Higher-Order Finite Element Methods

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CHAPMAN & HALL/CRC

A CRC Press Company
Boca Raton London New York Washington, D.C.

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Library of Congress Cataloging-in-Publication Data

Šolín, Pavel.
Higher-order finite element methods / Pavel Šolín, Karel Segeth, Ivo Doležel.
p. cm.
Includes bibliographical references and index.
ISBN 1-58488-438-X
1. Finite element method. I. Segeth, Karel. II. Doležel, Ivo. III. Title. IV. Series.

TA347.F5+.S68 2003
620′.001′51535—dc21 2003051470

Direct all inquiries to CRC Press LLC, 2000 N.W. Corporate Blvd., Boca Raton, Florida 33431.

Visit the CRC Press Web site at www.crcpress.com

International Standard Book Number 1-58488-438-X
Library of Congress Card Number 2003051470
Printed in the United States of America 1 2 3 4 5 6 7 8 9 0
Printed on acid-free paper

We dedicate this book to the memory of Prof. Jindřich Nečas (December 14, 1929 – December 5, 2002), an outstanding Czech mathematician and a world-renowned authority in the field of partial differential equations and modern functional analysis.

Prof. Jindřich Nečas contributed substantially to the development of modern functional analytic methods of solution to elliptic partial differential equations in his famous monograph *Les méthodes directes en théorie des équations elliptiques* (1967). He followed the modern Italian and French school and enhanced it with important results, for example, by a new "algebraic" proof of general inequalities of Korn's type and generalized regularity results. A few years later, in 1973, he published with collaborators the monograph *Spectral Analysis of Nonlinear Operators*, which aroused great interest. Prof. Nečas was always intrigued by the problem of regularity of solutions. Outstanding results in this field appeared in his book *Introduction to the Theory of Nonlinear Elliptic Equations* (1983, 1986).

From the very beginning Prof. Nečas devoted great effort to applications in mathematical physics and engineering. In 1967 he established a seminar on problems of continuum mechanics that continues to the present day. From this seminar came the monographs *Mathematical Theory of Elastic and Elastoplastic Bodies: An Introduction* (1981, 1983) and *Solution of Variational Inequalities in Mechanics* (1982). The latter book was translated into Russian (1986) and English (1988). Both these monographs also were directed toward numerical methods of solution based on the finite element method. This prompted P. G. Ciarlet and J. L. Lions to invite Prof. Nečas to write an article, "Numerical Methods for Unilateral Problems in Solid Mechanics," for their *Handbook of Numerical Analysis* (1996).

During the last two decades of his life Prof. Nečas' field of interest changed from solid to fluid mechanics, in particular to problems of transsonic flow. Using the method of entropic compactification and the method of viscosity, he achieved remarkable results that he published in his monograph *Écoulements de fluide: Compacité par entropie* (1989). Recent results of Prof. Nečas and his collaborators have been collected in the book *Weak and Measure Valued Solutions to Evolutionary PDE's* (1996).

Besides the above-mentioned monographs, Prof. Nečas initiated and published more than 180 papers in outstanding mathematical journals and conference proceedings.

An excellent teacher, Prof. Nečas influenced many students and colleagues with his never-ending enthusiasm. He organized lectures, seminars and two series of summer schools, and guided many students on the way to their diplomas and Ph.D. theses. They all will remember him with gratitude.

Both P. Šolín and K. Segeth were, at different times, students of J. Nečas.

Preface

The finite element method is one of the most popular tools for the numerical solution of engineering problems formulated in terms of partial differential equations. The latest developments in this field indicate that its future lies in adaptive higher-order methods, which successfully respond to the increasing complexity of engineering simulations and satisfy the overall trend of simultaneous resolution of phenomena with multiple scales.

Among various adaptive strategies for finite elements, the best results can be achieved using goal-oriented hp-adaptivity. Goal-oriented adaptivity is based on adaptation of the finite element mesh with the aim of improving the resolution of a specific quantity of interest (instead of minimizing the error of the approximation in some global norm), and hp-adaptivity is based on the combination of spatial refinements (h-adaptivity) with simultaneous variation of the polynomial order of approximation (p-adaptivity). There are nonacademic examples where the goal-oriented hp-adaptivity turned out to be the only way to resolve the problem on a required level of accuracy (see, e.g., [185]). Automatic hp-adaptivity belongs to the most advanced topics in the higher-order finite element technology and it is subject to active ongoing research. We refer the reader to works by Demkowicz et al. (see [162, 64, 62, 8, 122, 149, 172, 191] and references therein). The goal of this book is more modest – we present the basic principles of higher-order finite element methods and the technology of conforming discretizations based on hierarchic elements in spaces H^1, $\boldsymbol{H}(\mathrm{curl})$ and $\boldsymbol{H}(\mathrm{div})$. An example of an efficient and robust strategy for automatic goal-oriented hp-adaptivity is given in Chapter 6.

In the introductory Chapter 1 we review the aforementioned function spaces and their basic properties, define unisolvency of finite elements, formulate conformity requirements for finite elements in these spaces, introduce the basic steps in the finite element procedure, and present several families of orthogonal polynomials. Section 1.3 is devoted to the solution of a one-dimensional model problem on a mesh consisting of elements of arbitrary polynomial order. The technical simplicity of the one-dimensional case gives the reader the opportunity to encounter all the important features of higher-order finite element discretization at the same time.

A database of scalar and vector-valued hierarchic master elements of arbitrary order on the most commonly used reference domains in 2D and 3D is provided in Chapter 2. This chapter contains many formulae of higher-order shape functions and is intended for reference rather than for systematic reading. Chapter 3 discusses the basic principles of higher-order finite element methods in two and three spatial dimensions that the reader was first exposed to in Section 1.3. We begin with generalizing the standard nodal interpolation to higher-order hierarchic elements, and describe the design of reference maps based on the transfinite interpolation technique as well as their polynomial isoparametric approximation. We discuss an approach to the treatment of constrained approximations (approximations comprising “hanging nodes”) and mention selected software-technical aspects at the end of this chapter.

Chapter 4 is devoted to higher-order numerical quadrature in two and three spatial dimensions. Numerical quadrature lies at the heart of higher-order finite element codes and its proper implementation is crucial for their optimal performance. In particular the construction of integration points and weights for higher-order Gaussian numerical quadrature is not at all trivial, since they are not unique and the question of their optimal selection is extremely difficult. For illustration, each newly explored order of accuracy usually means a new paper in a journal of the numerical quadrature community. Tables of integration points and weights for all reference domains up to the order of accuracy $p = 20$ are available on the CD-ROM that accompanies this book.

Chapter 5 addresses the numerical solution of algebraic and ordinary differential equations resulting from the finite element discretization. We present an overview of contemporary direct and iterative methods for the solution of large systems of linear algebraic equations (such as matrix factorization, preconditioning by classical and block-iterative methods, multigrid techniques), and higher-order one-step and multistep schemes for evolutionary problems.

Chapter 6 presents several approaches to automatic mesh optimization and automatic h-, p- and hp-adaptivity based on the concept of *reference solutions*. Reference solutions are approximations of the exact solution that are substantially more accurate than the finite element approximation itself. We use reference solutions as robust error indicators to guide the adaptive strategies. We also find it useful to recall the basic principles of goal-oriented adaptivity and show the way goal-oriented adaptivity can be incorporated into standard adaptive schemes. The mathematical aspects are combined with intuitive explanation and illustrated with many examples and figures.

We assume that the reader has some experience with the finite element method – say that he/she can solve the Poisson equation ($-\triangle u = f$) in two spatial dimensions using piecewise-linear elements on a triangular mesh. Since it is our goal to make the book readable for both engineers and applied researchers, we attempt to avoid unnecessarily specific mathematical language whenever possible. Usually we prefer giving references to more difficult proofs rather than including them in the text. A somewhat deeper knowledge of mathematics (such as Sobolev spaces, embedding theorems, basic inequalities, etc.) is necessary to understand the theoretical results that accompany some of the finite element algorithms, but some of these can be skipped if the reader is interested only in implementation issues.

The first author is indebted to Prof. Leszek Demkowicz (ICES, The University of Texas at Austin) for many motivating discussions on theoretical issues related to the De Rham diagram, theory of higher-order finite elements and automatic *hp*-adaptivity. He further gratefully acknowledges the numerous suggestions of Prof. Jan Hesthaven (Division of Applied Mathematics, Brown University, Providence, RI), who despite his many other duties found time to read the whole manuscript. Especially noteworthy have been the ideas of Dr. Fabio Nobile (ICES, The University of Texas at Austin), who significantly influenced the structure of the first chapter. Deep appreciation goes to graduate student Denis Ridzal (Department of Computational and Applied Mathematics, Rice University, Houston, TX), who gave freely of his time in investigating the conditioning properties of higher-order shape functions for various types of finite elements in one and two spatial dimensions.

The authors would like to thank Prof. Ronald Cools (Departement Computerwetenschappen, Katholieke Universiteit Leuven, Belgium) for providing them with valuable information related to higher-order numerical quadrature and for his help with the review of Chapter 4. Many thanks are owed to Jan Haškovec (Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic), Dr. Petr Klouček (Department of Computational and Applied Mathematics, Rice University, Houston, TX), Dr. Dalibor Lukáš (Technical University of Ostrava, Czech Republic), Dr. Andreas Obereder (Institute of Industrial Mathematics, Johannes Kepler University, Linz, Austria), Dr. Tomáš Vejchodský (Mathematical Institute of the Academy of Sciences of the Czech Republic, Prague), and Martin Zítka (Faculty of Mathematics and Physics, Charles University, Prague) for their invaluable help with the review of the manuscript.

The authors would also like to thank Dr. Sunil Nair, Helena Redshaw, Jasmin Naim and Christine Andreasen (Chapman & Hall/CRC Press) for their friendly and efficient assistance during the final stage of the publishing process.

The work of the first author was sponsored partially by the Grant Agency of the Czech Republic under grants GP102/01/D114 and 102/01/0184, and partially by the TICAM Postdoctoral Fellowship Award. Several results from TICAM Reports No. 02-32 and No. 02-36 are included. The second and third authors acknowledge partial financial support of the Grant Agency of the Czech Republic under grants 201/01/1200 and 102/01/0184.

Our efforts could never have been successful without the understanding, patience and support of our families, for which we are deeply grateful.

Houston and Praha, March 2003 **P. Šolín, K. Segeth, I. Doležel**

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