15 Plastic bending of mild-steel beams

15.1 Introduction

We have seen that in the bending of a beam the greatest direct stresses occur in the extreme longitudinal fibres; when these stresses attain the yield-point values, or exceed the limit of proportionality, the distribution of stresses over the depth of the beams no longer remains linear, as in the case of elastic bending.

The general problem of the plastic bending of beams is complicated; plastic bending of a beam is governed by the forms of the stress–strain curves of the material in tension and compression. Mild steel, which is used extensively as a structural material, has tensile and compressive properties which lend themselves to a relatively simple treatment of the plastic bending of beams of this material. The tensile and compressive stress–strain curves for an annealed mild steel have the forms shown in Figure 15.1; in the elastic range Young's modulus is the same for tension and compression, and of the order of 300 MN/m^2. The yield point corresponds to a strain of the order 0.0015. When the strain corresponding with the upper yield point is exceeded straining takes place continuously at a constant lower yield stress until a strain of about 0.015 is attained; at this stage further straining is accompanied by an increase in stress, and the material is said to *strain-harden*. This region of strain-hardening begins at strains about ten times larger than the strains at the yield point of the material.

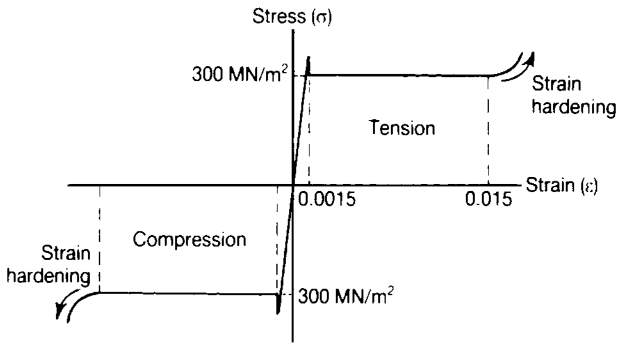


Figure 15.1 Tensile and compressive stress–strain curves of an annealed mild steel.

In applying these stress–strain curves to the plastic bending of mild-steel beams we simplify the problem by ignoring the upper yield point of the material; we assume the material is elastic, with a Young's modulus E, up to a yield stress σ_Y; Figure 15.2. We assume that the yield stress, σ_Y, and Young's modulus, E, are the same for tension and compression. These idealised stress–strain curves for tension and compression are then similar in form.

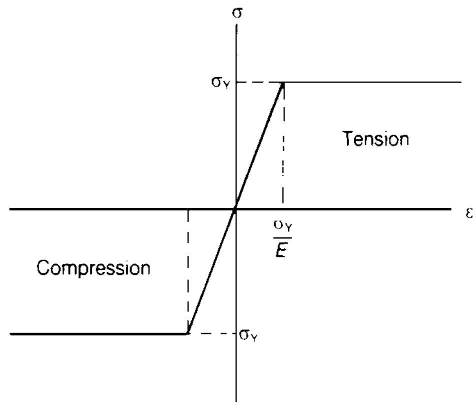


Figure 15.2 Idealized tensile and compressive stress–strain curves of annealed mild steel.

15.2 Beam of rectangular cross-section

As an example of the application of these idealised stress–strain curves for mild steel, consider the uniform bending of a beam of rectangular cross-section; b is the breadth of the cross-section and h its depth, Figure 15.3(i). Equal and opposite moments M are applied to the ends of a length of the beam. We found that in the elastic bending of a rectangular beam there is a linear distribution of direct stresses over a cross-section of the beam; an axis at the mid-depth of the cross-section is unstrained and therefore a neutral axis. The stresses are greatest in the extreme fibres of the beam; the yield stress, σ_Y, is attained in the extreme fibres, Figure 15.3(ii), when

$$M = \frac{2\sigma_Y I}{h} = M_Y \text{ (say)}$$

where I is the second moment of area of the cross-section about the axis of bending. But $I = bh^3/12$, and so

$$M_Y = \frac{1}{6} bh^2 \sigma_Y \tag{15.1}$$

As the beam is bent beyond this initial yielding condition, experiment shows that plane cross-sections of the beam remain nearly plane as in the case of elastic bending. The centroidal axis remains a neutral axis during inelastic bending, and the greatest strains occur in the extreme tension and compression fibres. But the stresses in these extreme fibres cannot exceed σ_Y, the yield stress; at an intermediate stage in the bending of the beam the central core is still elastic, but the extreme fibres have yielded and become plastic, Figure 15.3(iii).

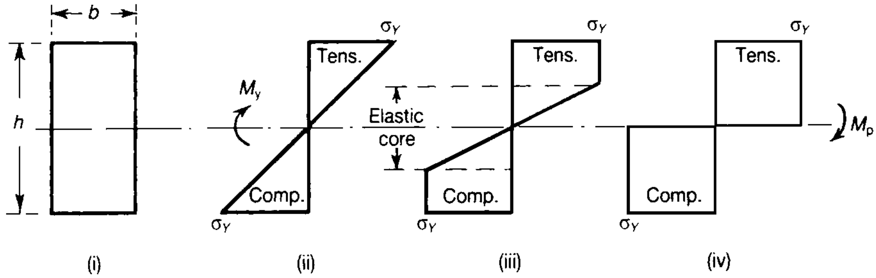


Figure 15.3 Stages in the elastic and plastic bending of a rectangular mild-steel beam.

If the curvature of the beam is increased the elastic core is diminished in depth; finally a condition is reached where the elastic core is reduced to negligible proportions, and the beam is more or less wholly plastic, Figure 15.3(iv); in this final condition there is still a central unstrained, or neutral, axis; fibres above the neutral axis are stressed to the yield point in tension, whereas fibres below the neutral axis to the yield point are in compression. In the ultimate fully plastic condition the resultant longitudinal tension in the upper half-depth of the beam is

$$\frac{1}{2} bh\sigma_Y$$

There is an equal resultant compression in the lower half-depth. There is, therefore, no resultant longitudinal thrust in the beam; the bending moment for this fully plastic condition is

$$M_P = \left(\frac{1}{2} bh\sigma_Y\right)\left(\frac{1}{2} h\right) = \frac{1}{4} bh^2\sigma_Y \tag{15.2}$$

This ultimate moment is usually called the *fully plastic moment* of the beam; comparing equations (15.1) and (15.2) we get

$$M_P = \frac{3}{2} M_Y \tag{15.3}$$

Thus plastic collapse of a rectangular beam occurs at a moment 50% greater than the bending moment at initial yielding of the beam.

15.3 Elastic–plastic bending of a rectangular mild-steel beam

In section 15.2 we introduced the concept of a fully plastic moment, M_P, of a mild-steel beam; this moment is attained when all longitudinal fibres of the beam are stressed into the plastic range of the material. Between the stage at which the yield stress is first exceeded and the ultimate stage at which the fully plastic moment is attained, some fibres at the centre of the beam are elastic and those remote from the centre are plastic. At an intermediate stage the bending is elastic–plastic.

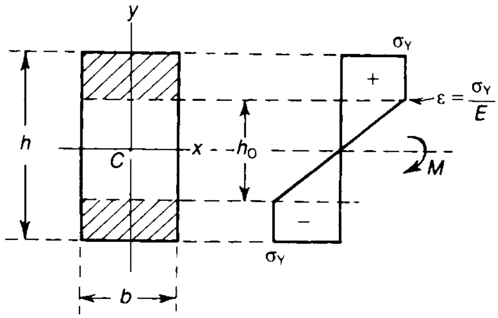


Figure 15.4 Elastic–plastic bending of a rectangular section beam.

Consider again a mild-steel beam of rectangular cross-section, Figure 15.4, which is bent about the centroidal axis Cx. In the elastic–plastic range, a central region of depth h_0 remains elastic; the yield stress σ_Y is attained in fibres beyond this central elastic core. If the central region of depth h_0 behaves as an elastic beam, the radius of curvature, R, is given by

$$\frac{2\sigma_Y}{h_0} = \frac{E}{R} \tag{15.4}$$

where E is Young's modulus in the elastic range of the material. Then

$$h_0 = \frac{2R\sigma_Y}{E} \tag{15.5}$$

Now, the bending moment carried by the elastic core of the beam is

$$M_1 = \sigma_Y \frac{bh_0^2}{6} \tag{15.6}$$

and the moment due to the stresses in the extreme plastic regions is

$$M_2 = \sigma_Y \left[\frac{bh^2}{4} - \frac{bh_0^2}{4}\right] \tag{15.7}$$

The total moment is, therefore,

$$M = M_1 + M_2 = \sigma_Y \frac{bh^2}{4} + \sigma_Y \left[\frac{bh_0^2}{6} - \frac{bh_0^2}{4}\right]$$

which gives

$$M = \sigma_Y \frac{bh^2}{4}\left[1 - \frac{h_0^2}{3h^2}\right] \tag{15.8}$$

But the fully plastic moment, M_P, of the beam is

$$M_P = \sigma_Y \frac{bh^2}{4}$$

Thus equation (15.8) may be written

$$M = M_P\left[1 - \frac{h_0^2}{3h^2}\right] \tag{15.9}$$

On substituting for h_0 from equation (15.5),

$$\frac{M}{M_P} = 1 - \frac{4}{3}\left(\frac{\sigma_Y}{E}\right)^2\left(\frac{R}{h}\right)^2 \tag{15.10}$$

At the onset of plasticity in the beam,

$$\frac{h}{R} = \frac{2\sigma_Y}{E} = \left(\frac{h}{R}\right)_Y \text{ (say)} \tag{15.11}$$

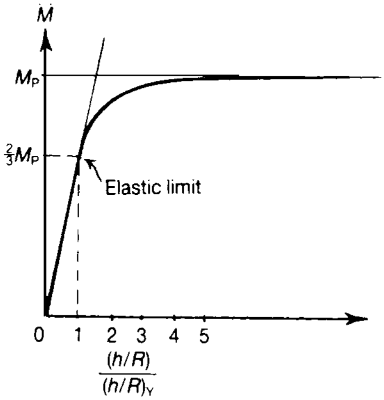
Then equation (15.10) may be written

$$\frac{M}{M_P} = 1 - \frac{1}{3}\frac{(h/R)_Y^2}{(h/R)^2} \tag{15.12}$$

Values of (M/M_P) for different values of $(h/R)/(h/R)_Y$ are given in Figure 15.5; the elastic limit of the beam is reached when

$$M = \frac{2}{3}M_P = M_Y \text{ (say)}$$

As M is increased beyond M_Y, the fully plastic moment M_P is approached rapidly with increase of curvature $(1/R)$ of the beam; M is greater than 99% of the fully plastic moment when the curvature is only five times as large as the curvature at the onset of plasticity.



$\frac{h/R}{(h/R)_Y}$	$\frac{M}{M_P}$
1	0.667
2	0.917
3	0.963
4	0.978
5	0.992

Figure 15.5 Moment–curvature relation for the elastic–plastic bending of a rectangular mild-steel beam.

15.4 Fully plastic moment of an I-section; shape factor

The cross-sectional dimensions of an I-section are shown in Figure 15.6; in the fully plastic condition, the centroidal axis Cx is a neutral axis of bending. The tensile fibres of the beam all carry the same stress σ_Y; the total longitudinal force in the upper flange is

$$\sigma_Y b t_f$$

and its moment about Cx is

$$\sigma_Y b t_f \left(\frac{1}{2}h - \frac{1}{2}t_f \right) = \frac{1}{2}\sigma_Y b t_f (h - t_f)$$

Similarly, the total force in the tensile side of the web is

$$\sigma_Y \left(\frac{h}{2} - t_f \right) t_w$$

and its moment about Cx is

$$\frac{1}{2}\sigma_Y \left(\frac{1}{2}h - t_f \right)^2 t_w = \frac{1}{8}\sigma_Y t_w (h - 2t_f)^2$$

The compressed longitudinal fibres contribute moments of the same magnitudes. The total moment carried by the beam is therefore

$$M_P = \sigma_Y \left[b t_f (h - t_f) + \frac{1}{4} t_w (h - 2t_f)^2 \right] \tag{15.13}$$

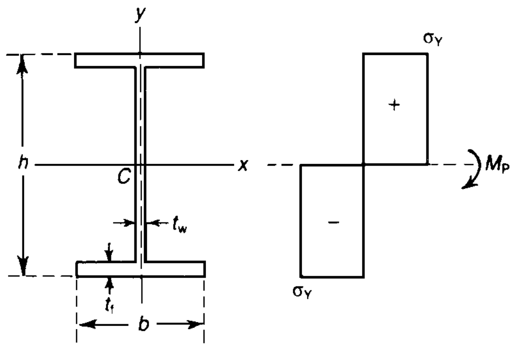


Figure 15.6 Fully plastic moment of an I-section beam.

In the case of elastic bending we defined the elastic section modulus, Z_e, as a geometrical property, which, when multiplied by the allowable bending stress, gives the allowable bending moment on the beam. In equation (15.13) suppose

$$Z_P = b t_f (h - t_f) + \frac{1}{4} t_w (h - 2t_f)^2 \tag{15.14}$$

Then Z_P is the *plastic section modulus* of the I-beam, and

$$M_P = \sigma_Y Z_P \tag{15.15}$$

As a particular case consider an I-section having dimensions:

$$h = 20 \text{ cm}, \qquad t_w = 0.70 \text{ cm}$$
$$b = 10 \text{ cm}, \qquad t_f = 1.00 \text{ cm}$$

Then

$$Z_P = (0.1)(0.010)(0.2 - 0.010) + \frac{1}{4}(0.007)(0.2 - 0.020)^2 = 0.247 \times 10^{-3} \text{ m}^3$$

The elastic section modulus is approximately

$$Z_e = 0.225 \times 10^{-3} \text{ m}^3$$

If M_Y is the bending moment at which the yield stress σ_Y is first reached in the extreme fibres of the beam, then

$$\frac{M_p}{M_Y} = \frac{Z_p}{Z_e} = \frac{0.247}{0.225} = 1.10 \tag{15.16}$$

Thus, in this case, the fully plastic moment is only 10% greater than the moment at initial yielding. The ratio (Z_p/Z_e) is sometimes called the *shape factor*.

15.5 More general case of plastic bending

In the case of the rectangular and I-section beams treated so far, the neutral axis of bending coincided with an axis of symmetry of the cross-section. For a section that is unsymmetrical about the axis of bending, the position of the neutral axis must be found first. The beam in Figure 15.7 has one axis of symmetry, *Oy*; the beam is bent into the fully plastic condition about *Ox*, which is perpendicular to *Oy*. The axis *Ox* is the neutral axis of bending; the total longitudinal force on the fibres above *Ox* is $A_1\sigma_Y$, where A_1 is the area of the cross-section of the beam above *Ox*. If A_2 is the area of the cross-section below *Ox*, the total longitudinal force on the fibres below *Ox* is $A_2\sigma_Y$. If there is no resultant longitudinal thrust in the beam, then

$$A_1\sigma_Y = A_2\sigma_Y$$

that is,

$$A_1 = A_2 \tag{15.17}$$

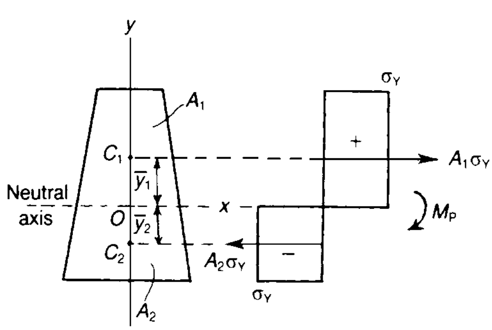


Figure 15.7 Plastic bending of a beam having one axis of symmetry in the cross-section, but unsymmetrical about the axis of bending.

The neutral axis *Ox* divides the beam cross-section into equal areas, therefore. If the total area of cross-section is A, then

$$A_1 = A_2 = \frac{1}{2}A$$

Then

$$A_1\sigma_Y = A_2\sigma_Y = \frac{1}{2}A\sigma_Y$$

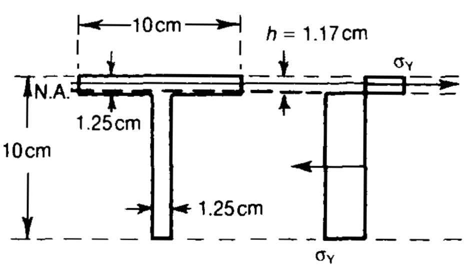
Suppose C_1 is the centroid of the area A_1 and C_2 the centroid of A_2; if the centroids C_1 and C_2 are distances $\bar{y}_1$ and $\bar{y}_2$, respectively, from the neutral axis Ox, then

$$M_P = \frac{1}{2}A\sigma_Y\left(\overline{y_1} + \overline{y_2}\right) \tag{15.18}$$

The plastic section modulus is

$$Z_P = \frac{M_P}{\sigma_Y} = \frac{1}{2}A\left(\overline{y_1} + \overline{y_2}\right) \tag{15.19}$$

Problem 15.1 A 10 cm by 10 cm T-section is of uniform thickness 1.25 cm. Estimate the plastic section modulus for bending about an axis perpendicular to the web.



Solution

The neutral axis of plastic bending divides the section into equal areas. If the neutral axis is a distance h below the extreme edge of the flange,

$$(0.1)h = (0.0875)(0.0125) + (0.1)(0.0125 - h)$$

Then

$$h = 0.0117 \text{ m}$$

Then

$$M_P = \frac{1}{2}(0.1)(0.0117)^2\sigma_Y + \frac{1}{2}(0.0875)(0.0008)^2\sigma_Y$$

$$+ \frac{1}{2}(0.0883)^2(0.0125)\sigma_Y$$

$$= (0.0557 \times 10^{-3})\sigma_Y$$

The plastic section modulus is then

$$Z_P = \frac{M_P}{\sigma_Y} = 0.0557 \times 10^{-3}\ \mathrm{m}^3$$

The elastic section modulus is

$$Z_e = 0.0311 \times 10^{-3}\ \mathrm{m}^3$$

Then

$$\frac{M_P}{M_Y} = \frac{Z_P}{Z_e} = \frac{0.0557}{0.0311} = 1.79$$

15.6 Comparison of elastic and plastic section moduli

For bending of a beam about a centroidal axis Cx, the elastic section modulus is

$$Z_e = \frac{I}{y_{\max}} \tag{15.20}$$

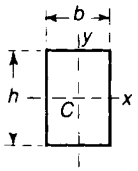
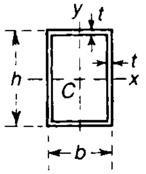
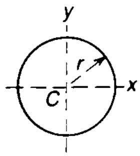
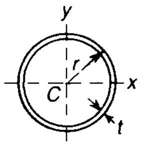
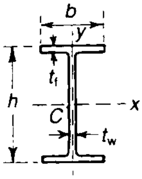
where I is the second moment of area of the cross-section about the axis of bending, and y_{max} is the distance of the extreme fibre from the axis of bending.

From equation (15.19) the plastic section modulus of a beam is

$$Z_P = \frac{1}{A}\left(\overline{y_1} + \overline{y_2}\right) \tag{15.21}$$

Values of Z_e and Z_p for some simple cross-sectional forms are shown in Table 15.1. In the solid rectangular and circular sections Z_p is considerably greater than Z_e; the difference between Z_p and Z_e is less marked in the case of thin-walled sections.

Table 15.1 Comparison of elastic and plastic section moduli for some simple cross-sectional forms

Cross-sectional form	Elastic section modulus, Z_e	Plastic section modulus, Z_p	Shape factor, $\frac{Z_p}{Z_e}$
 Solid rectangular section	Axis $Cy: \frac{1}{6}b^2h$ Axis $Cx: \frac{1}{6}bh^2$	Axis $Cy: \frac{1}{4}b^2h$ Axis $Cx: \frac{1}{4}bh^2$	1·5 1·5
 Thin-walled rectangular box of uniform wall-thickness, t	$t \ll h$; $t \ll b$ Axis $Cy: bt(h+\frac{1}{3}b)$ Axis $Cx: ht(b+\frac{1}{3}h)$	Axis $Cy: bt(h+\frac{1}{2}b)$ Axis $Cx: ht(b+\frac{1}{2}h)$	$\frac{h+\frac{1}{2}b}{h+\frac{1}{3}b}$ $\frac{b+\frac{1}{2}h}{b+\frac{1}{3}h}$
 Soild circular section	Axis Cy or Cx: $\frac{\pi r^3}{4}$	Axis Cy or Cx: $\frac{4r^3}{3}$	$\frac{16}{3\pi}$
 Thin-walled circular tube	$t \ll r$ Axis Cy or $Cx: \pi r^2 t$	$4r^2t$	$\frac{4}{\pi}$
 Thin-walled I-section	$t_f \ll b$; $t_w \ll h$ Axis $Cy: \frac{1}{3}b^2t$ Axis $Cx: h[bt_f+\frac{1}{6}ht_w]$	Axis $Cy: \frac{1}{2}b^2t$ Axis $Cx: h[bt_f+\frac{1}{4}ht_w]$	1·5 $\frac{bt_f+\frac{1}{4}ht_w}{bt_f+\frac{1}{6}ht_w}$

15.7 Regions of plasticity in a simply-supported beam

The mild-steel beam shown in Figure 15.8 has a rectangular cross-section; it is simply-supported at each end, and carries a central lateral load W. The variation of bending moment has the form shown in Figure 15.8(ii); the greatest bending moment occurs under the central load and has the value $WL/4$. From the preceding analysis we see that a section may take an increasing bending moment until the fully plastic moment M_P of the section is reached. The ultimate strength of the beam is reached therefore when

$$M_P = \frac{WL}{4} \tag{15.22}$$

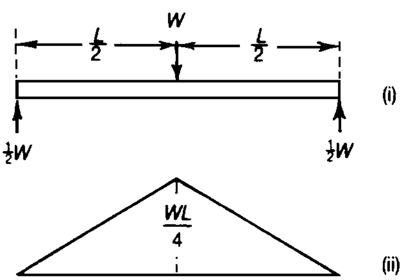


Figure 15.8 Plastic bending of a simply-supported beam.

If b is the breadth and h the depth of the rectangular cross-section, the bending moment, M_Y, at which the yield stress, σ_Y, is first attained in the extreme fibres is

$$M_Y = \sigma_Y \frac{bh^2}{6} = \frac{2}{3} M_P$$

At the ultimate strength of the beam

$$W = \frac{4M_P}{L} = \frac{4}{L}\left[\sigma_Y \frac{bh^2}{4}\right] \tag{15.23}$$

The beam is wholly elastic for a distance of

$$\frac{2}{3}\left(\frac{L}{2}\right) = \frac{1}{3}L \tag{15.24}$$

from each end support, Figure 15.9, as the bending moments in these regions are not greater than M_Y. The middle-third length of the beam is in an elastic–plastic state; in this central region consider a transverse section a–a of the beam, a distance z from the mid-length. The bending moment at this section is

$$M = \frac{1}{2}W\left(\frac{1}{2}L - z\right) \tag{15.25}$$

If W has attained its ultimate value given by equation (15.22),

$$M = \frac{2M_P}{L}\left(\frac{1}{2}L - z\right) \tag{15.26}$$

Suppose the depth of the elastic core of the beam at this section is h_0, Figure 15.9; then from equation (15.9),

$$M = M_P\left(1 - \frac{h_0^2}{3h^2}\right)$$

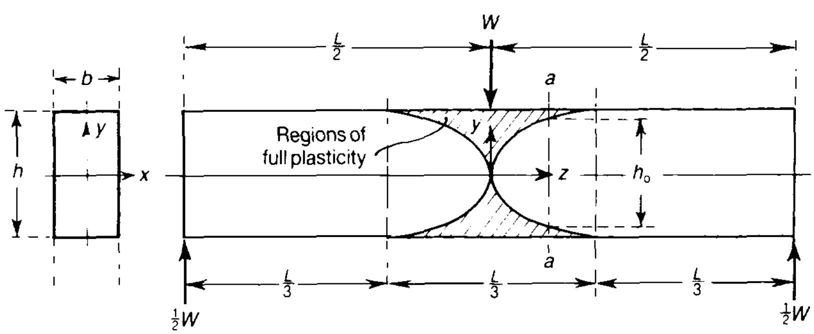


Figure 15.9 Regions of plasticity in a simply-supported beam carrying a distributed load; in the figure the depth of the beam is exaggerated.

On substituting this value of M into equation (15.26), we have

$$1 - \frac{h_0^2}{3h^2} = 1 - \frac{2z}{L} \tag{15.27}$$

and thus

$$h_0^2 = \frac{6h^2}{L}z \tag{15.28}$$

The total depth h_0, of the elastic core varies parabolically with z, therefore; from equation (15.28), $h_0 = h$ when $z = 1/6L$. The regions of full plasticity are wedge-shaped; the shapes of the regions developed in an actual mild-steel beam may be affected by, first, the stress-concentrations under the central load W, and, second, the presence of shearing stresses on sections such as a–a, Figure 15.9; equation (15.28) is true strictly for conditions of pure bending only.

For a simply-supported rectangular beam carrying a total uniformly distributed load W, Figure 15.10, the bending moment at the mid-length is

$$M_p = \frac{WL}{8}$$

at the ultimate load-carrying capacity of the beam. At a transverse section a–a, a distance z from the mid-length, the moment is

$$M = \frac{W}{8L}\left(L^2 - 4z^2\right) = \frac{M_p}{L^2}\left(L^2 - 4z^2\right) = M_p\left[1 - 4\left(\frac{z}{L}\right)^2\right] \tag{15.29}$$

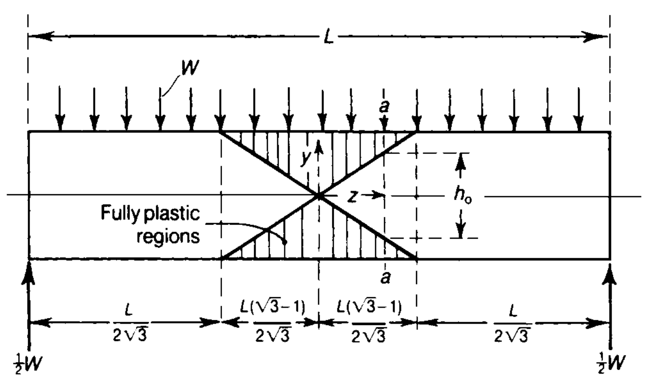


Figure 15.10 Regions of plasticity in a simply-supported beam carrying a distributed load; in the figure the depth of the beam is exaggerated.

From equation (15.9), the depth h_0 of the elastic core at the section a–a is given by

$$M = M_p\left[1 - \frac{h_0^2}{3h^2}\right]$$

Then

$$h_0^2 = 12h^2\left(\frac{z}{L}\right)^2$$

or

$$h_0 = 2\sqrt{3}\, h\left(\frac{z}{L}\right) \tag{15.30}$$

The limit of the wholly elastic length of the beam is given by $h = h_0$, or $z = L/(2\sqrt{3})$. The regions of plasticity near the mid-section are triangular-shaped, Figure 15.10.

15.8 Plastic collapse of a built-in beam

A uniform beam of length L is built-in at each end to rigid walls, and carries a uniformly distributed load w per unit length, Figure 15.11. If the material remains elastic, the bending moment at each end is $wL^2/12$, and at the mid-length $wL^2/24$. The bending moment is therefore greatest at the end supports; if yielding occurs first at a bending moment M_Y, then the lateral load at this stage is given by

$$M_Y = \frac{wL^2}{12} \tag{15.31}$$

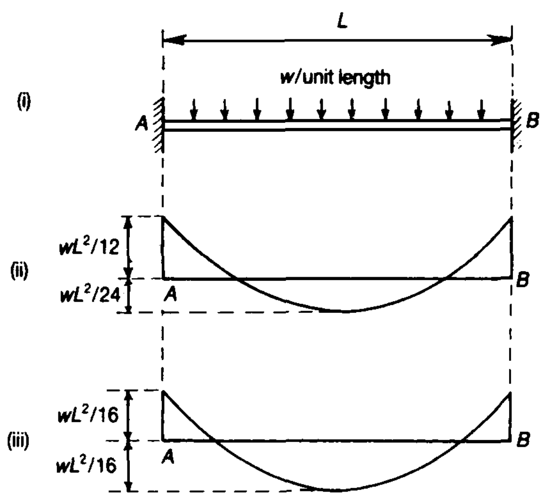


Figure 15.11 Plastic regions of a uniformly loaded built-in beam.

or

$$wL = \frac{12 M_Y}{L} \tag{15.32}$$

If the load w is increased beyond the limit of elasticity, plastic hinges first develop at the remote ends. The beam only becomes a mechanism when a third plastic hinge develops at the mid-length. On considering the statical equilibrium of a half-span of the beam we find that the moments at the ends and the mid-length, for plastic hinges at these sections, are

$$M_P = \frac{wL^2}{16} \tag{15.33}$$

or

$$wL = \frac{16M_P}{L} \tag{15.34}$$

Clearly, the load causing complete collapse is at least one-third greater than that at which initial yielding begins because M_P is greater than M_Y.

Another method of plastically analysing the beam of Figure 15.11 is by the *principle of virtual work* described in Chapter 17. In this case the beam is assumed to collapse in the form of a mechanism, when three plastic hinges form, as shown in Figure 15.12.

As the beam is encastré at both ends, it is statically indeterminate to the *second degree*, therefore *three hinges* are required to change it from a beam structure to a mechanism.

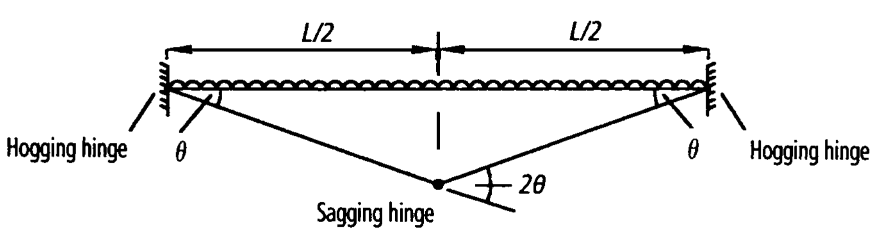


Figure 15.12 Plastic collapse of a beam.

Thus, because the beam cannot resist further loading at the three hinges, the slightest increase in load causes the hinges to rotate like 'rusty' hinges. Additionally, as the bending moment distribution is constant during this collapse, the curvature of the beam remains constant during collapse. Hence, for the purpose of analysis, the beam's two sections can be assumed to remain straight during collapse.

Work done by the three hinges during collapse

$$= M_p\theta + M_p\ 2\theta + M_p\theta \tag{15.35}$$

Work done by the distributed load

$$wL \times \frac{L}{4}\theta \tag{15.36}$$

Equating (15.35) and (15.36)

$$4M_p\theta = \frac{wL^2}{4}\theta \quad \text{or} \quad M_p = \frac{wL^2}{16} \tag{15.37}$$

which is identical to equation (15.33). This method of solution is discussed in greater detail in Chapter 17.

Further problems *(answers on page 693)*

15.2 A uniform mild-steel beam AB is 4 m long; it is built-in at A and simply-supported at B. It carries a single concentrated load at a point 1.5 m from A. if the plastic section modulus of the beam is 0.433×10^{-3} m^3, and the yield stress of the material is 235 MN/m^2, estimate the value of the concentrated load causing plastic collapse.

15.3 A uniform mild-steel beam is supported on four knife edges equally spaced a distance 8 m apart. Estimate the intensity of uniformly distributed lateral load over the whole length causing collapse, if the plastic section modulus of the beam is 1.690×10^{-3} m^3, and the yield stress of the material is 235 MN/m^2.

15.4 A uniform beam rests on three supports A, B and C with two spans each 5 m long. The collapse load is to be 100 kN per metre, and σ_Y = 235 MN/m^2. What will be a suitable mild-steel section using a shape factor 1.15?

15.5 If, in Problem 15.4, AB is 8 m and BC is 7 m, and the collapse loads are to be 100 kN/m on AB, 50 kN/m on BC, find a suitable mild-steel section I-beam, with σ_Y = 235 MN/m^2.

15.6 A continuous beam $ABCD$ has spans each 8 m long, it is 45 cm by 15 cm, with flanges 2.5 cm thick and web 1 cm thick. Find the collapse load if the whole beam carries a uniformly-distributed load. Which spans collapse? σ_Y = 235 MN/m^2.

15.7 A mild-steel beam 5 cm square section is subjected to a thrust of 200 kN acting in the plane of one of the principal axes, but may be eccentric. What eccentricity will cause the whole section to become plastic if σ_Y = 235 MN/m^2?