18 Buckling of columns and beams

18.1 Introduction

In all the problems treated in preceding chapters, we were concerned with the small strains and distortions of a stressed material. In certain types of problems, and especially those involving compressive stresses, we find that a structural member may develop relatively large distortions under certain critical loading conditions. Such structural members are said to *buckle*, or become *unstable*, at these critical loads.

As an example of elastic buckling, we consider firstly the buckling of a slender column under an axial compressive load.

18.2 Flexural buckling of a pin-ended strut

A perfectly straight bar of uniform cross-section has two axes of symmetry Cx and Cy in the cross-section on the right of Figure 18.1. We suppose the bar to be a flat strip of material, Cx being the weakest axis of the cross-section. End thrusts P are applied along the centroidal axis Cz of the bar, and EI its uniform flexural stiffness for bending about Cx.

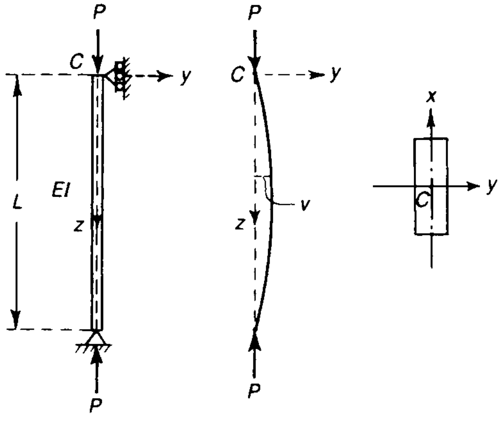


Figure 18.1 Flexural buckling of a pin-ended strut under axial thrust.

Now Cx is the weakest axis of bending of the bar, and if bowing of the compressed bar occurs we should expect bending to take place in the yz-plane. Consider the possibility that at some value of P, the end thrust, the strut can buckle laterally in the yz-plane. There can be no lateral deflections at the ends of the strut; suppose v is the displacement of the centre line of the bar parallel to Cy at any point. There can be no forces at the hinges parallel to Cy, as these would imply bending moments at the ends of the bar. The only two external forces are the end thrusts P, which are assumed to maintain their original line of action after the onset of buckling. The bending moment at any section of the bar is then

$$M = Pv \tag{18.1}$$

which is a sagging moment in relation to the axes Cz and Cy, in the sense of Section 13.2. But the moment–curvature relationship for the beam at any section is

$$M = -EI\frac{d^2v}{dz^2}$$

provided the deflection v is small. Thus

$$-EI\frac{d^2v}{dz^2} = Pv$$

Then

$$EI\frac{d^2v}{dz^2} + Pv = 0 \tag{18.2}$$

Put

$$\frac{P}{EI} = k^2 \tag{18.3}$$

Then

$$\frac{d^2v}{dz^2} + k^2v = 0 \tag{18.4}$$

The general solution of this differential equation is

$$v = A\cos kz + B\sin kz \tag{18.5}$$

where A and B are arbitrary constants. We have two boundary conditions to satisfy: at the ends $z = 0$ and $z = L$, $v = 0$. Then

$$A = 0 \quad \text{and} \quad B\sin kL = 0$$

Now consider the implications of the equation

$$B\sin kL = 0$$

which is derived from the boundary conditions. If $B = 0$, then both A and B are zero, and obviously the strut is undeflected.

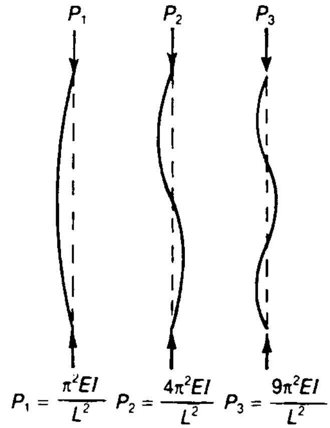


Figure 18.2 Modes of buckling of a pin-ended strut.

If, however, $\sin kL = 0$, B is indeterminate, and the strut may assume the form

$$v = B \sin kz$$

This is called a buckled condition of the strut. Obviously B is indeterminate when kL, assumes the values,

$$kL = \pi,\ 2\pi,\ \ldots \text{ etc.} \tag{18.6}$$

We need not consider the solution $kL = 0$, which implies $k = 0$, because the solution of the differential equation is not trigonometric in form when $k = 0$. Instability occurs when $kL\pi$, $= 2\pi$, etc.

$$\therefore P = k^2 EI = \frac{\pi^2 EI}{L^2},\ 4\pi^2 \frac{EI}{L^2} \text{ etc} \tag{18.7}$$

There are infinite number of values of P for instability, corresponding with various modes of buckling, Figure 18.2. The fundamental mode occurs at the lowest critical load

$$P_e = \pi^2 \frac{EI}{L^2} = \text{Euler load for pin-ended struts} \tag{18.8}$$

This is known as the *Euler formula* and corresponds with buckling in a single longitudinal half-wave. The critical load

$$P = 2^2\pi^2 \frac{EI}{L^2} = 4\pi^2 \frac{EI}{L^2} \tag{18.9}$$

corresponds with buckling in two longitudinal half-waves, and so on for higher modes. In practice the critical load P_e is never exceeded because high stresses develop at this load and collapse of the strut ensues. We are not therefore concerned with buckling loads higher than the lowest buckling load. For all practical purposes the buckling load of a pin-ended strut is given by equation (18.8).

At this load a perfectly straight pin-ended strut is in a state of *neutral equilibrium*; the small deflection

$$v = B \sin kz$$

is indeterminate, because B itself is indeterminate. Theoretically, the strut is in equilibrium at the load $\pi^2 EI/L^2$ for any small value of B, corresponding with a condition of *neutral equilibrium*; at this buckling load we should expect to be able to push the strut into any sinusoidal wave of small amplitude. This can be verified experimentally by compressing a long slender strip of material which remains elastic during bending.

At values of P less than $\pi^2 EI/L^2$ the strut is in a condition of *unstable equilibrium*; any small lateral disturbance produces motion and finally collapse of the strut. This, however, is a hypothetical situation as, in practice, the load $\pi^2 EI/L^2$ cannot be exceeded if the loads are static, and not applied suddenly.

The condition of neutral equilibrium at

$$P_e = \pi^2 \frac{EI}{L^2}$$

is only attained for small lateral displacements of the strut. When these displacements become large, the moment-curvature relation

$$M = -EI\frac{d^2 v}{dz^2}$$

is no longer valid; theoretically the problem becomes more difficult. The effect of large lateral displacements is to increase the flexural stiffness of the strut; in this case, provided the material remains elastic, end thrusts greater than $\pi^2 EI/L^2$ are attainable. If the thrust P is plotted against the lateral displacement v at any section, the P - v relation for a perfectly straight strut has the form shown in Figure 18.3(i), when account is taken of large deflections. Lateral deflections become possible only when

$$P \geq \frac{\pi^2 EI}{L^2}$$

This analysis is restricted to the hypothetical case of a perfectly straight strut. When the strut has small imperfections, displacements v are possible for all values of P (Figure 18.3(ii)), and the hypothetical condition of neutral equilibrium at

$$P = \frac{\pi^2 EI}{L^2}$$

is never attained. All materials have a limit of proportionality; when this is attained the flexural stiffness of the strut usually falls off rapidly. On the P–v diagram for the strut this corresponds with the development of a region of unstable equilibrium.

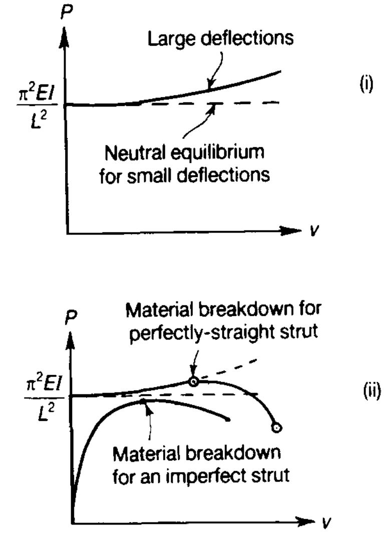


Figure 18.3 Large deflections and material breakdown of struts.

18.3 Rankine–Gordon formula

Predictions of buckling loads by the Euler formula is only reasonable for very long and slender struts that have very small geometrical imperfections. In practice, however, most struts suffer plastic knockdown and the experimentally obtained buckling loads are much less than the Euler predictions. For struts in this category, a suitable formula is the Rankine–Gordon formula which is a semi-empirical formula, and takes into account the crushing strength of the material, its Young's modulus and its slenderness ratio, namely L/k, where

L = length of the strut

k = least radius of gyration of the strut's cross-section

$$P_c = \sigma_c A \tag{18.10}$$

where

A = cross-sectional area

σ_c = crushing stress

Then

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c} \tag{18.11}$$

where

P_R = Rankine–Gordon buckling load
P_e = Euler buckling load

$$= \frac{\pi^2 EI}{L^2} \text{ for a pin-ended strut}$$

$$\therefore \frac{1}{P_R} = \frac{L_0^2}{\pi^2 EI} + \frac{1}{\sigma_{yc} \times A}$$

$$= \frac{L_0^2}{\pi^2 EAk^2} + \frac{1}{\sigma_{yc} A} \tag{18.12}$$

$$= \frac{L_0^2 \sigma_{yc} + \pi^2 Ek^2}{\pi^2 EAk^2 \sigma_{yc}}$$

or

$$P_R = \frac{\pi^2 EAk^2 \sigma_{yc}}{L_0^2 \sigma_{yc} + \pi^2 Ek^2}$$

$$= \frac{\sigma_{yc}}{L_0^2 \sigma_{yc} / \pi^2 EAk^2 + \pi^2 Ek^2 / \pi^2 EAk^2} \tag{18.13}$$

$$P_R = \frac{\sigma_{yc} \times A}{\left(\sigma_{yc} / \pi^2 E\right) \left(L_0 / k\right)^2 + 1}$$

Let

$$a = \frac{\sigma_{yc}}{\pi^2 E} \tag{18.14}$$

Then

$$P_R = \frac{\sigma_{yc} A}{1 + a(L_0 / K)^2} \qquad (18.15)$$

where a is the denominator constant in the Rankine–Gordon formula, which is dependent on the boundary conditions and material properties.

A comparison of the Rankine–Gordon and Euler formulae, for geometrically perfect struts, is given in Figure 18.4. Some typical values for $1/a$ and σ_{yc} are given in Table 18.1. Where L_0 is the effective length of the strut; see Section 18.4.

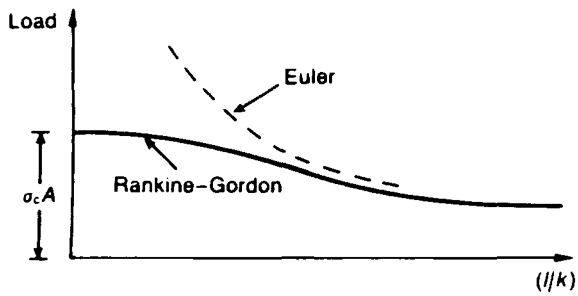


Figure 18.4 Comparison of Euler and Rankine–Gordon formulae.

Table 18.1 Rankine Constants

Material	1/a	σ_{yc}
Mild Steel	7500	300
Wrought Iron	8000	250
Cast Iron	18000	560
Timber	1000	35

18.4 Effects of geometrical imperfections

For intermediate struts with geometrical imperfections, the buckling load is further decreased, as shown in Figure 18.5.

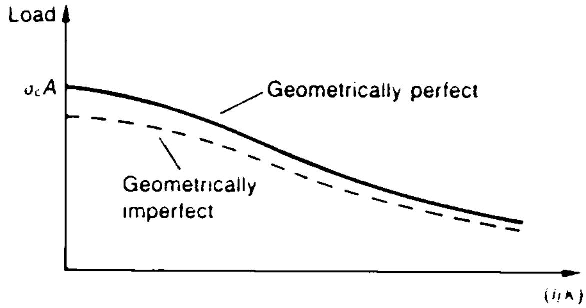


Figure 18.5 Rankine–Gordon loads for perfect and imperfect struts.

18.5 Effective lengths of struts

The theoretical buckling load for a pinned-ended strut is one-quarter of the theoretical buckling load of a fixed-ended strut and four times the theoretical buckling load for a strut fixed at one end and free at the other end; see Sections 18.10 to 18.12.

Table 18.2 Effective lengths of struts (L_0)

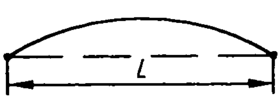
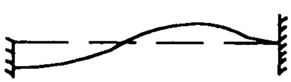
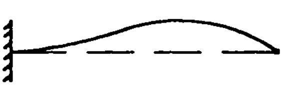
Type of Strut	Euler	BS449
	$L_0 = L$	$L_0 = L$
	$L_0 = L$	$L_0 = L$
	$L_0 = 0.5L$	$L_0 = 0.7L$
	$L_0 = 0.7L$	$L_0 = 0.85L$
	$L_0 = 2L$	$L_0 = 2L$

Table 18.2 gives the effective lengths of struts (L_0), which have actual lengths of L, for different boundary conditions, where BS449 allows for elastic relaxation at the ends of the strut.

18.6 Pin-ended strut with eccentric end thrusts

In practice it is difficult, if not impossible, to apply the end thrusts along the longitudinal centroidal axis Cz of a strut. We consider now the effect of an eccentrically applied compressive load P on a uniform strut of flexural stiffness EI and length L.

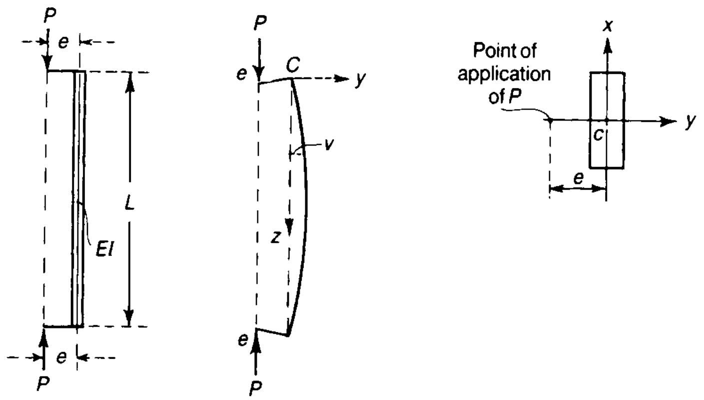


Figure 18.6 Eccentric loading of a strut.

Suppose the end thrusts are applied at a distance e from the centroid and on the axis Cy of the cross-section. We assume again that the cross-section is that of a flat rectangular strip, Cx being the weaker axis of bending. The end thrusts P are applied to rigid arms attached to the ends of the strut.

An end load P causes the straight strut to bend; suppose v is the displacement of any point on Cz from its original position. The bending moment at that section is

$$M = P(e+v)$$

which is a sagging moment in relation to the axes Cz and Cy. If v is small we have

$$M = -EI\frac{d^2v}{dz^2}$$

Thus

$$-EI\frac{d^2v}{dz^2} = P(e+v)$$

Then

$$EI\frac{d^2v}{dz^2} + Pv = -Pe$$

When $e = 0$, this differential equation reduces to that already derived for an axially loaded strut.

As before, put

$$k^2 = \frac{P}{EI}$$

Then

$$\frac{d^2v}{dz^2} + k^2v = -k^2e$$

The complete solution is

$$v = A \cos kz + B \sin kz - e$$

Now $v = 0$ at $z = 0$ and $z = L$, so that

$$A - e = 0, \quad \text{and} \quad A \cos kL + B \sin kL - e = 0$$

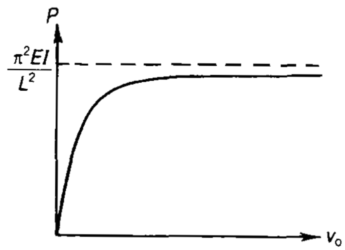


Figure 18.7 Deflections of an eccentrically loaded strut.

The first of these equations gives $A = e$, and the second gives

$$B = \frac{e(1 - \cos kL)}{\sin kL}$$

Then

$$v = e(\cos kz - 1) + \frac{e(1 - \cos kL)}{\sin kL} \sin kz \tag{18.16}$$

The displacement v at the mid-length, $z = \frac{1}{2}L$, is

$$v_0 = e\left[\left(\cos\frac{kL}{2} - 1\right) + \frac{1 - \cos kL}{\sin kL}\sin\frac{1}{2}kL\right]$$

$$= e\left[\frac{2\sin\frac{1}{2}kL\left(1 - \cos\frac{1}{2}kL\right)}{\sin kL}\right] \tag{18.17}$$

If $\sin \frac{1}{2} kL \neq 0$, we have

$$v_0 = e\left(\sec\frac{1}{2}kL - 1\right) \tag{18.18}$$

When $P = 0$,

$$\frac{1}{2}kL = \frac{L}{2}\sqrt{\frac{P}{EI}} = 0$$

and $v_0 = 0$. As P approaches $\pi^2 EI/L^2$, $\frac{1}{2}kL$ approaches $\pi/2$, and

$$\sec\frac{1}{2}kL \to \infty$$

Thus values of v_0 are possible from the onset of loading; the values of v_0 increase non-linearly with increases of P. The value of $P = \pi^2 EI/L^2$ is not attainable as this would imply an infinitely large value of v_0, and material breakdown would occur at some smaller value of P.

It is interesting to evaluate the longitudinal stresses at the mid-length of the strut; the largest lateral deflection occurs at this section, and the greatest bending moment also occurs at this section, therefore. The bending moment is

$$M = P(v_0 + e) = Pe\sec\frac{1}{2}kL \tag{18.19}$$

Suppose c is the distance from the centroidal axis Cx to the extreme fibres of the strut. Then the longitudinal bending stress set up by M is

$$\sigma_1 = \frac{Mc}{I} = \frac{Pec\sec\frac{1}{2}kL}{I} \tag{18.20}$$

The average longitudinal compressive stress set up by P is

$$\sigma_2 = \frac{P}{A} \tag{18.21}$$

where A is the cross-sectional area of the strut. Then the maximum longitudinal compressive stress is

$$\sigma_{max} = \sigma_2 + \sigma_1 = \frac{P}{A} + \frac{Pec}{I} \sec \frac{1}{2}kL \tag{18.22}$$

Suppose $I = Ar^2$, where r is the radius of gyration of the cross-section about Cx. Then

$$\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2} \sec \frac{1}{2}kL\right] \tag{18.23}$$

The minimum compressive stress is

$$\sigma_{min} = \frac{P}{A}\left[1 - \frac{ec}{r^2} \sec \frac{1}{2}kL\right] \tag{18.24}$$

The value of P giving rise to a maximum compressive stress σ is

$$P = \frac{A\sigma}{1 + \dfrac{ec}{r^2} \sec \dfrac{1}{2}kL} \tag{18.25}$$

However,

$$\frac{1}{2}kL = \frac{L}{2}\sqrt{\frac{P}{EI}}$$

and is therefore a function of P, so that the above equations must be solved by trial and error. A good approximation is derived as follows: let $\frac{1}{2}kL = \theta$, then for $0 < \theta < \frac{1}{2}\pi$

$$\sec \theta \doteqdot \frac{1 + 0.26\left(\dfrac{2\theta}{\pi}\right)^2}{1 - \left(\dfrac{2\theta}{\pi}\right)^2} = \frac{P_e + 0.26P}{P_e - P}$$

which leads to the following equation for P:

$$P^2\left(1 - 0.26\frac{ec}{r^2}\right) - P\left[P_e\left(1 + \frac{ec}{r^2}\right) + \sigma_A\right] + \sigma A P_e = 0$$

If $e = 0$, this has the roots $P = P_e$ or σA.

18.7 Initially curved pin-ended strut

In practice a strut cannot be made perfectly straight, and our analysis for the flexure of a compressed bar would become more realistic if account could be taken of the slight deviations from straightness of the centroidal axis of a strut.

Consider again a strut consisting of a flat strip of material. Suppose the centroidal longitudinal axis is initially curved, the lateral displacement at any point being v_0 from the axis Oz, Figure 18.8. Thrusts P are now applied at the ends of the strut and at the centroids of the end cross-sections.

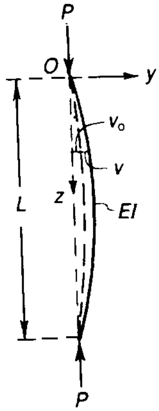


Figure 18.8 Initially curved strut.

The strut then bends further from its initial unloaded position. Suppose v is the additional lateral displacement at any section due to the application of P. If the ends of the strut are pinned there can be no lateral forces at the ends. The bending moment at any section of the strut is

$$M = P(v + v_0)$$

If the strut is initially unstressed then the bending moment at any section is proportioned to the change of curvature at that section. Then

$$M = -EI\frac{d^2v}{dz^2}$$

because the change of curvature is due only to the additional displacement v of the strut and not the total displacement $(v + v_0)$. Then

$$EI\frac{d^2v}{dz^2} + P(v + v_0) = 0$$

Put $P/EI = k^2$, as before. Then

$$\frac{d^2v}{dz^2} + k^2 v = -k^2 v_0$$

Suppose for the sake of simplicity that v_0 is sinusoidal in form; take

$$v_0 = a \sin \frac{\pi z}{L} \tag{18.26}$$

where a is a constant, and is the initial lateral displacement at the centre of the strut. Then

$$\frac{d^2v}{dz^2} + k^2 v = -k^2 a \sin \frac{\pi z}{L}$$

The general solution is

$$v = A \cos kz + B \sin kz + \frac{k^2 a}{\dfrac{\pi^2}{L^2} - k^2} \sin \frac{\pi z}{L}$$

If the ends are pinned we have

$$v = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = L$$

Then

$$A = 0 \quad \text{and} \quad B \sin kL = 0$$

If k is to assume any non-zero value we must have $B = 0$, so the relationship for v reduces to

$$v = \frac{k^2 a}{\dfrac{\pi^2}{L^2} - k^2} \sin \frac{\pi z}{L} \tag{18.27}$$

This may be written

$$v = \frac{a \sin \dfrac{\pi z}{L}}{\dfrac{\pi^2}{k^2 L^2} - 1} \tag{18.28}$$

But $k^2 = P/EI$, so on putting $\pi^2 EI/L^2 = P_e$, we have

$$v = \frac{a \sin \frac{\pi z}{L}}{\frac{P_e}{P} - 1} = \frac{v_0}{\frac{P_e}{P} - 1} \qquad (18.29)$$

Now P_e is the buckling load for the perfectly straight strut. The relation for v, which is the additional lateral displacement of the strut, shows that the effect of the end thrust P is to increase v_0 by the factor $1/[(P_e/P) - 1]$. Obviously as P approaches P_e, v tends to infinity. The additional displacement at the mid-length of the strut is

$$v_c = \frac{a}{\frac{P_e}{P} - 1} \qquad (18.30)$$

This relation between P and v_c has the form shown in Figure 18.9(i); v_c increases rapidly as P approaches P_e. Theoretically, the load P_e can only be attained at an infinitely large deflection. In practice material breakdown would occur before P_e could be attained, and at a finite displacement. We may write the relation for v_c in the form

$$P_e \frac{v_c}{P} - v_c = a \qquad (18.31)$$

This gives a linear relation between (v_c/P) and v_c, Figure 18.9. The negative intercept on the axis of v_c is equal to $(-a)$. If values of (v_c/P) and v_c are plotted in a strut test, it will be found that as the critical condition is approached these variables are related by a straight-line equation of the type discussed above. The slope of this straight line defines P_e, the buckling load for a perfectly-straight strut.

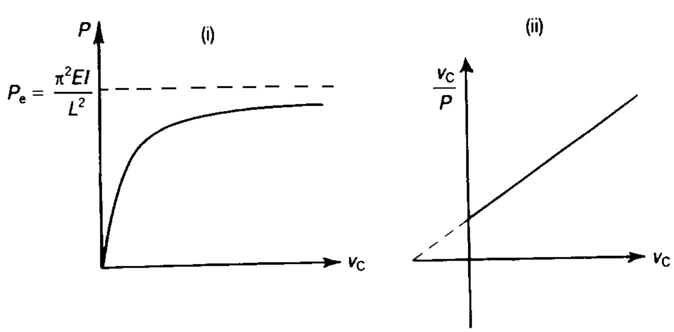


Figure 18.9 Deflections of an initially curved strut.

The P–v_c curve is asymptotic to the line $P = P_e$ if the material remains elastic. It is of considerable interest to evaluate the maximum longitudinal compressive stress in the strut. The maximum bending moment occurs at the mid-length, and has the value

$$M = P\left(a + v_c\right) = Pa\left[1 + \frac{1}{\frac{P_e}{P} - 1}\right] = Pa\left[\frac{P_e}{P_e - P}\right] \tag{18.32}$$

The maximum compressive stress occurs in an extreme fibre, and has the value

$$\sigma_{\max} = \frac{P}{A} + \frac{Pa\ P_e}{P_e - P}\left(\frac{c}{I}\right) = \frac{P}{A}\left[1 + \frac{P_e}{P_e - P}\left(\frac{ac}{r^2}\right)\right] \tag{18.33}$$

where A is the area of the cross-section, c is the distance from the centroidal axis to the extreme fibres, and r is the relevant radius of gyration of the cross-section. Now P/A is the average stress on the strut; if this is equal to σ, then

$$\sigma_{\max} = \sigma\left[1 + \frac{\sigma_e}{\sigma_e - \sigma}\left(\frac{ac}{r^2}\right)\right] \tag{18.34}$$

where

$$\sigma_e = \frac{P_e}{A} = \pi^2 E\left(\frac{r}{L}\right)^2 \tag{18.35}$$

Suppose $\dfrac{ac}{r^2} = \eta$. Then

$$\sigma_{\max} = \sigma\left[1 + \frac{\eta\sigma_e}{\sigma_e - \sigma}\right] \tag{18.36}$$

Then

$$\sigma_{\max} = (\sigma_e - \sigma) = \sigma\left[(1 + \eta)\,\sigma_e - \sigma\right]$$

Thus,

$$\sigma^2 - \sigma\left[\sigma_{\max} + (1 + \eta)\sigma_e\right] + \sigma_{\max}\,\sigma_e = 0$$

Then

$$\sigma = \frac{1}{2}\left[\sigma_{max} + (1+\eta)\sigma_e\right] - \sqrt{\frac{1}{4}\left[\sigma_{max} + (1+\eta)\sigma_e\right]^2 - \sigma_{max}\sigma_e} \tag{18.37}$$

We need not consider the positive square root, since we are only interested in the smaller of the two roots of the equation. This relation gives the value of average stress, σ, at which a maximum compressive stress σ_{max} would be attained for any value of η. If we are interested in the value of σ at which yield stress σ_Y of a mild-steel strut is attained, we have

$$\sigma = \frac{1}{2}\left[\sigma_y + (1+\eta)\sigma_e\right] - \sqrt{\frac{1}{4}\left[\sigma_y + (1+\eta)\sigma_e\right]^2 - \sigma_y\sigma_e} \tag{18.38}$$

18.8 Design of pin-ended struts

A commonly used structural material is mild steel. It has been found from tests on mild-steel pin-ended struts that failure of an initially-curved member takes place when the yield stress is first attained in one of the extreme fibres. From a wide range of tests Robertson concluded that the failing loads of mild-steel struts could be estimated if η is taken to be proportional to (L/r) the slenderness ratio of the strut; Robertson suggests that

$$\eta = 0.003\left(\frac{L}{r}\right) \tag{18.39}$$

This value of η gives

$$\sigma = \frac{1}{2}\left[\sigma_y + \left(1+0.003\frac{L}{r}\right)\sigma_e\right] - \sqrt{\frac{1}{4}\left[\sigma_y + \left(1+0.003\frac{L}{r}\right)\sigma_e\right]^2 - \sigma_y\sigma_e} \tag{18.40}$$

This represents a transition curve between yielding of the material for low slenderness ratios, Figure 18.10, and buckling at high slenderness ratios.

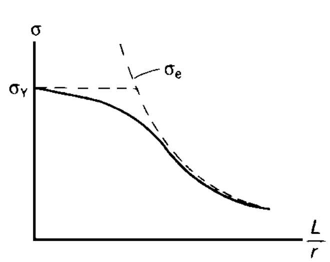


Figure 18.10 Effect of material breakdown on the buckling of a strut.

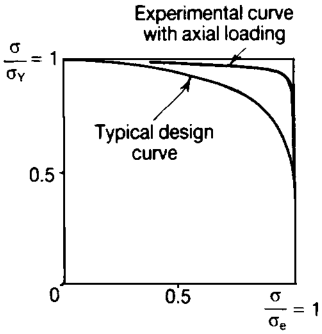


Figure 18.11 'Interaction' curves for practical struts.

In the case of mild-steel struts under true axial loading buckling occurs at σ_e the elastic buckling load or at σ_Y the yield stress. If true axial loading could be achieved in practice, all struts would fail at stresses that could be represented either by $\sigma/\sigma_Y = 1$, or $\sigma/\sigma_e = 1$. In a series of strut tests it is found that the test results are usually defined by a curve on the $\sigma/\sigma_Y - \sigma/\sigma_e$ diagram, Figure 18.11, and not by the two straight lines $\sigma/\sigma_Y = 1$ and $\sigma/\sigma_e = 1$. if the experimental technique is improved to give better axial-loading conditions the curve approaches these two straight lines. Any convenient transition curve on this diagram may be taken as a design curve for practical conditions of axial loading.

18.9 Strut with uniformly distributed lateral loading

In the preceding sections we considered the effects of end eccentricities and initial curvatures on the lateral bending of compressed struts; these produce lateral bending of the strut from the onset of compression.

A similar problem arises when a compressed strut carries a lateral load. Consider a pin-ended strut length L and uniform flexural stiffness EI, Figure 18.12. Suppose the axial thrust on the strut is P, and that there is a lateral load of uniform intensity w per unit length. At the ends of the strut there are lateral shearing forces ½wL.

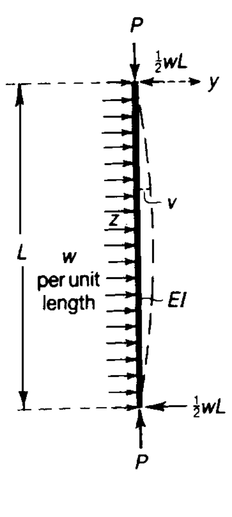


Figure 18.12 Laterally loaded struts.

If v is the lateral deflection at any point of the centroidal axis, then the bending moment at any section is

$$M = -EI\frac{d^2v}{dz^2} = Pv + \frac{1}{2}wLz - \frac{1}{2}wz^2$$

Then

$$\frac{d^2v}{dz^2} + \frac{Pv}{EI} = -\frac{w}{2EI}\left(Lz - z^2\right)$$

If $P/EI = k^2$, then

$$\frac{d^2v}{dz^2} + k^2v = -\frac{wk^2}{2P}\left(Lz - z^2\right)$$

The complete solution of this equation is

$$v = A\cos kz + B\sin kz - \frac{w}{2P}\left(Lz - z^2 + \frac{2}{k^2}\right)$$

in which A and B are arbitrary constants. Now, at $z = 0$ and $z = L$ we have $v = 0$, so

$$A - \frac{w}{Pk^2} = 0$$

and

$$A\cos kL + B\sin kL - \frac{w}{Pk^2} = 0$$

Then

$$A = \frac{w}{Pk^2}, \qquad B = \frac{w}{Pk^2}\left[\frac{1 - \cos kL}{\sin kL}\right]$$

Thus

$$v = \frac{w}{Pk^2}\left[\cos kz + \left(\frac{1-\cos kL}{\sin kL}\right)\sin kz - 1 - \frac{1}{2}k^2\left(Lz - z^2\right)\right] \tag{18.41}$$

The maximum value of v occurs at the mid-length, $z = \frac{1}{2}L$, and is given by

$$v_{\max} = \frac{w}{Pk^2}\left[\cos\frac{1}{2}kL + \left(\frac{1-\cos kL}{\sin kL}\right)\sin\frac{1}{2}kL - 1 - \frac{1}{8}k^2L^2\right] \tag{18.42}$$

This may be written

$$v_{\max} = \frac{w}{Pk^2}\left[\sec\frac{1}{2}kL - 1 - \frac{1}{8}k^2L^2\right] \tag{18.43}$$

The maximum bending moment also occurs at the mid-length, and has the value

$$M_{\max} = Pv_{\max} + \frac{1}{8}wL^2 \tag{18.44}$$

On substituting for $v_{\max}$, we have

$$M_{\max} = \frac{w}{k^2}\left[\sec\frac{1}{2}kL - 1 - \frac{1}{8}k^2L^2\right] + \frac{1}{8}wL^2 = \frac{w}{k^2}\left[\sec\frac{1}{2}kL - 1\right] \tag{18.45}$$

When P is small, k is also small, and

$$\sec\frac{1}{2}kL = \frac{1}{\cos\frac{1}{2}kL} \fallingdotseq \left[1 - \frac{1}{2}\left(\frac{1}{2}kL\right)^2 + \frac{1}{24}\left(\frac{1}{2}kL\right)^4\right]^{-1}$$

Thus, approximately,

$$\sec\frac{1}{2}kL \fallingdotseq 1 + \left[\frac{1}{8}(kL)^2 - \frac{1}{384}(kL)^4\right] + \left[\frac{1}{8}(kL)^2 - \frac{1}{384}(kL)^4\right]^2$$

$$= 1 + \frac{1}{8}(kL)^2 + \frac{5}{384}(kL)^4 \tag{18.46}$$

Then

$$v_{\max} = \frac{w}{Pk^2}\left[\frac{5}{384}k^4L^4\right] = \frac{5}{384}\frac{wL^4}{EI} \tag{18.47}$$

This agrees with the value of the central deflection of a laterally loaded beam without end thrust. Similarly, when k is small,

$$M_{\max} = \frac{wL^2}{8}\left[\frac{8\left(\sec\frac{1}{2}kL - 1\right)}{k^2L^2}\right] \tag{18.48}$$

the term in square brackets is the factor by which the bending moment due to w alone must be multiplied to give the correct bending moment.

18.10 Buckling of a strut with built-in ends

In the elastic buckling of struts, we have assumed so far that the ends of the strut are always hinged to some foundation. When the ends are supported so that no rotations can occur, Figure 18.13, then the relevant mode of instability for the lowest critical load involves points of contra flexure at the quarter points. The buckling load is therefore the same as that of a pin-ended strut of half the length. Then

$$P_{cr} = \frac{\pi^2 EI}{\left(\frac{1}{2}L\right)^2} = 4\pi^2 \frac{EI}{L^2}, \text{ where } L_0 = 0.5L \tag{18.49}$$

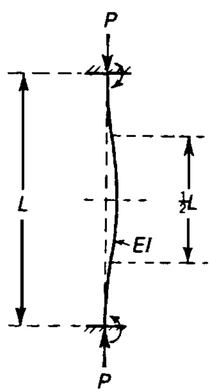


Figure 18.13 Buckling of a strut with built-in ends.

When the ends of the strut are built-in, no restraining moments are induced at the ends until the strut develops a buckled form.

18.11 Buckling of a strut with one end fixed and the other end free

When a vertical load P is applied to the free end of a vertical cantilever, AB, at the lowest critical load the laterally deflected form of the strut is a sinusoidal wave of length $2L$. If we consider the reflection of the buckled strut about A, Figure 18.14, then the strut of length $2L$ behaves as a pin-ended strut. The buckling load is

$$P_{cr} = \frac{\pi^2 EI}{(2L)^2} = \frac{\pi^2 EI}{4L^2}, \text{ where } L_0 = 2L \tag{18.50}$$

An important assumption in the preceding analysis is that the load at the free end of the cantilever is maintained in a vertical direction. If the load is always directed at A, that is its line of action is

BA, Figure 18.15 in the buckled form, then there is no restraining moment at A, and the cantilever behaves as a pin-ended strut. The buckling load is

$$P_{cr} = \pi^2 \frac{EI}{L^2} \qquad (18.51)$$

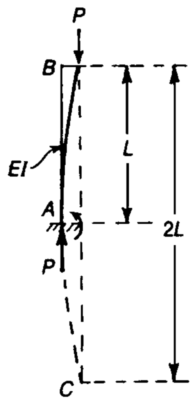


Figure 18.14 Buckling of a strut with one end free and the other built in.

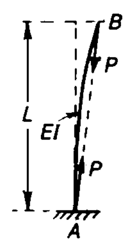


Figure 18.15 Thrust inclined to its original direction.

18.12 Buckling of a strut with one end pinned and the other end fixed

For other combinations of end conditions we are usually led to more involved calculations. A strut is pinned at its upper end and built-in to a rigid foundation at the lower end, Figure 18.16. In the buckled form of the strut a lateral shearing force F is induced at the upper end.

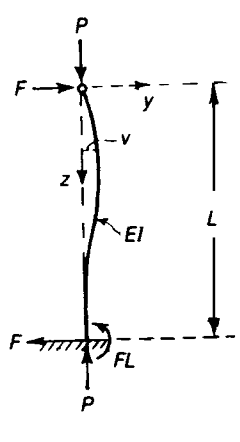


Figure 18.16 Strut with one end pinned and the other end fixed.

If v is the deflection of the central axis of the strut parallel to the y-axis, the bending moment at any section is

$$M = Pv - Fz$$

But

$$M = -EI \frac{d^2v}{dz^2}$$

Thus

$$-EI \frac{d^2v}{dz^2} = Pv - Fz$$

Put $k^2 = P/EI$. Then

$$\frac{d^2v}{dz^2} + k^2 v = \frac{Fk^2 z}{P}$$

The general solution is

$$v = A \cos kz + B \sin kz + \frac{F}{P} z$$

where A and B are arbitrary constants; the value of F is also unknown as yet, so there are three unknown constants in this equation. The boundary conditions are

$$v = 0, \quad \text{at} \quad z = 0$$

and $$v = 0 \quad \text{and} \quad \frac{dv}{dz} = 0, \quad \text{at} \quad z = L$$

These give

$$A = 0$$

$$B \sin kL + \frac{FL}{P} = 0$$

$$Bk \cos kL + \frac{F}{P} = 0$$

The last two of these equations give

$$\frac{B}{F} = -\frac{L}{P \sin kL} = -\frac{1}{Pk \cos kL}$$

Thus

$$kL \cos kL = \sin kL \qquad (18.52)$$

This equation gives the values of kL at which B and F are indeterminate, that is, at a condition of neutral equilibrium. The equation may be written

$$kL = \tan kL \qquad (18.53)$$

The smallest non-zero value of kL satisfying this equation is approximately equal to 4.49 (see Figure 18.17). This gives

$$P_{cr} = k^2 EI = 4.49^2 \frac{EI}{L^2} = 20.2 \frac{EI}{L^2}$$

We may derive an approximate value of kL in the following way: suppose kL is less than $3\pi/2$ by a small amount ε, then

$$kL = \frac{3\pi}{2} - \varepsilon$$

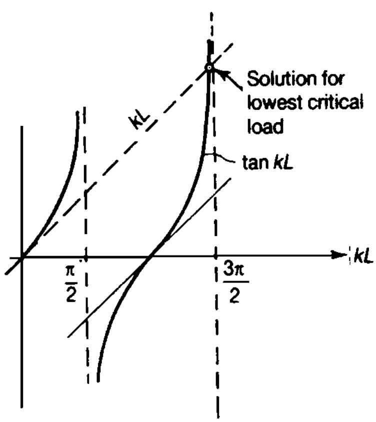


Figure 18.17 Graphical determination of buckling load.

Then we are interested in the roots of the equation

$$\frac{3\pi}{2} - \varepsilon = \tan\left(\frac{3\pi}{2} - \varepsilon\right)$$

If ε is small, then

$$\frac{3\pi}{2} - \varepsilon \fallingdotseq \cot \varepsilon \fallingdotseq \frac{1}{\varepsilon}\left(1 - \frac{1}{3}\varepsilon^2\right)$$

Approximately

$$\frac{3\pi}{2} = \frac{1}{\varepsilon}, \quad \text{or} \quad \varepsilon = \frac{2}{3\pi}$$

Then

$$kL = \frac{3\pi}{2} - \frac{2}{3\pi} = \frac{9\pi^2 - 4}{6\pi}$$

and

$$P_{cr} = k^2 EI = \left(\frac{9\pi^2 - 4}{6\pi}\right)^2 \frac{EI}{L^2} = 20.3 \frac{EI}{L^2} \tag{18.54}$$

where

$$L_0 = \sqrt{\pi^2 / 20.3} = 0.7$$

18.13 Flexural buckling of struts with other cross-sectional forms

In Section 18.2 we considered the strut to be in the form of a flat rectangular strip. We considered buckling to involve bending about the major axis Cx only, Figure 18.18. In the case of a flat rectangular strip the axis Cx is clearly the weaker axis of bending. In practice, structural sections rarely have this simple cross-sectional form, but frequently have I-sections, or angle sections, or circular sections.

In general, if the cross-sectional form of a strut has two axes of symmetry, we can consider flexural instability about these two axes independently. If an I-section has two axes of symmetry in the cross-section, Figure 18.19, flexural instability occurs usually about the axis of smaller stiffness, usually Cx. In a rectangular strut, Figure 18.19, the weaker bending axis is parallel to the longer sides. Circular cross-sectional forms have the property that any two mutually perpendicular diameters are principal centroidal axes; for these sections flexural instability is equally likely about any principal centroidal axis, Figure 18.19; when buckling occurs it is usually restricted to one plane. In making these statement we assume the ends of the strut are hinged about both axes Cy and Cz; this can be achieved in practice by loading through ball-ends. When the ends are not supported in the same way about Cy and Cx, then torsional effects may become important in the buckling behaviour.

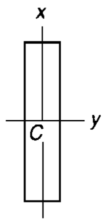


Figure 18.18 Narrow strip cross-section.

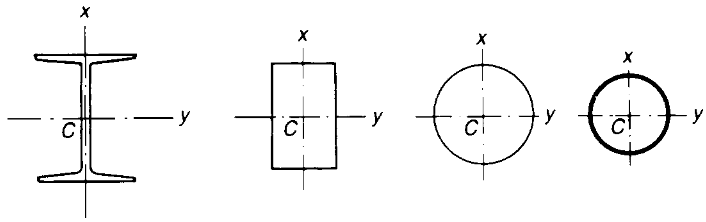


Figure 18.19 Cross-section with two axes of symmetry.

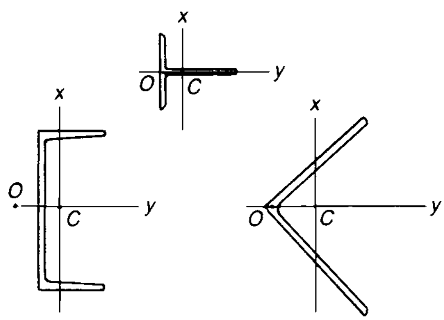


Figure 18.20 Cross-sections with only one axis of symmetry.

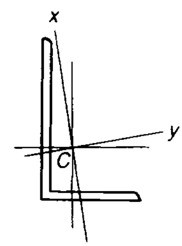


Figure 18.21 Unequal angle strut.

In the case of cross-sectional forms with only one axis of symmetry, *Cy*, say (Figure 18.20), torsional effects become important if the shear centre is not coincident with the centroid. This is true of channel sections, T-sections, and equal angle sections. Although for certain struts flexural instability occurs about the weaker principal axis *Cz*, in general twisting also occurs.

In the case of cross-sectional forms with no axes of symmetry, Figure 18.21, the buckled form always involves torsion, and the flexural buckling load has little meaning. This is true of unequal angle struts.

Problem 18.1 What thrust will a round steel rod take without buckling if it is 1.25 cm diameter, 2 m long, perfectly straight, and pin-jointed at the ends, the load being applied exactly along the axis of the rod?

<u>*Solution*</u>

We have

$$I = \frac{\pi(0.0125)^4}{64} = 1.20 \times 10^{-9}\ \mathrm{m}^4, \quad L = 2\ \mathrm{m}$$

Taking $E = 200\ \text{GN/m}^2$, we have

$$P_e = \frac{\pi^2 EI}{L^2} = 591\ N$$

18.14 Torsional buckling of a cruciform strut

We mentioned above that some struts are prone to torsional buckling effects. A cross-sectional form in which torsional instability occurs independently of any other form of buckling is a symmetrical cruciform section.

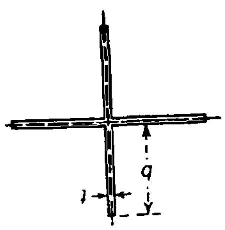


Figure 18.22 Cross-section of a cruciform strut.

The cruciform has four equally spaced limbs each of breadth b and uniform thickness t, Figure 18.22. Consider the section under a uniform compressive stress σ, Figure 18.23(i). We consider the possibility that the section may become unstable by twisting about the longitudinal axis Cz, Figure 18.23(ii); the stresses σ over the ends remain parallel to Cz during buckling.

Over any cross-section of the cruciform the stress is σ, acting parallel to Cz. Consider an elemental area δA of one limb at a distance x from the axis Cz, Figure 18.23(iii). If the relative twist between two cross-sections a distance δz apart is $\delta\theta$, then the force

$$\sigma \delta A$$

on the elemental area is statically equivalent to a force $\sigma\delta A$ acting along the twisted form of the strut and a small force

$$\sigma \delta A x \frac{d\theta}{dz}$$

acting in the plane of the cross-section. The inclined forces $\sigma\delta A$ on the two cross-sections are in equilibrium with each other, but the two forces $\sigma\delta Ax\,(d\theta/dz)$ give rise to a resultant torque at any cross-section. This torque is

$$4\int_0^b \sigma x^2 \frac{d\theta}{dx}\, dA = 4\sigma \frac{d\theta}{dz}\int_0^b x^2\, dA$$

since there are four limbs.

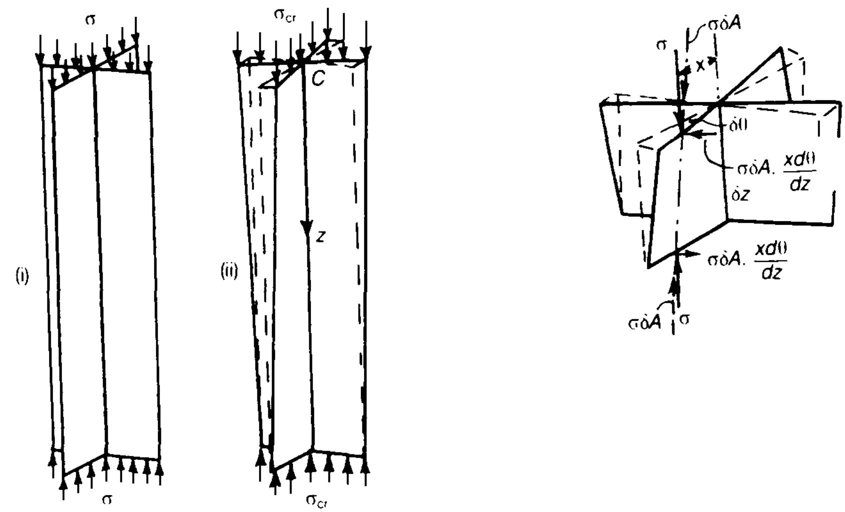


Figure 18.23 Torsional buckling of a cruciform column.

The geometrical quantity

$$4\int_0^b x^2 dA$$

is the polar second moment of area of the cross-section about Cz. The resultant torque at any cross-section is then

$$\sigma \frac{d\theta}{dz} J_0$$

where

$$J_0 = 4\int_0^b x^2\, dA = 4t\int_0^b x^2\, dz = \frac{4}{3}b^3 t$$

Now, we found in Chapter 16 that the torque-twist relation for a cruciform section is

$$\text{Torque} = GJ\frac{d\theta}{dz} = \frac{4}{3}Gbt^3\frac{d\theta}{dz}$$

In the case of the compressed cruciform, the twisted form can be maintained if

$$\sigma\frac{d\theta}{dz}J_0 = GJ\frac{d\theta}{dz}$$

Then

$$\sigma = G\left(\frac{J}{J_0}\right) = G\left(\frac{\frac{4}{3}bt^3}{\frac{4}{3}b^3t}\right) = G\left(\frac{t}{b}\right)^2 \qquad (18.55)$$

18.15 Modes of buckling of a cruciform strut

With a knowledge of the torsional and flexural buckling loads of a cruciform strut, we can estimate the range of struts, we can estimate the range of struts for which buckling is likely in the two modes.

If b is very much greater than t, and if all the limbs are similar in form, flexural buckling of a pin-ended strut is possible about any axis through the junction of the limbs, since the flexural stiffness is the same for all axes. For flexural instability the critical stress is

$$\sigma_f = \pi^2 \frac{EI}{AL^2} \qquad (18.56)$$

Now $I = 1/12\, t(2b)^3 = \frac{2}{3}b^3t$ and $A = 4bt$, and so

$$\sigma_f = \frac{\pi^2}{6}\frac{Eb^2}{L^2} \qquad (18.57)$$

Now, as we have seen, the torsional buckling stress is independent of L, and has the value

$$\sigma_t = G\left(\frac{t}{b}\right)^2 \qquad (18.58)$$

Then $\sigma_f > \sigma_t$ when

$$\frac{\pi^2}{6}\frac{Eb^2}{L^2} > G\left(\frac{t}{b}\right)^2$$

i.e. when

$$\frac{b^4}{L^2t^2} > \frac{6G}{\pi^2 E} = \frac{6}{2\pi^2(1+\nu)} = \frac{3}{\pi^2(1+\nu)} \qquad (18.59)$$

If $\nu = 0.3$, then

$$\frac{b^4}{L^2t^2} > \frac{3}{1.3\pi^2} = 0.234 \qquad (18.60)$$

Thus torsional buckling takes place when

$$\frac{b^2}{Lt} > \sqrt{0.234} = 0.484$$

i.e. when

$$\frac{Lt}{b^2} < 2.07$$

This condition may be written

$$\left(\frac{L}{b}\right) < 2.07\left(\frac{b}{t}\right) \qquad (18.61)$$

We can show the domains of flexural and torsional instability by plotting (L/b) against (b/t), Figure 18.24. For a practical material, yielding or material breakdown occurs when L/b and b/t approach zero; the lower left-hand corner is therefore the yielding domain. Above the straight line

$$\left(\frac{L}{b}\right) = 2.07\left(\frac{b}{t}\right)$$

buckling is by flexure, whereas below this line buckling is by torsion.

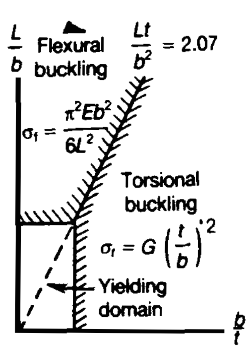


Figure 18.24 Modes of buckling of a cruciform strut.

18.16 Lateral buckling of a narrow beam

We have seen that the axial compression of a slender strut can lead to a condition of neutral equilibrium, in which at a certain compressive load a flexural form of deformation becomes possible. In the case of a cruciform strut we have shown that a form of neutral equilibrium involving torsion is possible under certain conditions.

Problems of structural instability are not restricted entirely to compression members, although there are many problems of this type. In the case of deep beams, for example, lateral buckling may occur, involving torsion and bending perpendicular to the plane of the depth of the beam. In general this problem is a complex one; however, we can determine some of the factors involved by studying the relatively simple case of the bending of a narrow deep beam.

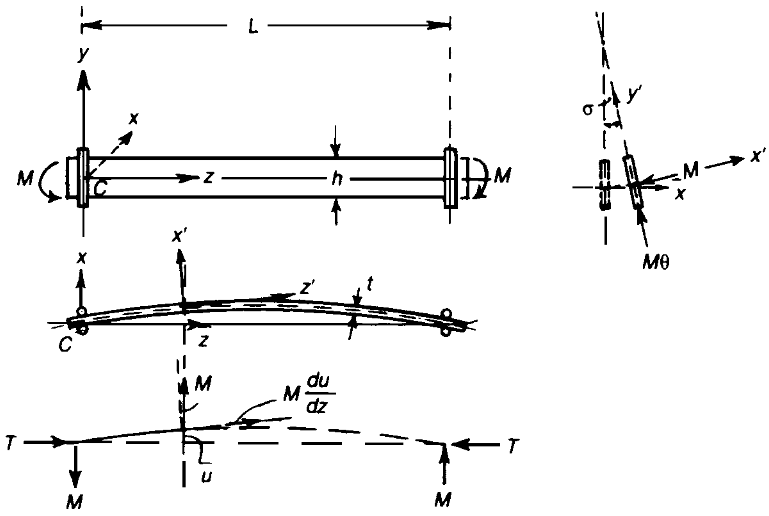


Figure 18.25 Lateral buckling of a narrow strip in pure bending.

A long rectangular strip has a depth h and thickness t, which is small compared with h, Figure 18.25. The principal centroidal axes are Cx, Cy and Cz. At the ends of the beam are vertical rollers which prevent twisting of the beam about a longitudinal axis. The distance between the end supports is L.

The beam is loaded with moments M applied at each end about axes parallel to Cx. Consider the possibility that the beam may become laterally unstable at some critical value of M. If $h >> t$ then bending displacements in the yz plane may be neglected. Suppose in the buckled form the principal centroidal axes at any cross-section are Cx', Cy' and Cz'. The lateral displacements parallel to Cx is u, and θ is the angle of twist about Cz at any cross-section. The moments M are assumed to be maintained along their original lines of action; the only other forces which may be induced at the ends are equal and opposite longitudinal torques T. The bending moment about the axis Cy' is then

$M\theta$

and as this gives rise to the curvature of the beam in the xz plane we have

$$M\theta = -EI_y \frac{d^2u}{dz^2}$$

Where EI_y is the bending stiffness of the beam about Cy. Again, for twisting about Cz'

$$T + M\frac{du}{dz} = GJ\frac{d\theta}{dz}$$

where GJ is the torsional stiffness about Cx. Differentiation of the second equations gives

$$M\frac{d^2U}{dz^2} = GJ\frac{d^2\theta}{dz^2}$$

Thus, on eliminating u,

$$M\theta = -EI_y \frac{GJ}{M}\frac{d^2\theta}{dz^2}$$

Then

$$\frac{d^2\theta}{dz^2} + \frac{M^2}{GJEI_y}\theta = 0$$

Put

$$k^2 = \frac{M^2}{GJEI_y} \qquad (18.62)$$

Then

$$\frac{d^2\theta}{dz^2} + k^2\theta = 0$$

The general solution is

$$\theta = A\cos kz + B\sin kz$$

where A and B are arbitrary constants. If $\theta = 0$ at $z = 0$, then $A = 0$. Further if $\theta = 0$ at $z = L$,

$$B \sin kL = 0$$

If $B = 0$, then both A and B are zero, and no buckling occurs; but B can be non-zero if

$$\sin kL = 0$$

We can disregard the root $kL = 0$, since the general solution is only valid if k is non-zero. The relevant roots are

$$kL = \pi, \quad 2\pi, \quad 3\pi \ldots \tag{18.63}$$

The smallest value of critical moment is

$$M_{cr} = k\sqrt{(GJ)(EI_y)} = \frac{\pi}{L}\sqrt{(GJ)(EI_y)}$$

Now, for a beam of rectangular cross-section,

$$GJ = \frac{1}{3}Ght^3, \quad EI_y = \frac{1}{12}Eht^3 \tag{18.64}$$

Then

$$M_{cr} = \frac{\pi}{L}\sqrt{\frac{1}{36}GEh^2t^6} = \frac{\pi}{L}\frac{ht^3}{6}\sqrt{GE} \tag{18.65}$$

If $G = E/2(1 + \nu)$ then

$$\sqrt{GE} = \sqrt{E^2/2(1+\nu)} = \frac{E}{\sqrt{2(1+\nu)}} \tag{18.66}$$

The maximum bending stress at the bending moment M_{cr} is

$$\sigma_{cr} = \frac{M_{cr}}{I_x}\frac{h}{2} = \frac{6M_{cr}}{h^2t} = \frac{\pi E}{\sqrt{2(1+\nu)}}\frac{t^2}{hL} \tag{18.67}$$

For a strip of given depth h and thickness t, the buckling stress σ_{cr} is proportional to the inverse of (L/t), which is sometimes referred to as the slenderness ratio of the beam.

Further problems *(answers on page 694)*

18.2 Calculate the buckling load of a pin-joined strut made of round steel rod 2 cm diameter and 4 m long.

18.3 Find the thickness of a round steel tubular strut 3.75 cm external diameter, 2 m long, pin-joined at the end, to withstand an axial load of 10 kN.

18.4 Calculate the buckling load of a strut built-in at both ends, the cross-section being a square 1 cm by 1 cm, and the length 2 m. Take $E = 200\ \mathrm{GN/m^2}$.

18.5 A steel scaffolding pole acts as a strut, but the load is applied eccentrically at 7.5 cm distance from the centre line with leverages in the same direction at top and bottom. The pole is tubular, 5 cm external diameter and 0.6 cm thick, 3 m in length between its ends which are not fixed in direction. If the steel has a yield stress of 300 $\mathrm{MN/m^2}$ and $E = 200\ \mathrm{GN/m^2}$, estimate approximately the load required to buckle the strut. *(RNEC)*

18.6 Two similar members of the same dimensions are connected together at their ends by two equal rigid links, the links being pin-jointed to the members. At the middle the members are rigidly connected by a distance piece. Equal couples are applied to the links, the axes of the couples being parallel to the pins of the joints. Show that buckling will occur in the top member if the couples M exceed a value given by the root of the equation

$$\tan \frac{1}{2}kL = \tanh \frac{1}{2}kL$$

where $k^2 = M/EId$. *(Cambridge)*