Three-dimensional stress analysis

6.1 Introduction

It will have become obvious to the reader by this stage of the book that there is but one further step to apply the general finite element procedure to fully three-dimensional problems of stress analysis. Such problems embrace clearly all the practical cases, though for some, the various two-dimensional approximations give an adequate and more economical ‘model’.

The simplest two-dimensional continuum element is a triangle. In three dimensions its equivalent is a tetrahedron, an element with four nodal corners†, and this chapter will deal with the basic formulation of such an element. Immediately, a difficulty not encountered previously is presented. It is one of ordering of the nodal numbers and, in fact, of a suitable representation of a body divided into such elements.

The first suggestions for the use of the simple tetrahedral element appear to be those of Gallagher et al. and Melosh. Argyris elaborated further on the theme and Rashid and Rockenhauser were the first to apply three-dimensional analysis to realistic problems.

It is immediately obvious, however, that the number of simple tetrahedral elements which has to be used to achieve a given degree of accuracy has to be very large. This will result in very large numbers of simultaneous equations in practical problems, which may place a severe limitation on the use of the method in practice. Further, the bandwidth of the resulting equation system becomes large, leading to increased use of iterative solution methods.

To realize the order of magnitude of the problems presented let us assume that the accuracy of a triangle in two-dimensional analysis is comparable to that of a tetrahedron in three dimensions. If an adequate stress analysis of a square, two-dimensional region requires a mesh of some $20 \times 20 = 400$ nodes, the total number of simultaneous equations is around 800 given two displacement variables at a node. (This is a fairly realistic figure.) The bandwidth of the matrix involves 20 nodes (Chapter 20), i.e., some 40 variables.

† The simplest polygonal shape which permits the approximation of the domain is known as the simplex. Thus a triangular and tetrahedral element constitute the simplex in two and three dimensions, respectively.
An equivalent three-dimensional region is that of a cube with $20 \times 20 \times 20 = 8000$ nodes. The total number of simultaneous equations is now some $24000$ as three displacement variables have to be specified. Further, the bandwidth now involves an interconnection of some $20 \times 20 = 400$ nodes or $1200$ variables.

Given that with direct solution techniques the computation effort is roughly proportional to the number of equations and to the square of the bandwidth, the magnitude of the problems can be appreciated. It is not surprising therefore that efforts to improve accuracy by use of complex elements with many degrees of freedom have been strongest in the area of three-dimensional analysis.\textsuperscript{6-10} The development and practical application of such elements will be described in the following chapters. However, the presentation of this chapter gives all the necessary ingredients of the formulation for three-dimensional elastic problems and so follows directly from the previous ones. Extension to more elaborate elements will be self-evident.

### 6.2 Tetrahedral element characteristics

#### 6.2.1 Displacement functions

Figure 6.1 illustrates a tetrahedral element $i, j, m, p$ in space defined by $x, y, z$ coordinates.

![Fig. 6.1 A tetrahedral volume. (Always use a consistent order of numbering, e.g., for $p$ count the other nodes in an anticlockwise order as viewed from $p$, giving the element as $ijmp$, etc.).]
The state of displacement of a point is defined by three displacement components, \( u, v, \) and \( w, \) in the directions of the three coordinates \( x, y, \) and \( z. \) Thus

\[
\mathbf{u} = \begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
\]  

Just as in a plane triangle where a linear variation of a quantity was defined by its three nodal values, here a linear variation will be defined by the four nodal values. In analogy to Eq. (4.3) we can write, for instance,

\[
u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z
\]

Equating the values of the displacement at the nodes we have four equations of the type

\[
u_i = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i + \alpha_4 z_i, \quad \text{etc.}
\]

from which \( \alpha_1 \) to \( \alpha_4 \) can be evaluated.

Again, it is possible to write this solution in a form similar to that of Eq. (4.5) by using a determinant form, i.e.,

\[
u = \frac{1}{6V} [(a_i + b_i x + c_i y + d_i z)u_i + (a_j + b_j x + c_j y + d_j z)u_j + (a_m + b_m x + c_m y + d_m z)u_m + (a_p + b_p x + c_p y + d_p z)u_p]
\]

with

\[
6V = \det \begin{vmatrix}
x_i & y_i & z_i \\
x_j & y_j & z_j \\
x_m & y_m & z_m \\
x_p & y_p & z_p
\end{vmatrix}
\]

in which, incidentally, the value \( V \) represents the volume of the tetrahedron. By expanding the other relevant determinants into their cofactors we have

\[
a_i = \det \begin{vmatrix}
x_j & y_j & z_j \\
x_m & y_m & z_m \\
x_p & y_p & z_p
\end{vmatrix}, \quad b_i = -\det \begin{vmatrix}
x_j & y_j & z_j \\
x_m & y_m & z_m \\
x_p & y_p & z_p
\end{vmatrix}, \quad c_i = -\det \begin{vmatrix}
x_j & 1 & z_j \\
x_m & 1 & z_m \\
x_p & 1 & z_p
\end{vmatrix}, \quad d_i = -\det \begin{vmatrix}
x_j & y_j & 1 \\
x_m & y_m & 1 \\
x_p & y_p & 1
\end{vmatrix}
\]

with the other constants defined by cyclic interchange of the subscripts in the order \( i, j, m, p. \)

The ordering of nodal numbers \( i, j, m, p \) must follow a 'right-hand' rule obvious from Fig. 6.1. In this the first three nodes are numbered in an anticlockwise manner when viewed from the last one.
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The element displacement is defined by the 12 displacement components of the nodes as

\[
\mathbf{a}^e = \begin{bmatrix} a_i \\ a_j \\ a_m \\ a_p \end{bmatrix}
\]

(6.6)

with

\[
\mathbf{a}_i = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}
\]

etc.

We can write the displacements of an arbitrary point as

\[
\mathbf{u} = [\mathbf{N}_i, \mathbf{N}_j, \mathbf{N}_m, \mathbf{N}_p] \mathbf{a}^e = \mathbf{N} \mathbf{a}^e
\]

(6.7)

with shape functions defined as

\[
N_i = \frac{a_i + b_i x + c_i y + d_i z}{6\nu}, \quad \text{etc.}
\]

(6.8)

and \(\mathbf{I}\) being a three by three identity matrix.

Once again the displacement functions used will obviously satisfy continuity requirements on interfaces between various elements. This fact is a direct corollary of the linear nature of the variation of displacement.

6.2.2 Strain matrix

Six strain components are relevant in full three-dimensional analysis. The strain matrix can now be defined as

\[
\mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{bmatrix} = \mathbf{S} \mathbf{u}
\]

(6.9)

following the standard notation of Timoshenko's elasticity text. Using Eqs (6.4)–(6.8) it is an easy matter to verify that

\[
\mathbf{\varepsilon} = \mathbf{S} \mathbf{N} \mathbf{a}^e = \mathbf{B} \mathbf{a}^e = [\mathbf{B}_i, \mathbf{B}_j, \mathbf{B}_m, \mathbf{B}_p] \mathbf{a}^e
\]

(6.10)
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in which

\[
B_i = \begin{bmatrix}
\frac{\partial N_i}{\partial x}, & 0, & 0 \\
0, & \frac{\partial N_i}{\partial y}, & 0 \\
0, & 0, & \frac{\partial N_i}{\partial z} \\
\frac{\partial N_i}{\partial y}, & \frac{\partial N_i}{\partial x}, & 0 \\
0, & \frac{\partial N_i}{\partial z}, & \frac{\partial N_i}{\partial y} \\
\frac{\partial N_i}{\partial z}, & 0, & \frac{\partial N_i}{\partial x}
\end{bmatrix} = \frac{1}{6V} \begin{bmatrix}
b_i, & 0, & 0 \\
0, & c_i, & 0 \\
0, & 0, & d_i \\
c_i, & b_i, & 0 \\
0, & d_i, & c_i \\
d_i, & 0, & b_i
\end{bmatrix}
\]

(6.11)

with other submatrices obtained in a similar manner simply by interchange of subscripts.

Initial strains, such as those due to thermal expansion, can be written in the usual way as a six-component vector which, for example, in an isotropic thermal expansion is simply

\[
\varepsilon_0 = \alpha \theta^e \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \alpha \theta^e \mathbf{m}
\]

(6.12)

with \( \alpha \) being the expansion coefficient and \( \theta^e \) the average element temperature rise.

### 6.2.3 Elasticity matrix

With complete anisotropy the \( \mathbf{D} \) matrix relating the six stress components to the strain components can contain 21 independent constants (see Sec. 4.2.3).

In general, thus,

\[
\sigma = \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{zx}
\end{bmatrix}
\]

\[
= \mathbf{D}(\varepsilon - \varepsilon_0) + \sigma_0
\]

(6.13)

Although no difficulty presents itself in computation when dealing with such materials, it is convenient to recapitulate here the \( \mathbf{D} \) matrix for an isotropic material. This, in terms of the usual elastic constants \( E \) (modulus) and \( \nu \) (Poisson’s ratio),
can be written as

\[
D = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{bmatrix}
1 - \nu, & \nu, & \nu, & 0, & 0, & 0 \\
1 - \nu, & \nu, & 0, & 0, & 0 \\
1 - \nu, & 0, & 0, & 0 \\
(1 - 2\nu)/2, & 0, & 0 \\
\text{Sym.} & (1 - 2\nu)/2, & 0 \\
(1 - 2\nu)/2, & \end{bmatrix}
\]

(6.14)

6.2.4 Stiffness, stress, and load matrices

The stiffness matrix defined by the general relationship (2.10) can now be explicitly integrated since the strain and stress components are constant within the element.

The general \( ij \) submatrix of the stiffness matrix will be a three by three matrix defined as

\[
K_{ij}^e = B_i^T D B_j V^e
\]

(6.15)

where \( V^e \) represents the volume of the elementary tetrahedron.

The nodal forces due to the initial strain become, similarly to Eq. (4.34),

\[
f_i^e = -B_i^T D e_0 V^e
\]

(6.16)

with a similar expression for forces due to initial stresses.

Distributed body forces can once again be expressed in terms of their \( b_x, b_y, \) and \( b_z \) components or in terms of the body force potential. Not surprisingly, it will once more be found that if the body forces are constant the nodal components of the total resultant are distributed in four equal parts [see Eq. (4.36)].

In fact, the similarity with the expressions and results of Chapter 4 is such that further explicit formulation is unnecessary. The reader will find no difficulty in repeating the various steps needed for the formulation of a computer program.

Fig. 6.2 A systematic way of dividing a three-dimensional object into 'brick'-type elements.
Fig. 6.3 Composite element with eight nodes and its subdivision into five tetrahedra by alternatives (a) or (b).
6.3 Composite elements with eight nodes

The division of a space volume into individual tetrahedra sometimes presents difficulties of visualization and could easily lead to errors in nodal numbering, etc., unless a fully automatic code is available. A more convenient subdivision of space is into eight-cornered brick elements (bricks being the natural way to build a universe!). By sectioning a three-dimensional body parallel sections can be drawn and, each one being subdivided into quadrilaterals, a systematic way of element definition could be devised as in Fig. 6.2.

Such elements could be assembled automatically from several tetrahedra and the process of creating these tetrahedra left to a simple logical program. For instance, Fig. 6.3 shows how a typical brick can be divided into five tetrahedra in two (and only two) distinct ways. Stresses could well be presented as averages for a whole brick-like element or as final nodal averages. We shall discuss again a rational procedure for stress recovery in Chapter 14.

In Fig. 6.4 a more convenient subdivision of a brick into six tetrahedra is shown. Here obviously the number of alternatives is very great; however (contrary to the
5-element subdivision) diagonals on adjacent faces of elements for a mesh type shown in Fig. 6.2 can always be made to match. Thus the 6-element subdivision creates a conforming approximation.

In later chapters it will be seen how the basic bricks can be obtained directly with more complex types of shape function.

6.4 Examples and concluding remarks

A simple, illustrative example of the application of simple, tetrahedral, elements is shown in Figs 6.5 and 6.6. Here the well-known Boussinesq problem of an elastic half-space with a point load is approximated by analysing a cubic volume of space. Use of symmetry is made to reduce the size of the problem and the boundary displacements are prescribed in a manner shown in Fig. 6.5. As zero displacements were prescribed at a finite distance below the load a correction obtained from the exact expression was applied before executing the plots shown in Fig. 6.6. Comparison of both stresses and displacement appears reasonable although it will be appreciated that the division is very coarse. However, even this trivial problem involved the solution of some 375 equations. More ambitious problems treated with simple tetrahedra are given in references 5 and 12. Figure 6.7, taken from the former, illustrates an analysis of a complex pressure vessel. Some 10,000 degrees of freedom are involved in this analysis. In Chapter 8 it will be seen how the use of complex elements permits a sufficiently accurate analysis to be performed with a much smaller total number of degrees of freedom for a very similar problem.

Fig. 6.5 The Boussinesq problem as one of three-dimensional stress analysis.
Fig. 6.6 The Boussineq problem: (a) vertical stresses ($\sigma_z$); (b) vertical displacements ($w$).
Fig. 6.7 A nuclear pressure vessel analysis using simple tetrahedral elements. Geometry, subdivision, and some stress results.
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Although we have in this chapter emphasized the easy visualization of a tetrahedral mesh through the use of brick-like subdivision, it is possible to generate automatically arbitrary tetrahedral meshes of great complexity with any prescribed mesh density distribution. The procedures follow the general pattern of automatic triangle generation\textsuperscript{13} to which we shall refer in Chapter 15 when discussing efficient, adaptively constructed meshes, but, of course, the degree of complexity introduced is much greater in three dimensions. Some details of such a generator are described by Peraire et al.\textsuperscript{14} and Fig. 6.8 illustrates an intersection of such an automatically generated mesh of tetrahedra for a specified mesh density in the exterior region on aircraft (a) and (b) an intersection of the mesh with the centreline plane.

Fig. 6.8 An automatically generated mesh of tetrahedra for a specified mesh density in the exterior region on aircraft (a) and (b) an intersection of the mesh with the centreline plane.
generated mesh with an outline of an aircraft. It is impractical to show the full plot of
the mesh which contains over 30,000 nodes. The important point to note is that such
meshes can be generated for any configuration which can be suitably described
gometrically. Although this example concerns aerodynamics rather than
elasticity, similar meshes can be generated in the latter context.

References

1963.
3. J.H. Argyris. Matrix analysis of three-dimensional elastic media – small and large
5. Y.R. Rashid and W. Rockenhauser. Pressure vessel analysis by finite element techniques.
7. B.M. Irons. Engineering applications of numerical integration in stiffness methods.
15. N.P. Weatherill, P.R. Eiseman, J. Hause, and J.F. Thompson. Numerical Grid Generation