Incompressible materials, mixed methods and other procedures of solution

12.1 Introduction

We have noted earlier that the standard displacement formulation of elastic problems fails when Poisson's ratio $\nu$ becomes 0.5 or when the material becomes incompressible. Indeed, problems arise even when the material is nearly incompressible with $\nu > 0.4$ and the simple linear approximation with triangular elements gives highly oscillatory results in such cases.

The application of a mixed formulation for such problems can avoid the difficulties and is of great practical interest as nearly incompressible behaviour is encountered in a variety of real engineering problems ranging from soil mechanics to aerospace engineering. Identical problems also arise when the flow of incompressible fluids is encountered.

In this chapter we shall discuss fully the mixed approaches to incompressible problems, generally using a two-field manner where displacement (or fluid velocity) $\mathbf{u}$ and the pressure $p$ are the variables. Such formulation will allow us to deal with full incompressibility as well as near incompressibility as it occurs. However, what we will find is that the interpolations used will be very much limited by the stability conditions of the mixed patch test. For this reason much interest has been focused on the development of so-called stabilized procedures in which the violation of the mixed patch test (or Babuška–Brezzi conditions) is artificially compensated. A part of this chapter will be devoted to such stabilized methods.

12.2 Deviatoric stress and strain, pressure and volume change

The main problem in the application of a ‘standard’ displacement formulation to incompressible or nearly incompressible problems lies in the determination of the mean stress or pressure which is related to the volumetric part of the strain (for isotropic materials). For this reason it is convenient to separate this from the total stress field and treat it as an independent variable. Using the ‘vector’ notation of stress, the mean stress or pressure is given by

$$p = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}\mathbf{m}^T\sigma$$

(12.1)
where $m$ for the general three-dimensional state of stress is given by

$$m = [1, 1, 1, 0, 0, 0]^T$$

For isotropic behaviour the 'pressure' is related to the volumetric strain, $\varepsilon_v$, by the bulk modulus of the material, $K$. Thus,

$$\varepsilon_v = \frac{p}{K}$$ (12.3)

For an incompressible material $K = \infty$ ($\nu \equiv 0.5$) and the volumetric strain is simply zero.

The deviatoric strain $\varepsilon^d$ is defined by

$$\varepsilon^d = \varepsilon - \frac{1}{3} m \varepsilon_v \equiv (I - \frac{1}{3} mm^T)\varepsilon = I_d \varepsilon$$ (12.4)

where $I_d$ is a deviatoric projection matrix which proves useful later and in Volume 2. In isotropic elasticity the deviatoric strain is related to the deviatoric stress by the shear modulus $G$ as

$$\sigma^d = I_d \sigma = 2G I_0 \varepsilon^d = 2G (I_0 - \frac{1}{3} mm^T)\varepsilon$$ (12.5)

where the diagonal matrix

$$I_0 = \frac{1}{2} \begin{bmatrix} 2 & & & \frac{1}{3} & & \frac{1}{3} \\ & 2 & & & \frac{1}{3} & \frac{1}{3} \\ & & 1 & & \frac{1}{3} & \frac{1}{3} \\ & & & 1 & & \frac{1}{3} \\ & & & & 1 & \frac{1}{3} \\ & & & & & 1 \end{bmatrix}$$

is introduced because of the vector notation. A deviatoric form for the elastic moduli of an isotropic material is written as

$$D_d = 2G (I_0 - \frac{1}{3} mm^T)$$ (12.6)

for convenience in writing subsequent equations.

The above relationships are but an alternate way of determining the stress strain relations shown in Chapters 2 and 4–6, with the material parameters related through

$$G = \frac{E}{2(1 + \nu)}$$

$$K = \frac{E}{3(1 - 2\nu)}$$ (12.7)

and indeed Eqs (12.5) and (12.3) can be used to define the standard $D$ matrix in an alternative manner.

### 12.3 Two-field incompressible elasticity ($u-p$ form)

In the mixed form considered next we shall use as variables the displacement $u$ and the pressure $p$. 
Now the equilibrium equation (11.22) is rewritten using (12.5), treating \( p \) as an independent variable, as
\[
\int_\Omega \delta \varepsilon^T D_{\varepsilon} \delta \varepsilon \, d\Omega + \int_\Omega \delta \varepsilon^T m \, d\Omega - \int_\Omega \delta u^T b \, d\Omega - \int_{\Gamma_r} \delta u^T t \, d\Gamma = 0
\] (12.8)
and in addition we shall impose a weak form of Eq. (12.3), i.e.,
\[
\int_\Omega \delta p \left[ m^T \varepsilon - \frac{p}{K} \right] \, d\Omega = 0
\] (12.9)
with \( \varepsilon = Su \). Independent approximation of \( u \) and \( p \) as
\[
u \approx \tilde{u} = N_u \tilde{u} \quad \text{and} \quad p \approx \tilde{p} = N_p \tilde{p}
\] (12.10)
immediately gives the mixed approximation in the form
\[
\begin{bmatrix}
A & C \\
C^T & -V
\end{bmatrix}
\begin{bmatrix}
\tilde{u} \\
\tilde{p}
\end{bmatrix} =
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}
\] (12.11)
where
\[
A = \int_\Omega B^T D_{\varepsilon} B \, d\Omega \quad C = \int_\Omega B^T m N_p \, d\Omega \quad V = \int_\Omega N_p^T \frac{1}{K} N_p \, d\Omega \quad f_1 = \int_\Omega N_u^T b \, d\Omega + \int_{\Gamma_r} N_u^T t \, d\Gamma \quad f_2 = 0
\] (12.12)

We note that for incompressible situations the equations are of the ‘standard’ form, see Eq. (11.14) with \( V = 0 \) (as \( K = \infty \)), but the formulation is useful in practice when \( K \) has a high value (or \( \nu \to 0.5 \)).

A formulation similar to that above and using the corresponding variational theorem was first proposed by Herrmann\(^1\) and later generalized by Key\(^2\) for anisotropic

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**Fig. 12.1** Incompressible elasticity \( u-p \) formulation. Discontinuous pressure approximation. (a) Single-element patch tests.
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Fig. 12.1 (continued) Incompressible elasticity $u-p$ formulation. Discontinuous pressure approximation.
(b) Multiple-element patch tests.

... elasticity. The arguments concerning stability (or singularity) of the matrices which we presented in Sec. 11.3 are again of great importance in this problem.

Clearly the mixed patch condition about the number of degrees of freedom now yields [see Eq. (11.18)]

$$n_u \geq n_p \quad (12.13)$$
and *is necessary* for prevention of locking (or instability) with the pressure acting now as the constraint variable of the lagrangian multiplier enforcing incompressibility.

In the form of a patch test this condition is most critical and we show in Figs 12.1 and 12.2 a series of such patch tests on elements with $C_0$ continuous interpolation of $u$ and either discontinuous or continuous interpolation of $p$. For each we have included all combinations of constant, linear and quadratic functions.

In the test we prescribe *all* the displacements on the boundaries of the patch and one pressure variable (as it is well known that in fully incompressible situations pressure will be indeterminate by a constant for the problem with all boundary displacements prescribed).

The single-element test is very stringent and eliminates most continuous pressure approximations whose performance is known to be acceptable in many situations. For this reason we attach more importance to the assembly test and it would appear that the following elements could be permissible according to the criteria of Eq. (12.13) (indeed all pass the B-B condition fully):

**Triangles:** T6/1; T10/3; T6/C3

**Quadrilaterals:** Q9/3; Q8/C4; Q9/C4

We note, however, that in practical applications quite adequate answers have been reported with Q4/1, Q8/3 and Q9/4 quadrilaterals, although severe oscillations of $p$ may occur. If full robustness is sought the choice of the elements is limited.\(^3\)

It is unfortunate that in the present 'acceptable' list, the linear triangle and quadrilateral are missing. This appreciably restricts the use of these simplest elements. A possible and indeed effective procedure here is to not apply the pressure constraint at the level of a single element but on an assembly. This was done by Herrmann in his original presentation\(^1\) where four elements were chosen for such a constraint as shown in Fig. 12.3(a). This composite 'element' passes the single-element (and multiple-element) patch tests but apparently so do several others fitting into this category. In Fig. 12.3(b) we show how a single triangle can be internally subdivided into three parts by the introduction of a central node. This coupled with constant pressure on the assembly allows the necessary count condition to be satisfied and a standard element procedure applies to the original triangle treating the central node as an internal variable. Indeed, the same effect could be achieved by the introduction of any other internal element function which gives zero value on the main triangle perimeter. Such a *bubble function* can simply be written in terms of the area coordinates (see Chapter 8) as

\[
L_1L_2L_3
\]

However, as we have stated before, the degree of freedom count is a necessary but not sufficient condition for stability and a direct rank test is always required. In particular it can be verified by algebra that the conditions stated in Sec. 11.3 are not fulfilled for this triple subdivision of a linear triangle (or the case with the bubble function) and thus

\[
Cp = 0 \text{ for some non-zero values of } p
\]

indicating instability.
Fig. 12.2 Incompressible elasticity $u–p$ formulation. Continuous ($C_0$) pressure approximation. (a) Single-element patch tests. (b) Multiple-element patch tests.
Fig. 12.3 Some simple combinations of linear triangles and quadrilaterals that pass the necessary patch test counts. Combinations (a), (c), and (d) are successful but (b) is still singular and not usable.
Fig. 12.4 Locking (zero displacements) of a simple assembly of linear triangles for which incompressibility is fully required \( (n_p = n_u = 24) \).

In Fig. 12.3(c) we show, however, that the same concept can be used with good effect for \( C_0 \) continuous \( p \). Similar internal subdivision into quadrilaterals or the introduction of bubble functions in quadratic triangles can be used, as shown in Fig. 12.3(d), with success.

The performance of all the elements mentioned above has been extensively discussed\(^5\)\(^\text{--}^\text{10} \) but detailed comparative assessment of merit is difficult. As we have observed, it is essential to have \( n_u \geq n_p \) but if near equality is only obtained in a large problem no meaningful answers will result for \( u \) as we observe, for example, in Fig. 12.4 in which linear triangles for \( u \) are used with the element constant \( p \). Here the only permissible answer is of course \( u = 0 \) as the triangles have to preserve constant volumes.

The ratio \( n_u/n_p \) which occurs as the field of elements is enlarged gives some indication of the relative performance, and we show this in Fig. 12.5. This approximates to the behaviour of a very large element assembly, but of course for any practical problem such a ratio will depend on the boundary conditions imposed.

We see that for the discontinuous pressure approximation this ratio for 'good' elements is 2–3 while for \( C_0 \) continuous pressure it is 6–8. All the elements shown in Fig. 12.5 perform very well, though two (Q4/1 and Q9/4) can on occasion lock when most boundary conditions are on \( u \).

### 12.4 Three-field nearly incompressible elasticity \( (u-p-\varepsilon_v) \) form

A direct approximation of the three-field form leads to an important method in finite element solution procedures for nearly incompressible materials which has sometimes been called the B-bar method. The methodology can be illustrated for the nearly
incompressible isotropic problem. For this problem the method often reduces to the same two-field form previously discussed. However, for more general anisotropic or inelastic materials and in finite deformation problems the method has distinct advantages as will be discussed further in Volume 2. The usual irreducible form (displacement method) has been shown to 'lock' for the nearly incompressible problem. As shown in Sec. 12.3, the use of a two-field mixed method can avoid this locking phenomenon when properly implemented (e.g., using the Q9/3 two-field form). Below we present an alternative which leads to an efficient and accurate implementation in many situations. For the development shown we shall assume

Fig. 12.5 The freedom index or infinite patch ratio for various u–p elements for incompressible elasticity ($\gamma = n_u/n_p$). (a) Discontinuous pressure. (b) Continuous pressure.
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that the material is isotropic linear elastic but it may be extended easily to include anisotropic materials.

Assuming an independent approximation to \( \varepsilon_v \) and \( p \) we can formulate the problem by use of Eq. (12.8) and the weak statement of relations (12.2) and (12.3) written as

\[
\begin{align*}
\int_{\Omega} \delta p \left[ m^T \mathbf{S} u - \varepsilon_v \right] \, d\Omega &= 0 \quad (12.14) \\
\int_{\Omega} \delta \varepsilon_v \left[ K \varepsilon_v - p \right] \, d\Omega &= 0 \quad (12.15)
\end{align*}
\]

If we approximate the \( u \) and \( p \) fields by Eq. (12.10) and

\[
\varepsilon_v \approx \tilde{\varepsilon}_v = N_p \tilde{e}_v
\]

we obtain a mixed approximation in the form

\[
\begin{bmatrix}
A & C & 0 \\
C^T & 0 & -\mathbf{E} \\
0 & -\mathbf{E}^T & \mathbf{H}
\end{bmatrix}
\begin{bmatrix}
\mathbf{\tilde{u}} \\
\mathbf{\tilde{p}} \\
\tilde{e}_v
\end{bmatrix} =
\begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}
\quad (12.17)
\]

where \( A, C, f_1, f_2 \) are given by Eq. (12.12) and

\[
\mathbf{E} = \int_{\Omega} N_v^T N_p \, d\Omega \quad f_3 = 0 \quad (12.18)
\]

with

\[
\mathbf{H} = \int_{\Omega} N_v^T K N_v \, d\Omega \quad (12.19)
\]

For completeness we give the variational theorem whose first variation gives Eqs (12.8), (12.14) and (12.15). First we define the strain deduced from the standard displacement approximation as

\[
\varepsilon_u = \mathbf{S} u \approx \mathbf{B} \tilde{u}
\]

The variational theorem is then given as

\[
\Pi = \frac{1}{2} \int_{\Omega} \left( \varepsilon_u^T \mathbf{D} \varepsilon_u + \varepsilon_v K \varepsilon_v \right) \, d\Omega + \int_{\Omega} p (\mathbf{m}^T \varepsilon_u - \varepsilon_v) \, d\Omega \\
- \int_{\Omega} \mathbf{u}^T \mathbf{b} \, d\Omega - \int_{\Gamma_t} \mathbf{u}^T \mathbf{t} \, d\Gamma \quad (12.21)
\]

### 12.4.1 The B-bar method for nearly incompressible problems

The second of (12.17) has the solution

\[
\tilde{\varepsilon}_v = \mathbf{E}^{-1} C^T \tilde{u} = \mathbf{W} \tilde{u}
\]

In the above we assume that \( \mathbf{E} \) may be inverted, which implies that \( N_v \) and \( N_p \) have the same number of terms. Furthermore, the approximations for the volumetric strain and pressure are constructed for each element individually and are not continuous
across element boundaries. Thus, the solution of Eq. (12.22) may be performed for each individual element. In practice \( N \) is normally assumed identical to \( N_p \) so that \( E \) is symmetric positive definite. The solution of the third of (12.17) yields the pressure parameters in terms of the volumetric strain parameters and is given by

\[ \hat{p} = E^{-T}H^T\hat{\varepsilon}_v \]  

(12.23)

Substitution of (12.22) and (12.23) into the first of (12.17) gives a solution that is in terms of displacements only. Accordingly,

\[ \tilde{A}\tilde{u} = f_i \]  

(12.24)

where for isotropy

\[ \tilde{A} = \int_\Omega B^T D_d B \, d\Omega + W^T H W \]

(12.25)

The solution of (12.24) yields the nodal parameters for the displacements. Use of (12.22) and (12.23) then gives the approximations for the volumetric strain and pressure.

The result given by (12.25) may be further modified to obtain a form that is similar to the standard displacement method. Accordingly, we write

\[ \tilde{A} = \int_\Omega \tilde{B}^T \tilde{D} \tilde{B} \, d\Omega \]  

(12.26)

where the strain-displacement matrix is now

\[ \tilde{B} = I_d B + \frac{1}{3} m N_p W \]  

(12.27)

For isotropy the modulus matrix is

\[ D = D_d + K m m^T \]  

(12.28)

We note that the above form is identical to a standard displacement model except that \( B \) is replaced by \( \tilde{B} \). The method has been discussed more extensively in references 11, 12 and 13.

The equivalence of (12.25) and (12.26) can be verified by simple matrix multiplication. Extension to treat general small strain formulations can be simply performed by replacing the isotropic \( D \) matrix by an appropriate form for the general material model. The formulation shown above has been implemented into an element included as part of the program referred to in Chapter 20. The elegance of the method is more fully utilized when considering non-linear problems, such as plasticity and finite deformation elasticity (see Volume 2).

We note that elimination starting with the third equation could be accomplished leading to a \( u-p \) two-field form using \( K \) as a penalty number. This is convenient for the case where \( p \) is continuous but \( \varepsilon_v \) remains discontinuous – this is discussed further in Sec. 12.7.3. Such an elimination, however, points out that precisely the same stability criteria operate here as in the two-field approximation discussed earlier.


12.5 Reduced and selective integration and its equivalence to penalized mixed problems

In Chapter 9 we mentioned the lowest order numerical integration rules that still preserve the required convergence order for various elements, but at the same time pointed out the possibility of a singularity in the resulting element matrices. In Chapter 10 we again referred to such low order integration rules, introducing the name 'reduced integration' for those that did not evaluate the stiffness exactly for simple elements and pointed out some dangers of its indiscriminate use due to resulting instability. Nevertheless, such reduced integration and selective integration (where the low order approximation is only applied to certain parts of the matrix) has proved its worth in practice, often yielding much more accurate results than the use of more precise integration rules. This was particularly noticeable in nearly incompressible elasticity (or Stokes fluid flow which is similar)\textsuperscript{14-16} and in problems of plate and shell flexure dealt with as a case of a degenerate solid\textsuperscript{17,18} (see Volume 2).

The success of these procedures derived initially by heuristic arguments proved quite spectacular — though some consider it somewhat verging on immorality to obtain improved results while doing less work! Obviously fuller justification of such processes is required.\textsuperscript{19} The main reason for success is associated with the fact that it provides the necessary singularity of the constraint part of the matrix [viz. Eqs (11.19)–(11.21)] which avoids locking. Such singularity can be deduced from a count of integration points,\textsuperscript{19,20} but it is simpler to show that there is a complete equivalence between reduced (of selective) integration procedures and the mixed formulation already discussed. This equivalence was first shown by Malkus and Hughes\textsuperscript{21} and later in a general context by Zienkiewicz and Nakazawa.\textsuperscript{22}

We shall demonstrate this equivalence on the basis of the nearly incompressible elasticity problem for which the mixed weak integral statement is given by Eqs (12.8) and (12.9). It should be noted, however, that equivalence holds only for the discontinuous pressure approximation.

The corresponding irreducible form can be written by satisfying the second of these equations exactly by implying

\[ p = K m^T \varepsilon \]  (12.29)

and substituting above into (12.8) as

\[ \int_\Omega \delta \varepsilon^T 2G \left( I_0 - \frac{1}{3} m^T m \right) \varepsilon \, d\Omega + \int_\Omega \delta \varepsilon^T m \varepsilon \, d\Omega = \int_\Gamma u^T b \, d\Gamma - \int_\Gamma u^T \bar{t} \, d\Gamma = 0 \]  (12.30)

On substituting

\[ u \approx \hat{u} = N_u \hat{u} \quad \text{and} \quad \varepsilon \approx \hat{\varepsilon} = S N_u \hat{u} = B \hat{u} \]  (12.31)

we have

\[ (A + \tilde{A}) \hat{u} = f_1 \]  (12.32)
where $A$ and $f_i$ are exactly as given in Eq. (12.12) and

$$\tilde{A} = \int_\Omega B^TmKm^TB \, d\Omega$$

(12.33)

The solution of Eq. (12.32) for $\tilde{u}$ allows the pressures to be determined at all points by Eq. (12.29). In particular, if we have used an integration scheme for evaluating (12.33) which samples at points $(\xi_k)$ we can write

$$p(\xi_k) = Km^Tc(\xi_k) = Km^TB(\xi_k)\tilde{u} = \sum_j N_{pj}(\xi_k)\tilde{p}_j$$

(12.34)

Now if we turn our attention to the penalized mixed form of Eqs (12.8)-(12.12) we note that the second of Eqn. (12.11) is explicitly

$$\int_\Omega N_p^T \left( m^T \tilde{B} \tilde{u} - \frac{1}{K} N_p \tilde{p} \right) \, d\Omega = 0$$

(12.35)

If a numerical integration is applied to the above sampling at the pressure nodes located at coordinate $(\xi_i)$, previously defined in Eq. (12.34), we can write for each scalar component of $N_p$

$$\sum_i N_{pi}(\xi_i) \left( m^T B(\xi_i)\tilde{u} - \frac{1}{K} N_p(\xi_i)\tilde{p} \right) W_i = 0$$

(12.36)

in which the summation is over all integration points $(\xi_i)$ and $W_i$ are the appropriate weights and jacobian determinants.

Now as

$$N_{pi}(\xi_k) = \delta_{jk}$$

if $\xi_i$ is at the pressure node $j$ and zero at other pressure nodes, Eq. (12.36) reduces simply to the requirement that at all pressure nodes

$$m^T B(\xi_i) \tilde{u} = \frac{1}{K} N_p(\xi_i)\tilde{p}$$

(12.37)

This is precisely the same condition as that given by Eq. (12.34) and the equivalence of the procedures is proved, providing the integrating scheme used for evaluating $\tilde{A}$ gives an identical integral of the mixed form of Eq. (12.35).

This is true in many cases and for these the reduced integration-mixed equivalence is exact. In all other cases this equivalence exists for a mixed problem in which an inexact rule of integration has been used in evaluating equations such as (12.35).

For curved isoparametric elements the equivalence is in fact inexact, and slightly different results can be obtained using reduced integration and mixed forms. This is illustrated in examples given in reference 23.

We can conclude without detailed proof that this type of equivalence is quite general and that with any problem of a similar type the application of numerical quadrature at $n_p$ points in evaluating the matrix $\tilde{A}$ within each element is equivalent to a mixed problem in which the variable $p$ is interpolated element-by-element using as $p$-nodal values the same integrating points.

The equivalence is only complete for the selective integration process, i.e., application of reduced numerical quadrature only to the matrix $\tilde{A}$, and ensures that this
matrix is singular, i.e., no locking occurs if we have satisfied the previously stated conditions \( n_u > n_p \).

The full use of reduced integration on the remainder of the matrix determining \( \mathbf{u} \), i.e., \( \mathbf{A} \), is only permissible if that remains non-singular – the case which we have discussed previously for the Q8/4 element.

It can therefore be concluded that all the elements with discontinuous interpolation of \( p \) which we have verified as applicable to the mixed problem (viz. Fig. 12.1, for instance) can be implemented for nearly incompressible situations by a penalized irreducible form using corresponding selective integration.†

In Fig. 12.6 we show an example which clearly indicates the improvement of displacements achieved by such reduced integration as the compressibility modulus \( K \) increases (or the Poisson ratio tends to 0.5). We note also in this example the dramatically improved performance of such points for stress sampling.

For problems in which the \( p \) (constraint) variable is continuously interpolated (\( C_0 \)) the arguments given above fail as quantities such as \( \mathbf{m}^T \mathbf{e} \) are not interelement continuous in the irreducible form.

A very interesting corollary of the equivalence just proved for (nearly) incompressible behaviour is observed if we note the rapid increase of order of integrating formulae with the number of quadrature points (viz. Chapter 9). For high order elements the number of quadrature points equivalent to the \( p \) constraint permissible for stability rapidly reaches that required for exact integration and hence their performance in nearly incompressible situations is excellent, even if exact integration is used. This was observed on many occasions\(^{24-26} \) and Sloan and Randolf\(^{27} \) have shown good performance with the quintic triangle. Unfortunately such high order elements pose other difficulties and are seldom used in practice.

A final remark concerns the use of 'reduced' integration in particular and of penalized, mixed, methods in general. As we have pointed out in Sec. 11.3.1 it is possible in such forms to obtain sensible results for the primary variable (\( \mathbf{u} \) in the present example) even though the general stability conditions are violated, providing some of the constraint equations are linearly dependent. Now of course the constraint variable (\( p \) in the present example) is not determinate in the limit.

This situation occurs with some elements that are occasionally used for the solution of incompressible problems but which do not pass our mixed patch test, such as Q8/4 and Q9/4 of Fig. 12.1. If we take the latter number to correspond to the integrating points these will yield acceptable \( \mathbf{u} \) fields, though not \( p \).

Figure 12.7 illustrates the point on an application involving slow viscous flow through an orifice – a problem that obeys identical equations to those of incompressible elasticity. Here elements Q8/4, Q8/3, Q9/4 and Q9/3 are compared although only the last completely satisfies the stability requirements of the mixed patch test. All elements are found to give a reasonable velocity (\( \mathbf{u} \)) field but pressures are acceptable only for the last one, with element Q8/4 failing to give results which can be plotted.\(^{3} \)

† The Q9/3 element would involve three-point quadrature which is somewhat unnatural for quadrilaterals. It is therefore better to simply use the mixed form here – and, indeed, in any problem which has non-linear behaviour between \( p \) and \( \mathbf{u} \) (see Volume 2).
Fig. 12.6 Sphere under internal pressure. Effect of numerical integration rules on results with different Poisson ratios.
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Fig. 12.7 Steady-state, low Reynolds number flow through an orifice. Note that pressure variation for element Q8/4 is so large it cannot be plotted. Solution with $u/p$ elements Q8/3, Q8/4, Q9/3, Q9/4.

It is of passing interest to note that a similar situation develops if four triangles of the T3/1 type are assembled to form a quadrilateral in the manner of Fig. 12.8. Although the original element locks, as we have previously demonstrated, a linear dependence of the constraint equation allows the assembly to be used quite effectively in many incompressible situations, as shown in reference 25.
12.6 A simple iterative solution process for mixed problems: Uzawa method

12.6.1 General

In the general remarks on the algebraic solution of mixed problems characterized by equations of the type [viz. Eq. (11.14)]

\[
\begin{bmatrix}
A & C \\
C^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
f_1 \\
f_2
\end{bmatrix}
\]  
(12.38)

we have remarked on the difficulties posed by the zero diagonal and the increased number of unknowns \((n_x + n_y)\) as compared with the irreducible form \((n_x \text{ or } n_y)\).

A general iterative form of solution is possible, however, which substantially reduces the cost.\(^{28}\) In this we solve successively

\[
y^{(k+1)} = y^{(k)} + pr^{(k)}
\]  
(12.39)

where \(r^{(k)}\) is the residual of the second equation computed as

\[
r^{(k)} = C^T x^{(k)} - f_2
\]  
(12.40)

and follow with solution of the first equation, i.e.,

\[
x^{(k+1)} = A^{-1}(f_1 - Cy^{(k+1)})
\]  
(12.41)

In the above \(p\) is a 'convergence accelerator matrix' and is chosen to be efficient and simple to use.

The algorithm is similar to that described initially by Uzawa\(^{29}\) and has been widely applied in an optimization context\(^{30-35}\).

Its relative simplicity can best be grasped when a particular example is considered.

12.6.2 Iterative solution for incompressible elasticity

In this case we start from Eq. (12.11) now written with \(V = 0\), i.e., complete incompressibility is assumed. The various matrices are defined in (12.12), resulting

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Fig. 12.8 A quadrilateral with intersecting diagonals forming an assembly of four T3/1 elements. This allows displacements to be determined for nearly incompressible behaviour but does not yield pressure results.
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in the form

\[
\begin{bmatrix}
A & C \\
C^T & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{u} \\
\tilde{p}
\end{bmatrix} = 
\begin{bmatrix}
f_1 \\
0
\end{bmatrix}
\]  

(12.42)

Now, however, for three-dimensional problems the matrix \(A\) is singular (as volumetric changes are not restrained) and it is necessary to augment it to make it non-singular. We can do this in the manner described in Sec. 11.3.1, or equivalently by the addition of a fictitious compressibility matrix, thus replacing \(A\) by

\[
\tilde{A} = A + \int_\Omega B^T(\lambda Gm m^T)B d\Omega
\]  

(12.43)

If the second matrix uses an integration consistent with the number of discontinuous pressure parameters assumed, then this is precisely equivalent to writing

\[
\tilde{A} = A + \lambda G C C^T
\]  

(12.44)

and is simpler to evaluate. Clearly this addition does not change the equation system.

The iteration of the algorithm (12.39)-(12.41) is now conveniently taken with the 'convergence accelerator' being simply defined as

\[
\rho = \lambda G I
\]  

(12.45)

We now have the iterative system given as

\[
\tilde{p}^{(k+1)} = \tilde{p}^{(k)} + \lambda G r^{(k)}
\]  

(12.46)

where

\[
r^{(k)} = C^T \tilde{u}^{(k)}
\]  

(12.47)

the residual of the incompressible constraint, and

\[
\tilde{u}^{(k+1)} = \tilde{A}^{-1}(f_1 - C \tilde{p}^{(k+1)})
\]  

(12.48)

In this \(\tilde{A}\) can be interpreted as the stiffness matrix of a compressible material with bulk modulus \(K = \lambda G\) and the process may be interpreted as the successive addition of volumetric 'initial' strains designed to reduce the volumetric strain to zero. Indeed this simple approach led to the first realization of this algorithm.\textsuperscript{36–38} Alternatively the process can be visualized as an amendment of the original equation (12.42) by subtracting the term \(p/(\lambda G)\) from each side of the second to give (this is often called an augmented lagrangian form)\textsuperscript{28,34}

\[
\begin{bmatrix}
A & C \\
C^T & 1/\lambda G - I
\end{bmatrix}
\begin{bmatrix}
\tilde{u} \\
\tilde{p}
\end{bmatrix} = 
\begin{bmatrix}
f_1 \\
1/\lambda G \tilde{p}
\end{bmatrix}
\]  

(12.49)

and adopting the iteration

\[
\begin{bmatrix}
\tilde{A} & C \\
C^T & 1/\lambda G - I
\end{bmatrix}
\begin{bmatrix}
\tilde{u}^{(k+1)} \\
\tilde{p}^{(k+1)}
\end{bmatrix} = 
\begin{bmatrix}
f_1 \\
1/\lambda G \tilde{p}^{(k)}
\end{bmatrix}
\]  

(12.50)

With this, on elimination, a sequence similar to Eqs (12.46)-(12.48) will be obtained provided \(\tilde{A}\) is defined by Eq. (12.44).
Starting the iteration from
\[ \tilde{u}^{(0)} = 0 \quad \text{and} \quad \bar{p}^{(0)} = 0 \]
in Fig. 12.9 we show the convergence of the maximum \( \text{div} \mathbf{u} \) computed at any of the integrating points used. We note that this convergence becomes quite rapid for large values of \( \lambda = (10^3 - 10^4) \).

For smaller \( \lambda \) values the process can be accelerated by using different \( \rho \) but for practical purposes the simple algorithm suffices for many problems, including applications in large strain. Clearly much better satisfaction of the incompressibility constraint can now be obtained than by the simple use of a 'large enough' bulk modulus or penalty parameter. With \( \lambda = 10^4 \), for instance, in five iterations the initial \( \text{div} \mathbf{u} \) is reduced from the value \( \sim 10^{-4} \) to \( 10^{-16} \), which is at the round-off limit of the particular computer used.

The reader will note that the solution improvement strategy discussed in Sec. 11.6 is indeed a similar example of the above iteration process.

Finally, we remind the reader that the above iterative process solves the equations of a mixed problem. Accordingly, it is fully effective only when the element used satisfies the stability and consistency conditions of the mixed patch test.
12.7 Stabilized methods for some mixed elements failing the incompressibility patch test

12.7.1 Introduction

It has been observed earlier in this chapter that many of the two field $u-p$ elements do not pass the stability conditions imposed by the mixed patch test at the incompressible limit (or the Babuška–Brezzi conditions). Here in particular we have such methods in which the displacement and pressure are interpolated in an identical manner (for instance, linear triangles, linear quadrilaterals, quadratic triangles, etc.) and many attempts for stabilization of such elements have been introduced.

The most obvious stabilized element can be directly achieved from the formulation suggested in Fig. 12.3(b) of the triangle with a displacement bubble introduced. If this internal displacement is eliminated, then we have a stable element which has a triangular shape with linear displacement and pressure interpolations from nodal values. However, alternatives to this exist and these form several categories. The first category is the introduction of non-zero diagonal terms by adding a least-square form to the Galerkin formulation. This was first suggested by Courant\textsuperscript{40} and it appears that Brezzi and Pitkaranta in 1984\textsuperscript{41} have produced an element of this kind. Numerous further suggestions have been proposed by Hughes \textit{et al.} between 1986 and 1989.\textsuperscript{42–44} More recently, an alternative proposal of achieving similar answers has been proposed by Oñate\textsuperscript{45} which gains the addition of diagonal terms by the introduction of so-called \textit{finite increment calculus} to the formulation.

There is, however, an alternative possibility introduced by time integration of the full incompressible formulation. Here many of the algorithms will yield, when steady-state conditions are recovered, a stabilized form. A number of such algorithms have been discussed by Zienkiewicz and Wu in 1991\textsuperscript{46} and more recently a very efficient method has appeared as a by-product of a fluid mechanics algorithm named the \textit{characteristic based split} (CBS) procedure\textsuperscript{47–50} which will be discussed at length in Volume 3.

In the latter algorithm there exists a free parameter. This parameter depends on the size of the time increment. In the other methods (with the exception of the bubble formulation) there is a weighting parameter applied to the additional terms introduced. We shall discuss each of these algorithms in the following subsections and compare the numerical results obtainable.

One may question, perhaps, that resort to stabilization procedures is not worthwhile in view of the relative simplicity of the full mixed form. But this is a matter practice will decide and is clearly in the hands of the analyst applying the necessary solutions.

12.7.2 Simple triangle with bubble eliminated

In Fig. 12.3(c) we indicated that the simple triangle with $C_0$ linear interpolation and an added bubble for the displacements $u$ together with continuous $C_0$ linear...
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interpolation for the pressure $p$ satisfied the count test part of the mixed patch test and can be used with success. Here we consider this element further to develop some understanding about its performance at the incompressible limit.

The displacement field with the bubble is written as

$$u \approx \tilde{u} = \sum_i N_i \tilde{u}_i + N_b \tilde{u}_b$$

(12.51)

where here

$$N_b = L_1 L_2 L_3$$

(12.52)

$\tilde{u}_i$ are nodal parameters of displacement and $\tilde{u}_b$ are parameters of the hierarchical bubble function. The pressures are similarly given by

$$p \approx \tilde{p} = \sum_i N_i \tilde{p}_i$$

(12.53)

where $\tilde{p}_i$ are nodal parameters of the pressure. In the above the shape functions are given by (e.g., see Eq. (8.34) and (8.32))

$$N_i = L_i = \frac{1}{2\Delta} (a_i + b_i x + c_i y)$$

(12.54)

where

$$a_i = x_j y_k - x_k y_j; \quad b_i = y_j - y_k; \quad c_i = x_k - y_j$$

$j, k$ are cyclic permutations of $i$ and

$$2\Delta = \det \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = a_1 + a_2 + a_3$$

The derivatives of the shape functions are thus given by

$$\frac{\partial N_i}{\partial x} = \frac{b_i}{2\Delta} \quad \text{and} \quad \frac{\partial N_i}{\partial y} = \frac{c_i}{2\Delta}$$

Similarly the derivatives of the bubble are given by

$$\frac{\partial N_b}{\partial x} = \frac{1}{2\Delta} (b_1 L_2 L_3 + b_2 L_3 L_1 + b_3 L_1 L_2)$$

$$\frac{\partial N_b}{\partial y} = \frac{1}{2\Delta} (c_1 L_2 L_3 + c_2 L_3 L_1 + c_3 L_1 L_2)$$

The strains may be expressed in terms of the above and the nodal parameters as†

$$\varepsilon = \sum_i \frac{1}{2\Delta} \begin{bmatrix} b_i & 0 \\ 0 & c_i \end{bmatrix} \tilde{u}_i + \sum_i \frac{L_j L_k}{2\Delta} \begin{bmatrix} b_i & 0 \\ 0 & c_i \end{bmatrix} \tilde{u}_b$$

(12.55)

where again $j, k$ are cyclic permutations of $i$.

† At this point it is also possible to consider the term added to the derivatives to be enhanced modes and delete the bubble mode from displacement terms.
Substituting the above strains into Eq. (12.12) and evaluating the integrals give

\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} & 0 \\
A_{21} & A_{22} & A_{23} & 0 \\
A_{31} & A_{32} & A_{33} & 0 \\
0 & 0 & 0 & A_{bb}
\end{bmatrix}
\]

where

\[
A_{ij} = \frac{G}{6\Delta} \begin{bmatrix}
(4b_ib_j + 3c_ic_j) & (3c_i b_j - 2b_j c_j) \\
(3b_i c_j - 2c_i b_j) & (3b_i b_j + 4c_i c_j)
\end{bmatrix}
\]

\[
A_{bb} = \frac{G}{2160\Delta} \begin{bmatrix}
(4b_i^T b + 3c_i^T c) & b_i^T c \\
3b_i^T b + 4c_i^T c & (3b_i^T b + 4c_i^T c)
\end{bmatrix}
\]

and

\[
b = [b_1, \ b_2, \ b_3] \quad \text{and} \quad c = [c_1, \ c_2, \ c_3]
\]

Note in the above that all terms except \(A_{bb}\) are standard displacement stiffnesses for the deviatoric part. Similarly,

\[
C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33} \\
C_{b1} & C_{b2} & C_{b3}
\end{bmatrix}
\]

where

\[
C_{ij} = \frac{1}{6} \begin{bmatrix}
b_j \\
c_j
\end{bmatrix} \quad \text{and} \quad C_{bj} = -\frac{1}{120} \begin{bmatrix}
b_j \\
c_j
\end{bmatrix}
\]

In all the above arrays \(i\) and \(j\) have values from 1 to 3 and \(b\) denotes the bubble mode.

We note that the bubble mode is decoupled from the other entries in the \(A\) array – it is precisely for this reason that the discontinuous constant pressure case shown in Fig. 12.3(b) cannot be improved by the addition of the internal parameters associated with \(\tilde{u}_b\). Also, the parameters \(\tilde{u}_b\) are defined separately for each element. Consequently, we may perform a partial solution at the element level to obtain the set of equations in the form Eq. (12.11) where now

\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}; \quad C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}; \quad V = \begin{bmatrix}
V_{11} & V_{12} & V_{13} \\
V_{21} & V_{22} & V_{23} \\
V_{31} & V_{32} & V_{33}
\end{bmatrix}
\]

with

\[
V_{ij} = \frac{b_i}{2\Delta} \begin{bmatrix}
\tau_{11} & \tau_{12} \\
\tau_{21} & \tau_{22}
\end{bmatrix} \begin{bmatrix}
b_i \\
c_i
\end{bmatrix} \frac{2\Delta}{c_i}
\]

and

\[
\tau = \frac{3\Delta^2}{10Go} \begin{bmatrix}
(3b_i^T b + 4c_i^T c) & -b_i^T c \\
-b_i^T c & (4b_i^T b + 3c_i^T c)
\end{bmatrix}
\]
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in which
\[
c = 12(b^Tb)^2 + 25(b^Tb)(c^Tc) + 12(c^Tc)^2 - (b^Tc)^2
\]

The reader may recognize the V array given above as that for the two-dimensional, steady heat equation with conductivity \( k = \tau \) and discretized by linear triangular elements. The direct reduction of the bubble matrix \( A_{bb} \) as given above leads to an anisotropic stabilization matrix \( \tau \). A diagonal form of the stabilization results if the weak form for the bubble terms is given by expressing the equilibrium equation in terms of the laplacian of each displacement component and the gradient of the pressure. This is permitted only for bubble terms which vanish identically on the boundary of each element. In Sec. 12.7.4 we indicate how such a reduction could be performed and leave as an exercise to the reader the construction of the weak form terms and the resulting diagonal matrix \( A_{bb} \). Numerical experiments indicate that very little difference is achieved between the two approaches. Since the construction of the diagonal form requires substitution of the constitutive equations into the equilibrium equation it is very limited in the type of applications which can be pursued (e.g., consideration of non-linear problems will preclude such simple substitution).

12.7.3 An enhanced strain stabilization

In the previous section we presented a simple two-field formulation using continuous \( u \) and \( p \) approximations together with added bubble modes to the displacements. For more general applications this form is not the most convenient. For example, if transient problems are considered the accelerations will also involve the bubble mode and affect the inertial terms. We will also find in the comparisons section that use of the above bubble is not fully effective in eliminating pressure oscillations in solutions. An alternative form is discussed in this section. In the alternative form we use a three-field approximation involving \( u, p \) and \( \varepsilon_v \) discussed in Sec. 12.4 together with an enhanced strain formulation as discussed in Sec. 11.5.3.

The enhanced strains are added to those computed from displacements as
\[
\tilde{\varepsilon} = \varepsilon_u + \varepsilon_v
\]
in which \( \varepsilon_v \) represents a set of enhanced strain terms. The internal strain energy is represented by
\[
W(\tilde{\varepsilon}, \varepsilon_v) = \frac{1}{2} (\tilde{\varepsilon}^T D_d \tilde{\varepsilon} + \varepsilon_v^T K \varepsilon_v)
\]

Using the above notation a Hu–Washizu type variational theorem for the deviatoric-spherical split may be written as
\[
\Pi_{me} = \int_\Omega \left[ W(\tilde{\varepsilon}, \varepsilon_v) + p(\mathbf{m}^T \tilde{\varepsilon} - \varepsilon_v) + \sigma^T (\varepsilon_u - \tilde{\varepsilon}) \right] d\Omega + \Pi_{ext}
\]
where \( \Pi_{ext} \) represents the terms associated with body and traction forces.
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After substitution for the mixed enhanced strain the last term in the integral simplifies as:

$$\int_{\Omega} \sigma^T (\varepsilon_u - \tilde{\varepsilon}) \, d\Omega = - \int_{\Omega} \sigma^T \varepsilon_e \, d\Omega$$  \hspace{1cm} (12.63)

Taking variations with respect to \( u, p, \varepsilon_v, \varepsilon_e \) and \( \sigma \) yields

$$\delta \Pi_{me} = \int_{\Omega} \delta u^T B^T [D_d \tilde{\varepsilon} + mp] \, d\Omega + \delta \Pi_{ext}$$

$$+ \int_{\Omega} \delta \varepsilon_v [K \varepsilon_v - p] \, d\Omega + \int_{\Omega} \delta p [m^T \tilde{\varepsilon} - \varepsilon_v] \, d\Omega$$

$$+ \int_{\Omega} \delta \varepsilon_e^T [D_d \tilde{\varepsilon} + mp - \sigma] \, d\Omega + \int_{\Omega} \delta \sigma^T \varepsilon_e \, d\Omega = 0$$  \hspace{1cm} (12.64)

Equal order interpolation with shape functions \( N \) are used to approximate \( u, p \) and \( \varepsilon_v \) as

$$u \approx \bar{u} = N\bar{u}$$

$$p \approx \bar{p} = N\bar{p}$$

$$\varepsilon_v \approx \bar{\varepsilon}_v = N\bar{\varepsilon}_v$$  \hspace{1cm} (12.65)

However, only approximations for \( u \) and \( p \) are \( C_0 \) continuous between elements. The approximation for \( \varepsilon_v \) may be discontinuous between elements. The stress \( \sigma \) in each element is assumed constant. Thus, only the approximation for \( \varepsilon_e \) remains to be constructed in such a way that Eq. (11.49) is satisfied. For the present we shall assume that this approximation may be represented by

$$\varepsilon_e \approx \bar{\varepsilon}_e = B_e \bar{\varepsilon}_e$$  \hspace{1cm} (12.66)

and will satisfy Eq. (11.49) so that the terms involving \( \sigma \) and its variation in Eq. (12.64) are zero and thus do not appear in the final discrete equations.

With the above approximations, Eq. (12.63) may be evaluated as

$$\begin{bmatrix} A_{uu} & A_{ue} & C_u & 0 \\ A_{eu} & A_{ee} & C_e & 0 \\ C_u^T & C_e^T & 0 & -E^T \\ 0 & 0 & -E & H \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{\varepsilon}_v \\ \bar{\varepsilon}_v \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

\hspace{1cm} (12.67)

where \( A_{uu} = A, C_u = C, f_i, E \) and \( H \) are as defined in Eqs (12.12), (12.18) and (12.19) and

$$A_{ue} = \int_{\Omega} BD_d B_e \, d\Omega = A_{eu}^T$$

$$A_{ee} = \int_{\Omega} B_e D_d B_e \, d\Omega$$

$$C_e = \int_{\Omega} B_e m N \, d\Omega$$  \hspace{1cm} (12.68)
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Since the approximations for $\varepsilon_v$ and $\varepsilon_e$ are discontinuous between elements we can again perform a partial solution for $\bar{\varepsilon}_e$ and $\bar{\alpha}_e$ using the second and fourth row of (12.67). After eliminating these variables from the first and third equation we again, as in the simple triangle with bubble eliminated, obtain a form identical to Eq. (12.11).

As an example we consider again the three-noded triangular element with linear approximations for $\mathbf{N}$ in terms of area coordinates $L_i$. We will construct enhanced strain terms from the derivatives of a function. The simplest such approximation is the bubble mode used in Sec. 12.7.2 where the function is given as

$$N_e(\xi) = L_1 L_2 L_3$$

(12.69)

and the enhanced strain part is given by

$$\varepsilon_e(L_i) = B_e(L_i) \bar{\alpha}_e$$

(12.70)

where $\bar{\alpha}_e$ are two enhanced strain parameters and $B_e$ is computed using Eq. (12.69) in the usual strain–displacement matrix

$$B_e = \begin{bmatrix}
\frac{\partial N_e}{\partial x} & 0 \\
0 & \frac{\partial N_e}{\partial y} \\
\frac{\partial N_e}{\partial y} & \frac{\partial N_e}{\partial x}
\end{bmatrix}$$

(12.71)

The result using Eq. (12.69) is identical to the bubble mode since here we are only considering static problems in the absence of body loads. If we considered the transient case or added body loads there would be a difference since the displacement in the enhanced form contains only the linear interpolations in $\mathbf{N}$.

While this is an admissible form we have noted above that it does not eliminate all oscillations for problems where strong pressure gradients occur. Accordingly, we also consider here an alternative form resulting from three enhanced functions

$$N^i_e = a L_i + L_j L_k$$

(12.72)

in which $i, j, k$ is a cyclic permutation and $a$ is a parameter to be determined. Note that this form only involves quadratic terms and thus gives linear strains which are fully consistent with the linear interpolations for $p$ and $\theta$. The derivatives of the enhanced function are given by

$$\frac{\partial N^i_e}{\partial x} = \frac{1}{2\Delta} \left[ a b_i + L_j b_k + L_k b_j \right]$$

$$\frac{\partial N^i_e}{\partial y} = \frac{1}{2\Delta} \left[ a c_i + L_j c_k + L_k c_j \right]$$

(12.73)

where

$$b_i = y_j - y_k \quad \text{and} \quad c_i = x_k - x_j$$

and $\Delta$ is the area of a triangular element. The requirement imposed by Eq. 11.49 gives $a = 1/3$. 
While the use of added enhanced modes leads to increased cost in eliminating the $\tilde{e}_v$ and $\tilde{\alpha}_c$ parameters in Eq. (12.67) the results obtained are free of pressure oscillations in the problems considered in Sec. 12.7.7. Furthermore, this form leads to improved consistency between the pressure and strain.

### 12.7.4 A pressure stabilization

In the first part of this chapter we separated the stress into the deviatoric and pressure components as

$$\sigma = \sigma^d + mp$$

Using the tensor form described in Appendix B this may be written in index form as

$$\sigma_{ij} = \sigma_{ij}^d + \delta_{ij} p$$

The deviatoric stresses are related to the deviatoric strains through the relation

$$\sigma_{ij}^d = 2G\varepsilon_{ij}^d = G\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k}\right)$$  \hspace{1cm} (12.74)

The equilibrium equations (in the absence of inertial forces) are:

$$\frac{\partial \sigma_{ij}^d}{\partial x_j} + \frac{\partial p}{\partial x_i} + b_j = 0$$

Substituting the constitutive equations for the deviatoric part yields the equilibrium form (assuming $G$ is constant)

$$G \left[ \frac{\partial^2 u_j}{\partial x_i \partial x_i} + \frac{1}{3} \frac{\partial^2 u_i}{\partial x_i \partial x_j} \right] + \frac{\partial p}{\partial x_j} + b_j = 0$$  \hspace{1cm} (12.75)

In intrinsic form this is given as

$$G[\nabla^2 u + \frac{1}{3} \nabla(div u)] + \nabla p + b = 0$$

where $\nabla^2$ is the laplacian operator and $\nabla$ the gradient operator. The constitutive equation (12.2) is expressed in terms of the displacement as

$$\varepsilon_v = \frac{\partial u_i}{\partial x_i} = div u = \frac{1}{K} p$$  \hspace{1cm} (12.76)

where $div(\cdot)$ is the divergence of the quantity. A single equation for pressure may be deduced from the divergence of the equilibrium equation. Accordingly, from Eq. (12.75) we obtain

$$\frac{4G}{3} \nabla^2(div u) + \nabla^2 p + div b = 0$$  \hspace{1cm} (12.77)

Upon noting (12.76) we obtain

$$\left(1 + \frac{4G}{3K}\right) \nabla^2 p + div b = 0$$  \hspace{1cm} (12.78)
Thus, in general, the pressure must satisfy a Poisson equation, or in the absence of body forces, a Laplace equation. We have noted the dangers of artificially raising the order of the differential equation in introducing spurious solutions, however, in the context of constructing approximate solutions to the incompressible problem the above is useful in providing additional terms to the weak form which otherwise would be zero. Brezzi and Pitkaranta suggested adding a weighted Eq. (12.78) to Eq. (12.8) and (on setting the body force to zero for simplicity) obtain

$$\int_{\Omega} \delta p \left( \mathbf{m}^T \mathbf{e} - \frac{1}{K} p \right) d\Omega + \beta \int_{\partial \Omega} \delta p \nabla^2 p d\Omega = 0$$  \hspace{1cm} (12.79)$$

The last term may be integrated by parts to yield a form which is more amenable to computation as

$$\int_{\Omega} \delta p \left( \mathbf{m}^T \mathbf{e} - \frac{1}{K} p \right) d\Omega + \beta \int_{\Omega} \frac{\partial \delta p}{\partial x_i} \frac{\partial p}{\partial x_i} d\Omega = 0$$  \hspace{1cm} (12.80)$$
in which the resulting boundary terms are ignored. Upon discretization using equal order linear interpolation on triangles for \( u \) and \( p \) we obtain a form identical to that for the bubble with the exception that \( \tau \) is now given by

$$\tau = \beta \mathbf{I}$$  \hspace{1cm} (12.81)$$

On dimensional considerations with the first term in Eq. (12.80) the parameter \( \beta \) should have a value proportional to \( L^2/F \), where \( L \) is length and \( F \) is force.

### 12.7.5 Galerkin least square method

In Chapter 3, Sec. 3.12.3 we introduced the Galerkin least square (GLS) approach as a modification to constructing a weak form. As a general scheme for solving the differential equations (3.1) by a finite element method we may write the GLS form as

$$\int_{\Omega} \delta \mathbf{u}^T \mathbf{A}(\mathbf{u}) d\Omega + \int_{\partial \Omega} \delta \mathbf{A}(\mathbf{u})^T \tau \mathbf{A}(\mathbf{u}) d\Omega = 0$$  \hspace{1cm} (12.82)$$

where the first term represents the normal Galerkin form and the added terms are computed for each element individually including a weight \( \tau \) to provide dimensional balance and scaling. Generally, the \( \tau \) will involve parameters which have to be selected for good performance. Discontinuous terms on boundaries between elements that arise from higher order terms in \( \mathbf{A}(\mathbf{u}) \) are commonly omitted.

The form given above has been used by Hughes as a means of stabilizing the fluid flow equations, which for the case of the incompressible Stokes problem coincide with those for incompressible linear elasticity. For this problem only the momentum equation is used in the least square terms. After substituting Eq. (12.75) into Eq. (12.76) the momentum equation may be written as (assuming that \( G \) and \( K \) are constant in each element)

$$G \frac{\partial^2 u_i}{\partial x_i^2} + \left( 1 + \frac{G}{3K} \right) \frac{\partial p}{\partial x_i} = 0$$  \hspace{1cm} (12.83)$$
A more convenient form results by using a single parameter defined as

\[
\bar{G} = \frac{G}{1 + G/3K}
\]

With this form the least square term to be appended to each element may be written as

\[
\int_{\Omega} \left( G \frac{\partial^{2} \delta u_{i}}{\partial x_{k}^{2}} + \frac{\partial \delta p}{\partial x_{k}} \right) \tau_{ij} \left( G \frac{\partial^{2} u_{j}}{\partial x_{m}^{2}} + \frac{\partial p}{\partial x_{m}} \right) d\Omega
\]

(12.85)

This leads to terms to be added to the standard Galerkin equations and is expressed as

\[
\begin{bmatrix}
A^{s}_{ij} & C^{s}_{ij} \\
C^{s}_{ij}^{T} & V^{s}_{ij}
\end{bmatrix}
\begin{bmatrix}
\bar{u} \\
\bar{p}
\end{bmatrix}
\]

where

\[
A^{s}_{ij} = \int_{\Omega} \bar{G}^{2} \nabla^{2} N_{i} \tau \nabla^{2} N_{j} d\Omega
\]

\[
C^{s}_{ij} = \int_{\Omega} \bar{G} \nabla^{2} N_{i} \tau \nabla N_{j} d\Omega
\]

\[
V^{s}_{ij} = \int_{\Omega} (\nabla N_{i})^{T} \tau \nabla N_{j} d\Omega
\]

and the operators on the shape functions are given in two dimensions by

\[
\nabla^{2} N_{i} = \frac{\partial^{2} N_{i}}{\partial x_{1}^{2}} + \frac{\partial^{2} N_{i}}{\partial x_{2}^{2}}
\]

\[
\nabla N_{i} = \begin{bmatrix}
\frac{\partial N_{i}}{\partial x_{1}} \\
\frac{\partial N_{i}}{\partial x_{2}}
\end{bmatrix}^{T}
\]

Note again that all infinite terms between elements are ignored.

For linear triangular elements the second derivatives of the shape functions are identically zero within the element and only the \( V \) term remains and is now nearly identical to the form obtained by eliminating the bubble mode. In the work of Hughes et al, \( \tau \) is given by

\[
\tau = -\frac{\alpha \bar{h}^{2}}{2G} I
\]

(12.86)

where \( \alpha \) is a parameter which is recommended to be of \( O(1) \) for linear triangles and quadrilaterals.

### 12.7.6 Incompressibility by time stepping

The fully incompressible case (i.e., \( K = \infty \)) has been studied by Zienkiewicz and Wu using various time stepping procedures. Their applications concern the solution of fluid problems in which the rate effects for the Stokes problem appear as first derivatives of time. We can consider such a method here as a procedure to obtain the static solutions of elasticity problems in the limit as the rate terms become zero. Thus, this approach is considered here as a method for either the Stokes problem or the case of static incompressible elasticity.
The governing equations for slightly compressible Stokes flow may be written as

\[
\rho_0 \frac{\partial u_i}{\partial t} - \frac{\partial \sigma_{ij}^d}{\partial x_j} - \frac{\partial p}{\partial x_i} = 0
\]  \hspace{1cm} (12.87)

\[
\frac{1}{\rho_0 c^2} \frac{\partial p}{\partial t} - \frac{\partial u_i}{\partial x_i} = 0
\]  \hspace{1cm} (12.88)

where \( \rho_0 \) is density (taken as unity in subsequent developments), \( c = (K/\rho_0)^{1/2} \) is the speed of compressible waves, \( p \) is the pressure (here taken as positive in tension), and \( u_i \) is a velocity (or for elasticity interpretations a displacement) in the \( i \)-coordinate direction. Note that the above form assumes some compressibility in order to introduce the pressure rate term. At the steady limit this term is not involved, consequently, the solution will correspond to the incompressible case. Deviatoric stresses \( \sigma_{ij}^d \) are related to deviatoric strains (or strain rates for fluids) as described by Eq. (12.74).

Zienkiewicz and Wu consider many schemes for integrating the above equations in time. Here we introduce only one of the forms, which will also be used in the solution of the fluid equations which include transport effects (see Volume 3). For the full fluid equations the algorithm is part of the characteristic based split (CBS) method.\(^{47–50}\)

The equations are discretized in time using the approximations \( u(t_n) \approx u^n \) and time derivatives

\[
\frac{\partial u_i}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t}
\]  \hspace{1cm} (12.89)

where \( \Delta t = t_{n+1} - t_n \). The time discretized equations are given by

\[
\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\partial \sigma_{ij}^{dn}}{\partial x_j} + \frac{\partial p^n}{\partial x_i} + \theta_2 \frac{\partial \Delta p}{\partial x_i}
\]  \hspace{1cm} (12.90)

\[
\frac{1}{c^2} \frac{p^{n+1} - p^n}{\Delta t} = \frac{u_i^n}{\partial x_i} + \theta_1 \frac{\partial \Delta u_i}{\partial x_i}
\]  \hspace{1cm} (12.91)

where \( \Delta p = p^{n+1} - p^n \); \( \Delta u_i = u_i^{n+1} - u_i^n \); \( \theta_1 \) can vary between 1/2 and 1; and \( \theta_2 \) can vary between 0 and 1. In all that follows we shall use \( \theta_1 = 1 \).

The form to be considered uses a split of the equations by defining an intermediate approximate velocity \( u_i^* \) at time \( t_{n+1} \) when integrating the equilibrium equation (12.90). Accordingly, we consider

\[
\frac{u_i^* - u_i^n}{\Delta t} = \frac{\partial \sigma_{ij}^{dn}}{\partial x_j}
\]  \hspace{1cm} (12.92)

\[
\frac{u_i^{n+1} - u_i^*}{\Delta t} = \frac{\partial p^n}{\partial x_i} + \theta_2 \frac{\partial \Delta p}{\partial x_i}
\]  \hspace{1cm} (12.93)

Differentiating the second of these with respect to \( x_i \) to get the divergence of \( u_i^{n+1} \) and combining with the discrete pressure equation (12.91) results in

\[
\frac{1}{c^2} \frac{\Delta p}{\Delta t} - \theta_2 \Delta t \frac{\partial^2 \Delta p}{\partial x_i \partial x_i} = \Delta t \frac{\partial^2 p^n}{\partial x_i \partial x_i} + \frac{\partial u_i^*}{\partial x_i}
\]  \hspace{1cm} (12.94)
Thus, the original problem has been replaced by a set of three equations which need to be solved successively.

Equations (12.92), (12.93) and (12.94) may be written in a weak form using as weighting functions \( \delta u^* \), \( \delta u \) and \( \delta p \), respectively (viz. Chapter 3). They are then discretized in space using the approximations

\[
\begin{align*}
\textbf{u}^n &\approx \textbf{u}^n = \textbf{N}_u \textbf{u}^n \quad \text{and} \quad \delta \textbf{u}^n \approx \delta \textbf{u} = \textbf{N}_u \delta \textbf{u}^n \\
\textbf{u}^* &\approx \textbf{u}^* = \textbf{N}_u \textbf{u}^* \quad \text{and} \quad \delta \textbf{u}^* \approx \delta \textbf{u}^* = \textbf{N}_u \delta \textbf{u}^* \\
p^n &\approx \textbf{p}^n = \textbf{N}_p \textbf{p}^n \quad \text{and} \quad \delta p \approx \delta \textbf{p} = \textbf{N}_p \delta \textbf{p}
\end{align*}
\]

with similar expressions for \( \textbf{u}^{n+1} \) and \( p^{n+1} \). The final discrete form is given by the three equation sets

\[
\begin{align*}
\frac{1}{\Delta t} \textbf{M}_u (\textbf{u}^* - \textbf{u}^n) &= -A \textbf{u}^n + \textbf{f}_1 \quad (12.95) \\
\frac{1}{\Delta t} \textbf{M}_u (\textbf{u}^{n+1} - \textbf{u}^*) &= -C^T (\textbf{p}^n + \theta_2 \Delta \textbf{p}) \\
\left[ \frac{1}{\Delta t} \textbf{M}_p + \theta_2 \Delta \textbf{H} \right] \Delta \textbf{p} &\approx -C \textbf{u}^* - \Delta t \textbf{H} \textbf{p}^n + \textbf{f}_3 \quad (12.97)
\end{align*}
\]

In the above we have integrated by parts all the terms which involve derivatives on deviator stress \( \sigma_{ij}^d \), pressure \( p \) and displacements (velocities). In addition we consider only the case where \( u_i^{n+1} = u_i^* = u_i \) on the boundary \( \Gamma_u \) (thus requiring \( \delta u_i = \delta u_i^* = 0 \) on \( \Gamma_u \)). Accordingly, the matrices are defined as

\[
\begin{align*}
\textbf{M}_u &= \int_\Omega \textbf{N}_u^T \textbf{N}_u \, d\Omega \\
\textbf{M}_p &= \int_\Omega \frac{1}{c^2} \textbf{N}_p^T \textbf{N}_p \, d\Omega \\
\textbf{A} &= \int_\Omega \textbf{B}^T \textbf{D}_d \textbf{B} \, d\Omega \\
\textbf{C} &= \int_\Omega \frac{\partial \textbf{N}_p}{\partial x_i} \textbf{N}_u \, d\Omega \\
\textbf{H} &= \int_\Omega \frac{\partial \textbf{N}_p}{\partial x_i} \frac{\partial \textbf{N}_p}{\partial x_j} \, d\Omega \\
\textbf{f}_1 &= \int_{\Gamma_i} \textbf{N}_u^T (\textbf{t} - k \textbf{n} \textbf{p}^n) \, d\Gamma \\
\textbf{f}_3 &= \int_{\Gamma_u} \textbf{N}_p^T \textbf{n}^T \textbf{u} \, d\Gamma
\end{align*}
\]

in which \( \textbf{D}_d \) are the deviatoric moduli defined previously. The parameter \( k \) denotes an option on alternative methods to split the boundary traction term and is taken as either zero or unity. We note that a choice of zero simplifies the computation of boundary contributions, however, some would argue that unity is more consistent with the integration by parts.

The boundary pressure acting on \( \Gamma_i \) is computed from the specified surface tractions \( (\textbf{t}_i) \) and the 'best' estimate for the deviator stress at step-\( n + 1 \) which is given by \( \sigma_{ij}^{d,*} \). Accordingly,

\[
\textbf{p}^{n+1} \approx n_i \textbf{t}_i - n_i \sigma_{ij}^{d,*} n_j
\]

is imposed at each node on the boundary \( \Gamma_i \).
In general we require that $\Delta t < \Delta t_{\text{crit}}$ where the critical time step is $h^2/2G$ (in which $h$ is the element size). Such a quantity is obviously calculated independently for each element and the lowest value occurring in any element governs the overall stability. It is possible and useful to use here the value of $\Delta t$ calculated for each element separately when calculating incompressible stabilizing terms in the pressure calculation and the overall time step elsewhere (we shall label the time increments multiplying $H$ in Eq. (12.97) as $\Delta t_{\text{int}}$). A ratio of $\gamma = \Delta t_{\text{int}}/\Delta t$ greater than unity improves considerably the stabilizing properties. As Eq. (12.97) has greater stability than Eqs (12.95) and (12.96), and for $\theta_2 \geq 1/2$ is unconditionally stable, we recommend that the time step used in this equation be $\gamma \Delta t_{\text{cr}}$ for each node. Generally a value of 2 is good as we shall show in the examples (for details see reference 50).

Equation (12.95) defines a value of $\bar{u}$ entirely in terms of known quantities at the $n$-step. If the mass matrix $M_u$ is made diagonal by lumping (see Chapter 17 and Appendix I) the solution is thus trivial. Such an equation is called explicit. The equation for $\Delta \bar{p}$, on the other hand depends on both $M_p$ and $H$ and it is not possible to make the latter diagonal easily. It is possible to make $M_p$ diagonal using a similar method as that employed for $M_u$. Thus, if $\theta_2$ is zero this equation will also be explicit, otherwise it is necessary to solve a set of algebraic equations and the method for this equation is called implicit. Once the value of $\Delta \bar{p}$ is known the solution for $\bar{u}^{n+1}$ is again explicit. In practice the above process is quite simple to implement, however, it is necessary to satisfy stability requirements by limiting the size of the time increment. This is discussed further in Chapter 18 and in reference 47. Here we only wish to show the limit result as the changes in time go to zero (i.e., for a constant in time load value) and when full incompressibility is imposed.

At the steady limit the solutions become

$$\bar{u}^n = \bar{u}^{n+1} = \bar{u} \quad \text{and} \quad \bar{p}^n = \bar{p}^{n+1} = \bar{p} \quad (12.99)$$

Eliminating $u^*$ the discrete equations reduce to the mixed problem

$$\begin{bmatrix} A & C \\ C^T & \Delta t(C^T M_u^{-1} C - \theta_1 H) \end{bmatrix} \begin{bmatrix} \bar{u} \\ \bar{p} \end{bmatrix} + \begin{bmatrix} f \\ 0 \end{bmatrix} = 0 \quad (12.100)$$

At the steady limit we again recover a term on the diagonal which stabilizes the solution. This term is again of a Laplace equation type – indeed, it is now the difference between two discrete forms for the Laplace equation. The term $C^T M_u^{-1} C$ makes the bandwidth of the resulting equations larger – thus this form is different from all the previous methods discussed above.

### 12.7.7 Comparisons

To provide some insight into the behaviour of the above methods we consider two example problems. The first is a problem often used to assess the performance of

\[\text{\textdagger} \text{It is possible to diagonalize the matrix by solving an eigenproblem as shown in Chapter 17 – for large problems this requires more effort than is practical.}\]
codes to solve steady-state Stokes flow problems – which is identical to the case for incompressible linear elasticity. The second example is a problem in nearly incompressible linear elasticity.

Example: Driven cavity A two-dimensional plane (strain) case is considered for a square domain with unit side lengths. The material properties are assumed to be fully incompressible ($v = 0.5$) with unit viscosity (elastic shear modulus, $G$, of unity). All boundaries of the domain are restrained in the $x$ and $y$ directions with the top boundary having a unit tangential velocity (displacement) at all nodes except the corner ones. Since the problem is incompressible it is necessary to prescribe the pressure at one point in the mesh – this is selected as the centre node along the bottom edge. The $10 \times 10$ element mesh of triangular elements (200 elements total) used for the comparison is shown in Fig. 12.10(a). The elements used for the analysis use linear velocity (displacement) and pressure on three-noded triangles. Results are presented for the horizontal velocity along the vertical centre line AA and for vertical velocity and pressure along the horizontal centre line BB. Three forms of stabilization are considered:

1. Galerkin least square (GLS) Brezzi–Pitkaranta (BP) where the effect of $\alpha$ on $\tau$ is assessed. The results for the horizontal velocity are given in Fig. 12.10(b) and for the vertical velocity and pressure in Figs 12.10(c) and (d), respectively. From the analysis it is assessed that the stabilization parameter $\tau$ should be about 0.5 to 1 (as also indicated by Hughes et al.\textsuperscript{44}). Use of lower values leads to excessive oscillation in pressure and use of higher values to strong dissipation of pressure results.

2. Cubic bubble (MINI) element stabilization. Results for vertical velocity are nearly indistinguishable from the GLS results as indicated in Fig. 12.11; however, those for pressure show oscillation. Such oscillation has also been observed by others along with some suggested boundary modifications.\textsuperscript{51} No free parameters exist for this element (except possible modification of the bubble mode used), thus, no artificial ‘tuning’ is possible. Use of more refined meshes leads to a strong decrease in the oscillation.

3. Enhanced strain stabilization with quadratic modes. In Fig. 12.11 we show results obtained using the enhanced formulation presented in Eq. (12.73). These results are free of oscillation in pressures and require no tuning parameters. For use in solving linear elasticity and Stokes problems they prove to be the most robust; however, when used with other material models there are limitations in their use.

4. The CBS algorithm. Finally in Fig. 12.11 we present results using the CBS solution which may be compared with GLS, $\alpha = 0.5$. Once again the reader will observe that with $\gamma = 2$, the results of CBS reproduce very closely those of GLS, $\alpha = 0.5$. However, in results for $\gamma = 1$ no oscillations are observed and they are quite reasonable. This ratio for $\gamma$ is where the algorithm gives excellent results in incompressible flow modelling as will be demonstrated further in results presented in Volume 3.

Example: Tension strip with slot As a second example we consider a plane strain linear problem on a square domain with a central slot. The domain is two units square and the central slot has a total width of 0.4 units and a height of 0.1 units. The ends of the slot are semicircular. Lateral boundaries have specified normal
displacement and zero tangential traction. The top and bottom boundaries are uniformly stretched by a uniform axial loading and lateral boundaries are maintained at zero displacement. We consider the linear elastic problem with elastic properties $E = 24$ and $\nu = 0.499995$; thus, giving a nearly incompressible situation. An unstructured mesh of triangles is constructed as shown in Fig. 12.12(b). Results for the pressure along the horizontal and vertical centre lines (i.e., the $x$ and $y$ axes) are presented in Figs 12.13(a) and 12.13(b) and the distribution of the vertical displacement is shown in Fig. 12.13(c). We note that the results for this problem cause very strong gradients in stress near the ends of the slot. The mesh used for the analysis is not highly refined in this region and hence results from different analyses can be
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Fig. 12.11 Vertical velocity and pressure for driven cavity problem.
expected to differ in this region. The results obtained using all formulations are similar in distribution. However, the bubble form does show some oscillations in pressure indicating that the stabilization achieved is not completely adequate. Results for the CBS algorithm show an oscillation in the pressure along the $x$-axis at the boundary of the slot. This is caused, we believe, by an inadequate resolution of the pressure condition at this point of the curved boundary. In general, however,
the results achieved with all forms are satisfactory and indicate that stabilized
methods may be considered for use in problems where constraints, such as incom-
pressibility, are encountered.

12.8 Concluding remarks

In this chapter we have considered in some detail the application of mixed methods to
incompressible problems and also we have indicated some alternative procedures.
The extension to non-isotropic problems and non-linear problems will be presented
in Volume 2, but will follow similar lines. In Volume 3 we shall note how important
the problem is in the context of fluid mechanics and it is there that much of the
attention to it has been given.

In concluding this chapter we would like to point out two matters:

1. The mixed formulation discovers immediately the non-robustness of certain
irreducible (displacement) elements and, indeed, helps us to isolate those which
perform well from those that do not. Thus, it has merit which as a test is applicable
to many irreducible forms at all times.

2. In elasticity, certain mixed forms work quite well at the near incompressible limit
without resort to splits into deviatoric and mean parts. These include the two-field
quadrilateral element of Pian–Sumihara and the enhanced strain quadrilateral
element of Simo–Rifai which were presented in the previous chapter. There we
noted how well such elements work for Poisson’s ratio approaching one-half as
compared to the standard irreducible element of a similar type.

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