The time dimension – semi-discretization of field and dynamic problems and analytical solution procedures

17.1 Introduction

In all the problems considered so far in this text conditions that do not vary with time were generally assumed. There is little difficulty in extending the finite element idealization to situations that are time dependent.

The range of practical problems in which the time dimension has to be considered is great. Transient heat conduction, wave transmission in fluids and dynamic behaviour of structures are typical examples. While it is usual to consider these various problems separately – sometimes classifying them according to the mathematical structure of the governing equations as ‘parabolic’ or ‘hyperbolic’ – we shall group them into one category to show that the formulation is identical.

In the first part of this chapter we shall formulate, by a simple extension of the methods used so far, matrix differential equations governing such problems for a variety of physical situations. Here a finite element discretization in the space dimension only will be used and a semi-discretization process followed (see Chapter 3). In the remainder of this chapter various analytical procedures of the solution for the resulting ordinary linear differential equation system will be dealt with. These form the basic arsenal of steady-state and transient analysis.

Chapter 18 will be devoted to the discretization of the time domain itself.

17.2 Direct formulation of time-dependent problems with spatial finite element subdivision

17.2.1 The ‘quasi-harmonic’ equation with time differential

In many physical problems the quasi-harmonic equation takes the form in which time derivatives of the unknown function \( \phi \) occur. In the three-dimensional case typically
we might have
\[
\frac{\partial^2}{\partial x} \left( k \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial \phi}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial \phi}{\partial z} \right) + \left( \bar{Q} - \mu \frac{\partial \phi}{\partial t} - \rho \frac{\partial^2 \phi}{\partial t^2} \right) = 0 \quad (17.1)
\]

In the above, quite generally, all the parameters may be prescribed functions of time, or in non-linear cases of \( \phi \), as well as of space \( x \), i.e.,
\[
k = k(x, \phi, t) \quad \bar{Q} = \bar{Q}(x, \phi, t) \quad \text{etc.} \quad (17.2)
\]

If a situation at a particular instant of time is considered, the time derivatives of \( \phi \) and all the parameters can be treated as \textit{prescribed functions of space coordinates}. Thus, at that instant the problem is precisely identified with those treated in Chapter 7 if the whole of the quantity in the last parentheses of Eq. (17.1) is identified as the source term \( Q \).

The finite element discretization of this in terms of \textit{space} elements has already been fully discussed and we found that with the prescription
\[
\phi = \sum N_i a_i = N a \quad (17.3)
\]

for each element, the standard form of assembled equations†

\[
K a + \bar{f} = 0 \quad (17.4)
\]

was obtained. Element contributions to the above matrices are defined in Chapter 7 and need not be repeated here except for that representing the 'load' term due to \( Q \). This is given by

\[
\bar{f} = - \int_\Omega N^T Q \, d\Omega \quad (17.5)
\]

Replacing \( Q \) by the last bracketed term of Eq. (17.1) we have

\[
\bar{f} = - \int_\Omega N^T \left( \bar{Q} - \mu \frac{\partial \phi}{\partial t} - \rho \frac{\partial^2 \phi}{\partial t^2} \right) \, d\Omega \quad (17.6)
\]

However, from Eq. (17.3) it is noted that \( \phi \) is approximated in terms of the nodal parameters \( a \). On substitution of this approximation we have

\[
\bar{f} = - \int_\Omega N^T Q \, d\Omega + \left( \int_\Omega N^T \mu N \, d\Omega \right) \frac{da}{dt} + \left( \int_\Omega N^T \rho N \, d\Omega \right) \frac{d^2 a}{dt^2} \quad (17.7)
\]

and on expanding Eq. (17.4) in its final assembled form we get the following \textit{matrix differential equation}:

\[
M \ddot{a} + C \dot{a} + K a + f = 0 \quad (17.8)
\]

\[
\dot{a} = \frac{da}{dt} \quad \ddot{a} = \frac{d^2 a}{dt^2} \quad (17.9)
\]

† We have replaced the matrix \( H \) of Chapter 7 by \( K \) to facilitate comparison with other transient equations.
in which all the matrices are assembled from element submatrices in the standard manner with submatrices \( K^e \) and \( f^e \) still given by relations (7) in Chapter 7 and

\[
C_{ij}^e = \int_{\Omega} N_i \mu N_j \, d\Omega \quad (17.10)
\]

\[
M_{ij}^e = \int_{\Omega} N_i \rho N_j \, d\Omega \quad (17.11)
\]

Once again these matrices are symmetric as seen from the above relations.

Boundary conditions imposed at any time instant are treated in the standard manner.

The variety of physical problems governed by Eq. (17.1) is so large that a comprehensive discussion of them is beyond the scope of this book. A few typical examples will, however, be quoted.

**Equation (17.1) with** \( \rho = 0 \)

This is the standard *transient heat conduction equation*\(^1,2\) which has been discussed in the finite element context by several authors.\(^3-6\) This same equation is applicable in other physical situations — one of these being the *soil consolidation equations*\(^7\) associated with *transient seepage forms*.\(^8\)

**Equation (17.1) with** \( \mu = 0 \)

Now the relationship becomes the famous *Helmholz wave equation* governing a wide range of physical phenomena. Electromagnetic waves,\(^9\) fluid surface waves\(^10\) and compression waves\(^11\) are but a few cases to which the finite element process has been applied.

**Equation (17.1) with** \( \mu \neq \rho \neq 0 \)

This damped wave equation is of yet more general applicability and has particular significance in fluid mechanics (wave) problems.

The reader will recognize that what we have done here is simply an application of the process of partial discretization described in Sec. 3.5. It is convenient, however, to perform the operations in the manner suggested above as all the matrices and discretization expressions obtained from steady-state analysis are immediately available.

### 17.2.2 Dynamic behaviour of elastic structures with linear damping

While in the previous section we have been concerned with, apparently, a purely mathematical problem, identical reasoning can be applied directly to the wide class of dynamic behaviour of elastic structures following precisely the general lines of Chapter 2.

When displacements of an elastic body vary with time two sets of additional forces are called into play. The first is the inertia force, which for an acceleration characterized by \( \ddot{u} \) can be replaced by its static equivalent

\[-\rho \ddot{u}\]
using the well-known d'Alembert principle. This is a force with components in directions identical to those of the displacement $\mathbf{u}$ and (generally) given per unit of volume. In this context $\rho$ is simply the mass per unit volume.

The second force is that due to (frictional) resistances opposing the motion. These may be due to microstructure movements, air resistance, etc., and are often related in a non-linear way to the velocity $\dot{\mathbf{u}}$. For simplicity of treatment, however, only a linear viscous-type resistance will be considered, resulting again in unit volume forces in an equivalent static problem of magnitude

$$-\mu \ddot{\mathbf{u}}$$

In the above $\mu$ is a set of viscosity parameters which can presumably be given numerical values.

The equivalent static problem, at any instant of time, is now discretized precisely in the manner of Chapter 2, but replacing the distributed body force $\mathbf{b}$ by its equivalent

$$\mathbf{b} - \rho \ddot{\mathbf{u}} - \mu \ddot{\mathbf{u}}$$

The element (nodal) forces given by Eq. (2.13) now become (excluding initial stress and strain contributions)

$$\mathbf{f}^e = -\int_{\Omega^e} \mathbf{N}^T \mathbf{b} \, d\Omega = -\int_{\Omega^e} \mathbf{N}^T \mathbf{b} \, d\Omega + \int_{\Omega^e} \mathbf{N}^T \rho \ddot{\mathbf{u}} \, d\Omega + \int_{\Omega^e} \mathbf{N}^T \mu \ddot{\mathbf{u}} \, d\Omega$$

(17.12)

in which the first force is that due to an external distributed body load and need not be considered further.

Substituting Eq. (17.12) into the general equilibrium equations we obtain finally, on assembly, the following matrix differential equation:

$$\mathbf{M} \ddot{\mathbf{u}} + \mathbf{C} \dot{\mathbf{u}} + \mathbf{K} \mathbf{u} + \mathbf{f} = 0$$

(17.13)

in which $\mathbf{K}$ and $\mathbf{f}$ are assembled stiffness and force matrices obtained by the usual addition of element stiffness coefficients and of element forces due to external specified loads, initial stresses, etc., in the manner fully described before. The new matrices $\mathbf{C}$ and $\mathbf{M}$ are assembled by the usual rule from element submatrices given by $\dagger$

$$C^e_{ij} = \int_{\Omega^e} \mathbf{N}^T_i \mu \mathbf{N}_j \, d\Omega$$

(17.14)

and

$$M^e_{ij} = \int_{\Omega^e} \mathbf{N}^T_i \rho \mathbf{N}_j \, d\Omega$$

(17.15)

The matrix $\mathbf{M}^e$ is known as the element mass matrix and the assembled matrix $\mathbf{M}$ as the system mass matrix. Similarly, the matrix $\mathbf{C}^e$ is known as the element damping matrix and the assembled matrix $\mathbf{C}$ as the system damping matrix.

It is of interest to note that in early attempts to deal with dynamic problems of his nature the mass of the elements was usually arbitrarily 'lumped' at nodes, always resulting in a diagonal matrix even if no actual concentrated masses existed. The

$\dagger$ For simplicity we shall only consider distributed inertia – concentrated mass and damping forces being a limiting case.
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fact that such a procedure was, in fact, unnecessary and apparently inconsistent was simultaneously recognized by Archer\textsuperscript{13} and independently by Leckie and Lindberg\textsuperscript{14} in 1963. The general presentation of the results given in Eq. (17.15) is due to Zienkiewicz and Cheung.\textsuperscript{15} The name consistent mass matrix has been coined for the mass matrix defined here, a term which may be considered to be unnecessary since it is the logical and natural consequence of the discretization process. By analogy the matrices $C^e$ and $C$ may be called consistent damping matrices.

For many computational processes the lumped mass matrix is, however, more convenient and economical. Many practitioners are today using such matrices exclusively – sometimes showing good accuracy. While with simple elements a physically obvious methodology of lumping is easy to devise, this is not the case with higher order elements and we shall return to the process of ‘lumping’ later.

Determination of the damping matrix $C$ is in practice difficult as knowledge of the viscous matrix $\mu$ is lacking. It is often assumed, therefore, that the damping matrix is a linear combination of stiffness and mass matrices, i.e.,

$$C = \alpha M + \beta K$$

(17.16)

Here the parameters $\alpha$ and $\beta$ are determined experimentally.\textsuperscript{12,16} Such damping is known as ‘Rayleigh damping’ and has certain mathematical advantages which we shall discuss later. On occasion $C$ may be completely specified and such approximation devices are not necessary.

It is perhaps worth recognizing that on occasion different shape functions need to be used to describe the inertia forces from those specifying the displacements $u$. For instance, in beams (Chapter 2) (also for plates considered in Chapter 4 of Volume 2) the full strain state is prescribed simply by defining $w$, the lateral displacement, as additional bending assumptions are introduced. When considering the inertia forces it may be desirable not only to include the simple lateral inertia force given by

$$-\rho A \frac{\partial^2 w}{\partial t^2}$$

(in which $\rho A$ is now the mass per unit length of the beam) but also to consider rotary inertia couples of the type

$$-\rho I \frac{\partial^2}{\partial t^2} \left( \frac{\partial w}{\partial x} \right)$$

in which $\rho I$ is the rotatory inertia. Now it will be necessary to describe a more generalized displacement $\mathbf{u}$:

$$\mathbf{u} = \begin{bmatrix} w \\ \frac{\partial w}{\partial x} \end{bmatrix} = \tilde{N} \mathbf{a}^r$$

in which $\tilde{N}$ will follow directly from the definition of $N$ which specifies only the $w$ component. Relations such as Eq. (17.15) are still valid, providing we replace $N$ by $\tilde{N}$ and put in place of $\rho$ the matrix

$$\begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix}$$
17.2.3 'Mass' or 'damping' matrices for some typical elements

It is impractical to present in an explicit form all the mass matrices for the various elements discussed in previous chapters. Some selected examples only will be discussed here.

**Plane stress and plane strain**

Using triangular elements discussed in Chapter 4 the matrix $N^e$ is defined as

$$N^e = \begin{bmatrix} N_i & N_j & N_k \end{bmatrix}$$

where

$$N_i^e = N_i I \quad \text{etc.}$$

and

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equation (4.8) gives the shape functions as

$$N_i = \frac{a_i + b_i x + c_i y}{2\Delta}, \quad \text{etc.}$$

where $\Delta$ is the area of the triangular element.

If the thickness of the element is $h$ and this is assumed to be constant within the element, we have, for the mass matrix, Eq. (17.15),

$$M^e = \rho h \int \int N^T N \, dx \, dy$$

or

$$M_{es}^e = \rho h I \int \int N_r N_s \, dx \, dy$$

If the relationships of Eq. (4.8) are substituted, it is easy to verify that

$$\int \int N_r N_s \, dx \, dy = \begin{cases} \frac{1}{12} \Delta & \text{when } r \neq s \\ \frac{1}{6} \Delta & \text{when } r = s \end{cases} \quad (17.17)$$

Thus taking the total mass of the element as

$$m = \rho h \Delta$$

the mass matrix becomes

$$M^e = \frac{m}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix} \quad (17.18)$$
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If the mass is lumped at the nodes in three equal parts the ‘lumped’ mass matrix contributed by the element is

\[
M' = \frac{m}{3} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(17.19)

Certainly both matrices differ considerably and yet in applications the results of the analysis are almost identical.

### 17.2.4 Mass ‘lumping’ or diagonalization

We have referred to the computational convenience of lumping of mass matrices and presenting these in diagonal form. On some occasions such lumping is physically obvious (see the linear triangle for instance), in others this is not the case and a ‘rational’ procedure is required. For matrices of the type given in Eq. (17.15) several alternative approximations have been developed as discussed in Appendix I. In all of these the essential requirement of mass preservation is satisfied, i.e.,

\[
\sum_i \tilde{M}_{ii} = \int_\Omega \rho \, d\Omega
\]

(17.20)

where \( \tilde{M}_{ii} \) is the diagonal of the lumped mass matrix \( \tilde{M} \).

Three main procedures exist (see Fig. 17.1):

1. the row sum method in which

\[
\tilde{M}_{ii} = \sum_j M_{ij}
\]

2. diagonal scaling in which

\[
\tilde{M}_{ii} = a M_{ii}
\]

with \( a \) adjusted so that Eq. (17.20) is satisfied\(^\text{17,18} \) and

3. evaluation of \( M \) using a quadrature involving only the nodal points and thus automatically yielding a diagonal matrix for standard finite element shape functions\(^\text{19,20} \) in which \( N_i = 0 \) for \( x = x_j, j \neq i \).

It should be remarked that Eq. (17.20) does not hold for hierarchical shape functions where no lumping procedure appears satisfactory.

The quadrature (numerical integration) process is mathematically most appealing but frequently leads to negative or zero lumped masses. Such a loss of positive definiteness is undesirable in some solution processes and cancels out the advantages of lumping. In Fig. 17.1 we show the effect of various lumping procedures on
triangular and quadrilateral elements of linear and quadratic type. It is clear from these that the optimal choice to lump the mass is by no means unique.

In general we would recommend the use of lumped matrices only as a convenient numerical device generally paid for by some loss of accuracy. An exception to this is for 'explicit' time integration of dynamics problems where the considerable efficiency of their use more than compensates for any loss in accuracy (see Chapter 18). In some problems of fluid mechanics (Volume 3) we shall indeed use lumping for an intermediate iterative step in getting the consistent solution. However, we note that it has occasionally been shown that lumping can improve accuracy of some problem by error cancellation. It can be shown that in the transient approximation the lumping process introduces additional dissipation of the 'stiffness matrix' form and this can help in cancelling out numerical oscillation.
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To demonstrate the nature of lumped and consistent mass matrices it is convenient to consider a typical one-dimensional problem specified by the equation

\[ \frac{\partial \phi}{\partial t} - \frac{\partial}{\partial x} \left( \mu \frac{\partial \phi}{\partial x} \right) - \frac{\partial}{\partial x} \left( k \frac{\partial \phi}{\partial x} \right) \]

Semi-discretization here gives a typical nodal equation \( i \) as

\[ (M_{ij} + H_{ij}) \ddot{a}_j + K_{ij} a_j = 0 \]

where

\[ M_{ij} = \int_{\Omega} N_i N_j \, dx \]
\[ H_{ij} = \int_{\Omega} \mu \frac{d N_i}{d x} \frac{d N_j}{d x} \, dx \]
\[ K_{ij} = \int_{\Omega} k \frac{d N_i}{d x} \frac{d N_j}{d x} \, dx \]

and it is observed that \( H \) and \( K \) have identical structure. With linear elements of constant size \( h \) the approximating equation at a typical node \( i \) (and surrounding nodes \( i-1 \) or \( i+1 \)) can be written as follows (as the reader can readily verify).

\[ M_{ij} \ddot{a}_j \equiv \frac{h}{6} (\ddot{a}_{i-1} + 4\ddot{a}_i + \ddot{a}_{i+1}) \]
\[ H_{ij} \ddot{a}_j \equiv \frac{\mu}{h} (-\ddot{a}_{i-1} + 2\ddot{a}_i - \ddot{a}_{i+1}) \]
\[ K_{ij} a_j \equiv \frac{k}{h} (-a_{i-1} + 2a_i - a_{i+1}) \]

If a lumped approximation is used for \( M \), that is \( \tilde{M} \), we have, simply by adding coefficients using the row sum method,

\[ \tilde{M}_{ij} \ddot{a}_j = h \dot{a}_i \]

The difference between the two expressions is

\[ \tilde{M}_{ij} \ddot{a}_j - M_{ij} \ddot{a}_j \equiv \frac{h}{6} (-\ddot{a}_{i-1} + 2\ddot{a}_i - \ddot{a}_{i+1}) \]

and is clearly identical to that which would be obtained by increasing \( \mu \) by \( h^2/6 \). As \( \mu \) in the above example can be considered as a viscous dissipation we note that the effect of using a lumped matrix is that of adding an extra amount of such viscosity and can often result in smoother (though probably less accurate) solutions.

Eigenvalues and analytical solution procedures

17.3 General classification

We have seen that as a result of semi-discretization many time-dependent problems can be reduced to a system of ordinary differential equations of the characteristic
Free response – eigenvalues for second-order problems and dynamic vibration

form given by

$$M\ddot{a} + C\dot{a} + Ka + f = 0$$  \hspace{1cm} (17.21)

In this, in general, all the matrices are symmetric (some cases involving non-symmetric matrices will be discussed in Volume 3, Chapter 2). This second-order system often becomes first order if $M$ is zero as, for instance, in transient heat conduction problems. We shall now discuss some methods of solution of such ordinary differential equation systems. In general, the above equations can be non-linear (if, for instance, stiffness matrices are dependent on non-linear material properties or if large deformations are involved) but here we shall concentrate on linear cases only.

Systems of ordinary linear differential equations can always in principle be solved analytically without the introduction of additional approximations. The remainder of this chapter will be concerned with such analytical processes. While such solutions are possible they may be so complex that further recourse has to be taken to the process of approximation; we shall deal with this matter in the next chapter. The analytical approach provides, however, an insight into the behaviour of the system which the authors always find helpful.

Some of the matter in this chapter will be an extension of standard well-known procedures used for the solution of differential equations with constant coefficients that are encountered in most studies of dynamics or mathematics. In the following we shall deal successively with:

1. determination of free response ($f = 0$)
2. determination of periodic response ($f(t)$ periodic)
3. determination of transient response ($f(t)$ arbitrary).

In the first two, initial conditions of the system are of no importance and a general solution is simply sought. The last, most important, phase presents a problem to which considerable attention will be devoted.

17.4 Free response – eigenvalues for second-order problems and dynamic vibration

17.4.1 Free dynamic vibration – real eigenvalues

If no damping or forcing terms exist in the dynamic problem of Eq. (17.21) it reduces to

$$M\ddot{a} + Ka = 0$$  \hspace{1cm} (17.22)

A general solution of such an equation may be written as

$$a = \tilde{a}\exp(i\omega t)$$

the real part of which simply represents a harmonic response as $\exp(i\omega t) \equiv \cos \omega t + i \sin \omega t$. Then on substitution we find that $\omega$ can be determined from

$$(-\omega^2 M + K)a = 0$$  \hspace{1cm} (17.23)
This is a general linear eigenvalue or characteristic value problem and for non-zero solutions the determinant of the above coefficient matrix must be zero:

$$| -\omega^2 M + K | = 0$$ (17.24)

Such a determinant will in general give \( n \) values of \( \omega^2 \) (or \( \omega_j, j = 1, 2, \ldots, n \)) when the size of the matrices \( K \) and \( M \) is \( n \times n \), providing the matrices \( K \) and \( M \) are symmetric positive definite.†

While the solution of Eq. (17.24) cannot determine the actual values of \( a \) we can find \( n \) vectors \( \vec{a}_j \) that give the proportions for the various terms. Such vectors are known as the normal modes of the system or eigenvectors and are made unique by normalizing so that

$$\vec{a}_j^T M \vec{a}_j = 1; \quad j = 1, 2, \ldots, n$$ (17.25)

At this stage it is useful to note the property of modal orthogonality, i.e., that

$$\vec{a}_i^T M \vec{a}_j = 0; \quad (i \neq j)$$ (17.26)

$$\vec{a}_i^T K \vec{a}_j = 0; \quad (i \neq j)$$ (17.27)

The proof of the above statement is simple. As Eq. (17.23) is valid for any mode we can write

$$\omega_i^2 M \vec{a}_i = K \vec{a}_i$$

$$\omega_j^2 M \vec{a}_j = K \vec{a}_j$$

Premultiplying the first by \( \vec{a}_j^T \) and the second by \( \vec{a}_i^T \) and noting the symmetry of \( M \) and \( K \) so that

$$\vec{a}_j^T M \vec{a}_i = \vec{a}_i^T M \vec{a}_j$$

$$\vec{a}_j^T K \vec{a}_i = \vec{a}_i^T K \vec{a}_j$$

the difference becomes

$$(\omega_i^2 - \omega_j^2) \vec{a}_i^T M \vec{a}_j = 0$$

and if \( \omega_i \neq \omega_j \) † the orthogonality condition for the matrix \( M \) has been proved. From this the orthogonality of the vectors with \( K \) follows immediately. The final condition

$$\vec{a}_i^T K \vec{a}_i = \omega^2$$

follows from Eq. (17.25) and a premultiplication of Eq. (17.23) for equation \( i \) by \( \vec{a}_i \).

### 17.4.2 Determination of eigenvalues

To find the actual eigenvalues it is seldom practicable to write the polynomial expanding the determinant given in Eq. (17.24) and alternative techniques have to

† A symmetric matrix is positive definite if all the diagonals of the triangular factors are positive, this is a usual case with structural problems – all roots of Eq. (17.24) are real positive numbers (for a proof see reference 1). These are known as the natural frequencies of the system. If only the \( M \) matrix is symmetric positive definite while \( K \) is symmetric positive semidefinite the roots are real and positive or zero.

† For any case where repeated frequencies occur we merely enforce the orthogonality by construction.
be developed. The discussion of such techniques is best left to specialist texts and indeed many standard computer programs exist as library routines.

Many extremely efficient procedures are available and the reader can find some interesting matter in references. In some processes the starting point is the standard eigenvalue problem given by

$$Hx = \lambda x$$

(17.28)

in which $H$ is a symmetric matrix and hence has real eigenvalues. Equation (17.23) can be written as

$$M^{-1}Ka = \omega^2 a$$

(17.29)

on inverting $M$ with $\lambda = \omega^2$, but symmetry is in general lost.

If, however, we write in triangular form

$$M = LL^T \quad \text{and} \quad M^{-1} = L^{-T}L^{-1}$$

in which $L$ is a lower triangular matrix (i.e., has all zero coefficients above the diagonal), Eq. (17.26) may now be written as

$$Ka = \omega^2 LL^T a$$

Calling

$$L\tilde{a} = x$$

(17.30)

and multiplying by $L^{-1}$ we have finally

$$Hx = \omega^2 x$$

(17.31)

in which

$$H = L^{-1}KL^{-T}$$

(17.32)

which is of the standard form of Eq. (17.30), as $H$ is now symmetric.

Having determined $\omega^2$ (all, or only a few of the selected smallest values corresponding to fundamental periods) the modes of $x$ are found, and hence by use of Eq. (17.30) the modes of $\tilde{a}$.

If the matrix $M$ is diagonal – as it will be if the masses have been ‘lumped’ – the procedure of deriving the standard eigenvalue problem is simplified and here appears the first advantage of the diagonalization, which we have discussed in Sec. 17.2.4.

17.4.3 Free vibration with the singular $K$ matrix

In static problems we have always introduced a suitable number of support conditions to allow the stiffness matrix $K$ to be inverted, or what is equivalent to solve the static equations uniquely. If such ‘support’ conditions are in fact not specified, as may well be the case with a rocket travelling in space, the arbitrary fixing of a minimum number of support conditions allows a static solution to be obtained without affecting the stresses. In dynamic situations such a fixing is not permissible and frequently one is faced with the problem of a free oscillation for which $K$ is singular and therefore does not possess unique triangular factors or an inverse.
To preserve the applicability of methods which require an inverse (e.g., methods based on inverse power iteration\textsuperscript{26}) a simple artifice is possible. Equation (17.23) is modified to

$$[(K + \alpha M) - (\omega^2 + \alpha)M] \ddot{\mathbf{a}} = \mathbf{0}$$

(17.33)

in which $\alpha$ is an arbitrary constant of the same order as the typical $\omega^2$ sought. The new matrix $(K + \alpha M)$ is no longer singular and can be factored (or inverted) for use in the standard eigensolution procedure to find $(\omega^2 + \alpha)$.

This simple but effective avoidance of an otherwise serious difficulty was first suggested by Cox\textsuperscript{28} and Jennings\textsuperscript{29}. Alternative methods of dealing with the above problem are given in references 30 and 31.

### 17.4.4 Reduction of the eigenvalue system

Independent of which technique is used to determine the eigenpairs of the system (17.23), the effort for $n \times n$ matrices is at least one order greater than that involved in an equivalent static situation. Further, while the number of eigenvalues of the real system is infinite, in practice, we are generally interested only in a relatively small number of the lower frequencies and it is possible to simplify the computation by reducing the size of the problem.

To achieve a reduced problem we assume that the unknown $\ddot{\mathbf{a}}$ can be expressed in terms of $m$ ($\ll n$) vectors $t_1, t_2, \ldots, t_m$ and corresponding participating factors $x_i$. We now write

$$\ddot{\mathbf{a}} = t_1 x_1 + t_2 x_2 + \cdots + t_m x_m = \mathbf{T} \mathbf{x}$$

(17.34)

Inserting Eq. (17.34) into Eq. (17.23) and premultiplying by $\mathbf{T}^T$ we have a reduced problem with only $m$ eigenpairs:

$$(\omega^*)^2 \mathbf{M}^* \mathbf{x} = \mathbf{K}^* \mathbf{x}$$

(17.35)

where

$$\mathbf{M}^* = \mathbf{T}^T \mathbf{M} \mathbf{T} \quad \mathbf{K}^* = \mathbf{T}^T \mathbf{K} \mathbf{T}$$

and $\omega^*$ are now eigenvalues of the reduced system, which for the appropriate choice of the $t_i$ vectors can be good approximations to the eigenvalues of the original system.

If by good fortune the trial vectors were to be chosen as eigenvectors of the original matrix the system would become diagonal and all eigenvalues (i.e., in this case $\omega^* = \omega$) could be determined by a trivial calculation. This indeed is what some iterative eigenproblem strategies attempt (e.g., subspace or Lanczos methods\textsuperscript{26,32}). It is also of course possible by physical insight to find vectors $\mathbf{t}$ that correspond closely to the principal modes of the movement (e.g., see reference 33).

### 17.4.5 Some examples

There are a variety of problems for which practical solutions exist, so only a few simple examples will be shown.
**Free response – eigenvalues for second-order problems and dynamic vibration**

**Fig. 17.2** Simply supported beam: (a) $\omega_1 = 3.8050$; (b) $\omega_2 = 59.2236$; (c) $\omega_3 = 290.0804$.

**Vibration of a simply supported beam**

Figure 17.2 shows the first three vibration modes of a simply supported beam with length 40 and rectangular cross-section of width 1 and depth 2 units. The elastic properties are $E = 30\,000$, $\nu = 0$, and $\rho = 0.1$ units. The beam is modelled using 9-noded quadrilateral elements of lagrangian type with the central node at the left end restrained in the $x$ and $y$ direction and the central node at the right end restrained only in the $y$ direction. The problem is also solved using a mesh with 1000 two-noded beam elements which include effects of transverse shearing deformation. In Table 17.1 we present the values for the first three frequencies obtained from the finite element analysis and compare to the value obtained from an exact solution for the beam without shear deformation.

**Vibration of an earth dam**

Figure 17.3 shows the vibration of a two-dimensional earth dam resting on a rigid foundation. The earth dam is modelled by linear triangular elements and includes the effects of different material layers.

**The ‘wave’ equation. Electromagnetic and fluid problems**

The basic dynamic equation (17.8) can be derived for a variety of non-structural problems. The eigenvalue problem once again occurs with ‘stiffness’ and ‘mass’ matrices now having alternate physical meanings.

| Table 17.1 Frequencies for a simply supported beam |
|-----------------|-----------------|-----------------|
| 9-noded element | 3.7785          | 59.2236         | 290.0804        |
| 2-noded element | 3.7787          | 59.2338         | 290.1774        |
| Beam theory     | 3.8050          | 60.8807         | 308.2080        |
A particular form of the more general equations discussed earlier is the well-known Helmholtz wave equation which, in two-dimensional form, is

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0
\]  

(17.36)

If the boundary conditions do not force a response, an eigenvalue problem results which has significance in several fields of physical science.

The first application is to electromagnetic fields. Figure 17.4 shows a modal shape of a field for a waveguide problem. Simple linear triangular elements are used here. More complex three-dimensional oscillations are also discussed in reference 9.

A similar equation also describes to a reasonable approximation the behaviour of shallow water waves in a body of water:

\[
\frac{\partial}{\partial x} \left( h \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( h \frac{\partial \psi}{\partial y} \right) + \frac{1}{g} \frac{\partial^2 \psi}{\partial t^2} = 0
\]  

(17.37)
Fig. 17.4 A ‘lunar’ waveguide,9 mode of vibration for electromagnetic field. Outer diameter = d, OO' = 0.13d, r = 0.29d, S = 0.055d, \( \theta = 22^\circ \).

Fig. 17.5 Oscillations of a natural harbour: contours of velocity amplitudes.10
in which \( h \) is the average water depth, \( \psi \) the surface elevation above average and \( g \) the gravity acceleration. The formulation of the last two problems are discussed in detail in Volume 3, Chapter 8.

Thus natural frequencies of bodies of water contained in harbours of varying depths may easily be found.\(^1\) Figure 17.5 shows the modal shape for a particular harbour.

**17.5 Free response – eigenvalues for first-order problems and heat conduction, etc.**

If in Eq. (17.21) \( M = 0 \), we have a form typical of the transient heat conduction equation [see Eq. (17.1)]. For free response we seek a solution of the homogeneous equation

\[
C\ddot{a} + Ka = 0
\]

Once again an exponential form can be used:

\[
a = \tilde{a} \exp(-\lambda t)
\]

Substituting we have

\[
(-\lambda C + K)\tilde{a} = 0
\]

which again gives an eigenvalue problem identical to that of Eq. (17.23). As \( C \) and \( K \) are usually positive definite, \( \lambda \) will be positive and real. The solution therefore represents simply an exponential decay term and is not really steady state. Combination of such terms, however, can be useful in the solution of initial value transient problems but is of little value *per se*.

**17.6 Free response – damped dynamic eigenvalues**

We shall now consider the full equation (17.21) for free response conditions. Writing

\[
M\ddot{a} + C\dot{a} + Ka = 0
\]

and substituting

\[
a = \tilde{a} \exp(\alpha t)
\]

we have the characteristic equation

\[
(\alpha^2 M + \alpha C + K)\tilde{a} = 0
\]

where \( \alpha \) and \( \tilde{a} \) will in general be found to be complex. The real part of the solution represents a decaying vibration.

The eigenvalue problem involved in Eq. (17.41) is more difficult than that arising in the previous sections. In solutions to date the problem is usually solved by splitting Eq. (17.40) into two first-order equations. This is accomplished by defining

\[
\tilde{a} = v
\]
and writing the split form as

\[
\begin{bmatrix}
M & 0 \\
0 & -M
\end{bmatrix}
\begin{bmatrix}
v \\
\dot{a}
\end{bmatrix} +
\begin{bmatrix}
C & K \\
M & 0
\end{bmatrix}
\begin{bmatrix}
v \\
a
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  

(17.43)

Now substituting

\[a = \bar{a} \exp(\alpha t) \quad v = \bar{v} \exp(\alpha t)\]

gives the general linear eigenproblem

\[
\left(\alpha \begin{bmatrix}
M & 0 \\
0 & -M
\end{bmatrix} + \begin{bmatrix}
C & K \\
M & 0
\end{bmatrix}\right)
\begin{bmatrix}
\bar{v} \\
\bar{a}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  

(17.44)

This form has been studied by Chen et al.\textsuperscript{34-36} Similar to the first-order problem, no steady-state solution exists and once more the concept of eigenvalues of the above kind is generally of importance only in modal analysis, as we shall see later.

### 17.7 Forced periodic response

If the forcing term in Eq. (17.21) is periodic or, more generally, if we can express it as

\[f = \bar{f} \exp(\alpha t)\]

(17.45)

where \(\alpha\) is complex, i.e.

\[\alpha = \alpha_1 + i \alpha_2\]

(17.46)

then a general solution can once more be written as

\[a = \bar{a} \exp(\alpha t)\]

(17.47)

Substituting the above in Eq. (17.21) gives

\[
(\alpha^2 M + \alpha C + K)\bar{a} \equiv \bar{K}\bar{a} = -\bar{f}
\]

(17.48)

which is no longer an eigenvalue problem but can be solved formally by inverting the matrix \(K\) as

\[\bar{a} = -\bar{K}^{-1}\bar{f}\]

(17.49)

The solution is thus precisely of the same form as that used for static problems but now, however, has to be determined in terms of complex quantities. Computer programs are available for operation of complex numbers but computation can be arranged in real numbers directly, noting that

\[
\exp(\alpha t) = \exp(\alpha_1 t) [\cos \alpha_2 t + i \sin \alpha_2 t]
\]

\[
\bar{f} = \bar{f}_1 + i \bar{f}_2
\]

\[
\bar{a} = \bar{a}_1 + i \bar{a}_2
\]

(17.50)

in which \(\alpha_1, \alpha_2, \bar{f}_1, \bar{f}_2, \bar{a}_1\) and \(\bar{a}_2\) are real quantities. Inserting the above into Eq. (17.48) we have

\[
\begin{bmatrix}
(\alpha_1^2 - \alpha_2^2) M + \alpha_1 C + K, & -2\alpha_1 \alpha_2 M - \alpha_2 C \\
-2\alpha_1 \alpha_2 M - \alpha_2 C, & -(\alpha_1^2 - \alpha_2^2) M - \alpha_1 C - K
\end{bmatrix}
\begin{bmatrix}
\bar{a}_1 \\
\bar{a}_2
\end{bmatrix} =
\begin{bmatrix}
\bar{f}_1 \\
-\bar{f}_2
\end{bmatrix}
\]

(17.51)
Equations (17.51) form a system in which all quantities are real and from which the response to any periodic input can be determined by direct solution. The system is no longer positive definite although it has been written in a form which is still symmetric.

With periodic input the solution after an initial transient is not sensitive to the initial conditions and the above solution represents the finally established response. It is valid for problems of dynamic structural and fluid-structure responses as well as for problems typical of heat conduction in which we simply put $M = 0$.

17.8 Transient response by analytical procedures

17.8.1 General

In the previous sections we have been concerned with steady-state general solutions which took no account of the initial conditions of the system or of the non-periodic form of the forcing terms. The response taking these features into account is essential if we consider, for instance, the earthquake behaviour of structures or the transient behaviour of the heat conduction problem. The solution of such general cases requires either a full-time discretization, which we shall discuss in detail in the next chapter, or the use of special analytical procedures. Here two broad possibilities exist:

1. the frequency response procedure
2. the modal analysis procedure.

We shall discuss these briefly.

17.8.2 Frequency response procedures

In Sec. 17.7 we have shown how the response of the system to any forcing terms of the general periodic type or in particular to a periodic forcing function

$$f = \bar{f}\exp(i\omega t)$$

(17.52)

can be obtained by solving a simple equation system. As a completely arbitrary forcing function can be represented approximately by a Fourier series or in the limit, exactly, as a Fourier integral, the response to such an input can be obtained by a synthesis of a curve representing the response of any quantity of interest, e.g., the displacement at a particular point, etc., to all frequencies ranging from zero to infinity. In fact only a limited number of such forcing frequencies has to be considered and a result can be synthesized efficiently by fast Fourier transform techniques. We shall not discuss the mathematical details for such procedures which can be found in standard texts on structural dynamics.\textsuperscript{12,16}

The technique of frequency response is readily adapted to problems where the damping matrix $C$ is of an arbitrary specified form. This is not the case with the more widely used modal decomposition procedures which are to be described in the next section.
By way of illustration we show in Fig. 17.6 the frequency response of an artificial harbour [see Eq. (17.37)] to an input of waves with different frequencies and damping due to the radiation of reflected waves which imposes a very particular form on the damping matrix. Details of this problem are given elsewhere (see also Volume 3). Similar techniques are frequently used in the analysis for the foundation response of structures where radiation of energy occurs.40

### 17.8.3 Modal decomposition analysis

This procedure is probably the most important and widely used in practice. Further, it provides an insight into the behaviour of the whole system, which is of value where strictly numerical processes are used. We shall therefore describe it in detail in the context of the general problem of Eq. (17.21), i.e.,

\[ M \ddot{a} + C \dot{a} + K a + f = 0 \]  \hspace{1cm} (17.53)

where \( f \) is an arbitrary function of time.

We have seen that the general solution for the free response is of the form

\[ a = \sum_{i=1}^{n} \bar{a}_i \exp(\alpha_i t) \]  \hspace{1cm} (17.54)

where \( \alpha_i \) are the (complex) eigenvalues and \( \bar{a}_i \) are the (complex) eigenvectors (Sec. 17.6). For forced response we shall assume that the solution can be written in a linear combination of modes as

\[ a = \sum_{i=1}^{n} \bar{a}_i y_i(t) = [\bar{a}_1, \bar{a}_2, \ldots]y(t) \]  \hspace{1cm} (17.55)

where the scalar mode participation factor \( y_i \) is now a function of time. This shows in a clear manner the proportions of each mode occurring. Such a decomposition of an arbitrary vector presents no restriction as all the modes are linearly independent vectors (with those for repeated frequencies being constructed to be linearly independent as mentioned in Sec. 17.4).

If expression (17.55) is substituted into Eq. (17.53) and the result is premultiplied by the complex conjugate transposed, \( \bar{a}_i^T \) \((i = 1, \ldots, n)\), then the result is simply a set of scalar, independent, equations

\[ m_i \ddot{y}_i + c_i \dot{y}_i + k_i y_i + f_i = 0 \]  \hspace{1cm} (17.56)

where

\[ m_i = \bar{a}_i^T M \bar{a}_i \]
\[ c_i = \bar{a}_i^T C \bar{a}_i \]
\[ k_i = \bar{a}_i^T K \bar{a}_i \]
\[ f_i = \bar{a}_i^T f \]  \hspace{1cm} (17.57)
The time dimension – semi-discretization of field and dynamic problems

Wave forcing frequency $\omega = k \sqrt{gh} = ka$, $h$ = depth of water

Fig. 17.6 Frequency response of an artificial harbour to an input of periodic wave.
as for true eigenvectors $\bar{a}_i$

$$\bar{a}_i^T \bar{M} \bar{a}_j = \bar{a}_i^T \bar{C} \bar{a}_j = \bar{a}_i^T \bar{K} \bar{a}_j = 0$$  \hspace{1cm} (17.58)

when $i \neq j$ (this result was proved in Sec. 17.4 for real eigenpairs but is valid generally for complex pairs, as could be verified by the reader).

Each scalar equation of (17.56) can be solved by elementary procedures independently and the total vector of response obtained by superposition following Eq. (17.57). In the general case, as we have shown in Sec. 17.6, the eigenpairs are complex and their determination is not simple.\textsuperscript{30} The more usual procedure is to use real eigenpairs corresponding to the solution of Eq. (17.22):

$$\bar{K} \bar{a} = \omega^2 \bar{M} \bar{a}$$  \hspace{1cm} (17.59)

Now repetition of procedures using the process described in Eqs (17.55)-(17.58) leads to decoupled equations with real variables $y$ only if

$$\bar{a}_i^T \bar{C} \bar{a}_j = 0; \quad i \neq j$$  \hspace{1cm} (17.60)

which generally does not occur as the eigenvectors now guarantee only orthogonality with $\bar{M}$ and $\bar{K}$ and not of the damping matrix. However, if the damping matrix $\bar{C}$ is of the form of Eq. (17.16), i.e., a linear combination of $\bar{M}$ and $\bar{K}$, such orthogonality will obviously occur. Unless the damping is of a definite form which requires special treatment, an assumption of orthogonality is made and Eq. (17.56) is assumed valid in terms of such eigenvectors.

From Eq. (17.59) we have

$$\bar{K} \bar{a}_i = \omega_i^2 \bar{M} \bar{a}_i$$  \hspace{1cm} (17.61)

and on premultiplying by $\bar{a}_i^T$ we obtain

$$k_i = \omega_i^2 m_i$$  \hspace{1cm} (17.62)

Writing the modal damping in the form

$$c_i = 2 \omega \xi_i$$  \hspace{1cm} (17.63)

(where $\xi_i$ represents the ratio of damping to its critical value) and assuming that the modes have been normalized so that $m_i = 1$ [see Eq. (17.25)], Eq. (17.56) can be rewritten in standard second order form:

$$\ddot{y}_i + 2 \omega_i \xi_i \dot{y}_i + \omega_i^2 y_i + f_i = 0$$  \hspace{1cm} (17.64)

A general solution can then be obtained by writing

$$y_i = \exp(-\xi_i \omega_i t) \left[ \frac{\dot{y}_{i0} + \xi_i \omega_i y_{i0}}{\omega} \sin \omega_i t + y_{i0} \cos \omega_i t \right]$$

$$+ \frac{1}{\omega_i} \int_0^t \exp(-\xi_i \omega_i [t - \tau]) \sin \omega_i (t - \tau) f_i(\tau) \, d\tau$$  \hspace{1cm} (17.65)

in which $\omega_i = \omega_i \sqrt{1 - \xi_i^2}$ and $y_{i0}, \dot{y}_{i0}$ are initial conditions computed from

$$y_{i0} = \bar{a}_i^T \bar{M} \bar{a}(0)$$

$$\dot{y}_{i0} = \bar{a}_i^T \bar{M} \dot{a}(0)$$  \hspace{1cm} (17.66)
The solution of Eq. (17.65) can be carried out by assuming the forcing function is given by linear interpolation between discrete time points $t_k$ and then evaluating the resulting integrals exactly. Alternatively, a numerical solution can be carried out and the response obtained. In practice, often a single calculation is carried out for each mode to determine the maximum responses and a suitable addition of these results is used. Such processes are described in standard texts and are used as procedures to calculate the bounds on behaviour of structures subjected to seismic loading.\textsuperscript{12,16,27}

### 17.8.4 Damping and participation of modes

The type of calculation implied in modal decomposition apparently necessitates the determination of all modes and eigenvalues, a task of considerable magnitude. In fact only a limited number of modes usually need to be taken into consideration as often the response to higher frequency is critically damped and insignificant.

To show that this is true consider the form of the damping matrices. In Sec. 17.2 [Eq. (17.16)] we have indicated that the damping matrix is often assumed as

\[ C = \alpha M + \beta K \]  

(17.67)

Indeed a form of this type is necessary for the use of modal decomposition, although other generalizations are possible.\textsuperscript{31,42} From the definition of $\xi_i$, the critical damping ratio in Eq. (17.63), we see that this can now be written as

\[ \xi_i = \frac{1}{\omega_i} \bar{a}_i^T (\alpha M + \beta K) \bar{a}_i = \frac{1}{2\omega_i} (\alpha + \beta \omega_i^2) \]  

(17.68)

Thus if the coefficient $\beta$ is of greater importance, as is the case with most structural damping, $\xi_i$ grows with $\omega_i$ and at high frequency an overdamped condition will arise.\textsuperscript{12} This is indeed fortunate as, in general, an infinite number of high frequencies exist which are not modelled by any finite element discretization.

We shall see in the next chapter that in the step-by-step recurrence computation the high frequencies often control the problem, and this effect needs to be ‘filtered out’ for realistic results.

### 17.9 Symmetry and repeatability

In concluding this chapter it is worth remarking that in dynamic calculation we have once again encountered all the general principles of assembly, etc., that are applicable to static problems. However, some aspects of symmetry and repeatability which were used previously (see Sec. 9.13) need amending. It is obviously possible for symmetric structures to vibrate in an unsymmetrical manner, for instance, and similarly a repeatable structure contains modes which are themselves non-repeatable. However, even here considerable simplification can still be made; details of this are discussed by Williams,\textsuperscript{43} Thomas\textsuperscript{44} and Evensen.\textsuperscript{45}
References

The time dimension – semi-discretization of field and dynamic problems


