plate are obtained by using the tangent moduli for the stress components as

\[
d M = \begin{pmatrix}
  d M_x \\
  d M_y \\
  d M_{xy}
\end{pmatrix} = \left[ 2 \int_0^{r/2} D_T^{(ps)} z^2 \, dz \right] L \nabla dw = D_T (L \nabla) \, dw
\]  

(4.126)

where \( D_T^{(ps)}(z) \) is the tangent modulus matrix of a plane stress material model at each lamina level \( z \), and \( D_T \) is the resulting bending tangent stiffness matrix of the plate.

The Newton–Raphson iteration for the displacement increment is computed as

\[
K_T^{(k)} \, d\bar{w}^{(k)} = \bar{\Phi}^{(k)}
\]  

(4.127)

with iterative updates

\[
\bar{w}^{(k+1)} = \bar{w}^{(k)} + d\bar{w}^{(k)}
\]  

(4.128)

until a suitable convergence criterion is satisfied. This follows precisely methods previously defined for solids.

### 4.20.2 Resultant constitutive models

A resultant yield function for plates with Huber–von Mises-type material is given by\(^{107}\)

\[
F(M) = \left( M_x^2 + M_y^2 - M_x M_y + 3 M_{xy}^2 \right) - M_u^2(k) \leq 0
\]  

(4.129)

where \( \kappa \) is an 'isotropic' hardening parameter and \( M_u \) denotes a uniaxial yield moment and which for homogeneous plates is generally given by

\[
M_u = \frac{1}{2} t^2 \sigma_y(k)
\]  

(4.130)

in which \( \sigma_y \) is the material uniaxial yield stress in tension (and compression). We observe that, in the absence of hardening, \( M_u \) is the moment that exists when the entire cross section is at a yield stress.

### 4.21 Concluding remarks – which elements?

The extensive bibliography of this chapter outlining the numerous approaches capable of solving the problems of thin, Kirchhoff, plate flexure shows both the importance of the subject in structural engineering – particularly as a preliminary to shell analysis – and the wide variety of possible approaches. Indeed, only part of the story is outlined here, as the next chapter, dealing with thick plate formulation, presents many practical alternatives of dealing with the same problem.

We hope that the presentation, in addition to providing a guide to a particular problem area, is useful in its direct extension to other fields where governing equations lead to \( C_1 \) continuity requirements.
Users of practical computer programs will be faced with a problem of 'which element' is to be used to satisfy their needs. We have listed in Table 4.3 some of the more widely known simple elements and compared their performance in Fig. 4.16. The choice is not always unique, and much more will depend on preferences and indeed extensions desired. As will be seen in Chapter 6 for general shell problems, triangular elements are an optimal choice for many applications and configurations. Further, such elements are most easily incorporated if adaptive mesh generation is to be used for achieving errors of predetermined magnitude.

References


34. B.M. Irons and A. Razzazque. Shape function formulation for elements other than displacement models. In C.A. Brebbia and H. Tottenham (eds), *Proc. of the International


References

