Inelastic and non-linear materials

3.1 Introduction

In Chapter 1 we presented a framework for solving general problems in solid mechanics. In this chapter we consider several classical models for describing the behaviour of engineering materials. Each model we describe is given in a strain-driven form in which a strain or strain increment obtained from each finite element solution step is used to compute the stress needed to evaluate the internal force, $\int B^T \sigma \, d\Omega$ as well as a tangent modulus matrix, or its approximation, for use in constructing the tangent stiffness matrix. Quite generally in the study of small deformation and inelastic materials (and indeed in some forms applied to large deformation) the strain (or strain rate) or the stress is assumed to split into an additive sum of parts. We can write this as

$$\varepsilon = \varepsilon^e + \varepsilon^i$$  \hspace{1cm} (3.1)

or

$$\sigma = \sigma^e + \sigma^i$$  \hspace{1cm} (3.2)

in which we shall generally assume that the elastic part is given by the linear model

$$\varepsilon^e = D^{-1} \sigma$$  \hspace{1cm} (3.3)

in which $D$ is the matrix of elastic moduli.

In the following sections we shall consider the problems of viscoelasticity, plasticity, and general creep in quite general form. By using these general types it is possible to present numerical solutions which accurately predict many physical phenomena. We begin with viscoelasticity, where we illustrate the manner in which we shall address the solution of problems given in a rate or differential form. This rate form of course assumes time dependence and all viscoelastic phenomena are indeed transient, with time playing an important part. We shall follow this section with a description of plasticity models in which times does not explicitly arise and the problems are time independent. However, we shall introduce for convenience a rate description of the behaviour. This is adopted to allow use of the same kind of algorithms for all forms discussed in this chapter.
Viscoelasticity – history dependence of deformation

3.2 Viscoelasticity – history dependence of deformation

Viscoelastic phenomena are characterized by the fact that the rate at which inelastic strains develop depends not only on the current state of stress and strain but, in general, on the full history of their development. Thus, to determine the increment of inelastic strain over a given time interval (or time step) it is necessary to know the state of stress and strain at all preceding times. In the computation process these can in fact be obtained and in principle the problem presents little theoretical difficulty. Practical limitations appear immediately, however, that each computation point must retain this history information – thus leading to very large storage demands. In the context of linear viscoelasticity, means of overcoming this limitation were introduced by Zienkiewicz et al.\textsuperscript{1} and White.\textsuperscript{2} Extensions to include thermal effects were also included in some of this early work.\textsuperscript{3} Further considerations which extend this approach are also discussed in earlier editions of this book.\textsuperscript{4,5}

3.2.1 Linear models for viscoelasticity

The representation of a constitutive equation for linear viscoelasticity may be given in the form of either a differential equation or an integral equation.\textsuperscript{6,7} In a differential model the constitutive equation may be written as a linear elastic part with an added series of partial strains $q$. Accordingly, we write

$$\sigma(t) = D_0 \varepsilon(t) + \sum_{m=1}^{M} D_m q^{(m)}(t)$$  \hspace{1cm} (3.4)

where for a linear model the partial stresses are solutions of the first-order differential equations

$$q^{(m)} + T_m q^{(m)} = \dot{\varepsilon}$$  \hspace{1cm} (3.5)

with $T_m$ a constant matrix of reciprocal relaxation times and $D_0$, $D_m$ constant moduli matrices. The presence of a split of stress as given by Eq. (3.2) is immediately evident in the above. Each of the forms in Eq. (3.5) represents an elastic response in series with a viscous response and is known as a Maxwell model. In terms of a spring–dashpot model, a representation for the Maxwell material is shown in Fig. 3.1(a) for a single stress component. Thus, the sum given by Eq. (3.4) describes a generalized Maxwell solid in which several elements are assembled in a parallel form and the $D_0$ term becomes a spring alone.

In an integral form the stress–strain behaviour may be written in a convolution form as

$$\sigma = D(t)\varepsilon(0) + \int_{0}^{t} D(t - t') \frac{\partial \varepsilon}{\partial t'} \, dt'$$  \hspace{1cm} (3.6)

where components of $D(t)$ are relaxation moduli functions.

Inverse relations may be given where the differential model is expressed as

$$\varepsilon(t) = J_0 \sigma(t) + \sum_{m=1}^{M} J_m r^{(m)}(t)$$  \hspace{1cm} (3.7)
where for a linear model the partial stresses $\mathbf{\tau}$ are solutions of
\[
\dot{\mathbf{\tau}}^{(m)} + \mathbf{V}_m^\dagger \mathbf{\tau}^{(m)} = \mathbf{\sigma}
\]
where $\mathbf{V}_m$ are constant reciprocal retardation time parameters and $\mathbf{J}_0$, $\mathbf{J}_m$ constant compliances (i.e. reciprocal moduli). Each partial stress corresponds to a solution in which a linear elastic and a viscous response are combined in parallel to describe a Kelvin model as shown in Fig. 3.1(b). The total model thus is a generalized Kelvin solid.

In an integral form the strain–stress constitutive relation may be written as
\[
\epsilon = \mathbf{J}(t)\mathbf{\sigma}(0) + \int_0^t \mathbf{J}(t - t') \frac{\partial \mathbf{\sigma}}{\partial t'} \, dt'
\]
where $\mathbf{J}(t)$ are known as creep compliance functions.

The parameters in the two forms of the model are related. For example, the creep compliances and relaxation moduli are related through
\[
\mathbf{J}(t) \mathbf{D}(0) + \int_0^t \mathbf{J}(t - t') \frac{\partial \mathbf{D}}{\partial t'} \, dt' = \mathbf{D}(t) \mathbf{J}(0) + \int_0^t \mathbf{D}(t - t') \frac{\partial \mathbf{J}}{\partial t'} \, dt' = \mathbf{I}
\]
as may easily be shown by applying, for example, Laplace transform theory to Eqs (3.6) and (3.9).

The above forms hold for isotropic and anisotropic linear viscoelastic materials. Solutions may be obtained by using standard numerical techniques to solve the constant coefficient differential or integral equations. Here we will proceed to describe a solution for the isotropic case where specific numerical schemes are presented. Generalization of the methods to the anisotropic case may be constructed by using a similar approach and is left as an exercise to the reader.

### 3.2.2 Isotropic models

To describe in more detail the ideas presented above we consider here isotropic models where we split the stress as
\[
\mathbf{\sigma} = \mathbf{s} + \mathbf{m} p \quad \text{with} \quad p = \frac{1}{2} \mathbf{m}^T \mathbf{\sigma}
\]
Viscoelasticity — history dependence of deformation

where \( s \) is the stress deviator,\(^*\) \( p \) is the mean (pressure) stress and, for a three-dimensional state of stress, \( m \) is given in Eq. (1.37). Similarly, a split of strain is expressed as

\[
\varepsilon = e + \frac{1}{3} m \theta \quad \text{with} \quad \theta = m^T e
\]

(3.12)

where \( e \) is the strain deviator and \( \theta \) is the volume change.

In the presentation given here, for simplicity we restrict the viscoelastic response to deviatoric parts and assume pressure–volume response is given by the linear elastic model

\[
p = K \theta
\]

(3.13)

where \( K \) is an elastic bulk modulus. A generalization to include viscoelastic behaviour in this component also may be easily performed by using the method described below for deviatoric components.

**Differential equation model**

The deviatoric part may be stated as differential equation models or in the form of integral equations as described above. In the differential equation model the constitutive equation may be written as

\[
s = 2G \left( \mu_0 e + \sum_{m=1}^{M} \mu_m q^{(m)} \right)
\]

(3.14)

in which \( \mu_m \) are dimensionless parameters satisfying

\[
\sum_{m=0}^{M} \mu_m = 1
\]

(3.15)

and dimensionless partial deviatoric strains \( q^{(m)} \) are obtained by solving

\[
q^{(m)} + \frac{1}{\lambda_m} q^{(m)} = \dot{e}
\]

(3.16)

in which \( \lambda_m \) are relaxation times. This form of the representation is again a generalized Maxwell model (a set of Maxwell models in parallel).

Each differential equation set may be solved numerically by using any of the finite-element-in-time methods described in Chapter 18 of Volume 1 (see Sec. 18.2). To solve numerically we first define a set of discrete points, \( t_k \), at which we wish to obtain the solution. For a time \( t_{n+1} \) we assume the solution at all previous points up to \( t_n \) are known. Using a simple single-step method the solution for each partial stress is given by:

\[
\left( 1 + \frac{\theta \Delta t}{\lambda_m} \right) q^{(m)}_{n+1} = \left( 1 - \frac{(1 - \theta) \Delta t}{\lambda_m} \right) q^{(m)}_n + e_{n+1} - e_n
\]

(3.17)

in which \( \Delta t = t_{n+1} - t_n \).

\(^*\) In Volume 1 \( \sigma^d \) was used to denote the deviatoric stress, and \( \varepsilon^d \) the deviatoric strain. Here we use the alternate notation \( s \) and \( e \) to avoid the extra superscript \( d \).
We note that this form of the solution is given directly in a strain-driven form. Accordingly, given the strain from any finite element solution step we can immediately compute the stresses by using Eqs (3.13), (3.14) and (3.17) in Eqs (3.11) and (3.12). Inserting the above into a Newton-type solution strategy requires the computation of the tangent moduli. The tangent moduli for the viscoelastic model are deduced from

\[
\mathbf{K}_T|_{n+1} = \frac{\partial \mathbf{s}_{n+1}}{\partial \mathbf{e}_{n+1}} = \frac{\partial \mathbf{s}_{n+1}}{\partial \mathbf{e}_{n+1}} + \mathbf{m} \frac{\partial p_{n+1}}{\partial e_{n+1}} \quad (3.18)
\]

The tangent part for the volumetric term is elastic and given by

\[
\mathbf{m} \frac{\partial p_{n+1}}{\partial e_{n+1}} = \mathbf{m} \frac{\partial p_{n+1}}{\partial e_{n+1}} \frac{\partial \theta_{n+1}}{\partial e_{n+1}} = K \mathbf{m} \mathbf{m}^T \quad (3.19)
\]

Similarly, the tangent part for the deviatoric term is deduced from Eq. (3.17) as

\[
\frac{\partial \mathbf{s}_{n+1}}{\partial \mathbf{e}_{n+1}} = 2G \left[ \mu_0 + \sum_{m=1}^{M} \frac{\mu_m}{1 + \frac{\theta \Delta t}{\lambda_m}} \right] \mathbf{I}_d \quad (3.20)
\]

where \( \mathbf{I}_d \) is defined in Eq. (1.37). Using the above, tangent moduli are expressed as

\[
\mathbf{K}_T|_{n+1} = K \mathbf{m} \mathbf{m}^T + 2G \left[ \mu_0 + \sum_{m=1}^{M} \frac{\mu_m}{1 + \frac{\theta \Delta t}{\lambda_m}} \right] \mathbf{I}_d \quad (3.21)
\]

and we note that the only difference from a linear elastic material is the replacement of the elastic shear modulus by the viscoelastic term

\[
G \rightarrow G \left[ \mu_0 + \sum_{m=1}^{M} \frac{\mu_m}{1 + \frac{\theta \Delta t}{\lambda_m}} \right]
\]

This relation is independent of stress and strain and hence when it is used with a Newton scheme it converges in one iteration (i.e. the residual of a second iteration is numerically zero).

The set of first-order differential equations (3.16) may be integrated exactly for specified strains, \( \mathbf{e} \). The integral for each term is given by

\[
\mathbf{q}^{(n)}(t) = \int_{-\infty}^{t} \exp \left[ - (t - t')/\lambda_m \right] \frac{\partial \mathbf{e}}{\partial t'} \, dt'
\]

An advantage to the differential equation form, however, is that it may be extended to include ageing or other nonlinear effects by making the parameters time or solution dependent. The exact solution to the differential equations for such a situation will then involve integrating factors, leading to more involved expressions. In the following parts of this section we consider the integral equation form and its numerical solution for linear viscoelastic behaviour. Models and their solutions for more general cases are left as an exercise for the reader.
Integral equation model
The integral equation form for the deviatoric stresses is expressed in terms of a relaxation modulus function which is defined by an idealized experiment in which, at time zero \((t = 0)\), a specimen is subjected to suddenly applied and constant strain, \(e_0\), and the stress response, \(s(t)\), is measured. For a linear material a unique relation is obtained which is independent of the magnitude of the applied strain. This relation may be written as

\[ s(t) = 2G(t)e_0 \]  

(3.23)

where \(G(t)\) is defined as the shear relaxation modulus function. A typical relaxation function is shown in Fig. 3.2. The function is shown on a logarithmic time scale since typical materials have time effects which cover wide ranges in time.

Using linearity and superposition for an arbitrary state of strain yields the integral equation specified as

\[ s(t) = \int_{-\infty}^{t} 2G(t - t') \frac{\partial e}{\partial t'} \, dt' \]  

(3.24)

We note that the above form is a generalization to the Maxwell material. However, the integral equation form may be specialized to the generalized Maxwell model by assuming the shear relaxation modulus function in a Prony series form

\[ G(t) = G \left[ \mu_0 + \sum_{m=1}^{M} \mu_m \exp\left(-t/\lambda_m\right) \right] \]  

(3.25)

where the \(\mu_m\) satisfy Eq. (3.15).

Solution to integral equation with Prony series
The solution to the viscoelastic model is performed for a set of discrete points \(t_k\). Thus, again assuming that all solutions are available up to time \(t_n\), we desire to
compute the next step for time $t_{n+1}$. Solution of the general form would require summation over all previous time steps for each new time; however, by using the generalized Maxwell model we may reduce the solution to a recursion formula in which each new solution is computed by a simple update of the previous solution.

We will consider a special case of the generalized Maxwell material in which the number of terms $M$ is equal to 1 [which defines a standard linear solid, Fig. 3.3(a)]. The addition of more terms is easily performed from the one-term solution. Accordingly, we take

$$G(t) = G_0 + G_1 \exp\left(-\frac{t}{\lambda_1}\right)$$

where $G_0 + G_1 = 1$. For the standard solid only a limited range of time can be considered, as can be observed from Fig. 3.3(b) for the model given by

$$G(t) = G_0 \left[0.15 + 0.85 \exp(-t)\right]$$

To consider a wider range it is necessary to use terms in which the $\lambda_m$ cover the total time by using at least one term for each decade of time (a decade being one unit on the log10 time scale).

Substitution of Eq. (3.26) into Eq. (3.24) yields

$$s(t) = 2G \int_{-\infty}^{t} \left[\mu_0 + \mu_1 \exp\left(- (t - t')/\lambda_1\right)\right] \frac{\partial \varepsilon}{\partial t'} \, dt'$$

which may be split and expressed as

$$s(t) = 2G \mu_0 \varepsilon(t) + 2G \mu_1 \int_{-\infty}^{t} \exp\left(- (t - t')/\lambda_1\right) \frac{\partial \varepsilon}{\partial t'} \, dt'$$

$$= 2G \left[\mu_0 \varepsilon(t) + \mu_1 q^{(1)}(t)\right]$$

where we note that $q^{(1)}$ is identical to the form given in Eq. (3.22). Thus use of a Prony series for $G(t)$ is identical to solving the differential equation model exactly.
In applications involving a linear viscoelastic model, it is usually assumed that the material is undisturbed until a time identified as zero. At time zero a strain may be suddenly applied and then varied over subsequent time. To evaluate a solution at time \( t_{n+1} \) the integral representation for the model may be simplified by dividing the integral into

\[
\int_{-\infty}^{t_{n+1}} (\cdot) \, dt' = \int_{-\infty}^{0^-} (\cdot) \, dt' + \int_{0^-}^{0^+} (\cdot) \, dt' + \int_{0^+}^{t_n} (\cdot) \, dt' + \int_{t_n}^{t_{n+1}} (\cdot) \, dt'
\]

In each analysis considered here the material is assumed to be unstrained before the time denoted as zero. Thus, the first term on the right-hand side is zero, the second term includes a jump term associated with \( e_0 \) at time zero, and the last two terms cover the subsequent history of strain. The result of this separation when applied to Eq. (3.27) gives the recursion

\[
q_{n+1}^{(1)} = \exp(-\Delta t/\lambda_1) q_n^{(1)} + \Delta q^{(1)}
\]

where

\[
\Delta q^{(1)} = \int_{t_n}^{t_{n+1}} \exp[-(t_{n+1} - t')/\lambda_1] \frac{\partial e}{\partial t'} \, dt'
\]

and \( q_0^{(1)} = e_0 \).

To obtain a numerical solution, we approximate the strain rate in each time increment by a constant to obtain

\[
\Delta q_{n+1}^{(1)} = \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \exp[-(t_{n+1} - t')/\lambda_1][e_{n+1} - e_n] \, dt'
\]

The integral may now be evaluated directly over each time step as

\[
\Delta q_{n+1}^{(1)} = \frac{\lambda_1}{\Delta t} \left[1 - \exp(-\Delta t/\lambda_1)\right][e_{n+1} - e_n] = \Delta q_{n+1}^{(1)} (e_{n+1} - e_n)
\]

This approximation is singular for zero time steps; however, the limit value at \( \Delta t = 0 \) is one. Thus, for small time steps a series expansion may be used to yield accurate values, giving

\[
\Delta q_{n+1}^{(1)} = 1 - \frac{1}{2} \left(\frac{\Delta t}{\lambda_1}\right) + \frac{1}{3!} \left(\frac{\Delta t}{\lambda_1}\right)^2 - \frac{1}{4!} \left(\frac{\Delta t}{\lambda_1}\right)^3 + \cdots
\]

Using a few terms for very small time increment ratios yields numerically correct answers (to computer precision). Once the time increment ratio is larger than a certain small value the representation given in Eq. (3.33) is used directly.

The above form gives a recursion which is stable for small and large time steps and produces very smooth transitions under variable time steps.

A numerical approximation to Eq. (3.32) in which the integrand of Eq. (3.31) is evaluated at \( t_{n+1}/2 \) has also been used with success. In the above recursion we note that a zero and infinite value of a time step produces a correct instantaneous and zero response, respectively, and thus is asymptotically accurate at both limits. The use of finite difference approximations on the differential equation form directly does not produce this property unless \( \theta = 1 \) and for this value is much less accurate than the solution given by Eq. (3.33).
Inelastic and non-linear materials

Using the recursion formula, the constitutive equation now has the simple form

\[ s_{n+1} = 2G\left[\mu_0 e_{n+1} + \mu_1 q_{n+1}^{(1)}\right] \tag{3.35} \]

The process may also be extended to include effects of temperature on relaxation times for use with thermorheologically simple materials.\(^3\)

The implementation of the above viscoelastic model into a Newton type solution process again requires the computation of a tangent tensor. Accordingly, for the deviatoric part we need to compute

\[ \frac{\partial s_{n+1}}{\partial e_{n+1}} = \frac{\partial s_{n+1}}{\partial e_{n+1}} I_d \tag{3.36} \]

The partial derivative with respect to the deviatoric stress follows from Eq. (3.35) as

\[ \frac{\partial s}{\partial e} = 2G \left[\mu_0 I + \mu_1 \frac{\partial q^{(1)}}{\partial e}\right] \tag{3.37} \]

Using Eq. (3.33) the derivative of the last term becomes

\[ \frac{\partial q^{(1)}}{\partial e} = \Delta q^{(1)}(\Delta t) I \tag{3.38} \]

Thus, the tangent tensor is given by

\[ \frac{\partial s_{n+1}}{\partial e_{n+1}} = 2G \left[\mu_0 + \mu_1 \Delta q^{(1)}_{n+1}(\Delta t)\right] I_d \tag{3.39} \]

Again, the only modification from a linear elastic material is the substitution of the elastic shear modulus by

\[ G \rightarrow G_0 + \mu_1 \Delta q^{(1)}_{n+1}(\Delta t) \tag{3.40} \]

We note that for zero \( \Delta t \) the full elastic modulus is recovered, whereas for very large increments the equilibrium modulus \( G_0 \) is used. Since the material is linear, use of this tangent modulus term again leads to convergence in one iteration (the second iteration produces a numerically zero residual).

The inclusion of more terms in the series reduces to evaluation of additional integral recursions. Computer storage is needed to retain the \( q_{n+1}^{(m)} \) for each solution (quadrature) point in the problem and each term in the series.

**Example: a thick-walled cylinder subjected to internal pressure**

To illustrate the importance of proper element selection when performing analyses in which material behaviour approaches a near incompressible situation we consider the case of internal pressure on a thick-walled cylinder. The material is considered to be isotropic and modelled by viscoelastic response in deviatoric stress–strain only. Material properties are: modulus of elasticity, \( E = 1000 \); Poisson’s ratio, \( \nu = 0.3 \); \( \mu_1 = 0.99 \); and \( \lambda_1 = 1 \). Thus, the viscoelastic relaxation function is given by

\[ G(t) = \frac{1000}{2.6} \left[0.01 + 0.99 \exp(-t)\right] \]

The ratio of the bulk modulus to shear modulus for instantaneous loading is given by \( K/G(0) = 2.167 \) and for long time loading by \( K/G(\infty) = 216.7 \) which indicates a
near incompressible behaviour for sustained loading cases (the effective Poisson ratio for infinite time is 0.498). The response for a suddenly applied internal pressure, \( p = 10 \), is computed to time 20 by using both displacement and the mixed element described in Chapter 1. Quadrilateral elements with four nodes (Q4) and nine nodes (Q9) are considered, and meshes with equivalent nodal forces are shown in Fig. 3.4. The exact solution to this problem is one-dimensional and, since all radial boundary conditions are traction ones, the stress distribution should be time independent. During the early part of the solution, when the response is still in the compressible range, the solutions from the two formulations agree well with this exact solution. However, during the latter part of the solution the answers from a displacement element diverge because of near incompressibility effects, whereas those from a mixed element do not. The distribution of quadrature point radial stresses at time \( t = 20 \) is shown in Fig. 3.5 where the highly oscillatory response of the displacement form is clearly evident. We note that extrapolation to reduced quadrature points

![Fig. 3.4](image-url)  
Fig. 3.4 Mesh and loads for internal pressure on a thick-walled cylinder: (a) four-noded quadrilaterals; (b) nine-noded quadrilaterals.

![Fig. 3.5](image-url)  
Fig. 3.5 Radial stress for internal pressure on a thick-walled cylinder: (a) mixed model; (b) displacement model.
would avoid these oscillations; however, use of fully reduced integration would lead to singularity in the stiffness matrix (as shown in Volume 1) and selective reduced integration is difficult to use with general non-linear material behaviour. Thus, for general applications the use of mixed elements is preferred.

### 3.2.3 Solution by analogies

The labour of step-by-step solutions for linear viscoelastic media can, on occasion, be substantially reduced. In the case of a homogeneous structure with linear isotropic viscoelasticity and constant Poisson ratio operator, the McHenry–Alfrey analogies allow single-step elastic solutions to be used to obtain stresses and displacements at a given time by the use of equivalent loads, displacements and temperatures.\(^9,10\)

Some extensions of these analogies have been proposed by Hilton and Russell.\(^11\) Further, when subjected to steady loads and when strains tend to a constant value at an infinite time, it is possible to determine the final stress distribution even in cases where the above analogies are not applicable. Thus, for instance, where the viscoelastic properties are temperature dependent and the structure is subject to a system of loads and temperatures which remain constant with time, long-term 'equivalent' elastic constants can be found and the problem solved as a single, non-homogeneous elastic one.\(^12\)

The viscoelastic problem is a particular case of a creep phenomenon to which we shall return in Sect. 3.3 using some other classical non-linear models to represent material behaviour.

### 3.3 Classical time-independent plasticity theory

Classical 'plastic' behaviour of solids is characterized by a non-unique stress–strain relationship which is independent of the rate of loading but does depend on loading sequence that may be conveniently represented as a process evolving in time. Indeed, one definition of plasticity is the presence of irrecoverable strains on load removal. If uniaxial behaviour of a material is considered, as shown in Fig. 3.6(a), a non-linear relationship on loading alone does not determine whether non-linear elastic or plastic behaviour is exhibited. Unloading will immediately discover the difference, with an elastic material following the same path and a plastic material showing a history-dependent different path. We have referred to non-linearity elasticity already in Sect. 1.2 [see Eq. (1.36)] and will not give further attention to it here as the techniques used for plasticity problems or non-linear elasticity show great similarity. Representation of non-linear elastic behaviour for finite deformation applications is more complex as we shall show in Chapter 10.

Some materials show a nearly ideal plastic behaviour in which a limiting yield stress, \(Y \) (or \(\sigma_y\)), exists at which the strains are indeterminate. For all stresses below such yield, a linear (or non-linear) elastic relationship is assumed, Fig. 3.6(b) illustrates this. A further refinement of this model is one of a hardening/softening plastic material in which the yield stress depends on some parameter \(\kappa\) (such as the
accumulated plastic strain $\varepsilon''_p$ [Fig. 3.6(c)]. It is with such kinds of plasticity that this section is concerned and for which much theory has been developed.\textsuperscript{13,14}

In a multiaxial rather than a uniaxial state of stress the concept of yield needs to be generalized. It is important to note that in the following development of results in a matrix form all nine tensor components are used instead of the six ‘engineering’ component form used previously. To distinguish between the two we introduce an underbar on the symbol for all nine-component forms. Thus, we shall use:

\begin{align}
\mathbf{\sigma} &= \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \sigma_{xy} & \sigma_{xz} & \sigma_{zx} \end{bmatrix}^T \\
\bar{\mathbf{\sigma}} &= \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \sigma_{xy} & \sigma_{xz} & \sigma_{zx} \end{bmatrix}^T \\
\mathbf{\varepsilon} &= \begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{xy} & \gamma_{xz} \end{bmatrix}^T \\
\bar{\mathbf{\varepsilon}} &= \begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z & \varepsilon_{xy} & \varepsilon_{xz} \end{bmatrix}^T
\end{align}

(3.41)

in which $\gamma_{ij} = 2\varepsilon_{ij}$. The transformations between the nine- and six-component forms needed later are obtained by using

\begin{align}
\mathbf{\varepsilon} &= \mathbf{P}\bar{\mathbf{\varepsilon}} \quad \text{and} \quad \mathbf{\sigma} &= \mathbf{P}^T\bar{\mathbf{\sigma}}
\end{align}

(3.42)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3_6.png}
\caption{Uniaxial behaviour of materials: (a) non-linear elastic and plastic behaviour; (b) ideal plasticity; (c) strain hardening plasticity.}
\end{figure}
Inelastic and non-linear materials

where

\[ \mathbf{P}^T = \frac{1}{2} \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix} \]

Accordingly, we first make all computations by using the nine ‘tensor’ components of stress and strain and only at the end do we reduce the computations to expressions in terms of the six independent ‘engineering’ quantities using \( \mathbf{P} \). This will permit final expressions for strain and equilibrium to be written in terms of \( \mathbf{B} \) as in all previous developments. In addition we note that:

\[ \mathbf{P}^T \mathbf{I} \mathbf{P} = \mathbf{P}^T \mathbf{P} = \mathbf{I}_0 \quad \text{with} \quad \mathbf{I}_0 = \frac{1}{2} \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 1 \\
2 & 1 & 1
\end{bmatrix} \quad (3.43) \]

(see Section 12.2, Volume 1).

3.3.1 Yield functions

It is quite generally postulated, as an experimental fact, that yielding can occur only if the stress satisfies the general yield criterion

\[ F(\mathbf{\sigma}, \mathbf{\kappa}, \mathbf{\kappa}) = 0 \quad (3.44) \]

Fig. 3.7 Yield surface and normality criterion in two-dimensional stress space.
where $\sigma$ denotes a matrix form with all nine components of stress, $\kappa$ represents kinematic hardening parameters and $\kappa$ an isotropic hardening parameter. We shall discuss these particular sets of parameters later but, of course, many other types of parameters also can be used to define hardening.

This yield condition can be visualized as a surface in an $n$-dimensional space of stress with the position and size of the surface dependent on the instantaneous value of the parameters $\kappa$ and $\kappa$ (Fig. 3.7).

### 3.3.2 Flow rule (normality principle)

Von Mises first suggested that basic behaviour defining the plastic strain increments is related to the yield surface. Heuristic arguments for the validity of the relationship proposed have been given by various workers in the field and at the present time the following hypothesis appears to be generally accepted for many materials; if $\varepsilon^p$ denotes the components of the plastic strain tensor the rate of plastic strain is assumed to be given by

$$\dot{\varepsilon}^p = \dot{\lambda} F_{,\sigma}$$

(3.45)

where the notation

$$F_{,\sigma} \equiv \frac{\partial F}{\partial \sigma}$$

(3.46)

is introduced. In the above, $\dot{\lambda}$ is a proportionality constant, as yet undetermined, often referred to as the ‘plastic consistency’ parameter. During sustained plastic deformation we must have

$$\dot{F} = 0 \quad \text{and} \quad \dot{\lambda} > 0$$

(3.47)

whereas during elastic loading/unloading $\dot{\lambda} = 0$ and $\dot{F} \neq 0$ leading to a general constraint condition in Kuhn–Tucker form

$$\dot{F} \dot{\lambda} = 0$$

(3.48)

The above rule is known as the normality principle because relation (3.45) can be interpreted as requiring the plastic strain rate components to be normal to the yield surface in the space of nine stress and strain dimensions.

Restrictions of the above rule can be removed by specifying separately a plastic flow rule potential

$$Q = Q(\sigma, \kappa)$$

(3.49)

which defines the plastic strain rate similarly to Eq. (3.45); that is, giving this as

$$\dot{\varepsilon}^p = \dot{\lambda} Q_{,\sigma}, \quad \dot{\lambda} > 0$$

(3.50)

* Some authors prefer to write Eq. (3.45) in an incremental form

$$d\varepsilon^p = d\lambda F_{,\sigma}$$

where then $d\varepsilon^p = \dot{\varepsilon}^p dt$, and $t$ is some pseudo-time variable. Here we prefer the rate form to permit use of common solution algorithms in which $d\varepsilon$ will denote an increment in a Newton-type solution. (Also note the difference in notation between a small increment ‘$d$’ and a differential ‘$d$’.)
The particular case of \( Q = F \) is known as associative plasticity. When this relation is not satisfied the plasticity is non-associative. In what follows this more general form will be considered initially (reductions to the associative case follow by simple substitution of \( Q = F \)).

The satisfaction of the normality rule for the associative case is essential for proving so called upper and lower bound theorems of plasticity as well as uniqueness. In the non-associative case the upper and lower bound do not exist and indeed it is not certain that the solutions are always unique. This does not prevent the validity of non-associated rules as it is well known that in frictional materials, for instance, uniqueness is seldom achieved but the existence of friction cannot be denied.

### 3.3.3 Hardening/softening rules

**Isotropic hardening**

The parameters \( \kappa \) and \( \kappa \) must also be determined from rate equations and define hardening (or softening) of the plastic behaviour of the material. The evolution of \( \kappa \), govern the size of the yield surface is commonly related to the rate of plastic work or directly to the consistency parameter. If related to the rate of plastic work \( \kappa \) has dimensions of stress and a relation of the type

\[
\dot{\kappa} = \sigma^T \varepsilon_p = Y(\kappa) \dot{\varepsilon}_p^n
\]

is used to match behaviour to a uniaxial tension or compression result. The slope

\[
A = \frac{\partial Y}{\partial \kappa}
\]

provides a modulus defining instantaneous isotropic hardening.

In the second approach \( \kappa \) is dimensionless (e.g., an accumulated plastic strain\(^{14} \)) and is related directly to the consistency parameter using

\[
\dot{\kappa} = [(\varepsilon_p^T \varepsilon_p)^{1/2}] = \lambda[Q^T \sigma, Q]^{1/2}
\]

A constitutive equation is then introduced to match uniaxial results. For example, a simple linear form is given by

\[
\sigma_{j}(\kappa) = \sigma_{j0} + H_{0}\kappa
\]

where \( H_{0} \) is a constant isotropic hardening modulus.

**Kinematic hardening**

A classical procedure to represent kinematic hardening was introduced by Prager\(^{24} \) and modified by Ziegler.\(^{25} \) Here the stress in each yield surface is replaced by a linear relation in terms of a 'back stress' \( \mathbf{k} \) as

\[
\varepsilon' = \sigma - \kappa
\]

with the yield function now given as

\[
F(\sigma - \kappa, \kappa) = F(\varepsilon', \kappa) = 0
\]
Classical time-independent plasticity theory

during plastic behaviour. We note that with this approach derivatives of the yield surface differ only by a sign and are given by

\[ F_\frac{\partial}{\partial t} = F_\frac{\partial}{\partial \dot{x}} = -F_\frac{\partial}{\partial \dot{\sigma}} \]  

(3.56)

Accordingly, the yield surface will now translate, and if isotropic hardening is present will also expand or contract, during plastic loading.

A rate equation may be specified most directly by introducing a conjugate work variable \( \dot{\beta} \) from which the hardening parameter \( \kappa \) is deduced by using a hardening potential \( \mathcal{H} \). This may be stated as

\[ \kappa = -\mathcal{H}_{\dot{\beta}} \]  

(3.57)

which is completely analogous to use of an elastic energy to relate \( \sigma \) and \( \dot{\varepsilon} \). A rate equation may be expressed now as

\[ \dot{\beta} = \lambda Q_\kappa \]  

(3.58)

It is immediately obvious that here also we have two possibilities. Using \( Q \) in the above expression defines a non-associative hardening, whereas replacing \( Q \) by \( F \) would give an associative hardening. Thus for a fully associative model we require that \( F \) be used to define both the plastic potential and the hardening. In such a case the relations of plasticity also may be deduced by using the principle of maximum plastic dissipation.\(^{13,14,26,27}\) A quadratic form for the hardening potential may be adopted and written as

\[ \mathcal{H} = \frac{1}{2} \beta^T H_k \beta \]  

(3.59)

in which \( H_k \) is assumed to be an invertible set of constant hardening parameters. Now \( \beta \) may be eliminated to give the simple rate form

\[ \dot{\kappa} = -\lambda H_k \frac{\partial Q}{\partial \kappa} = -\lambda H_k Q_\kappa \]  

(3.60)

Use of a linear shift in relation (3.54) simplifies this, noting Eq. (3.56), to

\[ \dot{\kappa} = \lambda H_k Q_\frac{\sigma}{\dot{\beta}} \]  

(3.61)

In our subsequent discussion we shall usually assume a general quadratic model for both elastic and hardening potentials. For a more general treatment the reader is referred to references 14 and 28.

Another approach to kinematic hardening was introduced by Armstrong and Frederick\(^{29}\) and provides a means of retaining smoother transitions from elastic to inelastic behaviour during cyclic loading. Here the hardening is given as

\[ \dot{\kappa} = \lambda [H_k Q_\frac{\sigma}{\dot{\beta}} - H_{NL} \kappa] \]  

(3.62)

Applications of this approach are presented by Chaboche\(^{30,31}\) and numerical comparisons to a simpler approach using a generalized plasticity model\(^{32,33}\) are given by Auricchio and Taylor.\(^{34}\)

Many other approaches have been proposed to represent classical hardening behaviour and the reader is referred to the literature for additional information and discussion.\(^{19–21,35–37}\) A physical procedure utilizing directly the finite element
method is available to obtain both ideal plasticity and hardening. Here several ideal plasticity components, each with different yield stress, are put in series and it will be found that both hardening and softening behaviour can be obtained easily retaining the properties so far described. This approach was named by many authors as an ‘overlay’ model\textsuperscript{38,39} and by others is described as a ‘sublayer’ model.

There are of course many other possibilities to define change in surfaces during the process of loading and unloading. Here frictional soils present one of the most difficult materials to model and for the non-associative case we find it convenient to use the generalized plasticity method described in Sect. 3.6.

### 3.3.4 Plastic stress–strain relations

To construct a constitutive model for plasticity, the strains are assumed to be divisible into elastic and plastic parts given as

\[
\varepsilon = \varepsilon^e + \varepsilon^p
\]  

(3.63)

For linear elastic behaviour, the elastic strains are related to stresses by a symmetric $9 \times 9$ matrix of constants $\mathbf{D}$. Differentiating Eq. (3.63) and incorporating the plastic relation (3.50) we obtain

\[
\dot{\varepsilon} = \mathbf{D}^{-1} \dot{\sigma} + \dot{\lambda} \mathbf{Q} \sigma
\]  

(3.64)

The plastic strain (rate) will occur only if the ‘elastic’ stress changes

\[
\dot{\sigma}^e \equiv \mathbf{D} \dot{\varepsilon}
\]  

(3.65)

tends to put the stress outside the yield surface, that is, is in the plastic loading direction. If, on the other hand, this stress change is such that unloading occurs then of course no plastic straining will be present, as illustrated for the one-dimensional case in Fig 3.6. The test of the above relation is therefore crucial in differentiating between loading and unloading operations and underlines the importance of the straining path in computing stress changes.

When plastic loading is occurring the stresses are on the yield surface given by Eq. (3.44). Differentiating this we can therefore write

\[
\dot{F} = \frac{\partial F}{\partial \sigma_x} \dot{\sigma}_x + \frac{\partial F}{\partial \sigma_y} \dot{\sigma}_y + \cdots + \frac{\partial F}{\partial \xi} \dot{x} + \frac{\partial F}{\partial \xi_y} \dot{y} + \cdots + \frac{\partial F}{\partial \xi} \dot{k} = 0
\]

or

\[
\dot{F} = F_{\sigma}^T \dot{\sigma} + F_{k}^T \dot{k} - H_i \dot{\lambda} = 0
\]  

(3.66)

in which we make the substitution

\[
H_i \dot{\lambda} = -\frac{\partial F}{\partial \xi} \dot{k} = - F_{\sigma} \dot{k}
\]  

(3.67)

where $H_i$ denotes an isotropic hardening modulus.
For the case where kinematic hardening is introduced, using Eq. (3.54) we can substitute Eq. (3.61) and modify Eq. (3.64) to

\[
\mathbf{D} \dot{\epsilon} = \ddot{\xi} + (\mathbf{D} + \mathbf{H}_k) \dot{\lambda} Q_\ddot{\xi}
\]  
(3.68)

Similarly, introducing Eq. (3.56) into Eq. (3.66) we obtain

\[
\dot{F} = F^T \dot{\xi} - H_i \dot{\lambda} = 0
\]  
(3.69)

Equations (3.68) and (3.69) now can be written in matrix form as

\[
\begin{pmatrix}
\mathbf{D} \dot{\epsilon} \\
0
\end{pmatrix} = 
\begin{bmatrix}
1 \\
F^T_\ddot{\xi}
\end{bmatrix}
\begin{pmatrix}
(\mathbf{D} + \mathbf{H}_k) Q_\ddot{\xi} \\
-H_i
\end{pmatrix}
\begin{pmatrix}
\ddot{\xi} \\
\dot{\lambda}
\end{pmatrix}
\]  
(3.70)

The indeterminate constant \( \dot{\lambda} \) can now be eliminated (taking care not to multiply or divide by \( H_i \) or \( \mathbf{H}_k \) which are zero in ideal plasticity). To accomplish the elimination we solve the first set of Eq. (3.70) for \( \ddot{\xi} \), giving

\[
\ddot{\xi} = \mathbf{D} \dot{\epsilon} - (\mathbf{D} + \mathbf{H}_k) Q_\ddot{\xi} \dot{\lambda}
\]

and substitute into the second, yielding the expression

\[
F^T_\ddot{\xi} \mathbf{D} \dot{\epsilon} - [H_i + F^T_\ddot{\xi}(\mathbf{D} + \mathbf{H}_k) Q_\ddot{\xi}] \dot{\lambda} = 0
\]

Equation (3.64) now results in an explicit expansion that determines the stress changes in terms of imposed strain changes. Using Eq. (3.43) this may now be reduced to a form in which only six-independent components are present and expressed as

\[
\dot{\sigma} = \mathbf{D}^{*\text{p}} \dot{\epsilon}
\]  
(3.71)

and

\[
\mathbf{D}^{*\text{p}} = \mathbf{P}^T \mathbf{D} \mathbf{P} - \frac{1}{H^*} \mathbf{P}^T Q_\ddot{\xi} F^T_\ddot{\xi} \mathbf{D} \mathbf{P}
\]  
(3.72)

where

\[
H^* = H_i + F^T_\ddot{\xi}(\mathbf{D} + \mathbf{H}_k) Q_\ddot{\xi}
\]

The elasto-plastic matrix \( \mathbf{D}^{*\text{p}} \) takes the place of the elasticity matrix \( \mathbf{D}_I^T \) in a continuum rate formulation. We note that in the absence of kinematic hardening it is possible to make reductions to the six-component form for all the computations at the very beginning. However, the manner in which the back stress enters the computation is not the same as that for the plastic strain and would be necessary to scale the two differently to make the general reduction. Thus, for the developments reported here we prefer to carry out all calculations using the full nine-component form (or, in the case of plane stress, to follow a four-component form) and make final reductions using Eq. (3.72).

* We shall show this step in more detail below for the \( J_2 \) plasticity model. In general, however, the final result involves only the usual form of the \( \mathbf{D} \) matrix and six independent components from the derivative of the yield function.
For a generalization of the above concepts to a yield surface possessing 'corners' where $Q_o$ is indeterminate, the reader is referred to the work of Koiter\textsuperscript{17} or the multiple surface treatments in Simo and Hughes\textsuperscript{14}.

An alternative procedure exists here simply by smoothing the corners. We shall refer to it later in the context of the Mohr–Coulomb surface often used in geomechanics and the procedure can be applied to any form of yield surface.

The continuum elasto-plastic matrix is symmetric only when plasticity is associative and when kinematic hardening is symmetric. In general, non-associative materials present stability difficulties, and special care is needed to use them effectively. Similar difficulties occur if the hardening moduli are negative which, in fact, leads to a softening behaviour. This is addressed further in Secs 3.11 and 3.12.

The elasto-plastic matrix given above is defined even for ideal plasticity when $H_i$ and $\mathbf{H}_k$ are zero. Direct use of the continuum tangent in an incremental finite element context where the rates are approximated by

$$\dot{\mathbf{e}}_{n+1} \Delta t \approx \Delta \mathbf{e}_{n+1} \quad \text{and} \quad \dot{\sigma}_{n+1} \Delta t \approx \Delta \sigma_{n+1}$$

was first made by Yamada et al.\textsuperscript{40} and Zienkiewicz et al.\textsuperscript{41} However, this approach does not give quadratic convergence when used in the Newton–Raphson scheme. For the associative case we can introduce a \textit{discrete time integration algorithm} in order to develop an exact (numerically consistent) tangent which does produce quadratic convergence when used in the Newton–Raphson iterative algorithm.

### 3.4 Computation of stress increments

We have emphasized that with the use of iterative procedures within a particular increment of loading, it is important to compute always the stresses as

$$\mathbf{\sigma}_{n+1}^k = \mathbf{\sigma}_n + \Delta \mathbf{\sigma}_n^k$$

(3.73)

corresponding to the total change in displacement parameters $\Delta \mathbf{a}_{n}^k$ and hence the total strain change

$$\Delta \mathbf{e}_{n}^k = \mathbf{B} \Delta \mathbf{a}_{n}^k \quad \Delta \mathbf{a}_{n}^k = \sum_{i=0}^{k} \mathbf{d} \mathbf{a}_{n}^i$$

(3.74)

which has accumulated in all previous iterations within the step. This point is of considerable importance as constitutive models with path dependence (namely, plasticity-type models) have different responses for loading and unloading. If a decision on loading/unloading is based on the increment $\mathbf{d} \mathbf{a}_{n}^i$ erroneous results will be obtained. Such decisions must \textit{always} be performed with respect to the total increment $\Delta \mathbf{a}_{n}^k$.

In terms of the elasto-plastic modulus matrix given by Eq. (3.72) this means that the stresses have to be integrated as

$$\mathbf{\sigma}_{n+1}^k = \mathbf{\sigma}_n + \int_{0}^{\Delta \mathbf{e}_{n}^k} \mathbf{D}_{cp} \mathbf{de}$$

(3.75)
Incorporating into $D_{\varepsilon}$ the dependence on variables in a manner corresponding to a linear increase of $\Delta \varepsilon^k_n$ (or $\Delta a_n^k$). Here, of course, all other rate equations have to be suitably integrated, though this generally presents little additional difficulty.

Various procedures for integration of Eq. (3.75) have been adopted and can be classified into explicit and implicit categories.

### 3.4.1 Explicit methods

In explicit procedures either a direct integration process is used or some form of the Runge–Kutta process is adopted. In the former the known increment $\Delta \varepsilon^k_n$ is subdivided into $m$ intervals and the integral of Eq. (3.75) is replaced by direct summation, writing

\[
\Delta \sigma^k_n = \frac{1}{m} \sum_{j=0}^{m-1} D^*_{(n+j/m)} \Delta \varepsilon^k_n
\]

where $D^*_{(n+j/m)}$ denotes the tangent matrix computed for stresses and hardening parameters updated from the previous increment in the sum.

This procedure, originally introduced in reference 43 and described in detail in references 44 and 45, is known as subincrementation. Its accuracy increases with the number of subincrements, $m$, used. In general it is difficult \textit{a priori} to decide on this number, and accuracy of prediction is not easy to determine.

Such integration will generally result in the stress change departing from the yield surface by some margin. In problems such as those of ideal plasticity where the yield surface forms a meaningful limit a proportional scaling of stresses (or return map) has been practiced frequently to obtain stresses which are on the yield surface at all times.\textsuperscript{45,46} In this process the effects of integrating the evolution equation for hardening must also be treated.

A more precise explicit procedure is provided by use of a Runge–Kutta method. Here, first an increment of $\Delta \varepsilon/2$ is applied in a single-step explicit manner to obtain

\[
\Delta \sigma_{n+1/2} = \frac{1}{2} D^*_n \Delta \varepsilon_n
\]

using the initial elasto-plastic matrix. This increment of stress (and corresponding $\kappa_{n+1/2}$) is evaluated to compute $D^*_{n+1/2}$ and finally we evaluate

\[
\Delta \sigma_n = D^*_{n+1/2} \Delta \varepsilon_n
\]

This process has a second-order accuracy and, in addition, can give an estimate of errors incurred as

\[
\Delta \sigma_n = 2 \Delta \sigma_{n+1/2}
\]

If such stress errors exceed a certain norm the size of the increment can be reduced. This approach is particularly useful for integration of non-associative models or models without yield functions where "tangent" matrices are simply evaluated (see Sect. 3.6).
3.4.2 Implicit methods

The integration of Eq. (3.75) can, of course, be written in an implicit form. For instance, we could write in place of Eq. (3.75), during each iteration $k$, that

$$\Delta \sigma_{n+1}^k = [(1 - \theta) D_n^* + \theta D_{n+1}^{*k}] \Delta \varepsilon_{n+1}^k$$

(3.80)

where here $D_n^*$ denotes the value of the tangential matrix at the beginning of the time step and $D_{n+1}^{*k}$ the current estimate to the tangential matrix at the end of the step.

This non-linear equation set could be solved by any of the procedures previously described; however, derivatives of the tangent matrix are quite complex and in any case a serious error is committed in the approximate form of Eq. (3.80). Further, there is no guarantee that the stresses do not depart from the yield surface.

Return map algorithm

In 1964 a very simple algorithm was introduced simultaneously by Maenchen and Sacks\textsuperscript{47} and by Wilkins\textsuperscript{48}. This algorithm uses a two-step process to compute the new stress and was originally implemented in an explicit time integration form, thus requiring no explicit construction of an elasto-plastic tangent matrix; however, later its versatility and robustness was demonstrated for implicit solutions.\textsuperscript{49,50} The steps of the algorithm are:

1. Perform a predictor step in which the entire increment of strain (for the present discussion we omit the iteration counter $k$ for simplicity)

$$\varepsilon_{n+1} = \varepsilon_n + \Delta \varepsilon_n$$

is used to compute trial stresses (denoted by superscript TR) assuming elastic behaviour. Accordingly,

$$\sigma_{n+1}^{TR} = D(\varepsilon_{n+1} - \varepsilon_n^p)$$

(3.81)

where only an elastic modulus $D$ is required.

2. Evaluate the yield function in terms of the trial stress and the values of the plastic parameters at the previous time:

$$F(\sigma^{TR}, \kappa_n) = \begin{cases} 
\leq 0, & \text{elastic} \\
> 0, & \text{plastic} 
\end{cases}$$

(3.82)

(a) For an elastic value of $F$ set the current stress to the trial value, accordingly

$$\sigma_{n+1} = \sigma_{n+1}^{TR}, \quad \kappa_{n+1} = \kappa_n \quad \text{and} \quad \kappa_{n+1} = \kappa_n$$

(b) For a plastic state solve a discretized set of plasticity rate equations (namely, using any appropriate time integration method as described in Chapter 18 of Volume 1) such that the final value of $F_{n+1}$ is zero.

A plastic correction can be most easily developed by returning to the original Eq. (3.64) and writing the relation for stress increment as

$$\Delta \sigma_n = D(\Delta \varepsilon_n - \Delta \varepsilon_n^p)$$

(3.83)
Now integrating the plastic strain relation (3.50) using a form similar to that in Eq. (3.80) yields

\[ \Delta \varepsilon^p_n = \Delta \lambda [(1 - \theta) \mathbf{Q}_n + \theta \mathbf{Q}_{n+1}] \tag{3.84} \]

where \( \Delta \lambda \) represents an approximation to the change in consistency parameter over the time increment. Kinematic hardening is included by integrating Eq. (3.60) as

\[ \Delta \mathbf{k}_n = \Delta \lambda \mathbf{H}_k [(1 - \theta) \mathbf{Q}_n + \theta \mathbf{Q}_{n+1}] \tag{3.85} \]

Finally, during the plastic solution we enforce

\[ F_{n+1} = 0 \tag{3.86} \]

thus ensuring that final values at \( t_{n+1} \) satisfy the yield condition exactly.

The above solution process is particularly simple for \( \theta = 1 \) (backward difference or Euler implicit) and now, eliminating \( \Delta \varepsilon^p_n \), we can write the above non-linear system in residual form

\[
\begin{align*}
\mathbf{R}_\sigma^i &= \Delta \varepsilon_n - \mathbf{D}^{-1} \Delta \varepsilon^p_n - \Delta \lambda \mathbf{Q}_n^{i+1} \\
\mathbf{R}_\kappa^i &= -\mathbf{H}_k^{-1} \Delta \mathbf{k}_n - \Delta \lambda \mathbf{Q}_k^{i+1} \\
\mathbf{r}^i &= -F_{n+1}^i
\end{align*}
\]

and seek solutions which satisfy \( \mathbf{R}_\sigma^i = 0, \mathbf{R}_\kappa^i = 0 \) and \( \mathbf{r}^i = 0 \). Any of the general iterative schemes described in Chapter 2 can now be used. In particular, the full Newton–Raphson process is convenient. Noting that \( \Delta \varepsilon_n \) is treated here as a specified constant (actually, the \( \Delta \varepsilon_n^p \) from the current finite element solution), we can write, on linearization

\[
\begin{bmatrix}
\mathbf{D}^{-1} + \Delta \lambda \mathbf{Q}_\sigma & \Delta \lambda \mathbf{Q}_{\sigma\kappa} & \mathbf{Q}_\sigma \\
\Delta \lambda \mathbf{Q}_{\kappa\sigma} & \mathbf{H}_k^{-1} + \Delta \lambda \mathbf{Q}_{\kappa\kappa} & \mathbf{Q}_\kappa \\
\mathbf{F}_\sigma^T & \mathbf{F}_\kappa^T & -\mathbf{H}_k_{n+1}^{-1}
\end{bmatrix}
\begin{bmatrix}
d\sigma^i \\
d\kappa^i \\
d\lambda^i
\end{bmatrix} =
\begin{bmatrix}
\mathbf{R}_\sigma^i \\
\mathbf{R}_\kappa^i \\
\mathbf{r}^i
\end{bmatrix}
\]

where \( H_i \) is the same hardening parameter as that obtained in Eq. (3.67). Some complexity is introduced by the presence of the second derivatives of \( Q \) in Eq. (3.87) and the term may be omitted for simplicity (although at the expense of asymptotic quadratic convergence in the Newton–Raphson iteration). Analytical forms of such second derivatives are available for frequently used potential surfaces.\(^{14,28,49–51}\) Appendix A also presents results for second derivatives of stress invariants.

It is important to note that the requirement that \( F_{n+1} = -\mathbf{r}^i [\text{Eq. (3.87)}] \) ensures that the \( \mathbf{r}^i \) residual measures precisely the departure from the yield surface. This measure is not available for any of the tangential forms if \( \mathbf{D}_{sp}^p \) is adopted.

For the solution it is only necessary to compute \( d\lambda^i \) and update as

\[ \Delta \lambda^i = \sum_{j=0}^{i} d\lambda^j \tag{3.88} \]
This solution process can be done in precisely the same way as was done in establish-
ing Eq. (3.72). Thus, a solution may be constructed by defining the following:

\[
\mathbf{R} = \begin{bmatrix} \mathbf{R}_f \\ \mathbf{R}_\kappa \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} \sigma \\ \kappa \end{bmatrix}
\]

\[
\nabla F = \begin{bmatrix} F_\sigma \\ F_\kappa \end{bmatrix}, \quad \nabla Q = \begin{bmatrix} Q_\sigma \\ Q_\kappa \end{bmatrix}
\]

\[
\mathbf{A} = \begin{bmatrix} \mathbf{D}^{-1} & 0 \\ 0 & H_\kappa^{-1} \end{bmatrix} + \Delta \lambda^i \begin{bmatrix} Q_{\sigma\sigma} & Q_{\sigma\kappa} \\ Q_{\kappa\sigma} & Q_{\kappa\kappa} \end{bmatrix}
\]

and expressing Eq. (3.87) as

\[
d_s^i = \mathbf{A}^{-1} \mathbf{R}^i - \frac{1}{A^i} \mathbf{A}^{-1} \nabla Q^i \left[ (\nabla F^i)^T \mathbf{A}^{-1} \mathbf{R}^i - r^i \right]
\]

where

\[
A^* = H_f + (\nabla F^i)^T \mathbf{A}^{-1} \nabla Q^i
\]

Immediately, we observe that at convergence \(\mathbf{R}^i = 0\) and \(r^i = 0\), thus, here we obtain a zero stress increment. At this point we have computed a stress state \(\sigma_{n+1}\) which satisfies the yield condition exactly. However, this stress, when substituted back into the finite element residual [e.g. Eq. (1.24) or (1.44)] may not satisfy the equili-
brum condition and it is now necessary to compute a new iteration \(k\) and obtain a new
strain increment \(d_{\varepsilon^k}\) from which the process is repeated. We note that inserting
this new increment into Eq. (3.87) will again result in a non-zero value for \(\mathbf{R}\), but that
\(\mathbf{R}_\kappa\) and \(r\) remain zero until subsequent iterations. Thus, Eq. (3.90) provides directly
now the required tangent matrix \(\mathbf{D}_{\varepsilon_p}\) from

\[
\begin{bmatrix} d\sigma \\ d\kappa \end{bmatrix} = \left[ \mathbf{D}_{\varepsilon_p} \right] \begin{bmatrix} d\varepsilon \\ 0 \end{bmatrix} = \begin{bmatrix} d\varepsilon^i_p \\ 0 \end{bmatrix}
\]

Thus, we find the tangent matrix \(\mathbf{D}_{\varepsilon_p}\) is obtained from the upper diagonal block of Eq.
(3.92). We note that this development also follows exactly the procedure for computing \(\mathbf{D}_{\varepsilon_p}\) in Eq. (3.72). At this stage the terms may once again be reduced to their six-
component form using \(\mathbf{P}\) as indicated in Eq. (3.42).

Some remarks on the above algorithm are in order:

1. For non-associative plasticity (namely, \(Q \neq F\)) the return direction is not normal
to the yield surface. In this case no solution may exist for some strain increments
(in general, arbitrary selection of \(F\) and \(Q\) forms in non-associative does not assure
stability) and the iteration process will not converge.
2. For associative plasticity the normality principle is valid, requiring a convex yield
surface. In this case the above iteration process always converges for a hardening
material.
3. Convergence of the finite element equations may not always occur if more than
one quadrature point changes from elastic to plastic or from plastic to elastic in
subsequent iterations.
Based on these comments it is evident that no universal method exists that can be used with the many alternatives which can occur in practice. In the next several sections we illustrate some formulations which employ the alternatives we have discussed above.

3.5 **Isotropic plasticity models**

We consider here some simple cases for isotropic plasticity-type models in which both a yield function and a flow rule are used. For an isotropic material linear elastic response may be expressed by moduli defined with two parameters. Here we shall assume these to be the bulk and shear moduli, as used previously in the viscoelastic section (Sec. 3.2). Accordingly, the stress at any discrete time $t_{n+1}$ is computed from elastic strains in matrix form as

$$
\sigma_{n+1} = \rho_{n+1} m + s_{n+1} = K m m^T e_{n+1} + 2G(1 - \frac{1}{3} m m^T) e_{n+1}^e
$$

$$
= D (e_{n+1} - e_{n+1}^p) \tag{3.93}
$$

where the elastic modulus matrix for an isotropic material is given in the simple form

$$
D = K m m^T + 2G(1 - \frac{1}{3} m m^T) \tag{3.94}
$$

and $I$ is the $9 \times 9$ identity matrix and $m$ is the nine-component matrix

$$
m = [1 1 1 0 0 0 0 0 0]^T
$$

Using Eqs (3.42) and (3.43) immediately reduces the above to

$$
D = K m m^T + 2G(I_0 - \frac{1}{3} m m^T) \tag{3.95}
$$

The above relation yields the stress at the current time provided we know the current total strain and the current plastic strain values. The total strain is available from the finite element equations using the current value of nodal displacements, and the plastic strain is assumed to be computed with use of one of the algorithms given above. In the discussion to follow we consider relations for various classical yield surfaces.

3.5.1 **Isotropic yield surfaces**

The general procedures outlined in the previous section allow determination of the tangent matrices for almost any yield surface applicable in practice. For an isotropic material all functions can be represented in terms of the three stress invariants:

$$
I_1 = \sigma_{ii} = m^T \sigma
$$

$$
2J_2 = s_{ij} s_{ij} = \mathbf{s}^T \mathbf{s} = |\mathbf{s}|^2 \tag{3.96}
$$

$$
3J_3 = s_{ij} s_{jk} s_{ki} = \det \mathbf{s}
$$

where we can observe that definition of all the invariants is most easily performed in indicial notation.

* Appendix A presents a summary of invariants and their derivatives.