4

Plate bending approximation: thin (Kirchhoff) plates and $C_1$ continuity requirements

4.1 Introduction

The subject of bending of plates and indeed its extension to shells was one of the first to which the finite element method was applied in the early 1960s. At that time the various difficulties that were to be encountered were not fully appreciated and for this reason the topic remains one in which research is active to the present day. Although the subject is of direct interest only to applied mechanicians and structural engineers there is much that has more general applicability, and many of the procedures which we shall introduce can be directly translated to other fields of application.

Plates and shells are but a particular form of a three-dimensional solid, the treatment of which presents no theoretical difficulties, at least in the case of elasticity. However, the thickness of such structures (denoted throughout this and later chapters as $t$) is very small when compared with other dimensions, and complete three-dimensional numerical treatment is not only costly but in addition often leads to serious numerical ill-conditioning problems. To ease the solution, even long before numerical approaches became possible, several classical assumptions regarding the behaviour of such structures were introduced. Clearly, such assumptions result in a series of approximations. Thus numerical treatment will, in general, concern itself with the approximation to an already approximate theory (or mathematical model), the validity of which is restricted. On occasion we shall point out the shortcomings of the original assumptions, and indeed modify these as necessary or convenient. This can be done simply because now we are granted more freedom than that which existed in the 'pre-computer' era.

The thin plate theory is based on the assumptions formalized by Kirchhoff in 1850, and indeed his name is often associated with this theory, though an early version was presented by Sophie Germain in 1811. A relaxation of the assumptions was made by Reissner in 1945 and in a slightly different manner by Mindlin in 1951. These modified theories extend the field of application of the theory to thick plates and we shall associate this name with the Reissner–Mindlin postulates.

It turns out that the thick plate theory is simpler to implement in the finite element method, though in the early days of analytical treatment it presented more difficulties. As it is more convenient to introduce first the thick plate theory and by imposition of
additional assumptions to limit it to thin plate theory we shall follow this path in the present chapter. However, when discussing numerical solutions we shall reverse the process and follow the historical procedure of dealing with the thin plate situations first in this chapter. The extension to thick plates and to what turns out always to be a mixed formulation will be the subject of Chapter 5.

In the thin plate theory it is possible to represent the state of deformation by one quantity $w$, the lateral displacement of the middle plane of the plate. Clearly, such a formulation is irreducible. The achievement of this irreducible form introduces second derivatives of $w$ in the strain definition and continuity conditions between elements have now to be imposed not only on this quantity but also on its derivatives ($C_1$ continuity). This is to ensure that the plate remains continuous and does not 'kink'. Thus at nodes on element interfaces it will always be necessary to use both the value of $w$ and its slopes (first derivatives of $w$) to impose continuity.

Determination of suitable shape functions is now much more complex than those needed for $C_0$ continuity. Indeed, as complete slope continuity is required on the interfaces between various elements, the mathematical and computational difficulties often rise disproportionately fast. It is, however, relatively simple to obtain shape functions which, while preserving continuity of $w$, may violate its slope continuity between elements, though normally not at the node where such continuity is imposed. If such chosen functions satisfy the 'patch test' (see Chapter 10, Volume 1) then convergence will still be found. The first part of this chapter will be concerned with such 'non-conforming' or 'incompatible' shape functions. In later parts new functions will be introduced by which continuity can be restored. The solution with such 'conforming' shape functions will now give bounds to the energy of the correct solution, but, on many occasions, will yield inferior accuracy to that achieved with non-conforming elements. Thus, for practical usage the methods of the first part of the chapter are often recommended.

The shape functions for rectangular elements are the simplest to form for thin plates and will be introduced first. Shape functions for triangular and quadrilateral elements are more complex and will be introduced later for solutions of plates of arbitrary shape or, for that matter, for dealing with shell problems where such elements are essential.

The problem of thin plates is associated with fourth-order differential equations leading to a potential energy function which contains second derivatives of the unknown function. It is characteristic of a large class of physical problems and, although the chapter concentrates on the structural problem, the reader will find that the procedures developed also will be equally applicable to any problem which is of fourth order.

The difficulty of imposing $C_1$ continuity on the shape functions has resulted in many alternative approaches to the problems in which this difficulty is side-stepped. Several possibilities exist. Two of the most important are:

1. independent interpolation of rotations $\theta$ and displacement $w$, imposing continuity as a special constraint, often applied at discrete points only;

* If 'kinking' occurs the second derivative or curvature becomes infinite and squares of infinite terms occur in the energy expression.

† Later we show that even slope discontinuity at the node may be used.
2. the introduction of lagrangian variables or indeed other variables to avoid the
necessity of $C_1$ continuity.

Both approaches fall into the class of mixed formulations and we shall discuss these
brieﬂy at the end of the chapter. However, a fuller statement of mixed approaches will
be made in the next chapter where both thick and thin approximations will be dealt
with simultaneously.

4.2 The plate problem: thick and thin formulations

4.2.1 Governing equations

The mechanics of plate action is perhaps best illustrated in one dimension, as shown
in Fig. 4.1. Here we consider the problem of cylindrical bending of plates. In this
problem the plate is assumed to have inﬁnite extent in one direction (here assumed
the $y$ direction) and to be loaded and supported by conditions independent of $y$. In
this case we may analyse a strip of unit width subjected to some stress resultants
$M_x$, $P_x$, and $S_x$, which denote $x$-direction bending moment, axial force and transverse

![Diagram of plate bending](image)

Fig. 4.1 Displacements and stress resultants for a typical beam.
shear force, respectively. For cross-sections that are originally normal to the middle plane of the plate we can use the approximation that at some distance from points of support or concentrated loads plane sections will remain plane during the deformation process. The postulate that sections normal to the middle plane remain plane during deformation is thus the \textit{first} and most important assumption of the theory of plates (and indeed shells). To this is added the \textit{second} assumption. This simply observes that the direct stresses in the normal direction, \( z \), are small, that is, of the order of applied lateral load intensities, \( q \), and hence direct strains in that direction can be neglected. This ‘inconsistency’ in approximation is compensated for by assuming plane stress conditions in each lamina.

With these two assumptions it is easy to see that the total state of deformation can be described by displacements \( u_0 \) and \( w_0 \) of the middle surface \((z = 0)\) and a rotation \( \theta_x \) of the normal. Thus the local displacements in the directions of the \( x \) and \( z \) axes are taken as

\[
  u(x, z) = u_0(x) - z \theta_x(x) \quad \text{and} \quad w(x, z) = w_0(x)
\]

(4.1)

Immediately the strains in the \( x \) and \( z \) directions are available as

\[
  \varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z \frac{\partial \theta_x}{\partial x}
\]

\[
  \varepsilon_z = 0
\]

\[
  \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\theta_x + \frac{\partial w_0}{\partial x}
\]

(4.2)

For the cylindrical bending problem a state of linear elastic, plane stress for each lamina yields the stress-strain relations

\[
  \sigma_x = \frac{E}{1 - \nu^2} \varepsilon_x \quad \text{and} \quad \tau_{xz} = G \gamma_{xz}
\]

The stress resultants are obtained as

\[
  P_x = \int_{-l/2}^{l/2} \sigma_x \, dz = B \frac{\partial u_0}{\partial x}
\]

\[
  S_x = \int_{-l/2}^{l/2} \tau_{xz} \, dz = \kappa G t \left( \frac{\partial w_0}{\partial x} - \theta_x \right)
\]

\[
  M_x = - \int_{-l/2}^{l/2} \sigma_x z \, dz = D \frac{\partial \theta_x}{\partial x}
\]

(4.3)

where \( B \) is the in-plane plate stiffness and \( D \) the bending stiffness computed from

\[
  B = \frac{Et}{1 - \nu^2} \quad \text{and} \quad D = \frac{Et^3}{12(1 - \nu^2)}
\]

(4.4)

with \( \nu \) Poisson’s ratio, \( E \) and \( G \) direct and shear elastic moduli, respectively.*

* A constant \( \kappa \) has been added here to account for the fact that the shear stresses are not constant across the section. A value of \( \kappa = 5/6 \) is exact for a rectangular, homogeneous section and corresponds to a parabolic shear stress distribution.
Three equations of equilibrium complete the basic formulation. These equilibrium
equations may be computed directly from a differential element of the plate or by
integration of the local equilibrium equations. Using the latter approach and assum-
ing zero body and inertial forces we have for the axial resultant

\[
\int_{-t/2}^{t/2} \left[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \right] dz = 0
\]

\[
\frac{\partial}{\partial x} \int_{-t/2}^{t/2} \sigma_x dz + \tau_{xz} \big|_{t/2} - \tau_{xz} \big|_{-t/2} = 0
\] (4.5)

where the shear stress on the top and bottom of the plate are assumed to be zero.
Similarly, the shear resultant follows from

\[
\int_{-t/2}^{t/2} \tau_{xz} dz + \sigma_z \big|_{t/2} - \sigma_z \big|_{-t/2} = 0
\] (4.6)

\[
\frac{\partial S_x}{\partial x} + q_z = 0
\]

where the transverse loading \(q_z\) arises from the resultant of the normal traction on the
top and/or bottom surfaces. Finally, the moment equilibrium is deduced from

\[
- \int_{-t/2}^{t/2} z \left[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} \right] dz = 0
\]

\[
- \frac{\partial}{\partial x} \int_{-t/2}^{t/2} z \sigma_x dz + \int_{-t/2}^{t/2} \tau_{xz} dz = 0
\] (4.7)

\[
\frac{\partial M_x}{\partial x} + S_x = 0
\]

In the elastic case of a plate it is easy to see that the in-plane displacements and forces,
\(u_0\) and \(P_x\), decouple from the other terms and the problem of lateral deformations can
be dealt with separately. We shall thus only consider bending in the present chapter,
returning to the combined problem, characteristic of shell behaviour, in later chapters.

Equations (4.1)–(4.7) are typical for thick plates, and the thin plate theory adds an
additional assumption. This simply neglects the shear deformation and puts \(G = \infty\).
Equation (4.3) thus becomes

\[
\frac{\partial w_0}{\partial x} - \theta_x = 0
\] (4.8)

This thin plate assumption is equivalent to stating that the normals to the middle
plane remain normal to it during deformation and is the same as the well-known
Bernoulli–Euler assumption for thin beams. The thin, constrained theory is very
Plate bending approximation

Fig. 4.2 Support (end) conditions for a plate or a beam. Note: the conventionally illustrated simple support leads to infinite displacement – reality is different.

widely used in practice and proves adequate for a large number of structural problems, though, of course, should not be taken literally as the true behaviour near supports or where local load action is important and is three dimensional.

In Fig. 4.2 we illustrate some of the boundary conditions imposed on plates (and beams) and immediately note that the diagrammatic representations of simple support as a knife edge would lead to infinite displacements and stresses. Of course, if a rigid bracket is added in the manner shown this will alter the behaviour to that which we shall generally assume.

The one-dimensional problem of plates and the introduction of thick and thin assumptions translate directly to the general theory of plates. In Fig. 4.3 we illustrate the extensions necessary and write, in place of Eq. (4.1) (assuming \( u_0 \) and \( v_0 \) to be zero)

\[
\begin{align*}
  u &= -z \theta_x(x,y) \\
  v &= -z \theta_y(x,y) \\
  w &= w_0(x,y)
\end{align*}
\]

where we note that displacement parameters are now functions of \( x \) and \( y \).
Fig. 4.3 Definitions of variables for plate approximations: (a) displacements and rotation; (b) stress resultants.

The strains may now be separated into bending (in-plane components) and transverse shear groups and we have, in place of Eq. (4.2),

$$\varepsilon = \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = -z \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{pmatrix} \theta_x \\ \theta_y \end{pmatrix} \equiv -zL\theta$$

(4.10)

and

$$\gamma = \begin{pmatrix} \gamma_{xz} \\ \gamma_{yz} \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{pmatrix} - \begin{pmatrix} \theta_x \\ \theta_y \end{pmatrix} = \nabla w - \theta$$

(4.11)
Plate bending approximation

We note that now in addition to normal bending moments $M_x$ and $M_y$, now defined by expression (4.3) for the $x$ and $y$ directions, respectively, a twisting moment arises defined by

$$M_{xy} = -\int_{-\phi/2}^{\phi/2} \tau_{xy} \, dz$$  \hspace{1cm} (4.12)

Introducing appropriate constitutive relations, all moment components can be related to displacement derivatives. For isotropic elasticity we can thus write, in place of Eq. (4.3),

$$\mathbf{M} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \mathbf{D} \mathbf{L} \mathbf{\theta}$$  \hspace{1cm} (4.13)

where, assuming plane stress behaviour in each layer,

$$\mathbf{D} = D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1 - \nu)/2 \end{bmatrix}$$  \hspace{1cm} (4.14)

where $\nu$ is Poisson’s ratio and $D$ is defined by the second of Eqs (4.4). Further, the shear force resultants are

$$\mathbf{S} = \begin{bmatrix} S_x \\ S_y \end{bmatrix} = \mathbf{a}(\nabla w - \mathbf{\theta})$$  \hspace{1cm} (4.15)

For isotropic elasticity (though here we deliberately have not related $G$ to $E$ and $\nu$ to allow for possibly different shear rigidities)

$$\mathbf{a} = \kappa G t \mathbf{I}$$  \hspace{1cm} (4.16)

where $\mathbf{I}$ is a $2 \times 2$ identity matrix.

Of course, the constitutive relations can be simply generalized to anisotropic or inhomogeneous behaviour such as can be manifested if several layers of materials are assembled to form a composite. The only apparent difference is the structure of the $\mathbf{D}$ and $\mathbf{a}$ matrices, which can always be found by simple integration.

The governing equations of thick and thin plate behaviour are completed by writing the equilibrium relations. Again omitting the ‘in-plane’ behaviour we have, in place of Eq. (4.6),

$$\begin{bmatrix} \frac{\partial}{\partial x}, & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} S_x \\ S_y \end{bmatrix} + q = \nabla^T \mathbf{S} = \nabla \mathbf{S} = \nabla \begin{bmatrix} S_x \\ S_y \end{bmatrix} + q = 0$$  \hspace{1cm} (4.17)

and, in place of Eq. (4.7),

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} + \begin{bmatrix} S_x \\ S_y \end{bmatrix} = \mathbf{L}^T \mathbf{M} + \mathbf{S} = \mathbf{0}$$  \hspace{1cm} (4.18)
The plate problem: thick and thin formulations

Equations (4.13)–(4.18) are the basis from which the solution of both thick and thin plates can start. For thick plates any (or all) of the independent variables can be approximated independently, leading to a mixed formulation which we shall discuss in Chapter 5 and also briefly in Sec. 4.16 of this chapter.

For thin plates in which the shear deformations are suppressed Eq. (4.15) is rewritten as
\[ \nabla w - \theta = 0 \] (4.19)
and the strain-displacement relations (4.10) become
\[ \varepsilon = -zL\nabla w = -z\begin{bmatrix} \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial y^2} \\ 2 \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} = -z\kappa \] (4.20)
where \( \kappa \) is the matrix of changes in curvature of the plate. Using the above form for the thin plate, both irreducible and mixed forms can now be written. In particular, it is an easy matter to eliminate \( M, S \) and \( \theta \) and leave only \( w \) as the variable.

Applying the operator \( \nabla^T \) to expression (4.17), inserting Eqs (4.13) and (4.17) and finally replacing \( \theta \) by the use of Eq. (4.19) gives a scalar equation
\[ (L\nabla)^TDL\nabla w - q = 0 \] (4.21)
where, using Eq. (4.20),
\[ (L\nabla)^T = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial y^2} & 2 \frac{\partial^2}{\partial x \partial y} \end{bmatrix} \]

In the case of isotropy with constant bending stiffness \( D \) this becomes the well-known biharmonic equation of plate flexure
\[ D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) - q = 0 \] (4.22)

4.2.2 The boundary conditions

The boundary conditions which have to be imposed on the problem (see Figs 4.2 and 4.4) include the following classical conditions.

1. Fixed boundary, where displacements on restrained parts of the boundary are given specified values.* These conditions are expressed as
\[ w = \bar{w}; \quad \theta_n = \bar{\theta}_n \quad \text{and} \quad \theta_s = \bar{\theta}_s \]

* Note that in thin plates the specification of \( w \) along \( s \) automatically specifies \( \theta \), by Eq. (4.19), but this is not the case in thick plates where the quantities are independently prescribed.
Plate bending approximation

120

Fig. 4.4 Boundary traction and conjugate displacement. Note: the simply supported condition requiring $M_n = 0, \theta_s = 0$ and $w = 0$ is identical at a corner node to specifying $\theta_n = \theta_s = 0$, that is, a clamped support. This leads to a paradox if a curved boundary (a) is modelled as a polygon (b).

Here $n$ and $s$ are directions normal and tangential to the boundary curve of the middle surface. A clamped edge is a special case with zero values assigned.

2. **Traction boundary**, where stress resultants $M_n, M_{ns}$ and $S_n$ (conjugate to the displacements $\theta_n, \theta_s$ and $w$) are given prescribed values. A free edge is a special case with zero values assigned.

3. **‘Mixed’ boundary conditions**, where both traction and displacements can be specified. Typical here is the simply supported edge (see Fig. 4.2). For this, clearly, $M_n = 0$ and $w = 0$, but it is less clear whether $M_{ns}$ or $\theta_s$ needs to be given. Specification of $M_{ns} = 0$ is physically a more acceptable condition and does not lead to difficulties. This should always be adopted for thick plates.

In thin plates $\theta_s$ is automatically specified from $w$ and we shall find certain difficulties, and indeed anomalies, associated with this assumption. For instance, in Fig. 4.4 we see how a specification of $\theta_s = 0$ at corner nodes implicit in thin plates formally leads to the prescription of all boundary parameters, which is identical to boundary conditions of a clamped plate for this point.

4.2.3 The irreducible, thin plate approximation

The thin plate approximation when cast in terms of a single variable $w$ is clearly irreducible and is in fact typical of a displacement formulation. The equations (4.17) and (4.18) can be written together as

$$(L \nabla)^T M - q = 0$$

and the constitutive relation (4.13) can be recast by using Eq. (4.19) as

$$M = D L \nabla w$$
The plate problem: thick and thin formulations

The derivation of the finite element equations can be obtained either from a weak form of Eq. (4.23) obtained by weighting with an arbitrary function (say \( v = N \bar{v} \)) and integration by parts (done twice) or, more directly, by application of the virtual work equivalence. Using the latter approach we may write the internal virtual work for the plate as

\[
\delta \Pi_{\text{int}} = \int_{\Omega} (\delta \varepsilon)^T D \varepsilon \, d\Omega = \int_{\Omega} \delta w (L \nabla)^T D (L \nabla) w \, d\Omega \quad (4.25)
\]

where \( \Omega \) denotes the area of the plate reference (middle) surface and \( D \) is the plate stiffness, which for isotropy is given by Eq. (4.14).

Similarly the external work is given by

\[
\delta \Pi_{\text{ext}} = \int_{\Omega} \delta w \, q \, d\Omega + \int_{\Gamma_n} \delta \theta_n \, M_n \, d\Gamma + \int_{\Gamma_r} \delta \theta_s \, M_{ns} \, d\Gamma + \int_{\Gamma_i} \delta w \, \bar{S}_n \, d\Gamma \quad (4.26)
\]

where \( M_n, M_{ns}, \bar{S}_n \) are specified values and \( \Gamma_n, \Gamma_r, \Gamma_i \) are parts of the boundary where each component is specified. For thin plates with straight edges Eq. (4.19) gives immediately \( \theta_s = \partial w / \partial s \) and thus the last two terms above may be combined as

\[
\int_{\Gamma_r} \delta \theta_s \, M_{ns} \, d\Gamma + \int_{\Gamma_i} \delta w \, \bar{S}_n \, d\Gamma = \int_{\Gamma_r} \delta w \left( \bar{S}_n - \frac{\partial \bar{M}_{ns}}{\partial s} \right) \, d\Gamma + \sum_i \delta w_i \, R_i \quad (4.27)
\]

where \( R_i \) are concentrated forces arising at locations where corners exist (see Fig 4.2).\(^2\)

Substituting into Eqs (4.25) and (4.26) the discretization

\[
w = N a \quad (4.28)
\]

where \( a \) are appropriate parameters, we can obtain for a linear case standard displacement approximation equations

\[
K a = f \quad (4.29)
\]

with

\[
K a = \left( \int_{\Omega} B^T D B \, d\Omega \right) a = \int_{\Omega} B^T M \, d\Omega \quad (4.30)
\]

and

\[
f = \int_{\Omega} N^T q \, d\Omega + f_b \quad (4.31)
\]

where \( f_b \) is the boundary contribution to be discussed later and

\[
M = D B a \quad (4.32)
\]

with

\[
B = (L \nabla) N \quad (4.33)
\]

It is of interest, and indeed important to note, that when tractions are prescribed to non-zero values the force term \( f_b \) includes all prescribed values of \( M_n, M_{ns} \) and \( S_n \) irrespective of whether the thick or thin formulation is used. The reader can verify that this term is

\[
f_b = \int_{\Gamma} (N_n^T \bar{M}_n + N_{ns}^T \bar{M}_{ns} + N^T \bar{S}_n) \, d\Gamma \quad (4.34a)
\]
Plate bending approximation

where \( \bar{M}_n, \bar{M}_n, \) and \( S_n \) are prescribed values and for thin plates [though, of course, relation (4.34a) is valid for thick plates also]:

\[
N_n = \frac{\partial N}{\partial n} \quad \text{and} \quad N_s = \frac{\partial N}{\partial s}
\] (4.34b)

The reader will recognize in the above the well-known ingredients of a displacement formulation (see Chapter 2 of Volume 1, and Chapter 1 of this volume) and the procedures are almost automatic once \( N \) is chosen.

4.2.4 Continuity requirement for shape functions (\( C_1 \) continuity)

In Sections 4.3–4.13 we will be concerned with the above formulation [starting from Eqs (4.24) and (4.26)], and the presence of the second derivatives indicates quite clearly that we shall need \( C_1 \) continuity of the shape functions for the irreducible, thin plate, formulation. This continuity is difficult to achieve and reasons for this are given below.

To ensure the continuity of both \( w \) and its normal slope across an interface we must have both \( w \) and \( \frac{\partial w}{\partial n} \) uniquely defined by values of nodal parameters along such an interface. Consider Fig. 4.5 depicting the side 1–2 of a rectangular element. The normal direction \( n \) is in fact that of \( y \) and we desire \( w \) and \( \frac{\partial w}{\partial y} \) to be uniquely determined by values of \( w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \) at the nodes lying along this line.

Following the principles expounded in Chapter 8 of Volume 1, we would write along side 1–2,

\[
w = A_1 + A_2 x + A_3 y + \cdots \quad (4.35)
\]

and

\[
\frac{\partial w}{\partial y} = B_1 + B_2 x + B_3 y + \cdots \quad (4.36)
\]

with a number of constants in each expression just sufficient to determine a unique solution for the nodal parameters associated with the line.

Thus, for instance, if only two nodes are present a cubic variation of \( w \) should be permissible noting that \( \frac{\partial w}{\partial x} \) and \( w \) are specified at each node. Similarly, only a linear, or two-term, variation of \( \frac{\partial w}{\partial y} \) would be permissible.

Fig. 4.5 Continuity requirement for normal slopes.
The plate problem: thick and thin formulations

Note, however, that a similar exercise could be performed along the side placed in the $y$ direction preserving continuity of $\partial w/\partial x$ along this. Along side 1–2 we thus have $\partial w/\partial y$, depending on nodal parameters of line 1–2 only, and along side 1–3 we have $\partial w/\partial x$, depending on nodal parameters of line 1–3 only. Differentiating the first with respect to $x$, on line 1–2 we have $\partial^2 w/\partial x \partial y$, depending on nodal parameters of line 1–2 only, and similarly, on line 1–3 we have, $\partial^2 w/\partial y \partial x$, depending on nodal parameters of line 1–3 only.

At the common point, 1, an inconsistency arises immediately as we cannot automatically have there the necessary identity for continuous functions

$$\frac{\partial^2 w}{\partial x \partial y} \equiv \frac{\partial^2 w}{\partial y \partial x}$$

for arbitrary values of the parameters at nodes 2 and 3. It is thus impossible to specify simple polynomial expressions for shape functions ensuring full compatibility when only $w$ and its slopes are prescribed at corner nodes.$^9$

Thus if any functions satisfying the compatibility are found with the three nodal variables, they must be such that at corner nodes these functions are not continuously differentiable and the cross-derivative is not unique. Some such functions are discussed in the second part of this chapter.$^{10-16}$

The above proof has been given for a rectangular element. Clearly, the arguments can be extended for any two arbitrary directions of interface at the corner node 1.

A way out of this difficulty appears to be obvious. We could specify the cross-derivative as one of the nodal parameters. This, for an assembly of rectangular elements, is convenient and indeed permissible. Simple functions of that type have been suggested by Bogner et al.$^{17}$ and used with some success. Unfortunately, the extension to nodes at which a number of element interfaces meet with different angles (Fig. 4.6) is not, in general, permissible. Here, the continuity of cross-derivatives in several sets of orthogonal directions implies, in fact, a specification of all second derivatives at a node.

This, however, violates physical requirements if the plate stiffness varies abruptly from element to element, for then equality of moments normal to the interfaces cannot be maintained. However, this process has been used with some success in homogeneous plate situations$^{18-25}$ although Smith and Duncan$^{18}$ comment adversely on the effect of imposing such excessive continuities on several orders of higher derivatives.

Fig. 4.6 Nodes where elements meet in arbitrary directions.
Plate bending approximation

The difficulties of finding compatible displacement functions have led to many attempts at ignoring the complete slope continuity while still continuing with the other necessary criteria. Proceeding perhaps from a naive but intuitive idea that the imposition of slope continuity at nodes only must, in the limit, lead to a complete slope continuity, several successful, 'non-conforming', elements have been developed.11,26-40

The convergence of such elements is not obvious but can be proved either by application of the patch test or by comparison with finite difference algorithms. We have discussed the importance of the patch test extensively in Chapter 11 of Volume 1 and additional details are available in references 41-43.

In plate problems the importance of the patch test in both design and testing of elements is paramount and this test should never be omitted. In the first part of this chapter, dealing with non-conforming elements, we shall repeatedly make use of it. Indeed, we shall show how some of the most successful elements currently used have developed via this analytical interpretation.44-49

Non-conforming shape functions

4.3 Rectangular element with corner nodes (12 degrees of freedom)

4.3.1 Shape functions

Consider a rectangular element of a plate $ijkl$ coinciding with the $xy$ plane as shown in Fig. 4.7. At each node, $n$, displacements $a_n$ are introduced. These have three components: the first a displacement in the $z$ direction, $w_n$, the second a rotation about the $x$ axis, $(\theta_x)_n$ and the third a rotation about the $y$ axis $(\theta_y)_n$. *

The nodal displacement vectors are defined below as $a_i$. The element displacement will, as usual, be given by a listing of the nodal displacements, now totalling twelve:

$$\mathbf{a}^e = \begin{bmatrix} a_i \\ a_j \\ a_k \\ a_l \end{bmatrix}, \quad \mathbf{a}_i = \begin{bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{bmatrix}$$ (4.38)

A polynomial expression is conveniently used to define the shape functions in terms of the 12 parameters. Certain terms must be omitted from a complete fourth-order

* Note that we have changed here the convention from that of Fig. 4.3 in this chapter. This allows transformations needed for shells to be carried out in an easier manner. However, when manipulating the equations of Chapter 5 we shall return to the original definitions of Fig. 4.3. Similar difficulties are discussed by Hughes,46 and a simple transformation is as follows:

$$\mathbf{\theta} = \mathbf{T}\theta \quad \text{where} \quad \mathbf{T} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
polynomial. Writing
\[ w = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 + \alpha_{11} x y^3 + \alpha_{12} y^3 \]
\[ \equiv \mathbf{Pa} \] (4.39)

has certain advantages. In particular, along any \( x \) constant or \( y \) constant line, the displacement \( w \) will vary as a cubic. The element boundaries or interfaces are composed of such lines. As a cubic is uniquely defined by four constants, the two end values of slopes and the two displacements at the ends will therefore define the displacements along the boundaries uniquely. As such end values are common to adjacent elements continuity of \( w \) will be imposed along any interface.

It will be observed that the gradient of \( w \) normal to any of the boundaries also varies along it in a cubic way. (Consider, for instance, values of the normal \( \partial w / \partial y \) along a line on which \( x \) is constant.) As on such lines only two values of the normal slope are defined, the cubic is not specified uniquely and, in general, a discontinuity of normal slope will occur. The function is thus 'non-conforming'.

The constants \( \alpha_1 \) to \( \alpha_{12} \) can be evaluated by writing down the 12 simultaneous equations linking the values of \( w \) and its slopes at the nodes when the coordinates take their appropriate values. For instance,
\[ w_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \cdots \]
\[ \left( \frac{\partial w}{\partial y} \right)_i = \hat{\theta}_{yi} = \alpha_3 + \alpha_5 x_i + \cdots \]
\[ - \left( \frac{\partial w}{\partial x} \right)_i = \hat{\theta}_{xi} = -\alpha_2 - \alpha_5 y_i - \cdots \]

Listing all 12 equations, we can write, in matrix form,
\[ \mathbf{a}^e = \mathbf{Ca} \] (4.40)
where $C$ is a $12 \times 12$ matrix depending on nodal coordinates, and $\mathbf{a}$ is a vector of the 12 unknown constants. Inverting we have

$$\mathbf{a} = C^{-1} \mathbf{a}^c$$  \hspace{1cm} (4.41)

This inversion can be carried out by the computer or, if an explicit expression for the stiffness, etc., is desired, it can be performed algebraically. This was in fact done by Zienkiewicz and Cheung.\textsuperscript{26}

It is now possible to write the expression for the displacement within the element in a standard form as

$$\mathbf{u} \equiv w = \mathbf{N} \mathbf{a}^c = \mathbf{P} C^{-1} \mathbf{a}^c$$  \hspace{1cm} (4.42)

where

$$\mathbf{P} = (1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, xy^3)$$

The form of the $\mathbf{B}$ is obtained directly from Eqs (4.28) and (4.33). We thus have

$$\mathbf{L} \nabla \mathbf{w} = \begin{bmatrix} +2\alpha_4 & +6\alpha_7 x & +2\alpha_8 y + 6\alpha_{11} xy \\ +2\alpha_6 & +2\alpha_9 x & +6\alpha_{10} y + 6\alpha_{12} xy \\ +2\alpha_5 & +4\alpha_8 x & +4\alpha_9 y + 6\alpha_{11} x^2 + 6\alpha_{12} y^2 \end{bmatrix}$$

We can write the above as

$$\mathbf{L} \nabla \mathbf{w} = \mathbf{Q} \mathbf{a} = \mathbf{Q} C^{-1} \mathbf{a}^c = \mathbf{B} \mathbf{a}^c$$  \hspace{1cm} and thus $\mathbf{B} = \mathbf{Q} C^{-1}$  \hspace{1cm} (4.43)

in which

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 & 0 & 6xy & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & 6y & 0 & 6xy \\ 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 & 6x^2 & 6y^2 \end{bmatrix}$$  \hspace{1cm} (4.44)

It is of interest to remark now that the displacement function chosen does in fact permit a state of constant strain (curvature) to exist and therefore satisfies one of the criteria of convergence stated in Volume 1.\textsuperscript{*}

An explicit form of the shape function $\mathbf{N}$ was derived by Melosh\textsuperscript{36} and can be written simply in terms of normalized coordinates. Thus, we can write for any node

$$\mathbf{N}_i^T = \frac{1}{8} (1 + \xi_0)(1 + \eta_0) \begin{bmatrix} 2 + \xi_0 + \eta_0 - \xi_0^2 - \eta_0^2 \\ b \eta_i(1 - \eta_0^2) \\ -a \xi_i(1 - \xi_0^2) \end{bmatrix}$$  \hspace{1cm} (4.45)

with normalized coordinates defined as:

$$\xi = \frac{x - x_c}{a}$$  \hspace{1cm} where $\xi_0 = \xi \xi_i$,

$$\eta = \frac{y - y_c}{b}$$  \hspace{1cm} where $\eta_0 = \eta \eta_i$.

\textsuperscript{*} If $\alpha_7$ to $\alpha_{12}$ are zero, then the ‘strain’ defined by second derivatives is constant. By Eq. (4.40), the corresponding $\mathbf{a}^c$ can be found. As there is a unique correspondence between $\mathbf{a}^c$ and $\mathbf{a}$ such a state is therefore unique. All this presumes that $C^{-1}$ does in fact exist. The algebraic inversion shows that the matrix $C$ is never singular.
This form avoids the explicit inversion of $C$; however, for simplicity we pursue the direct use of polynomials to deduce the stiffness and load matrices.

### 4.3.2 Stiffness and load matrices

Standard procedures can now be followed, and it is almost superfluous to recount the details. The stiffness matrix relating the nodal forces (given by lateral force and two moments at each node) to the corresponding nodal displacement is

$$
K^e = \int_{x_1}^{x_2} B^T D B \, dx \, dy
$$

or, substituting Eq. (4.43) into this expression,

$$
K^e = C^{-T} \left( \int_{-b}^{b} \int_{-a}^{a} Q^T DQ \, dx \, dy \right) C^{-1}
$$

The terms not containing $x$ and $y$ have now been moved from the operation of integrating. The term within the integral sign can be multiplied out and integrated explicitly without difficulty if $D$ is constant.

The external forces at nodes arising from distributed loading can be assigned ‘by inspection’, allocating specific areas as contributing to any node. However, it is more logical and accurate to use once again the standard expression (4.31) for such an allocation.

The contribution of these forces to each of the nodes is

$$
f_i = \begin{cases} f_{w_i} \\ f_{\theta_{x_i}} \\ f_{\theta_{y_i}} \end{cases} = \int_{-b}^{b} \int_{-a}^{a} N^T q \, dx \, dy
$$

or, by Eq. (4.42),

$$
f_i = -C^{-T} \int_{-b}^{b} \int_{-a}^{a} P^T q \, dx \, dy
$$

The integral is again evaluated simply. It will now be noted that, in general, all three components of external force at any node will have non-zero values. This is a result that the simple allocation of external loads would have missed. The nodal load vector for uniform loading $q$ is given by

$$
f_1 = \frac{1}{12} q ab \begin{cases} 3 \\ b \\ -a \end{cases}, \quad f_2 = \frac{1}{12} q ab \begin{cases} 3 \\ -b \\ -a \end{cases}, \quad f_3 = \frac{1}{12} q ab \begin{cases} 3 \\ b \\ a \end{cases}, \quad f_4 = \frac{1}{12} q ab \begin{cases} 3 \\ -b \\ a \end{cases}
$$

The vector of nodal plate forces due to initial strains and initial stresses can be found in a similar way. It is necessary to remark in this connection that initial strains, such as may be due to a temperature rise, is seldom confined in its effects on curvatures. Usually, direct (in-plane) strains in the plate are introduced additionally, and the complete problem can be solved only by consideration of the plane stress problem as well as that of bending.
4.4 Quadrilateral and parallelogram elements

The rectangular element developed in the preceding section passes the patch test\textsuperscript{4} and is always convergent. However, it cannot be easily generalized into a quadrilateral shape. Transformation of coordinates of the type described in Chapter 9 of Volume 1 can be performed but unfortunately now it will be found that the constant curvature criterion is violated. As expected, such elements behave badly but by arguments given in Chapter 9 of Volume 1 convergence may still occur providing the patch test is passed in the curvilinear coordinates. Henshell \textit{et al.}\textsuperscript{40} studied the performance of such an element (and also some of a higher order) and concluded that reasonable accuracy is attainable. Their paper gives all the details of transformations required for an isoparametric mapping and the resulting need for numerical integration.

Only for the case of a parallelogram is it possible to achieve states of constant curvature exclusively using functions of $\xi$ and $\eta$ and the patch test is satisfied. For a parallelogram the local coordinates can be related to the global ones by the explicit expression (Fig. 4.8)

$$\xi = \frac{x - y \cot \alpha}{a}$$
$$\eta = \frac{y \csc \alpha}{b}$$

(4.51)

and all expressions for the stiffness and loads can therefore also be derived directly. Such an element is suggested in the discussion in reference 26, and the stiffness matrices have been worked out by Dawe.\textsuperscript{28} A somewhat different set of shape functions was suggested by Argyris.\textsuperscript{29}

![Fig. 4.8 Parallelogram element and skew coordinates.](image)

4.5 Triangular element with corner nodes (9 degrees of freedom)

At first sight, it would seem that once again a simple polynomial expansion could be used in a manner identical to that of the previous section. As only nine independent
movements are imposed, only nine terms of the expansion are permissible. Here an immediate difficulty arises as the full cubic expansion contains 10 terms [Eq. (4.39) with $\alpha_{11} = \alpha_{12} = 0$] and any omission has to be made arbitrarily. To retain a certain symmetry of appearance all 10 terms could be retained and two coefficients made equal (for example $\alpha_8 = \alpha_9$) to limit the number of unknowns to nine. Several such possibilities have been investigated but a further, much more serious, problem arises. The matrix corresponding to $C$ of Eq. (4.40) becomes singular for certain orientations of the triangle sides. This happens, for instance, when two sides of the triangle are parallel to the $x$ and $y$ axes respectively.

An 'obvious' alternative is to add a central node to the formulation and eliminate this by static condensation. This would allow a complete cubic to be used, but again it was found that an element derived on this basis does not converge to correct answers. Difficulties of asymmetry can be avoided by the use of area coordinates described in Sec. 8.8 of Volume 1. These are indeed nearly always a natural choice for triangles, see (Fig. 4.9).

### 4.5.1 Shape functions

As before we shall use polynomial expansion terms, and it is worth remarking that these are given in area coordinates in an unusual form. For instance,

$$\alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3$$

(4.52)

gives the three terms of a complete linear polynomial and

$$\alpha_1 L_1^2 + \alpha_2 L_2^2 + \alpha_3 L_3^2 + \alpha_4 L_1 L_2 + \alpha_5 L_2 L_3 + \alpha_6 L_3 L_1$$

(4.53)

gives all six terms of a quadratic (containing within it the linear terms).\(^*\) The 10 terms of a cubic expression are similarly formed by the products of all possible cubic

\(^*\) However, it is also possible to write a complete quadratic as

$$\alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 + \alpha_4 L_1 L_2 + \alpha_5 L_2 L_3 + \alpha_6 L_3 L_1$$
and so on, for higher orders. This has the advantage of explicitly stating all retained terms of polynomials of lower order.
Plate bending approximation

Fig. 4.10 Some basic functions in area coordinate polynomials.

combinations, that is,

\[ L_1, L_2, L_3, L_1^2L_2, L_1L_3, L_2L_3, L_2^2L_1, L_2L_1, L_1^2L_2, L_1L_2L_3 \]  

(4.54)

For a 9 degree-of-freedom element any of the above terms can be used in a suitable combination, remembering, however, that only nine independent functions are needed and that constant curvature states have to be obtained. Figure 4.10 shows some functions that are of importance. The first [Fig. 4.10(a)] gives one of three functions representing a simple, unstrained rotation of the plate. Obviously, these must be available to produce the rigid body modes. Further, functions of the type \( L_1^2L_2 \), of which there are six in the cubic expression, will be found to take up a form similar (though not identical) to Fig. 4.10(b).

The cubic function \( L_1L_2L_3 \) is shown in Fig 4.10(c), illustrating that this is a purely internal (bubble) mode with zero values and slopes at all three corner nodes (though slopes are not zero along edges). This function could thus be useful for a nodeless or internal variable but will not, in isolation, be used as it cannot be prescribed in terms of corner variables. It can, however, be added to any other basic shape in any proportion, as indicated in Fig. 4.10(b).
The functions of the second kind are of special interest. They have zero values of \( w \) at all corners and indeed always have zero slope in the direction of one side. A linear combination of two of these (for example \( L_2^2L_1 \) and \( L_2^3L_3 \)) are capable of providing any desired slopes in the \( x \) and \( y \) directions at one node while maintaining all other nodal slopes at zero.

For an element envisaged with 9 degrees of freedom we must ensure that all six quadratic terms are present. In addition we select three of the cubic terms. The quadratic terms ensure that a constant curvature, necessary for patch test satisfaction, is possible. Thus, the polynomials we consider are

\[
\mathbf{P} = [L_1, L_2, L_3, L_1L_2, L_2L_3, L_3L_1, L_1^2L_2, L_2^2L_3, L_3^2L_1]
\]

and we write the interpolation as

\[
w = \mathbf{P} \alpha
\]

where \( \alpha \) are parameters to be expressed in terms of nodal values. The nine nodal values are denoted as

\[
(\hat{w}_i, \quad \hat{\theta}_{xi}, \quad \hat{\theta}_{yi}) = \left( \hat{w}_i, \quad \frac{\partial \hat{w}_i}{\partial y}, \quad -\frac{\partial \hat{w}_i}{\partial x} \right); \quad i = 1, 2, 3
\]

Upon noting that

\[
\frac{\partial}{\partial x} \begin{bmatrix} \frac{\partial L_1}{\partial x} & \frac{\partial L_2}{\partial x} & \frac{\partial L_3}{\partial x} \\ \frac{\partial L_1}{\partial y} & \frac{\partial L_2}{\partial y} & \frac{\partial L_3}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial L_1} \\ \frac{\partial}{\partial L_2} \\ \frac{\partial}{\partial L_3} \end{bmatrix}
\]

where

\[
2\Delta = b_1c_2 - b_2c_1
\]

\[
b_i = y_j - y_k
\]

\[
c_i = x_k - x_j
\]

with \( i, j, k \) a cyclic permutation of indices (see Chapter 9 of Volume 1), we now determine the shape function by a suitable inversion [see Sec. 4.3.1, Eq. (4.42)], and write for node \( i \)

\[
\mathbf{N}_i^T = \left\{ \begin{array}{c} 3L_i^2 - 2L_i^3 \\ L_i^2 (b_iL_k - b_kL_i) + \frac{1}{2} (b_i - b_k)L_1L_2L_3 \\ L_i^2 (c_iL_k - c_kL_i) + \frac{1}{2} (c_i - c_k)L_1L_2L_3 \end{array} \right\}
\]

Here the term \( L_1L_2L_3 \) is added to permit constant curvature states.

The computation of stiffness and load matrices can again follow the standard patterns, and integration of expressions (4.30) and (4.31) can be done exactly using the general integrals given in Fig. 4.9. However, numerical quadrature is generally used and proves equally efficient (see Chapter 9 of Volume 1). The stiffness matrix requires computation of second derivatives of shape functions and these may be
conveniently obtained from
\[
\begin{bmatrix}
\frac{\partial^2 N_i}{\partial x^2} & \frac{\partial^2 N_i}{\partial x \partial y} \\
\frac{\partial^2 N_i}{\partial y \partial x} & \frac{\partial^2 N_i}{\partial y^2}
\end{bmatrix} = \frac{1}{4\Delta^2} \begin{bmatrix}
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{bmatrix} \begin{bmatrix}
\frac{\partial^2 N_i}{\partial L_1^2} & \frac{\partial^2 N_i}{\partial L_1 \partial L_2} & \frac{\partial^2 N_i}{\partial L_1 \partial L_3} \\
\frac{\partial^2 N_i}{\partial L_2 \partial L_1} & \frac{\partial^2 N_i}{\partial L_2^2} & \frac{\partial^2 N_i}{\partial L_2 \partial L_3} \\
\frac{\partial^2 N_i}{\partial L_3 \partial L_1} & \frac{\partial^2 N_i}{\partial L_3 \partial L_2} & \frac{\partial^2 N_i}{\partial L_3^2}
\end{bmatrix} \begin{bmatrix}
b_1 & c_1 \\
b_2 & c_2 \\
b_3 & c_3
\end{bmatrix}
\]
(4.58)

in which \(N_i\) denotes any of the shape functions given in Eq. (4.57).

The element just derived is one first developed in reference 11. Although it satisfies the constant strain criterion (being able to produce constant curvature states) it unfortunately does not pass the patch test for arbitrary mesh configurations. Indeed, this was pointed out in the original reference (which also was the one in which the patch test was mentioned for the first time). However, the patch test is fully satisfied with this element for meshes of triangles created by three sets of equally spaced straight lines. In general, the performance of the element, despite this shortcoming, made the element quite popular in practical applications.

It is possible to amend the element shape functions so that the resulting element passes the patch test in all configurations. An early approach was presented by Kikuchi and Ando by replacing boundary integral terms in the virtual work statement of Eq. (4.26) by
\[
\delta \Pi_{\text{ext}} = \int_{\Omega} \delta w q \, d\Omega + \sum_e \delta \left[ \int_{\Gamma_e} \left( \frac{\partial w}{\partial n} - \frac{\partial \bar{w}}{\partial n} \right) M_n(w) \, d\Gamma \right] \\
+ \int_{\Gamma_e} \left[ \delta w S_n + \frac{\partial \delta w}{\partial s} \bar{M}_{ns} \right] d\Gamma + \int_{\Gamma_n} \frac{\partial \delta \bar{w}}{\partial n} \bar{M}_n \, d\Gamma
\]
(4.59)
in which, \(\Gamma_e\) is the boundary of each element \(e\), \(M_n(w)\) is the normal moment computed from second derivatives of the \(w\) interpolation, and \(s\) is the tangent direction along the element boundaries. The interpolations given by Eq. (4.57) are \(C_0\) conforming and have slopes which match those of adjacent elements at nodes. To correct the slope incompatibility between nodes, a simple interpolation is introduced along each element boundary segment as
\[
\frac{\partial \bar{w}}{\partial n} = \left( 1 - s' \right) \left[ \frac{\partial w}{\partial x} \bigg|_{j} m_j + \frac{\partial w}{\partial y} \bigg|_{j} n_j \right] + s' \left[ \frac{\partial w}{\partial x} \bigg|_{k} m_k + \frac{\partial w}{\partial y} \bigg|_{k} n_k \right]
\]
(4.60)
where \(s'\) is 0 at node \(j\) and 1 at node \(k\), and \(m_i\) and \(n_i\) are direction cosines with respect to the \(x\) and \(y\) axes, respectively. The above modification requires boundary integrals in addition to the usual area integrals; however, the final result is one which passes the patch test.

Bergen and Samuelsson also show a way of producing elements which pass the patch test, but a successful modification useful for general application with elastic and inelastic material behaviour is one derived by Specht. This modification uses three fourth-order terms in place of the three cubic terms of the equation preceding
Triangular element of the simplest form (6 degrees of freedom)

Eq. (4.55). The particular form of these is so designed that the patch test criterion which we shall discuss in detail later in Sec. 4.7 is identically satisfied. We consider now the nine polynomial functions given by

\[ P = [L_1, L_2, L_3, L_1L_2, L_2L_3, L_3L_1, \]

\[ L_1^2L_2 + \frac{1}{2} L_1L_2L_3 \{3(1 - \mu_3)L_1 - (1 + 3\mu_3)L_2 + (1 + 3\mu_3)L_3\}, \]

\[ L_2^2L_3 + \frac{1}{2} L_1L_2L_3 \{3(1 - \mu_1)L_2 - (1 + 3\mu_1)L_3 + (1 + 3\mu_1)L_1\}, \]

\[ L_3^2L_1 + \frac{1}{2} L_1L_2L_3 \{3(1 - \mu_2)L_3 - (1 + 3\mu_2)L_1 + (1 + 3\mu_2)L_2\} \quad (4.61) \]

where

\[ \mu_i = \frac{l_i^2 - l_j^2}{l_i^2} \quad (4.62) \]

and \( l_i \) is the length of the triangle side opposite node \( i \).\(^*\)

The modified interpolation for \( w \) is taken as

\[ w = Pa \quad (4.63) \]

and, on identification of nodal values and inversion, the shape functions can be written explicitly in terms of the components of the vector \( P \) defined by Eq. (4.61) as

\[ N_i^T = \begin{pmatrix} \frac{P_i - P_{i+3} + P_{k+3} + 2(P_{i+6} - P_{k+6})}{b_j(P_{k+6} - P_{k+3}) - b_k P_{i+6}} \\ -c_j(P_{k+6} - P_{k+3}) - c_k P_{i+6} \end{pmatrix} \quad (4.64) \]

where \( i, j, k \) are the cyclic permutations of \( 1, 2, 3 \).

Once again, stiffness and load matrices can be determined either explicitly or using numerical quadrature. The element derived above passes all the patch tests and performs excellently.\(^4\) Indeed, if the quadrature is carried out in a ‘reduced’ manner using three quadrature points (see Volume 1, Table 9.2 of Sec. 9.11) then the element is one of the best triangles with 9 degrees of freedom that is currently available, as we shall show in the section dealing with numerical comparisons.

### 4.6 Triangular element of the simplest form (6 degrees of freedom)

If conformity at nodes (\( C_0 \) continuity) is to be abandoned, it is possible to introduce even simpler elements than those already described by reducing the element inter-connections. A very simple element of this type was first proposed by Morley.\(^3\) In this element, illustrated in Fig. 4.11, the interconnections require continuity of the displacement \( w \) at the triangle vertices and of normal slopes at the element mid-sides.

\* The constants \( \mu_i \) are geometric parameters occurring in the expression for normal derivatives. Thus on side \( l_i \) the normal derivative is given by

\[ \frac{\partial}{\partial n} = \frac{l_i}{4\Delta} \left[ \frac{\partial}{\partial L_j} + \frac{\partial}{\partial L_k} - 2 \frac{\partial}{\partial L_i} + \mu \left( \frac{\partial}{\partial L_k} - \frac{\partial}{\partial L_j} \right) \right] \]
With 6 degrees of freedom the expansion can be limited to quadratic terms alone, which one can write as
\[ w = [L_1, \ L_2, \ L_3, \ L_1L_2, \ L_2L_3, \ L_3L_1]a \] (4.65)

Identification of nodal variables and inversion leads to the following shape functions:

for corner nodes
\[ N_i = L_i - L_i(1 - L_i) - \frac{b_i b_k - c_i c_k}{b_j^2 + c_j^2} L_j(1 - L_j) - \frac{b_i b_j - c_i c_j}{b_k^2 + c_k^2} L_k(1 - L_3) \] (4.66)

and for ‘normal gradient’ nodes
\[ N_{i+3} = \frac{2\Delta}{\sqrt{b_i^2 + c_i^2}} L_i(1 - L_i) \] (4.67)

where the symbols are identical to those used in Eq. (4.56) and \( i, j, k \) are a cyclic permutation of 1, 2, 3.

Establishment of stiffness and load matrices follows the standard pattern and we find that once again the element passes fully all the patch tests required. This simple element performs reasonably, as we shall show later, though its accuracy is, of course, less than that of the preceding ones.

It is of interest to remark that the moment field described by the element satisfies exactly interelement equilibrium conditions on the normal moment \( M_n \), as the reader can verify. Indeed, originally this element was derived as an equilibrating one using the complementary energy principle, and for this reason it always gives an upper bound on the strain energy of flexure. This is the simplest possible element as it simply represents the minimum requirements of a constant moment field. An explicit form of stiffness routines for this element is given by Wood.

4.7 The patch test – an analytical requirement

The patch test in its different forms (discussed fully in Chapters 10 and 11 of Volume 1) is generally applied numerically to test the final form of an element. However, the basic requirements for its satisfaction by shape functions that violate compatibility can be forecast accurately if certain conditions are satisfied in the choice of such functions. These conditions follow from the requirement that for constant strain states the virtual work done by internal forces acting at the discontinuity must be zero. Thus if the
The patch test – an analytical requirement

Trawctions acting on an element interface of a plate are (see Fig. 4.4)

\[ M_{nt}, \quad M_{ns}, \quad \text{and} \quad S_n \]  

(4.68)

and if the corresponding mismatch of virtual displacements are

\[ \Delta \theta_n = \Delta \left( \frac{\partial w}{\partial n} \right), \quad \Delta \theta_s = \Delta \left( \frac{\partial w}{\partial s} \right) \quad \text{and} \quad \Delta w \]

(4.69)

then ideally we would like the integral given below to be zero, as indicated, at least for the constant stress states:

\[
\int_{\Gamma_v} M_n \Delta \theta_n \, d\Gamma + \int_{\Gamma_v} M_{ns} \Delta \theta_s \, d\Gamma + \int_{\Gamma_v} S_n \Delta w \, d\Gamma = 0 \quad (4.70)
\]

The last term will always be zero identically for constant \( M_n, M_{ns}, M_{xy} \) fields as then \( S_n = S_s = 0 \) [in the absence of applied couples, see Eq. (4.18)] and we can ensure the satisfaction of the remaining conditions if

\[
\int_{\Gamma_v} \Delta \theta_n \, d\Gamma = 0 \quad \text{and} \quad \int_{\Gamma_v} \Delta \theta_s \, d\Gamma = 0 \quad (4.71)
\]

is satisfied for each straight side \( \Gamma_v \) of the element.

For elements joining at vertices where \( \partial w/\partial n \) is prescribed, these integrals will be identically zero only if anti-symmetric cubic terms arise in the departure from linearity and a quadratic variation of normal gradients is absent, as shown in Fig. 4.12(a). This is the motivation for the rather special form of shape function basis chosen to describe the incompatible triangle in Eq. (4.61), and here the first condition of Eq. (4.71) is automatically satisfied. The satisfaction of the second condition of Eq. (4.71) is always ensured if the function \( w \) and its first derivatives are prescribed at the corner nodes.

For the purely quadratic triangle of Sec. 4.6 the situation is even simpler. Here the gradients can only be linear, and if their value is prescribed at the element mid-side as shown in Fig. 4.11(b) the integral is identically zero.

The same arguments apparently fail when the rectangular element with the function basis given in Eq. (4.42) is examined. However, the reader can verify by direct

![Fig. 4.12 Continuity condition for satisfaction of patch test \[ \int (\partial w/\partial n) \, ds = 0 \]; variation of \( \partial w/\partial n \) along side. (a) Definition by corner nodes (linear component compatible); (b) definition by one central node (constant component compatible).](image-url)
Plate bending approximation

Fig. 4.13 A square plate with clamped edges; uniform load \( q \), square elements.

### Table 4.1 Computed central deflection of a square plate for several meshes (rectangular elements)

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Total number of nodes</th>
<th>Simply supported plate</th>
<th>Clamped plate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \alpha^* )</td>
<td>( \beta^i )</td>
</tr>
<tr>
<td>2 x 2</td>
<td>9</td>
<td>0.003446</td>
<td>0.001480</td>
</tr>
<tr>
<td>4 x 4</td>
<td>25</td>
<td>0.003939</td>
<td>0.0012327</td>
</tr>
<tr>
<td>8 x 8</td>
<td>81</td>
<td>0.004033</td>
<td>0.0021929</td>
</tr>
<tr>
<td>16 x 16</td>
<td>169</td>
<td>0.004050</td>
<td>0.001715</td>
</tr>
<tr>
<td>Series (Timoshenko)</td>
<td>169</td>
<td>0.004062</td>
<td>0.001160</td>
</tr>
</tbody>
</table>

* \( w_{max} = qL^4/D \) for uniformly distributed load \( q \). \( w_{max} = 3PL^2/D \) for central concentrated load \( P \). Note: Subdivision of whole plate given for mesh.

### Table 4.2 Corner supported square plate

<table>
<thead>
<tr>
<th>Method</th>
<th>Mesh</th>
<th>Point 1</th>
<th>Point 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( w )</td>
<td>( M_x )</td>
</tr>
<tr>
<td>Finite element</td>
<td>2 x 2</td>
<td>0.0126</td>
<td>0.139</td>
</tr>
<tr>
<td></td>
<td>4 x 4</td>
<td>0.0165</td>
<td>0.149</td>
</tr>
<tr>
<td></td>
<td>6 x 6</td>
<td>0.0173</td>
<td>0.150</td>
</tr>
<tr>
<td>Marcus</td>
<td>2 x 2</td>
<td>0.0180</td>
<td>0.154</td>
</tr>
<tr>
<td></td>
<td>4 x 4</td>
<td>0.0170</td>
<td>0.140</td>
</tr>
<tr>
<td>Ballesteros and Lee</td>
<td>6 x 6</td>
<td>( qL^4/D )</td>
<td>( qL^2 )</td>
</tr>
<tr>
<td>Multiplier</td>
<td></td>
<td>( qL^4/D )</td>
<td>( qL^2 )</td>
</tr>
</tbody>
</table>

Note: point 1, centre of side; point 2, centre of plate.
Fig. 4.14 A skew, curved, bridge with beams and non-uniform thickness; plot of principal moments under dead load.
Plate bending approximation

algebra that the integrals of Eqs (4.71) are identically satisfied. Thus, for instance,
\[ \int_{-a}^{a} \frac{\partial w}{\partial y} \, dx = 0 \quad \text{when} \quad y = \pm b \]
and \( \partial w / \partial y \) is taken as zero at the two nodes (i.e. departure from prescribed linear variations only is considered).

The remarks of this section are verified in numerical tests and lead to an intelligent, \textit{a priori}, determination of conditions which make shape functions convergent for incompatible elements.

4.8 Numerical examples

The various plate bending elements already derived – and those to be derived in subsequent sections – have been used to solve some classical plate bending problems. We first give two specific illustrations and then follow these with a general convergence study of elements discussed.

Fig. 4.15 Castleton railway bridge: general geometry and details of finite element subdivision. (a) Typical actual section; (b) idealization and meshing.
Figure 4.13 shows the deflections and moments in a square plate clamped along its edges and solved by the use of the rectangular element derived in Sec. 4.3 and a uniform mesh. Table 4.1 gives numerical results for a set of similar examples solved with the same element, and Table 4.2 presents another square plate with more complex boundary conditions. Exact results are available here and comparisons are made.

Figures 4.14 and 4.15 show practical engineering applications to more complex shapes of slab bridges. In both examples the requirements of geometry necessitate the use of a triangular element – with that of reference 11 being used here. Further, in both examples, beams reinforce the slab edges and these are simply incorporated in the analysis on the assumption of concentric behaviour.

Finally in Fig. 4.16(a)–(d) we show the results of a convergence study of the square plate with simply supported and clamped edge conditions for various triangular and

![Diagram](image)

**Fig. 4.15** (Continued) Castleton railway bridge: general geometry and details of finite element subdivision. (c) moment components (ton ft $^2$) under uniform load of 150 lb $^2$ with computer plot of contours.
Plate bending approximation

(a) Triangular elements

(a) Rectangular elements
Numerical examples

(b) Triangular elements

Fig. 4.16 (a) Simply supported uniformly loaded square plate; (b) simply supported square plate with concentrated central load.
Plate bending approximation

(c) Triangular elements

(c) Rectangular elements
Fig. 4.16 (Continued) (c) clamped uniformly loaded square plate, (d) clamped square plate with concentrated central load. Percentage error in central displacement (see Table 4.3 for key).
Plate bending approximation

Table 4.3  List of elements for comparison of performance in Fig. 4.16: (a) 9 degree-of-freedom triangles; (b) 12 degree-of-freedom rectangles; (c) 16 degree-of-freedom rectangle

<table>
<thead>
<tr>
<th>Code</th>
<th>Reference</th>
<th>Symbol</th>
<th>Description and comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>BCIZ 1</td>
<td>☐</td>
<td>Displacement, non-conforming (fails patch test)</td>
</tr>
<tr>
<td></td>
<td>Bazeley et al. 11</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>PAT</td>
<td>△</td>
<td>Displacement, non-conforming</td>
</tr>
<tr>
<td></td>
<td>Specht 99</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>BCIZ 2</td>
<td>□</td>
<td>Displacement, conforming</td>
</tr>
<tr>
<td></td>
<td>Bazeley et al. 11</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(HCT)</td>
<td>☐</td>
<td>Discrete Kirchhoff</td>
</tr>
<tr>
<td></td>
<td>Clough and Tocher 10</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>DKT</td>
<td>○</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Stricklin et al. 59 and Dhatt 60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>ACM</td>
<td>△</td>
<td>Displacement, non-conforming</td>
</tr>
<tr>
<td></td>
<td>Zienkiewicz and Cheung 26</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Q19</td>
<td>○</td>
<td>Displacement, conforming</td>
</tr>
<tr>
<td></td>
<td>Clough and Felippa 13</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>DKQ</td>
<td>□</td>
<td>Displacement, conforming</td>
</tr>
<tr>
<td></td>
<td>Batoz and Ben Tohar 61</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>BF</td>
<td>☐</td>
<td>Displacement conforming</td>
</tr>
<tr>
<td></td>
<td>Bogner et al. 17</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

rectangular elements and two load types. This type of diagram is conventionally used for assessing the behaviour of various elements, and we show on it the performance of the elements already described as well as others to which we shall refer to later. Table 4.3 gives the key to the various element 'codes' which include elements yet to be described.55–58

![Graph](image)

**Fig. 4. 17** Rate of convergence in energy norm versus degree of freedom for three elements: the problem of a slightly skewed, simply supported plate (80°) with uniform mesh subdivision.7
Singular shape functions for the simple triangular element

The comparison singles out only one displacement and each plot uses the number of mesh divisions in a quarter of the plate as abscissa. It is therefore difficult to deduce the convergence rate and the performance of elements with multiple nodes. A more convenient plot gives the energy norm $||u||$, versus the number of degrees of freedom $N$ on a logarithmic scale. We show such a comparison for some elements in Fig. 4.17 for a problem of a slightly skewed, simply supported plate. It is of interest to observe that, owing to the singularity, both high- and low-order elements converge at almost identical rates (though, of course, the former give better overall accuracy). Different rates of convergence would, of course, be obtained if no singularity existed (see Chapter 14 of Volume 1).

Conforming shape functions with nodal singularities

4.9 General remarks

It has already been demonstrated in Sec. 4.3 that it is impossible to devise a simple polynomial function with only three nodal degrees of freedom that will be able to satisfy slope continuity requirements at all locations along element boundaries. The alternative of imposing curvature parameters at nodes has the disadvantage, however, of imposing excessive conditions of continuity (although we will investigate some of the elements that have been proposed from this class). Furthermore, it is desirable from many points of view to limit the nodal variables to three quantities only. These, with simple physical interpretation, allow the generalization of plate elements to shells to be easily interpreted also.

It is, however, possible to achieve $C_1$ continuity by provision of additional shape functions for which, in general, second-order derivatives have non-unique values at nodes. Providing the patch test conditions are satisfied, convergence is again assured.

Such shape functions will be discussed now in the context of triangular and quadrilateral elements. The simple rectangular shape will be omitted as it is a special case of the quadrilateral.

4.10 Singular shape functions for the simple triangular element

Consider for instance either of the following sets of functions:

\[
\varepsilon_{jk} = \frac{L_i L_j^2 L_k^2 (L_k - L_i)}{(L_i + L_j)(L_j + L_k)} \tag{4.72}
\]

or

\[
\varepsilon_{jk} = \frac{L_i L_j^2 L_k^2 (1 + L_i)}{(L_i + L_j)(L_j + L_k)} \tag{4.73}
\]

in which once again $i, j, k$ are a cyclic permutation of 1, 2, 3. Both have the property that along two sides ($i-j$ and $i-k$) of a triangle (Fig. 4.18) their values and the values