Chapter 4

Higher-order numerical quadrature

The numerical quadrature technique (sometimes referred to as *cubature* in three spatial dimensions) lies at the heart of all higher-order finite element codes. When evaluating the approximate variational formulation (1.22) on the reference element, one has to numerically integrate higher-order shape functions and their products, their derivatives, sometimes nonlinearities coming from the reference maps and other higher-order terms. The order of accuracy of the numerical integration should correspond to the highest polynomial order that appears behind the integral sign, otherwise one risks a loss of accuracy of the whole scheme and other disagreeable side-effects. Sometimes exact integration is not possible (e.g., because the variational formulation contains nonpolynomial terms) – in such cases one has to be very careful and choose the order of accuracy of the quadrature a little higher rather than a little lower.

We recommend the utmost care in the evaluation of matrices that have to be inverted: the safest way is to evaluate them exactly, which often means to use quadrature formulae at least twice as accurate as the polynomial order of the finite elements. When one underestimates the precision of evaluation of these matrices, their inversion can produce unexpected errors or they even may become singular.

In this chapter we introduce several types of higher-order quadrature schemes related to the reference domains K_a, K_q, K_t, K_B, K_T and K_P from Chapter 2. In our implementations we usually prefer the Gauss quadrature because of its efficiency but we will also mention other standard techniques.

Judgment of quality of quadrature schemes is a risky business – generally it cannot be said that one of the quadrature rules is *better* than the others. Obviously, results for polynomials up to the order n obtained by various quadrature schemes of the same order of accuracy n are *by definition* the same. But we may obtain dramatically different results by applying the same rules to polynomials of order higher than n, nonpolynomial functions or, in the worst case, to functions that oscillate. A limited-order numerical integration of oscillating functions may give arbitrary results. It is the responsibility of everyone to choose quadrature rules that fit as well as possible the nature of the solved problem.

All tables of integration points and weights that will be presented in the following can also be found on the companion CD-ROM, including many more that were too long to be included here. All of them are sufficient for polynomials of the order $p = 20$ or higher, except for economical Gauss quadrature on the reference brick K_B (where to the best of our knowledge the highest known order of accuracy is $p = 19$). Still higher (in fact almost unlimited) order of accuracy can be achieved by means of product rules based on one-dimensional Gauss quadrature.

4.1 One-dimensional reference domain K_a

The basic ideas of the numerical quadrature will be illustrated on one-dimensional schemes. Let $g(y)$ be a function continuous in interval $[a,b]$, $a < b$. The numerical quadrature of order n on this interval is defined as

$$\int_a^b g(y)\mathrm{d}y \approx \sum_{k=0}^{n} A_{n,k} g(y_{n,k}), \tag{4.1}$$

where the symbols $A_{n,k}$ and $y_{n,k}$, $k = 0, 1, \ldots, n$ denote the quadrature coefficients and nodes, respectively. The nodes $y_{n,k}$, $k = 0, 1, \ldots, n$ are assumed distinct. Putting

$$y = c\xi + d, \qquad c = \frac{b-a}{2}, \qquad d = \frac{b+a}{2}, \tag{4.2}$$

substituting into (4.1) and rearranging, we get a formula corresponding to the one-dimensional reference domain $K_a = (-1, 1)$,

$$\int_{-1}^{1} f(\xi)\mathrm{d}\xi \approx \sum_{k=0}^{n} w_{n,k} f(\xi_{n,k}) \tag{4.3}$$

where $f(\xi) = g(c\xi + d)$ and $w_{n,k} = A_{n,k}/c$. Symbols $w_{n,k}$ are called *weights*.

There are a number of possibilities for choosing suitable weights $w_{n,k}$ and nodes $\xi_{n,k}$ for the numerical quadrature of the function $f(\xi)$. Specific characteristics will be discussed in the following.

Selection of shape functions

The problem of determining the integration points and weights is crucial for all types of quadrature rules. In general we can use various systems of linearly independent functions (not only polynomials) whose integrals can be determined analytically. The choice of such functions usually does not matter as long as the order of accuracy is reasonably small. In this case probably the easiest choice is the monomials ξ^i. For higher-order monomials, however, the inversion of the system matrix becomes problematic. The reason is roundoff errors in the evaluation of higher-order monomials for arguments close to zero.

Probably the most natural choice is to use either the Legendre polynomials or H^1-hierarchic shape functions.

4.1.1 Newton-Cotes quadrature

The quadrature rules of the Newton-Cotes type are generally based on the summation of weighted function values at equidistantly distributed integration points. The $(n+1)$-point Newton-Cotes closed quadrature formula (of accuracy n) for polynomials of the n-th order is given by (4.3), where

$$\xi_{n,k} = -1 + kh_n, \quad k = 0, 1, \ldots, n, \tag{4.4}$$

$h_n = 2/n$ and $n > 0$.

The integration weights may be determined by several different methods based, for example, on the Taylor expansion of $f(\xi)$, Lagrange polynomials or Vandermonde matrix. We will illustrate the last method.

Let $P(\xi)$ be a polynomial of order n expressed as

$$P(\xi) = \sum_{k=0}^{n} p_{n,k}\xi^k \tag{4.5}$$

and let us put

$$\int_{-1}^{1} P(\xi)\mathrm{d}\xi = \sum_{k=0}^{n} \frac{p_{n,k}}{k+1}[1-(-1)^{k+1}] = \sum_{k=0}^{n} w_{n,k}P(\xi_{n,k}). \tag{4.6}$$

Comparison of terms on both sides of (4.6) containing the individual coefficients $p_{n,k}$ leads to a system of linear algebraic equations of the form

$$\begin{aligned}
w_{n,0} + w_{n,1} + \ldots + w_{n,n} &= [1-(-1)^1]/1 = 2, \\
w_{n,0}\xi_{n,0} + w_{n,1}\xi_{n,1} + \ldots + w_{n,n}\xi_{n,n} &= [1-(-1)^2]/2 = 0, \\
&\vdots \\
w_{n,0}\xi_{n,0}^k + w_{n,1}\xi_{n,1}^k + \ldots + w_{n,n}\xi_{n,n}^k &= [1-(-1)^{k+1}]/k+1, \\
&\vdots \\
w_{n,0}\xi_{n,0}^n + w_{n,1}\xi_{n,1}^n + \ldots + w_{n,n}\xi_{n,n}^n &= [1-(-1)^{n+1}]/n+1.
\end{aligned} \tag{4.7}$$

After rearranging the system and using substitution (4.4) for $\xi_{n,k}$ we obtain

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2 & \ldots & n \\ 0 & 1^2 & 2^2 & \ldots & n^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 1^n & 2^n & \ldots & n^n \end{pmatrix} \begin{pmatrix} w_{n,0} \\ w_{n,1} \\ w_{n,2} \\ \ldots \\ w_{n,n} \end{pmatrix} = \begin{pmatrix} 2n^0/1 \\ 2n^1/2 \\ 2n^2/3 \\ \ldots \\ 2n^n/(n+1) \end{pmatrix}. \tag{4.8}$$

The system is characterized by the Vandermonde matrix that is regular and, thus, invertible. Moreover, the weights $w_{n,k}$, $k = 0, \ldots, n$ depend only on parameter n. On the other hand, the Vandermonde matrix is not well-conditioned and for $n > 7$ some weights are negative, which can lead to round-off problems during the evaluation of the right-hand side of (4.6). Therefore, the closed Newton-Cotes formulae are mostly used only for low values of n.

Integration points and weights for lower values of n are shown in Tables 4.1 – 4.7. As the integration points are symmetric with respect to zero, we list only the positive ones. Notice that in one dimension the $(n+1)$-point closed Newton-Cotes quadrature rule has the order of accuracy n. In general, each closed Newton-Cotes formula is exact for all polynomials whose order is by one degree less than the order of the derivative in its error term. For even values of n the integration is exact for all polynomials of order $n + 1$.

TABLE 4.1: Closed Newton-Cotes quadrature on K_a, order $n = 1$ (trapezoidal rule).

Point #	$\pm\,\xi$-Coordinate	Weight
1.	1	1

TABLE 4.2: Closed Newton-Cotes quadrature on K_a, order $n = 2$ (Simpson's 1/3 rule).

Point #	$\pm\,\xi$-Coordinate	Weight
1.	1	1/3
2.	0	4/3

TABLE 4.3: Closed Newton-Cotes quadrature on K_a, order $n = 3$ (Simpson's 3/8 rule).

Point #	$\pm\,\xi$-Coordinate	Weight
1.	1	1/4
2.	1/3	3/4

TABLE 4.4: Closed Newton-Cotes quadrature on K_a, order $n = 4$ (Bode's rule).

Point #	$\pm\ \xi$-Coordinate	Weight
1.	1	7/45
2.	1/2	32/45
3.	0	12/45

TABLE 4.5: Closed Newton-Cotes quadrature on K_a, order $n = 5$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	1	19/144
2.	3/5	75/144
3.	1/5	50/144

TABLE 4.6: Closed Newton-Cotes quadrature on K_a, order $n = 6$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	1	41/420
2.	2/3	216/420
3.	1/3	27/420
4.	0	272/420

TABLE 4.7: Closed Newton-Cotes quadrature on K_a, order $n = 7$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	1	751/8640
2.	5/7	3577/8640
3.	3/7	1323/8640
4.	1/7	2989/8640

In a similar way one can obtain *open Newton-Cotes quadrature formulae* approximating the integral only by function values at internal points of the interval $[a, b]$ (i.e., at the points $\xi_{n,1}, \xi_{n,2}, \ldots, \xi_{n,n-1}$), while the points $\xi_{n,0} = a$ and $\xi_{n,n} = b$ are omitted. These formulae can be used, for example, when values $f(a)$ and $f(b)$ are unavailable. As their errors are much greater than errors of the closed Newton-Cotes formulae, we will not discuss them in detail.

4.1.2 Chebyshev quadrature

The quadrature rules of the Chebyshev type are based on the summation of equally weighted function values at nonequidistantly distributed integration points. The $(n+1)$-point Chebyshev quadrature rule for the one-dimensional reference domain $K_a = (-1, 1)$ reads

$$\int_{-1}^{1} f(\xi)\mathrm{d}\xi \approx \frac{2}{n} \sum_{k=1}^{n} f(\xi_{n,k}). \tag{4.9}$$

Notice that the uniform weight $2/n$ is determined from the integration of constant functions. The integration points (abscissas) are obtained after inserting sufficiently many linearly independent functions with known integrals (e.g., the Legendre polynomials or the one-dimensional H^1-hierarchic shape functions for reasons mentioned earlier) into (4.9) and resolving the resulting system of nonlinear algebraic equations. It can be shown that these abscissas may be obtained by using terms up to y^n in the Maclaurin series of the function

$$s_n(z) = exp\left\{\frac{n}{2}\left[-2 + \ln[(1-z^2)(1-z^{-2})]\right]\right\}. \tag{4.10}$$

Then the abscissas are determined as roots of the function

$$C_n(\xi) = \xi^n s_n\left(\frac{1}{\xi}\right). \tag{4.11}$$

The roots are all real only for $n < 8$ and $n = 9$. These values of n represent the only permissible orders for the Chebyshev quadrature. The corresponding functions $C_n(\xi)$ follow:

$$\begin{aligned}
C_2(\xi) &= \frac{1}{3}(3\xi^2 - 1) \\
C_3(\xi) &= \frac{1}{2}(2\xi^3 - \xi) \\
C_4(\xi) &= \frac{1}{45}(45\xi^4 - 30\xi^2 + 1) \\
C_5(\xi) &= \frac{1}{72}(72\xi^5 - 60\xi^3 + 7\xi) \\
C_6(\xi) &= \frac{1}{105}(105\xi^6 - 105\xi^4 + 21\xi^2 - 1) \\
C_7(\xi) &= \frac{1}{6480}(6480\xi^7 - 7560\xi^5 + 2142\xi^3 - 149\xi) \\
C_9(\xi) &= \frac{1}{22400}(22400\xi^9 - 33600\xi^7 + 15120\xi^5 - 2280\xi^3 + 53\xi)
\end{aligned} \tag{4.12}$$

In the one-dimensional case the n-point Chebyshev quadrature rules achieve $(n+1)$-th order of accuracy.

Let us list the integration points for the permitted values of n calculated using Mathematica (and compared with [170]) in Tables 4.8 – 4.14. As the integration points are symmetric with respect to zero, again we list only the positive ones.

TABLE 4.8: Chebyshev quadrature on K_a, order $n + 1 = 3$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.57735 02691 89625 76450 91488	1

TABLE 4.9: Chebyshev quadrature on K_a, order $n + 1 = 4$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.70710 67811 86547 52440 08444	2/3
2.	0.00000 00000 00000 00000 00000	2/3

TABLE 4.10: Chebyshev quadrature on K_a, order $n + 1 = 5$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.79465 44722 91766 12295 55309	1/2
2.	0.18759 24740 85079 89986 01393	1/2

TABLE 4.11: Chebyshev quadrature on K_a, order $n + 1 = 6$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.83249 74870 00981 87589 25836	2/5
2.	0.37454 14095 53581 06558 60444	2/5
3.	0.00000 00000 00000 00000 00000	2/5

TABLE 4.12: Chebyshev quadrature on K_a, order $n+1=7$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.86624 68181 07820 59138 35981	1/3
2.	0.42251 86537 61111 52911 85464	1/3
3.	0.26663 54015 16704 72033 15346	1/3

TABLE 4.13: Chebyshev quadrature on K_a, order $n+1=8$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.88386 17007 58049 03570 42241	2/7
2.	0.52965 67752 85156 81138 50475	2/7
3.	0.32391 18105 19907 63751 96731	2/7
4.	0.00000 00000 00000 00000 00000	2/7

TABLE 4.14: Chebyshev quadrature on K_a, order $n+1=10$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.91158 93077 28434 47366 49486	2/9
2.	0.60101 86553 80238 07142 81279	2/9
3.	0.52876 17830 57879 99326 01816	2/9
4.	0.16790 61842 14803 94306 80319	2/9
5.	0.00000 00000 00000 00000 00000	2/9

REMARK 4.1 Expressions of the type

$$\frac{f(\xi)}{\sqrt{1-\xi^2}} \tag{4.13}$$

can be integrated by means of the Gauss-Chebyshev explicit formula

$$\int_{-1}^{1} \frac{f(\xi)}{\sqrt{1-\xi^2}} \mathrm{d}\xi \approx \frac{\pi}{n} \sum_{i=1}^{n} f\left[\cos\left(\frac{(2i-1)\pi}{2n}\right)\right]. \tag{4.14}$$

□

4.1.3 Lobatto (Radau) quadrature

The quadrature rules of the Lobatto (Radau) type are based on the summation of weighted function values at nonequidistantly distributed integration points containing the endpoints of the interval of integration.

The n-point Lobatto (Radau) quadrature rule for the one-dimensional reference domain $K_a = (-1, 1)$ reads

$$\int_{-1}^{1} f(\xi)\mathrm{d}\xi \approx w_{n,1}f(-1) + \sum_{i=2}^{n-1} w_{n,i}f(\xi_{n,i}) + w_{n,n}f(1). \tag{4.15}$$

Analogously as for the Chebyshev rules, the integration points and weights are obtained after inserting sufficiently many linearly independent functions with known integrals into (4.15) and resolving the resulting system of nonlinear algebraic equations. Notice that for the n-point rule we have $n-2$ unknown points and n weights. Thus we need $2n-2$ equations and consequently this quadrature rule achieves in one spatial dimension only the order of accuracy $2n-3$.

The unknown abscissas $\xi_{n,i}$ are the roots of the polynomial $L'_{n-1}(\xi)$, where $L_{n-1}(\xi)$ is the Legendre polynomial of order $n-1$. The weights of the unknown abscissas are expressed by

$$w_{n,k} = \frac{2}{n(n-1)L^2_{n-1}(\xi_i)}, \quad k = 2, \ldots, n-1 \tag{4.16}$$

while the weights at the endpoints are

$$w_{n,1} = w_{n,n} = \frac{2}{n(n-1)}. \tag{4.17}$$

In Tables 4.15 – 4.20 we show the integration points and weights for a few selected orders of accuracy computed using Mathematica (and compared with [126]). As the integration points are symmetric with respect to zero, we list only the positive ones, with the corresponding weights. The weights for symmetric integration points are equal. Additional Lobatto (Radau) quadrature rules up to the order $p = 21$ can be found on the companion CD-ROM.

TABLE 4.15: Lobatto (Radau) quadrature on K_a, order $2n-3=3$.

Point #	$\pm$ ξ-Coordinate	Weight
1.	1.00000 00000 00000 00000 00000	0.33333 33333 33333 33333 33333
2.	0.00000 00000 00000 00000 00000	1.33333 33333 33333 33333 33333

TABLE 4.16: Lobatto (Radau) quadrature on K_a, order $2n-3=5$.

Point #	$\pm$ ξ-Coordinate	Weight
1.	1.00000 00000 00000 00000 00000	0.16666 66666 66666 66666 66667
2.	0.44721 35954 99957 93928 18347	0.83333 33333 33333 33333 33333

TABLE 4.17: Lobatto (Radau) quadrature on K_a, order $2n-3=7$.

Point #	$\pm$ ξ-Coordinate	Weight
1.	1.00000 00000 00000 00000 00000	0.10000 00000 00000 00000 00000
2.	0.65465 36707 07977 14379 82925	0.54444 44444 44444 44444 44444
3.	0.00000 00000 00000 00000 00000	0.71111 11111 11111 11111 11111

TABLE 4.18: Lobatto (Radau) quadrature on K_a, order $2n-3=9$.

Point #	$\pm$ ξ-Coordinate	Weight
1.	1.00000 00000 00000 00000 00000	0.06666 66666 66666 66666 66667
2.	0.76505 53239 29464 69285 10030	0.37847 49562 97846 98031 66128
3.	0.28523 15164 80645 09631 41510	0.55485 83770 35486 35301 67205

TABLE 4.19: Lobatto (Radau) quadrature on K_a, order $2n-3=11$.

Point #	$\pm$ ξ-Coordinate	Weight
1.	1.00000 00000 00000 00000 00000	0.04761 90476 19047 61904 76190
2.	0.83022 38962 78566 92987 20322	0.27682 60473 61565 94801 07004
3.	0.46884 87934 70714 21380 37719	0.43174 53812 09862 62341 78710
4.	0.00000 00000 00000 00000 00000	0.48761 90476 19047 61904 76190

TABLE 4.20: Lobatto (Radau) quadrature on K_a, order $2n-3=13$. See the companion CD-ROM for additional Lobatto (Radau) quadrature rules up to the order $p=21$.

Point #	$\pm$ ξ-Coordinate	Weight
1.	1.00000 00000 00000 00000 00000	0.03571 42857 14285 71428 57143
2.	0.87174 01485 09606 61533 74457	0.21070 42271 43506 03938 29921
3.	0.59170 01814 33142 30214 45107	0.34112 26924 83504 36476 42407
4.	0.20929 92179 02478 86876 86573	0.41245 87946 58703 88156 70530

4.1.4 Gauss quadrature

The quadrature rules of the Gauss type are based on the summation of weighted function values on nonequidistantly distributed integration points. The n-point Gauss quadrature rule for the one-dimensional reference domain $K_a = (-1, 1)$ reads

$$\int_{-1}^{1} f(\xi)\mathrm{d}\xi \approx \sum_{i=1}^{n} w_{n,i} f(\xi_{n,i}) \tag{4.18}$$

Analogously as for the Chebyshev and Lobatto (Radau) rules, the integration points and weights can be obtained after inserting sufficiently many linearly independent functions with known integrals and resolving the resulting system of nonlinear algebraic equations. Since we have $2n$ unknown parameters at our disposal (n integration points $\xi_{n,i}$ and n weights $w_{n,i}$), the resulting formula will be accurate for all polynomials of order $2n-1$ and lower.

It can be shown that the integration points are roots of the Legendre polynomials $L_n(\xi)$, whose values are sufficiently well tabulated. Hence, the complexity of the problem reduces to the level of Newton-Cotes quadrature rules, since with known points the nonlinear system comes over to a system of linear algebraic equations. The analysis leads even further; it is known that the weights $w_{n,i}$ can be expressed as

$$w_{n,i} = \frac{2}{(1-\xi_{n,i}^2)L_n'(\xi)^2}, \quad i = 1, \ldots, n \tag{4.19}$$

Let us list the integration points and weights for a few selected n-point rules, which again were computed in Mathematica and compared with [1], in Tables 4.21 – 4.35. As the integration points are symmetric with respect to zero, we list only the positive ones. As usual, symmetric integration points have identical weights. Additional Gauss quadrature rules up to the order $p = 127$ can be found on the companion CD-ROM.

TABLE 4.21: Gauss quadrature on K_a, order $2n-1=3$.

Point #	$\pm$ ξ-Coordinate	Weight
1.	0.57735 02691 89625 76450 91488	1.00000 00000 00000 00000 00000

TABLE 4.22: Gauss quadrature on K_a, order $2n-1=5$.

Point #	$\pm$ ξ-Coordinate	Weight
1.	0.00000 00000 00000 00000 00000	0.88888 88888 88888 88888 88889
2.	0.77459 66692 41483 37703 58531	0.55555 55555 55555 55555 55556

TABLE 4.23: Gauss quadrature on K_a, order $2n-1=7$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.33998 10435 84856 26480 26658	0.65214 51548 62546 14262 69361
2.	0.86113 63115 94052 57522 39465	0.34785 48451 37453 85737 30639

TABLE 4.24: Gauss quadrature on K_a, order $2n-1=9$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.00000 00000 00000 00000 00000	0.56888 88888 88888 88888 88889
2.	0.53846 93101 05683 09103 63144	0.47862 86704 99366 46804 12915
3.	0.90617 98459 38663 99279 76269	0.23692 68850 56189 08751 42640

TABLE 4.25: Gauss quadrature on K_a, order $2n-1=11$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.23861 91860 83196 90863 05017	0.46791 39345 72691 04738 98703
2.	0.66120 93864 66264 51366 13996	0.36076 15730 48138 60756 98335
3.	0.93246 95142 03152 02781 23016	0.17132 44923 79170 34504 02961

TABLE 4.26: Gauss quadrature on K_a, order $2n-1=13$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.00000 00000 00000 00000 00000	0.41795 91836 73469 38775 51020
2.	0.40584 51513 77397 16690 66064	0.38183 00505 05118 94495 03698
3.	0.74153 11855 99394 43986 38648	0.27970 53914 89276 66790 14678
4.	0.94910 79123 42758 52452 61897	0.12948 49661 68869 69327 06114

TABLE 4.27: Gauss quadrature on K_a, order $2n-1=15$.

Point #	$\pm\ \xi$-Coordinate	Weight
1.	0.18343 46424 95649 80493 94761	0.36268 37833 78361 98296 51504
2.	0.52553 24099 16328 98581 77390	0.31370 66458 77887 28733 79622
3.	0.79666 64774 13626 73959 15539	0.22238 10344 53374 47054 43560
4.	0.96028 98564 97536 23168 35609	0.10122 85362 90376 25915 25314

TABLE 4.28: Gauss quadrature on K_a, order $2n-1=17$.

Point #	± ξ-Coordinate	Weight
1.	0.00000 00000 00000 00000 00000	0.33023 93550 01259 76316 45251
2.	0.32425 34234 03808 92903 85380	0.31234 70770 40002 84006 86304
3.	0.61337 14327 00590 39730 87020	0.26061 06964 02935 46231 87429
4.	0.83603 11073 26635 79429 94298	0.18064 81606 94857 40405 84720
5.	0.96816 02395 07626 08983 55762	0.08127 43883 61574 41197 18922

TABLE 4.29: Gauss quadrature on K_a, order $2n-1=19$.

Point #	± ξ-Coordinate	Weight
1.	0.14887 43389 81631 21088 48260	0.29552 42247 14752 87017 38930
2.	0.43339 53941 29247 19079 92659	0.26926 67193 09996 35509 12269
3.	0.67940 95682 99024 40623 43274	0.21908 63625 15982 04399 55349
4.	0.86506 33666 88984 51073 20967	0.14945 13491 50580 59314 57763
5.	0.97390 65285 17171 72007 79640	0.06667 13443 08688 13759 35688

TABLE 4.30: Gauss quadrature on K_a, order $2n-1=21$.

Point #	± ξ-Coordinate	Weight
1.	0.00000 00000 00000 00000 00000	0.27292 50867 77900 63071 44835
2.	0.26954 31559 52344 97233 15320	0.26280 45445 10246 66218 06889
3.	0.51909 61292 06811 81592 57257	0.23319 37645 91990 47991 85237
4.	0.73015 20055 74049 32409 34163	0.18629 02109 27734 25142 60980
5.	0.88706 25997 68095 29907 51578	0.12558 03694 64904 62463 46940
6.	0.97822 86581 46056 99280 39380	0.05566 85671 16173 66648 27537

TABLE 4.31: Gauss quadrature on K_a, order $2n-1=23$.

Point #	± ξ-Coordinate	Weight
1.	0.12523 34085 11468 91547 24414	0.24914 70458 13402 78500 05624
2.	0.36783 14989 98180 19375 26915	0.23349 25365 38354 80876 08499
3.	0.58731 79542 86617 44729 67024	0.20316 74267 23065 92174 90645
4.	0.76990 26741 94304 68703 68938	0.16007 83285 43346 22633 46525
5.	0.90411 72563 70474 85667 84659	0.10693 93259 95318 43096 02547
6.	0.98156 06342 46719 25069 05491	0.04717 53363 86511 82719 46160

TABLE 4.32: Gauss quadrature on K_a, order $2n-1=31$.

Point #	± ξ-Coordinate	Weight
1.	0.09501 25098 37637 44018 53193	0.18945 06104 55068 49628 53967
2.	0.28160 35507 79258 91323 04605	0.18260 34150 44923 58886 67637
3.	0.45801 67776 57227 38634 24194	0.16915 65193 95002 53818 93121
4.	0.61787 62444 02643 74844 66718	0.14959 59888 16576 73208 15017
5.	0.75540 44083 55003 03389 51012	0.12462 89712 55533 87205 24763
6.	0.86563 12023 87831 74388 04679	0.09515 85116 82492 78480 99251
7.	0.94457 50230 73232 57607 79884	0.06225 35239 38647 89286 28438
8.	0.98940 09349 91649 93259 61542	0.02715 24594 11754 09485 17806

TABLE 4.33: Gauss quadrature on K_a, order $2n-1=39$.

Point #	$\pm$ ξ-Coordinate	Weight
1.	0.07652 65211 33497 33375 46404	0.15275 33871 30725 85069 80843
2.	0.22778 58511 41645 07808 04962	0.14917 29864 72603 74678 78287
3.	0.37370 60887 15419 56067 25482	0.14209 61093 18382 05132 92983
4.	0.51086 70019 50827 09800 43641	0.13168 86384 49176 62689 84945
5.	0.63605 36807 26515 02545 28367	0.11819 45319 61518 41731 23774
6.	0.74633 19064 60150 79261 43051	0.10193 01198 17240 43503 67501
7.	0.83911 69718 22218 82339 45291	0.08327 67415 76704 74872 47581
8.	0.91223 44282 51325 90586 77524	0.06267 20483 34109 06356 95065
9.	0.96397 19272 77913 79126 76661	0.04060 14298 00386 94133 10400
10.	0.99312 85991 85094 92478 61224	0.01761 40071 39152 11831 18620

TABLE 4.34: Gauss quadrature on K_a, order $2n-1=47$.

Point #	$\pm$ ξ-Coordinate	Weight
1.	0.06405 68928 62605 62608 50431	0.12793 81953 46752 15697 40562
2.	0.19111 88674 73616 30915 86398	0.12583 74563 46828 29612 13754
3.	0.31504 26796 96163 37438 67933	0.12167 04729 27803 39120 44632
4.	0.43379 35076 26045 13848 70842	0.11550 56680 53725 60135 33445
5.	0.54542 14713 88839 53565 83756	0.10744 42701 15965 63478 25773
6.	0.64809 36519 36975 56925 24958	0.09761 86521 04113 88826 98807
7.	0.74012 41915 78554 36424 38281	0.08619 01615 31953 27591 71852
8.	0.82000 19859 73902 92195 39499	0.07334 64814 11080 30573 40336
9.	0.88641 55270 04401 03421 31543	0.05929 85849 15436 78074 63678
10.	0.93827 45520 02732 75852 36490	0.04427 74388 17419 80616 86027
11.	0.97472 85559 13094 98198 39199	0.02853 13886 28933 66318 13078
12.	0.99518 72199 97021 36017 99974	0.01234 12297 99987 19954 68057

TABLE 4.35: Gauss quadrature on K_a, order $2n-1=63$. See the companion CD-ROM for additional Gauss quadrature rules up to the order $p=127$.

Point #	$\pm$ ξ-Coordinate	Weight
1.	0.04830 76656 87738 31623 48126	0.09654 00885 14727 80056 67648
2.	0.14447 19615 82796 49348 51864	0.09563 87200 79274 85941 90820
3.	0.23928 73622 52137 07454 46032	0.09384 43990 80804 56563 91802
4.	0.33186 86022 82127 64977 99168	0.09117 38786 95763 88471 28686
5.	0.42135 12761 30635 34536 41194	0.08765 20930 04403 81114 27715
6.	0.50689 99089 32229 39002 37475	0.08331 19242 26946 75522 21991
7.	0.58771 57572 40762 32904 07455	0.07819 38957 87070 30647 17409
8.	0.66304 42669 30215 20097 51152	0.07234 57941 08848 50622 53994
9.	0.73218 21187 40289 68038 74267	0.06582 22227 76361 84683 76501
10.	0.79448 37959 67942 40696 30973	0.05868 40934 78535 54714 52836
11.	0.84936 76137 32569 97013 36930	0.05099 80592 62376 17619 61632
12.	0.89632 11557 66052 12396 53072	0.04283 58980 22226 68065 68787
13.	0.93490 60759 37739 68917 09191	0.03427 38629 13021 43310 26877
14.	0.96476 22555 87506 43077 38119	0.02539 20653 09262 05945 57526
15.	0.98561 15115 45268 33540 01750	0.01627 43947 30905 67060 51706
16.	0.99726 38618 49481 56354 49811	0.00701 86100 09470 09660 04071

4.2 Reference quadrilateral K_q

In this section we present selected higher-order quadrature rules for the reference quadrilateral domain K_q.

4.2.1 Composite Gauss quadrature

Let us start with a technique that is easiest to implement – quadrature formulae based on the Cartesian product of two one-dimensional quadrature rules in the axial directions ξ_1 and ξ_2. Consider the formula

$$\int_{K_a} f(\xi)\mathrm{d}\xi \approx \sum_{i=1}^{M_a} w_{g_a,i} f(y_{g_a,i}), \tag{4.20}$$

where $y_{g_a,i}, w_{g_a,i}$ are Gauss integration points and weights on the one-dimensional reference domain $K_a = (-1, 1)$ that integrate exactly all polynomials of the order p and lower. It is easy to see that the product formula

$$\int_{K_a}\int_{K_a} g(\xi_1, \xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 \approx \sum_{i=1}^{M_a}\sum_{j=1}^{M_a} w_{g_a,i}w_{g_a,j}g(y_{g_a,i}, y_{g_a,j}) \tag{4.21}$$

is of the order p for polynomials of two independent variables ξ_1, ξ_2.

REMARK 4.2 In addition to its simple implementation, the product formula (4.21) has one more advantage – it can easily be generalized to polynomials with different orders of approximation in the axial directions ξ_1 and ξ_2. Such polynomials may appear naturally as a consequence of p-anisotropic refinements of quadrilateral elements, that may occur, e.g., within boundary and internal layers. Recall that in Chapter 2 the master element shape functions for the reference quadrilateral domain were designed to allow for p-anisotropy. $\square$

However, for *complete* polynomials of order p (with generally $n = (p+1)(p+2)/2$ nonzero terms), much more efficient formulae are known. We introduce them in Paragraph 4.2.2.

4.2.2 Economical Gauss quadrature

In this paragraph we introduce Gauss quadrature rules that require fewer integration points than their product counterparts from Paragraph 4.2.1. Some of them are even known to require *the minimum number* of integration points. For this reason sometimes one calls these quadrature rules *economical*. We design them for complete polynomials only, starting from the symmetry of the reference quadrilateral K_q.

First let us have a look at the integral

$$\int_{-1}^{1}\int_{-1}^{1} \xi_1^j \xi_2^k \, \mathrm{d}\xi_2 \, \mathrm{d}\xi_1, \tag{4.22}$$

where j and k are nonnegative integers. As long as j or k is odd, the integral is equal to zero and the corresponding terms need not be taken into account for calculation. If j and k are zero or even, its value is $4/(j+1)/(k+1)$. Moreover, due to the symmetry of the reference domain K_q with respect to the axes $\xi_1 = \xi_2$ and $\xi_1 = -\xi_2$, it is sufficient to consider only terms in which $j \geq k$.

To give an example, the analysis of complete polynomials of an even order p requires us to consider only the terms $1, \xi_1^2, \ldots, \xi_1^p, \xi_1^2\xi_2^2, \ldots, \xi_1^{p-2}\xi_2^2, \xi_1^4\xi_2^4, \ldots,$ $\xi_1^{p-4}\xi_2^4$, ..., etc. Denoting the reduced number of terms by m, for $p = 4$ we have $m = 4$ while $n = (p+1)(p+2)/2 = 15$. For $p = 6$ we have $m = 6$ while $n = 28$, for $p = 20$ it is $m = 36$ while $n = 231$. We see that the symmetry considerations are essential.

4.2.3 Tables of Gauss quadrature points and weights

Dunavant [73] provides a useful overview of minimum numbers of Gaussian quadrature points for quadrilaterals in Table 4.36.

TABLE 4.36: Minimum numbers of quadrature points for the Gauss quadrature over quadrilaterals.

Polyn. order	Minimum num. of nonsym. points	Minimum num. of sym. points	Achieved num. of sym. points
1	1	1	1
2	3	4	4
3	4	4	4
4	6	8	8
5	7	8	8
6	10	12	12
7	12	12	12
8	15	20	20
9	17	20	20
10	21	25	25
11	24	25	25
12	28	36	36
13	31	36	36
14	36	44	45
15	40	44	45
16	45	56	60
17	49	56	60
18	55	68	72
19	60	68	72
20	65	84	88
21	71	84	88

Dunavant divides the integration points into four groups with different numbers of unknowns and tests their best choices. The results obtained by solving the corresponding systems of nonlinear equations are given in Tables 4.37 – 4.40 below. Additional economical Gauss quadrature rules up to the order $p = 21$ can be found on the companion CD-ROM.

TABLE 4.37: Gauss quadrature on K_q, order $p = 0, 1$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	Weight
1.	0.00000 00000 00000	0.00000 00000 00000	4.00000 00000 00000

TABLE 4.38: Gauss quadrature on K_q, order $p = 2, 3$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	Weight
1.	0.57735 02691 89626	0.57735 02691 89626	1.00000 00000 00000
2.	0.57735 02691 89626	-0.57735 02691 89626	1.00000 00000 00000
3.	-0.57735 02691 89626	0.57735 02691 89626	1.00000 00000 00000
4.	-0.57735 02691 89626	-0.57735 02691 89626	1.00000 00000 00000

TABLE 4.39: Gauss quadrature on K_q, order $p = 4, 5$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	Weight
1.	0.68313 00510 63973	0.00000 00000 00000	0.81632 65306 12245
2.	-0.68313 00510 63973	0.00000 00000 00000	0.81632 65306 12245
3.	0.00000 00000 00000	0.68313 00510 63973	0.81632 65306 12245
4.	0.00000 00000 00000	-0.68313 00510 63973	0.81632 65306 12245
5.	0.88191 71036 88197	0.88191 71036 88197	0.18367 34693 87755
6.	0.88191 71036 88197	-0.88191 71036 88197	0.18367 34693 87755
7.	-0.88191 71036 88197	0.88191 71036 88197	0.18367 34693 87755
8.	-0.88191 71036 88197	-0.88191 71036 88197	0.18367 34693 87755

TABLE 4.40: Gauss quadrature on K_q, order $p = 6, 7$. See the companion CD-ROM for additional economical Gauss quadrature rules up to the order $p = 21$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	Weight
1.	0.92582 00997 72551	0.00000 00000 00000	0.24197 53086 41975
2.	-0.92582 00997 72551	0.00000 00000 00000	0.24197 53086 41975
3.	0.00000 00000 00000	0.92582 00997 72551	0.24197 53086 41975
4.	0.00000 00000 00000	-0.92582 00997 72551	0.24197 53086 41975
5.	0.80597 97829 18599	0.80597 97829 18599	0.23743 17746 90630
6.	0.80597 97829 18599	-0.80597 97829 18599	0.23743 17746 90630
7.	-0.80597 97829 18599	0.80597 97829 18599	0.23743 17746 90630
8.	-0.80597 97829 18599	-0.80597 97829 18599	0.23743 17746 90630
9.	0.38055 44332 08316	0.38055 44332 08316	0.52059 29166 67394
10.	0.38055 44332 08316	-0.38055 44332 08316	0.52059 29166 67394
11.	-0.38055 44332 08316	0.38055 44332 08316	0.52059 29166 67394
12.	-0.38055 44332 08316	-0.38055 44332 08316	0.52059 29166 67394

4.3 Reference triangle K_t

In this section we present selected higher-order quadrature techniques for the reference triangular domain K_t. Algorithmically simpler but less efficient schemes are mentioned in Paragraphs 4.3.1 and 4.3.2. More efficient are of course economical Gauss quadrature rules that we introduce in Paragraph 4.3.3.

4.3.1 Translation of quadrature to the ref. quadrilateral K_q

This is probably the easiest way to perform numerical quadrature on K_t with practically unlimited order of accuracy. The idea of the technique consists in a transformation of the integrated function to the reference quadrilateral K_q and consequent application of the composite Gauss quadrature rules discussed in Paragraph 4.2.1. One can improve the efficiency of this type of quadrature by using the economical Gauss quadrature on K_q (see Paragraph 4.2.2) instead of the product rules.

Consider a function g defined on K_t. Intuitively the procedure can be viewed as "stretching" the function g to be defined on K_q so that the volume under g is conserved. The following proposition defines the technique precisely.

PROPOSITION 4.1
Let $g(\boldsymbol{\xi})$ be a continuous bounded function defined on the reference triangle K_t. Then

$$\int_{K_t} g(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \int_{K_q} \frac{1-y_2}{2} g\left(-1 + \frac{1-y_2}{2}(y_1+1), y_2\right)\, d\boldsymbol{y}. \tag{4.23}$$

PROOF Consider the (degenerate) mapping

$$\boldsymbol{\xi}(\boldsymbol{y}) : \boldsymbol{y} \to \boldsymbol{\xi}(\boldsymbol{y}) = \begin{pmatrix} -1 + \frac{1-y_2}{2}(y_1+1) \\ y_2 \end{pmatrix}$$

that transforms K_q to K_t. Its Jacobian

$$\det\left(\frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}\boldsymbol{y}}\right) = \frac{1-y_2}{2}$$

is positive except for the upper edge $y_2 = 1$ where it vanishes. In spite of this the standard Substitution Theorem can be applied, and (4.23) follows immediately. ∎

REMARK 4.3 Notice that the transformation $\boldsymbol{\xi}(\boldsymbol{y})$ produces an additional linear factor $(1 - y_2)/2$ behind the integral sign. Hence, one has to increase the order of integration in the y_2-direction by one. ▯

4.3.2 Newton-Cotes quadrature

The idea behind the construction of the Newton-Cotes quadrature rules for the reference triangle is the same as it was for the one-dimensional reference domain K_a – summation of weighted function values on equidistributed integration points.

Let us consider an integer number $n \geq 2$. The $n(n+1)/2$ equidistributed integration points $\boldsymbol{\xi}_{n,k_1,k_2} = [\xi_{1,n,k_1,k_2}, \xi_{2,n,k_1,k_2}]$; $k_1 = 1, 2, \ldots, n$; $k_2 = 1, 2, \ldots, n + 1 - k_1$, can be chosen as

$$\begin{aligned} \xi_{1,n,k_1,k_2} &= -1 + (k_1 - 1)h_n, \\ \xi_{2,n,k_1,k_2} &= -1 + (k_2 - 1)h_n, \end{aligned} \tag{4.24}$$

where $h_n = 2/(n-1)$. The distribution of integration points is illustrated for $n = 2, 3, 5$ in Figure 4.1.

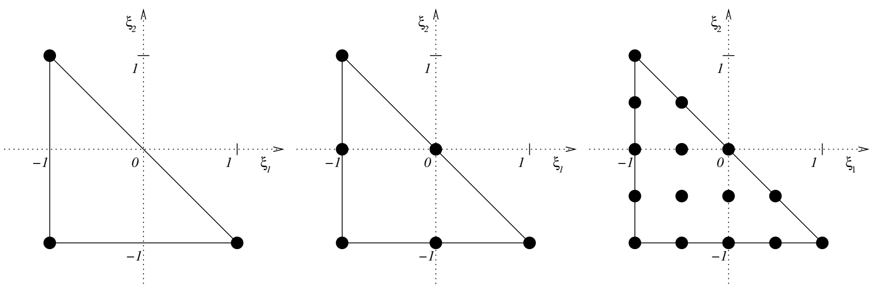

FIGURE 4.1: Newton-Cotes integration points for the reference triangle K_t, $n = 2, 3, 5$.

The $[n(n+1)/2]$-point Newton-Cotes quadrature rule for the reference triangle K_t reads

$$\int_{-1}^{1} \int_{-1}^{1-\xi_1} f(\xi_1, \xi_2) \,\mathrm{d}\xi_2 \,\mathrm{d}\xi_1 \approx \sum_{k_1=1}^{n} \sum_{k_2=1}^{n+1-k_1} w_{n,k_1,k_2} f(\xi_{1,n,k_1,k_2}, \xi_{2,n,k_1,k_2}) \tag{4.25}$$

with the coordinates of the integration points defined in (4.24).

Thus, we have $n(n+1)/2$ unknown integration weights w_{n,k_1,k_2} which can be computed by inserting $n(n+1)/2$ linearly independent polynomials f_k of the order k, $k = 0, 1, \ldots, n-1$, into (4.25) and by solving the resulting linear system. One can use, e.g., the H^1-hierarchic shape functions defined in Paragraph 2.2.3. Notice that the $[n(n+1)/2]$-point Newton-Cotes quadrature rule has the order of accuracy $n-1$.

REMARK 4.4 Conditioning of the linear system for the integration weights can be improved by a choice of more sophisticated nodal points. Instead of the equidistributed points shown in Figure 4.1, one may want to use the *Gauss-Lobatto* points (see Figure 1.2). ▯

4.3.3 Gauss quadrature

We have seen already in Paragraph 4.2.2 that although the Gauss quadrature rules in 2D are designed following the same principles as in one spatial dimension, the calculation of the corresponding integration points and weights is much more difficult.

The fundamental equation for the construction of the integration points and weights for the reference triangle K_t reads

$$\int_{-1}^{1}\int_{-1}^{1-\xi_1} f(\xi_1,\xi_2)\,\mathrm{d}\xi_2\,\mathrm{d}\xi_1 \approx \sum_{k=1}^{m} w_k f(\xi_{1,k},\xi_{2,k}), \tag{4.26}$$

where m denotes the number of integration points. Each point is characterized by three unknowns: w_k, $\xi_{1,k}$ and $\xi_{2,k}$. In order to illustrate complications we have to face when determining their values, let us briefly analyze the formula for complete polynomials of the order $p = 1$ and $p = 2$.

The first-order polynomial $f(\xi_1,\xi_2) = a_0 + a_1\xi_1 + a_2\xi_2$ does not cause any difficulties yet. In this case $m = 1$ and we easily obtain three equations,

$$w_1 = 2, w_1\xi_{1,1} = -2/3, w_1\xi_{2,1} = -2/3, \tag{4.27}$$

the solution of which is

$$w_1 = 2, \xi_{1,1} = -1/3, \xi_{2,1} = -1/3. \tag{4.28}$$

The number of terms of complete quadratic polynomial is 6 and the minimum number of the Gaussian points is 2 (leading to 6 unknowns). The system of equations for the coordinates and weights now reads

$$\begin{aligned} w_1 + w_2 &= 2, \\ w_1\xi_{1,1} + w_1\xi_{1,2} &= -2/3, \end{aligned} \tag{4.29}$$

$$
\begin{aligned}
w_1\xi_{2,1} + w_1\xi_{2,2} &= -2/3,\\
w_1\xi_{1,1}\xi_{2,1} + w_1\xi_{1,2}\xi_{2,2} &= 0,\\
w_1\xi_{1,1}^2 + w_1\xi_{1,2}^2 &= 2/3,\\
w_1\xi_{2,1}^2 + w_1\xi_{2,2}^2 &= 2/3.
\end{aligned}
$$

These equations, unfortunately, are not independent and an unambiguous solution does not exist. That is why at least three points instead of two have to be chosen to ensure that (4.26) is valid.

The same holds for polynomials of higher degrees. Moreover, the number n of terms of a complete polynomial of order p is often not divisible by three. For $p = 3$, for example, $n = (p+1)(p+2)/2 = 10$, so that we have to use at least four Gaussian points.

Difficulties of this kind are common for problems associated with solution of systems of nonlinear algebraic equations, and make the calculation of the integration points and weights for higher values of p practically infeasible by standard methods or mathematical software.

Lyness [133] proposed a sophisticated algorithm based on determining the points and weights within an *equilateral triangle* in polar coordinates, taking advantage of its multiple symmetries. The algorithm provides the minimum number of Gaussian points for any polynomial order p and strongly reduces the size of the nonlinear system. Dunavant [74] extended the algorithm and calculated the points and weights up to the order $p = 20$ (some weights being negative and some points lying outside the triangle). Position of integration points and values of the weights corresponding to other triangles can be obtained by a simple affine transformation.

The fact that some of the weights are negative means that the stability of the quadrature will tend to decrease when integrating oscillatory functions whose polynomial behavior exceeds the order of accuracy of the quadrature formulae. In this case the schemes still can be used, but one has to combine them with spatial refinements of the reference element. If oscillations (or other excessive nonlinearities) in the integrated functions are expected, an application of *adaptive formulae* that compare results from several refinement levels may be a good idea.

4.3.4 Tables of Gauss integration points and weights

Dunavant [74] provides an overview of minimum numbers of quadrature points for Gaussian quadrature over triangles in Table 4.41.

Tables 4.42 – 4.48 give an overview of Gaussian integration points and weights calculated in [133, 74] and transformed to the reference triangle K_t. Additional Gauss quadrature rules up to the order $p = 20$ can be found on the CD-ROM included with this book. A survey of numerical quadrature over triangles can be found in [132].

TABLE 4.41: Minimum numbers of quadrature points for the Gauss quadrature over triangles.

Polyn. order	Known min. num. of points	Predicted min. num. of points	Achieved num. of points
1	1		1
2	3		3
3	4		4
4	6		6
5	7		7
6	12		12
7	13		13
8	16		16
9	19		19
10	24		25
11		27	27
12		33	33
13		36	37
14		40	42
15		45	48
16		51	52
17		55	61
18		63	70
19		67	73
20		73	79

TABLE 4.42: Gauss quadrature on K_t, order $p = 1$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	Weight
1.	-0.33333 33333 33333	-0.33333 33333 33333	2.00000 00000 00000

TABLE 4.43: Gauss quadrature on K_t, order $p = 2$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	Weight
1.	-0.66666 66666 66667	-0.66666 66666 66667	0.66666 66666 66667
2.	-0.66666 66666 66667	0.33333 33333 33333	0.66666 66666 66667
3.	0.33333 33333 33333	-0.66666 66666 66667	0.66666 66666 66667

TABLE 4.44: Gauss quadrature on K_t, order $p = 3$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	Weight
1.	-0.33333 33333 33333	-0.33333 33333 33333	-1.12500 00000 00000
2.	-0.60000 00000 00000	-0.60000 00000 00000	1.04166 66666 66667
3.	-0.60000 00000 00000	0.20000 00000 00000	1.04166 66666 66667
4.	0.20000 00000 00000	-0.60000 00000 00000	1.04166 66666 66667

TABLE 4.45: Gauss quadrature on K_t, order $p = 4$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	Weight
1.	-0.10810 30181 68070	-0.10810 30181 68070	0.44676 31793 56022
2.	-0.10810 30181 68070	-0.78379 39636 63860	0.44676 31793 56022
3.	-0.78379 39636 63860	-0.10810 30181 68070	0.44676 31793 56022
4.	-0.81684 75729 80458	-0.81684 75729 80458	0.21990 34873 10644
5.	-0.81684 75729 80458	0.63369 51459 60918	0.21990 34873 10644
6.	0.63369 51459 60918	-0.81684 75729 80458	0.21990 34873 10644

TABLE 4.46: Gauss quadrature on K_t, order $p = 5$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	Weight
1.	-0.33333 33333 33333	-0.33333 33333 33333	0.45000 00000 00000
2.	-0.05971 58717 89770	-0.05971 58717 89770	0.26478 83055 77012
3.	-0.05971 58717 89770	-0.88056 82564 20460	0.26478 83055 77012
4.	-0.88056 82564 20460	-0.05971 58717 89770	0.26478 83055 77012
5.	-0.79742 69853 53088	-0.79742 69853 53088	0.25187 83610 89654
6.	-0.79742 69853 53088	0.59485 39707 06174	0.25187 83610 89654
7.	0.59485 39707 06174	-0.79742 69853 53088	0.25187 83610 89654

TABLE 4.47: Gauss quadrature on K_t, order $p = 6$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	Weight
1.	-0.50142 65096 58180	-0.50142 65096 58180	0.23357 25514 52758
2.	-0.50142 65096 58180	0.00285 30193 16358	0.23357 25514 52758
3.	0.00285 30193 16358	-0.50142 65096 58180	0.23357 25514 52758
4.	-0.87382 19710 16996	-0.87382 19710 16996	0.10168 98127 40414
5.	-0.87382 19710 16996	0.74764 39420 33992	0.10168 98127 40414
6.	0.74764 39420 33992	-0.87382 19710 16996	0.10168 98127 40414
7.	-0.37929 50979 32432	0.27300 49982 42798	0.16570 21512 36748
8.	0.27300 49982 42798	-0.89370 99003 10366	0.16570 21512 36748
9.	-0.89370 99003 10366	-0.37929 50979 32432	0.16570 21512 36748
10.	-0.37929 50979 32432	-0.89370 99003 10366	0.16570 21512 36748
11.	0.27300 49982 42798	-0.37929 50979 32432	0.16570 21512 36748
12.	-0.89370 99003 10366	0.27300 49982 42798	0.16570 21512 36748

TABLE 4.48: Gauss quadrature on K_t, order $p = 7$. See the companion CD-ROM for additional Gauss quadrature rules up to the order $p = 20$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	Weight
1.	-0.33333 33333 33333	-0.33333 33333 33333	-0.29914 00889 35364
2.	-0.47930 80678 41920	-0.47930 80678 41920	0.35123 05148 66416
3.	-0.47930 80678 41920	-0.04138 38643 16160	0.35123 05148 66416
4.	-0.04138 38643 16160	-0.47930 80678 41920	0.35123 05148 66416
5.	-0.86973 97941 95568	-0.86973 97941 95568	0.10669 44712 17676
6.	-0.86973 97941 95568	0.73947 95883 91136	0.10669 44712 17676
7.	0.73947 95883 91136	-0.86973 97941 95568	0.10669 44712 17676
8.	-0.37426 90079 90252	0.27688 83771 39620	0.15422 75217 80514
9.	0.27688 83771 39620	-0.90261 93691 49368	0.15422 75217 80514
10.	-0.90261 93691 49368	-0.37426 90079 90252	0.15422 75217 80514
11.	-0.37426 90079 90252	-0.90261 93691 49368	0.15422 75217 80514
12.	0.27688 83771 39620	-0.37426 90079 90252	0.15422 75217 80514
13.	-0.90261 93691 49368	0.27688 83771 39620	0.15422 75217 80514

4.4 Reference brick K_B

This section is devoted to higher-order numerical quadrature on the reference brick domain K_B. The product geometry of K_B is analogous to the geometry of the reference quadrilateral K_q, and therefore also the quadrature schemes exhibit common features. Again we will mention a simple and less efficient product scheme with practically unlimited order of accuracy, and several more economical Gaussian quadrature rules.

4.4.1 Composite Gauss quadrature

The simplest quadrature rules can be constructed by combining one-dimensional Gauss formulae in the three axial directions ξ_1, ξ_2, ξ_3. Let the quadrature rule

$$\int_{K_a} f(\xi)\, d\xi \approx \sum_{i=1}^{M_a} w_{g_a,i} f(y_{g_a,i}), \tag{4.30}$$

where $y_{g_a,i}, w_{g_a,i}$ are Gauss integration points and weights corresponding to the one-dimensional reference domain K_a introduced in Paragraph 4.1.4, integrate exactly all polynomials of the order p and lower. It is easy to see that the formula

$$\int_{K_a^3} g(\xi_1, \xi_2, \xi_3)\, d\xi_1\, d\xi_2\, d\xi_3 \approx \sum_{i=1}^{M_a} \sum_{j=1}^{M_a} \sum_{k=1}^{M_a} w_{g_a,i} w_{g_a,j} w_{g_a,k} g(y_{g_a,i}, y_{g_a,j}, y_{g_a,k}) \tag{4.31}$$

has the order of accuracy p for functions of three independent variables ξ_1, ξ_2, ξ_3 defined in K_B. The formula (4.31) can easily be generalized to polynomials with different directional orders of approximation.

Similarly as for quadrilaterals, more efficient formulae can be used when integrating *complete polynomials* of the order p (with generally $n = (p+1)(p+2)(p+3)/6$ nonzero terms). The formulae that we are going to introduce in the next paragraph were derived in [75].

4.4.2 Economical Gauss quadrature

The construction of economical Gauss quadrature rules for complete polynomials starts from the manifold symmetry of the reference brick K_B (nine symmetry planes). Let us first have a look at the integral

$$\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \xi_1^j \xi_2^k \xi_3^l \, d\xi_3\, d\xi_2\, d\xi_1, \tag{4.32}$$

where j, k and l are positive integers. As long as j, k or l is odd, the integral is equal to zero. Moreover, due to the symmetry with respect to various planes it is sufficient to consider terms in which $j \geq k$ and $k \geq l$ only.

4.4.3 Tables of Gauss integration points and weights

Dunavant [75] provides an overview of minimum numbers of quadrature points for odd polynomial orders in Table 4.49.

TABLE 4.49: Minimum numbers of quadrature points for the Gaussian quadrature over bricks.

Polyn. order	Min. num. of nonsym. points	Min. num. of sym. points	Achieved num. of sym. points
1	1	1	1
3	4	6	6
5	10	14	14
7	20	27	27
9	35	52	53
11	56	77	89
13	84	127	151
15	120	175	235
17	165	253	307
19	220	333	435

Dunavant divides the integration points into seven groups with different numbers of unknowns and tests to determine the best choice. The results obtained by solving the corresponding systems of nonlinear equations are given in Tables 4.50 – 4.53. A list of economical Gauss quadrature rules up to the order $p = 19$ with five more decimal digits can be found on the CD-ROM included with this book.

TABLE 4.50: Gauss quadrature on K_B, order $p = 0, 1$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	ξ_3-Coordinate	Weight
1.	0.00000 00000	0.00000 00000	0.00000 00000	8.00000 00000

TABLE 4.51: Gauss quadrature on K_B, order $p = 2, 3$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	ξ_3-Coordinate	Weight
1.	1.00000 00000	0.00000 00000	0.00000 00000	1.33333 33333
2.	-1.00000 00000	0.00000 00000	0.00000 00000	1.33333 33333
3.	0.00000 00000	1.00000 00000	0.00000 00000	1.33333 33333
4.	0.00000 00000	-1.00000 00000	0.00000 00000	1.33333 33333
5.	0.00000 00000	0.00000 00000	1.00000 00000	1.33333 33333
6.	0.00000 00000	0.00000 00000	-1.00000 00000	1.33333 33333

TABLE 4.52: Gauss quadrature on K_B, order $p = 4, 5$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	ξ_3-Coordinate	Weight
1.	0.79582 24257	0.00000 00000	0.00000 00000	0.88642 65927
2.	-0.79582 24257	0.00000 00000	0.00000 00000	0.88642 65927
3.	0.00000 00000	0.79582 24257	0.00000 00000	0.88642 65927
4.	0.00000 00000	-0.79582 24257	0.00000 00000	0.88642 65927
5.	0.00000 00000	0.00000 00000	0.79582 24257	0.88642 65927
6.	0.00000 00000	0.00000 00000	-0.79582 24257	0.88642 65927
7.	0.75878 69106	0.75878 69106	0.75878 69106	0.33518 00554
8.	0.75878 69106	-0.75878 69106	0.75878 69106	0.33518 00554
9.	0.75878 69106	0.75878 69106	-0.75878 69106	0.33518 00554
10.	0.75878 69106	-0.75878 69106	-0.75878 69106	0.33518 00554
11.	-0.75878 69106	0.75878 69106	0.75878 69106	0.33518 00554
12.	-0.75878 69106	-0.75878 69106	0.75878 69106	0.33518 00554
13.	-0.75878 69106	0.75878 69106	-0.75878 69106	0.33518 00554
14.	-0.75878 69106	-0.75878 69106	-0.75878 69106	0.33518 00554

TABLE 4.53: Gauss quadrature on K_B, order $p = 6, 7$. See the companion CD-ROM for additional economical Gauss quadrature rules up to the order $p = 19$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	ξ_3-Coordinate	Weight
1.	0.00000 00000	0.00000 00000	0.00000 00000	0.78807 34827
2.	0.84841 80014	0.00000 00000	0.00000 00000	0.49936 90023
3.	-0.84841 80014	0.00000 00000	0.00000 00000	0.49936 90023
4.	0.00000 00000	0.84841 80014	0.00000 00000	0.49936 90023
5.	0.00000 00000	-0.84841 80014	0.00000 00000	0.49936 90023
6.	0.00000 00000	0.00000 00000	0.84841 80014	0.49936 90023
7.	0.00000 00000	0.00000 00000	-0.84841 80014	0.49936 90023
8.	0.65281 64721	0.65281 64721	0.65281 64721	0.47850 84494
9.	0.65281 64721	-0.65281 64721	0.65281 64721	0.47850 84494
10.	0.65281 64721	0.65281 64721	-0.65281 64721	0.47850 84494
11.	0.65281 64721	-0.65281 64721	-0.65281 64721	0.47850 84494
12.	-0.65281 64721	0.65281 64721	0.65281 64721	0.47850 84494
13.	-0.65281 64721	-0.65281 64721	0.65281 64721	0.47850 84494
14.	-0.65281 64721	0.65281 64721	-0.65281 64721	0.47850 84494
15.	-0.65281 64721	-0.65281 64721	-0.65281 64721	0.47850 84494
16.	0.00000 00000	1.10641 28986	1.10641 28986	0.03230 37423
17.	0.00000 00000	-1.10641 28986	1.10641 28986	0.03230 37423
18.	0.00000 00000	1.10641 28986	-1.10641 28986	0.03230 37423
19.	0.00000 00000	-1.10641 28986	-1.10641 28986	0.03230 37423
20.	1.10641 28986	0.00000 00000	1.10641 28986	0.03230 37423
21.	-1.10641 28986	0.00000 00000	1.10641 28986	0.03230 37423
22.	1.10641 28986	0.00000 00000	-1.10641 28986	0.03230 37423
23.	-1.10641 28986	0.00000 00000	-1.10641 28986	0.03230 37423
24.	1.10641 28986	1.10641 28986	0.00000 00000	0.03230 37423
25.	-1.10641 28986	1.10641 28986	0.00000 00000	0.03230 37423
26.	1.10641 28986	-1.10641 28986	0.00000 00000	0.03230 37423
27.	-1.10641 28986	-1.10641 28986	0.65281 64721	0.03230 37423

4.5 Reference tetrahedron K_T

In this paragraph we introduce several higher-order quadrature schemes for the reference tetrahedral domain K_T. Of course we are mostly interested in economical Gauss quadrature rules, but for higher polynomial orders ($p > 9$) their design is extremely difficult. In Paragraph 4.5.2 we will present the best selection of optimal and suboptimal Gauss quadrature rules that we found.

Frequently used product rules, which are based on a degenerate mapping from K_B to K_T and the Substitution Theorem, will be described in Paragraph 4.5.1. This time we will not address explicitly the Newton-Cotes quadrature formulae which can be constructed analogously as in the triangular case (Paragraph 4.3.2).

4.5.1 Translation of quadrature to the reference brick K_B

This approach is relevant for extremely high orders of accuracy where better quadrature rules are not available. One also can use it as a quick fix for debugging purposes or when for another reason the efficiency of quadrature procedures is not important.

Transforming the integrated function to the reference brick K_B and applying either the composite or economical Gauss quadrature formulae (Paragraphs 4.4.1 and 4.4.2), one ends up with simple quadrature rules for K_T. The idea of the degenerate transform is the same as it was in 2D with the reference quadrilateral K_q and the reference triangle K_t.

PROPOSITION 4.2

Let $g(\boldsymbol{\xi})$ be a continuous bounded function defined on the reference tetrahedron K_T. Then

$$\int_{K_T} g(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \int_{K_B} \frac{(1-y_2)(1-y_3)^2}{8} g \begin{pmatrix} -1 + \frac{(1-y_2)(1-y_3)}{4}(y_1+1) \\ -1 + \frac{(1+y_2)(1-y_3)}{2} \\ y_3 \end{pmatrix} d\boldsymbol{y}. \tag{4.33}$$

PROOF Consider the (degenerate) mapping

$$\boldsymbol{\xi}(\boldsymbol{y}) : \boldsymbol{y} \to \boldsymbol{\xi}(\boldsymbol{y}) = \begin{pmatrix} -1 + \frac{(1-y_2)(1-y_3)}{4}(y_1+1) \\ -1 + \frac{(1+y_2)(1-y_3)}{2} \\ y_3 \end{pmatrix}$$

that transforms K_B to K_T. Its Jacobian

$$\det\left(\frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}\boldsymbol{y}}\right) = \frac{(1-y_2)(1-y_3)^2}{8}$$

is positive except for faces $y_2 = 1, y_3 = 1$ where it vanishes. Hence, (4.33) immediately follows from the Substitution Theorem. ∎

REMARK 4.5 Notice that the transformation $\boldsymbol{\xi}(\boldsymbol{y})$ produces an additional polynomial factor $(1-y_2)(1-y_3)^2/2$. One has to increase the order of integration by one and two in the y_2- and y_3-direction, respectively. ∎

4.5.2 Economical Gauss quadrature

Economical Gauss quadrature formulae for complete polynomials of the order $p \leq 9$ (with generally $n = (p+1)(p+2)(p+3)/6$ nonzero terms) have been derived for tetrahedra in [123] and others.

The construction of Gauss quadrature points and weights is based on topological symmetries within the tetrahedron with vertices $w_1 = [0,0,0]$, $w_2 = [1,0,0]$, $w_3 = [0,1,0]$, $w_4 = [0,0,1]$. We transform the results found in the literature to the reference tetrahedron K_T. The transform $\boldsymbol{\xi} = \boldsymbol{\xi}(\boldsymbol{y})$ is given by

$$\xi_1 = 2y_1 - 1, \quad \xi_2 = 2y_2 - 1, \quad \xi_3 = 2y_3 - 1, \tag{4.34}$$

where $\boldsymbol{y} = (y_1, y_2, y_3)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ stand for the original and new coordinates, respectively. The corresponding Jacobian is constant,

$$\det\left(\frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}\boldsymbol{y}}\right) = 8.$$

The quadrature rules for $p \leq 7$ are presented in Tables 4.55 – 4.61.

The complexity of the problem increases rapidly as $p > 9$. For odd $p \geq 11$ one may construct suboptimal Gaussian rules based on combinatorial formulae of variable order for n-simplices derived in [101]. We evaluated these formulae by Mathematica using a notebook that the reader can utilize to obtain quadrature rules for high odd p's. The parameters `t1` and `t2` at its end are set to the polynomial orders for which we evaluate the integration points and weights. For example, for polynomial orders 1, 3, 5, ..., 21 we set `t1 := 1` and `t2 := 11`, i.e., `s = 2t - 1`, where `s` is the polynomial order and `t` is a counter from `t1` to `t2`. The notebook as well as the tabulated data (with five more decimal digits) are provided on the CD-ROM included with this book.

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                2^\((2  s)\)\)/\((\(i!\)\ \(\((2  s + 4 - i)\)!\))\)\ ; \
        ww = Abs[w]; b = s - i; d = 2  s + 4 - 2  i;
        f = \(\(\((3 + b)\)!\)/\(3!\)\)/\(b!\); \
        sezn = Compositions[b, 4]; \ \[IndentingNewLine]Do[
          a1 = sezn[\([k, 2]\)]; a2 = sezn[\([k, 3]\)];
          a3 = sezn[\([k, 4]\)];
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          b2 = 2 \((\((2  a2 + 1)\)/d)\) - 1;
          b3 = 2 \((\((2  a3 + 1)\)/d)\) - 1; \[IndentingNewLine]g1 =
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4.5.3 Tables of Gauss integration points and weights

Table 4.54 gives an overview of numbers of integration points for the Gaussian quadrature over tetrahedra.

TABLE 4.54: Minimum numbers of quadrature points for the Gauss quadrature over tetrahedra.

Polynomial order	Known minimum number of points	Achieved number of points
1	1	1
2	4	4
3	5	5
4	11	11
5	14	14
6	24	24
7	28	31
8	40	43
9	52	53
10	68	
11		126
12		
13		210
14		
15		330
16		
17		495
18		
19		715

TABLE 4.55: Gauss quadrature on K_T, order $p = 1$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	ξ_3-Coordinate	Weight
1.	-0.50000 00000	-0.50000 00000	-0.50000 00000	1.33333 33333

TABLE 4.56: Gauss quadrature on K_T, order $p = 2$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	ξ_3-Coordinate	Weight
1.	-0.72360 67977	-0.72360 67977	-0.72360 67977	0.33333 33333
2.	0.17082 03932	-0.72360 67977	-0.72360 67977	0.33333 33333
3.	-0.72360 67977	0.17082 03932	-0.72360 67977	0.33333 33333
4.	-0.72360 67977	-0.72360 67977	0.17082 03932	0.33333 33333

TABLE 4.57: Gauss quadrature on K_T, order $p = 3$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	ξ_3-Coordinate	Weight
1.	-0.50000 00000	-0.50000 00000	-0.50000 00000	-1.06666 66666
2.	-0.66666 66666	-0.66666 66666	-0.66666 66666	0.60000 00000
3.	-0.66666 66666	-0.66666 66666	0.00000 00000	0.60000 00000
4.	-0.66666 66666	0.00000 00000	-0.66666 66666	0.60000 00000
5.	0.00000 00000	-0.66666 66666	-0.66666 66666	0.60000 00000

TABLE 4.58: Gauss quadrature on K_T, order $p = 4$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	ξ_3-Coordinate	Weight
1.	-0.50000 00000	-0.50000 00000	-0.50000 00000	-0.10524 44444
2.	-0.85714 28571	-0.85714 28571	-0.85714 28571	0.06097 77777
3.	-0.85714 28571	-0.85714 28571	0.57142 85714	0.06097 77777
4.	-0.85714 28571	0.57142 85714	-0.85714 28571	0.06097 77777
5.	0.57142 85714	-0.85714 28571	-0.85714 28571	0.06097 77777
6.	-0.20119 28476	-0.20119 28476	-0.79880 71523	0.19911 11111
7.	-0.20119 28476	-0.79880 71523	-0.20119 28476	0.19911 11111
8.	-0.79880 71523	-0.20119 28476	-0.20119 28476	0.19911 11111
9.	-0.20119 28476	-0.79880 71523	-0.79880 71523	0.19911 11111
10.	-0.79880 71523	-0.20119 28476	-0.79880 71523	0.19911 11111
11.	-0.79880 71523	-0.79880 71523	-0.20119 28476	0.19911 11111

TABLE 4.59: Gauss quadrature on K_T, order $p = 5$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	ξ_3-Coordinate	Weight
1.	-0.81452 94993	-0.81452 94993	-0.81452 94993	0.09799 07241
2.	0.44358 84981	-0.81452 94993	-0.81452 94993	0.09799 07241
3.	-0.81452 94993	0.44358 84981	-0.81452 94993	0.09799 07241
4.	-0.81452 94993	-0.81452 94993	0.44358 84981	0.09799 07241
5.	-0.37822 81614	-0.37822 81614	-0.37822 81614	0.15025 05676
6.	-0.86531 55155	-0.37822 81614	-0.37822 81614	0.15025 05676
7.	-0.37822 81614	-0.86531 55155	-0.37822 81614	0.15025 05676
8.	-0.37822 81614	-0.37822 81614	-0.86531 55155	0.15025 05676
9.	-0.09100 74082	-0.09100 74082	-0.90899 25917	0.05672 80277
10.	-0.09100 74082	-0.90899 25917	-0.09100 74082	0.05672 80277
11.	-0.90899 25917	-0.09100 74082	-0.09100 74082	0.05672 80277
12.	-0.09100 74082	-0.90899 25917	-0.90899 25917	0.05672 80277
13.	-0.90899 25917	-0.09100 74082	-0.90899 25917	0.05672 80277
14.	-0.90899 25917	-0.90899 25917	-0.09100 74082	0.05672 80277

TABLE 4.60: Gauss quadrature on K_T, order $p = 6$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	ξ_3-Coordinate	Weight
1.	-0.57079 42574	-0.57079 42574	-0.57079 42574	0.05323 03336
2.	-0.28761 72275	-0.57079 42574	-0.57079 42574	0.05323 03336
3.	-0.57079 42574	-0.28761 72275	-0.57079 42574	0.05323 03336
4.	-0.57079 42574	-0.57079 42574	-0.28761 72275	0.05323 03336
5.	-0.91865 20829	-0.91865 20829	-0.91865 20829	0.01343 62814
6.	0.75595 62487	-0.91865 20829	-0.91865 20829	0.01343 62814
7.	-0.91865 20829	0.75595 62487	-0.91865 20829	0.01343 62814
8.	-0.91865 20829	-0.91865 20829	0.75595 62487	0.01343 62814
9.	-0.35532 42197	-0.35532 42197	-0.35532 42197	0.07380 95753
10.	-0.93402 73408	-0.35532 42197	-0.35532 42197	0.07380 95753
11.	-0.35532 42197	-0.93402 73408	-0.35532 42197	0.07380 95753
12.	-0.35532 42197	-0.35532 42197	-0.93402 73408	0.07380 95753
13.	-0.87267 79962	-0.87267 79962	-0.46065 53370	0.06428 57142
14.	-0.87267 79962	-0.46065 53370	-0.87267 79962	0.06428 57142
15.	-0.87267 79962	-0.87267 79962	0.20601 13295	0.06428 57142
16.	-0.87267 79962	0.20601 13295	-0.87267 79962	0.06428 57142
17.	-0.87267 79962	-0.46065 53370	0.20601 13295	0.06428 57142
18.	-0.87267 79962	0.20601 13295	-0.46065 53370	0.06428 57142
19.	-0.46065 53370	-0.87267 79962	-0.87267 79962	0.06428 57142
20.	-0.46065 53370	-0.87267 79962	0.20601 13295	0.06428 57142
21.	-0.46065 53370	0.20601 13295	-0.87267 79962	0.06428 57142
22.	0.20601 13295	-0.87267 79962	-0.46065 53370	0.06428 57142
23.	0.20601 13295	-0.87267 79962	-0.87267 79962	0.06428 57142
24.	0.20601 13295	-0.46065 53370	-0.87267 79962	0.06428 57142

TABLE 4.61: Gauss quadrature on K_T, order $p = 7$. See the companion CD-ROM for additional Gauss quadrature rules up to the order $p = 21$.

Point #	ξ_1-Coordinate	ξ_2-Coordinate	ξ_3-Coordinate	Weight
1.	0.00000 00000	0.00000 00000	-1.00000 00000	0.00776 01410
2.	0.00000 00000	-1.00000 00000	0.00000 00000	0.00776 01410
3.	-1.00000 00000	0.00000 00000	0.00000 00000	0.00776 01410
4.	-1.00000 00000	-1.00000 00000	0.00000 00000	0.00776 01410
5.	-1.00000 00000	0.00000 00000	-1.00000 00000	0.00776 01410
6.	0.00000 00000	-1.00000 00000	-1.00000 00000	0.00776 01410
7.	-0.50000 00000	-0.50000 00000	-0.50000 00000	0.14611 37877
8.	-0.84357 36153	-0.84357 36153	-0.84357 36153	0.08479 95321
9.	-0.84357 36153	-0.84357 36153	0.53072 08460	0.08479 95321
10.	-0.84357 36153	0.53072 08460	-0.84357 36153	0.08479 95321
11.	0.53072 08460	-0.84357 36153	-0.84357 36153	0.08479 95321
12.	-0.75631 35666	-0.75631 35666	-0.75631 35666	-0.50014 19209
13.	-0.75631 35666	-0.75631 35666	0.26894 07000	-0.50014 19209
14.	-0.75631 35666	0.26894 07000	-0.75631 35666	-0.50014 19209
15.	0.26894 07000	-0.75631 35666	-0.75631 35666	-0.50014 19209
16.	-0.33492 16711	-0.33492 16711	-0.33492 16711	0.03913 14021
17.	-0.33492 16711	-0.33492 16711	-0.99523 49866	0.03913 14021
18.	-0.33492 16711	-0.99523 49866	-0.33492 16711	0.03913 14021
19.	-0.99523 49866	-0.33492 16711	-0.33492 16711	0.03913 14021
20.	-0.80000 00000	-0.80000 00000	-0.60000 00000	0.22045 85537
21.	-0.80000 00000	-0.60000 00000	-0.80000 00000	0.22045 85537
22.	-0.80000 00000	-0.80000 00000	0.20000 00000	0.22045 85537
23.	-0.80000 00000	0.20000 00000	-0.80000 00000	0.22045 85537
24.	-0.80000 00000	-0.60000 00000	0.20000 00000	0.22045 85537
25.	-0.80000 00000	0.20000 00000	-0.60000 00000	0.22045 85537
26.	-0.60000 00000	-0.80000 00000	-0.80000 00000	0.22045 85537
27.	-0.60000 00000	-0.80000 00000	0.20000 00000	0.22045 85537
28.	-0.60000 00000	0.20000 00000	-0.80000 00000	0.22045 85537
29.	0.20000 00000	-0.80000 00000	-0.60000 00000	0.22045 85537
30.	0.20000 00000	-0.80000 00000	-0.80000 00000	0.22045 85537
31.	0.20000 00000	-0.60000 00000	-0.80000 00000	0.22045 85537

4.6 Reference prism K_P

Finally we will deal with numerical quadrature on the reference prism K_P. To the best of our knowledge, no economical Gauss quadrature rules related to complete higher-order polynomials for this type of domain are known. Therefore we have to stay with product formulae combining the triangular and one-dimensional Gauss quadrature rules.

4.6.1 Composite Gauss quadrature

Let the quadrature rule

$$\int_{K_a} f(\xi) \mathrm{d}\xi \approx \sum_{i=1}^{M_a} w_{g_a,i} f(y_{g_a,i}), \tag{4.35}$$

where $y_{g_a,i}, w_{g_a,i}$ are Gauss integration points and weights on the one-dimensional reference domain $K_a = (-1, 1)$, integrate exactly all polynomials of the order p and lower. Further, let the quadrature rule

$$\int_{K_t} g(\xi_1, \xi_2)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2 \approx \sum_{i=1}^{M_t} w_{g_t,i} g(y_{g_t,1,i}, y_{g_t,2,i}), \tag{4.36}$$

where $Y_{g_t,i} = [y_{g_t,1,i}, y_{g_t,2,i}]$ are Gauss integration points and weights for the reference triangle K_t, integrate exactly all polynomials of the order p and lower. It is easy to see that the formula

$$\int_{K_a} \int_{K_t} h(\xi_1, \xi_2, \xi_3)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3 \approx \sum_{i=1}^{M_a} \sum_{j=1}^{M_t} w_{g_a,i} w_{g_t,j} h(y_{g_t,1,j}, y_{g_t,2,j}, y_{g_a,i}) \tag{4.37}$$

integrates exactly all polynomials (of three independent variables ξ_1, ξ_2, ξ_3) of the order p and lower on the reference prism K_P.

Notice that we can also use other higher-order quadrature rules for the reference domains K_a and K_t in order to produce composite quadrature rules for the reference prism K_P in the way presented.