14 Built-in and continuous beams

14.1 Introduction

In all our investigations of the stresses and deflections of beams having two supports, we have supposed that the supports exercise no constraint on bending of the beam, i.e. the axis of the beam has been assumed free to take up any inclination to the line of supports. This has been necessary because, without knowing how to deal with the deformation of the axis of the beam, we were not in a position to find the bending moments on a beam when the supports constrain the direction of the axis. We shall now investigate this problem. When the ends of a beam are fixed in direction so that the axis of the beam has to retain its original direction at the points of support, the beam is said to be built-in or direction fixed.

Consider a straight beam resting on two supports A and B (Figure 14.1) and carrying vertical loads. If there is no constraint on the axis of the beam, it will become curved in the manner shown by broken lines, the extremities of the beam rising off the supports.

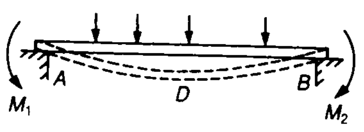


Figure 14.1 Beam with end couples.

In order to make the ends of the beam lie flat on the horizontal supports, we shall have to apply couples as shown by M_1 and M_2. If the beam is firmly built into two walls, or bolted down to two piers, or in any way held so that the axis cannot tip up at the ends in the manner indicated, the couples such as M_1 and M_2 are supplied by the resistance of the supports to deformation. These couples are termed *fixed-end moments*, and the main problem of the built-in beam is the determination of these couples; when we have found these we can draw the bending moment diagram and calculate the stresses in the usual way. The couples M_1 and M_2 in Figure 14.1 must be such as to produce curvature in the opposite direction to that caused by the loads.

14.2 Built-in beam with a single concentrated load

We may deduce the bending moments in a built-in beam under any conditions of lateral loading from the case of a beam under a single concentrated lateral load.

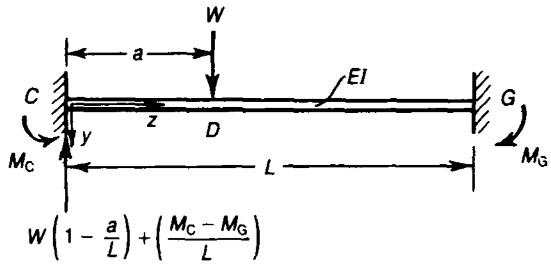


Figure 14.2 Built-in beam carrying a single lateral load.

Consider a uniform beam, of flexural stiffness EI, and length L, which is built-in to end supports C and G, Figure 14.2. Suppose a concentrated vertical load W is applied to the beam at a distance a from C. If M_C and M_G are the restraining moments at the supports, then the vertical reaction is at C is

$$W\left(1 - \frac{a}{L}\right) + \frac{1}{L}\left(M_C - M_G\right)$$

The bending moment in the beam at a distance z from C is therefore

← – – – – – – – – – – – – $z <= a$ – – – – – – – – – – – – → ← – – – $a < z <= L$ – – – →

$$M = \left\{W\left(1 - \frac{a}{L}\right) + \frac{1}{L}\left(M_C - M_G\right)\right\} z - M_C \qquad -W\left[z - a\right]$$

Then, for the deflected form of the beam, the displacement is given by

← – – – – – – – – – – – – $z <= a$ – – – – – – – – – – – – → ← – – – $a < z <= L$ – – – →

$$EI\frac{d^2v}{dz^2} = -\left\{W\left(1 - \frac{a}{L}\right) + \frac{1}{L}\left(M_C - M_G\right)\right\} z + M_C \qquad + W\left[z - a\right] \tag{14.1}$$

or

$$EI\frac{dv}{dz} = -\left\{W\left(1 - \frac{a}{L}\right) + \frac{1}{L}\left(M_C - M_G\right)\right\} \frac{z^2}{2} + M_C z + A \qquad + \frac{W}{2}\left[z - a\right]^2 \tag{14.2}$$

and

$$EIv = -\left\{W\left(1 - \frac{a}{L}\right) + \frac{1}{L}\left(M_C - M_G\right)\right\} \frac{z^3}{6} + \frac{M_C z^2}{2} + Az + B \qquad + \frac{W}{6}\left[z - a\right]^3 \tag{14.3}$$

Two suitable boundary conditions are:

when $\quad z = 0, \; v = dv/dz = 0$

As the Macaulay brackets will be negative when these boundary conditions are substituted, the terms on the right of equations (14.2) and (14.3) can be ignored, hence

$$A = B = 0$$

Two other boundary conditions are:

at $\quad z = L, \; v = dv/dz = 0,$

which on substituting into equations (14.2) and (14.3) give the following two simultaneous equations:

$$-\left[W\left(1 - \frac{a}{L}\right) + \frac{1}{L}\left(M_C - M_G\right)\right]\frac{L^2}{2} + M_C L + \frac{W}{2}\left(L - a\right)^2 = 0$$

$$-\left[W\left(1 - \frac{a}{L}\right) + \frac{1}{L}\left(M_C - M_G\right)\right]\frac{L^3}{6} + \frac{M_C L^2}{6} + \frac{W}{6}\left(L - a\right)^3 = 0$$

These simultaneous equations give

$$M_C = Wa\left(\frac{L - a}{L}\right)^2 \tag{14.4}$$

$$M_G = W(L - a)\left(\frac{a}{L}\right)^2 \tag{14.5}$$

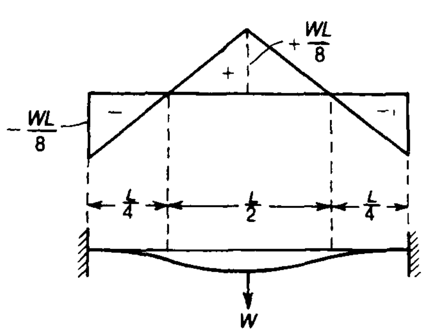


Figure 14.3 Variation in bending moment in a built-in beam carrying a concentrated load at mid-length.

M_C and M_G are referred to as the *fixed-end moments* of the beam; M_C is measured anticlockwise, and M_G clockwise.

In the particular case when the load W is applied at the mid-length, $a = ½L$, and

$$M_C = M_G = \frac{WL}{8}$$

The bending moment in the beam vary linearly from hogging moments of $WL/8$ at each end to a sagging moment of $WL/8$ at the mid-length, Figure 14.3. There are points of contraflexure, or zero bending moment, at distances $L/4$ from each end.

14.3 Fixed-end moments for other loading conditions

The built-in beam of Figure 14.4 carries a uniformly distributed load of w per unit length over the section of the beam from $z = a$ to $z = b$.

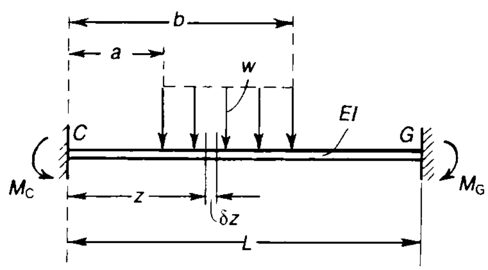


Figure 14.4 Distributed load over part of the span of a built-in beam.

Consider the loading on an elemental length δz of the beam; the vertical load on the element is $w\delta z$, and this induces a retraining moment at C of amount

$$\delta M_C = w\delta z \frac{z(L - z)^2}{L^2}$$

from equation (14.4).

The total moment at C due to all loads is

$$M_C = \int_a^b \frac{w}{L^2} z(L - z)^2 dz$$

which gives

$$M_C = \frac{w}{L^2}\left[\frac{L^2}{2}\left(b^2 - a^2\right) - \frac{2L}{3}\left(b^3 - a^3\right) + \frac{1}{4}\left(b^4 - a^4\right)\right] \tag{14.6}$$

M_G may be found similarly. When the load covers the whole of the span, $a = 0$ and $b = L$, and equation (14.6) reduces to

$$M_C = \frac{wL^2}{12} \tag{14.7}$$

In this particular case, $M_G = M_C$; the variation of bending moment is parabolic, and of the form shown in Figure 14.5; the bending moment at the mid-length is $wL^2/24$, so the fixed-end moments are also the greatest bending moments in the beam.

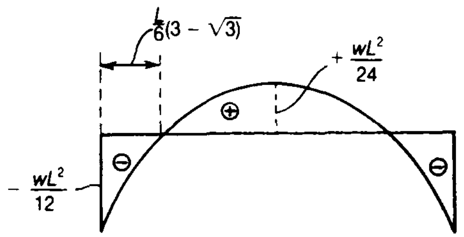


Figure 14.5 Variation of bending moment in a built-in beam carrying a uniformly distributed load over the whole span.

The points of contraflexure, or points of zero bending moment, occur at a distance

$$\frac{L}{6}\left(3 - \sqrt{3}\right) \tag{14.8}$$

from each end of the beam.

When a built-in beam carries a number of concentrated lateral loads, W_1, W_2, and W_3, Figure 14.6, the fixed-end moments are found by adding together the fixed-end moments due to the loads acting separately. For example,

$$M_C = \sum_{r = 1,2,3} W_r a_r \left(\frac{L - a_r}{L}\right)^2 \tag{14.9}$$

for the case shown in Figure 14.6.

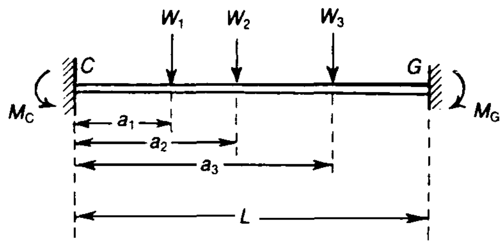


Figure 14.6 Built-in beam carrying a number of concentrated loads.

We may treat the case of a concentrated couple M_0, applied a distance a from the end C, Figure 14.7, as a limiting case of two equal and opposite loads W a small distance δa apart. The fixed-end moment at C is

$$M_C = -\frac{Wa}{L^2}(L - a)^2 + \frac{W(a + \delta a)}{L^2}(L - a - \delta a)^2$$

If δa is small,

$$M_C \doteq -\frac{Wa}{L^2}(L - a)^2 + \frac{W}{L^2}\left[a(L - a)^2 + \delta a\,(L - a)(L - 3a)\right]$$

which gives

$$M_C = \frac{W\delta a}{L^2}(L - a)(L - 3a)$$

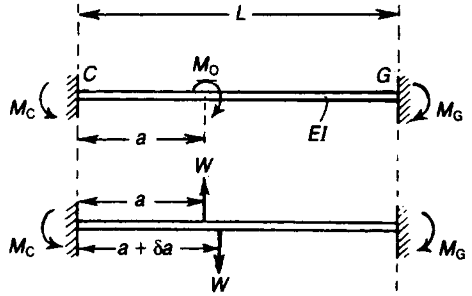


Figure 14.7 Built-in beam carrying a concentrated couple.

But if δa is small, M_0 is statically equivalent to the couple $W\delta a$, and

$$M_C = \frac{M_0}{L^2}(L - a)(L - 3a) \tag{14.10}$$

Similarly,

$$M_G = \frac{M_0}{L^2}a(2L - 3a) \tag{14.11}$$

14.4 Disadvantages of built-in beams

The results we have obtained above show that a beam which has its ends firmly fixed in direction is both stronger and stiffer than the same beam with its ends simply-supported. On this account it might be supposed that beams would always have their ends built-in whenever possible; in practice it is not often done. There are several objections to built-in beams: in the first place a small subsidence of one of the supports will tend to set up large stresses, and, in erection, the supports must be aligned with the utmost accuracy; changes of temperature also tend to set up large stresses. Again, in the case of live loads passing over bridges, the frequent fluctuations of bending moment, and vibrations, would quickly tend to make the degree of fixing at the ends extremely uncertain.

Most of these objections can be obviated by employing the double cantilever construction. As the bending moments at the ends of a built-in beam are of opposite sign to those in the central part of the beam, there must be points of inflexion, i.e. points where the bending moment is zero. At these points a hinged joint might be made in the beam, the axis of the hinge being parallel to the bending axis, because there is no bending moment to resist. If this is done at each point of inflexion, the beam will appear as a central girder freely supported by two end cantilevers; the bending moment curve and deflection curve will be exactly the same as if the beam were solid and built in. With this construction the beam is able to adjust itself to changes of temperature or subsistence of the supports.

14.5 Effect of sinking of supports

When the ends of a beam are prevented from rotating but allowed to deflect with respect to each other, bending moments are set up in the beam. The uniform beam of Figure 14.8 is displaced so that no rotations occur at the ends but the remote end is displaced downwards an amount δ relative to C.

The end reactions consist of equal couples M_C and equal and opposite shearing forces $2M_C/L$, because the system is antisymmetric about the mid-point of the beam. The half-length of the beam behaves as a cantilever carrying an end load $2M_CL$; then, from equation (13.18),

$$\frac{1}{2}\delta = \frac{(2M_C/L)(L/2)^2}{3EI} = \frac{M_CL^2}{12EI}$$

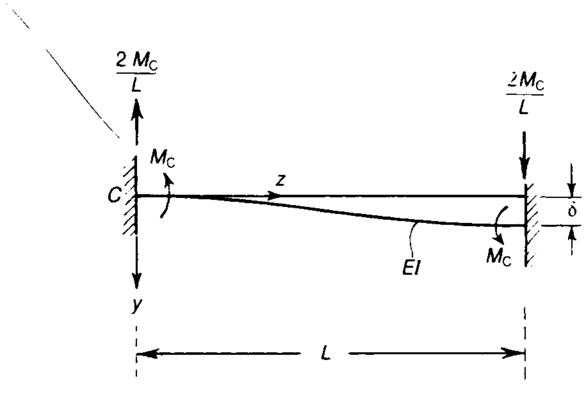


Figure 14.8 End moments induced by the sinking of the supports of a built-in beam.

Therefore

$$M_C = \frac{6EI\delta}{L^2} \tag{14.12}$$

For a downwards deflection δ, the induced end moments are both anticlockwise; these moments must be superimposed on the fixed-end moments due to any external lateral loads on the beam.

Problem 14.1 A horizontal beam 6 m long is built-in at each end. The elastic section modulus is 0.933×10^{-3} m^3. Estimate the uniformly-distributed load over the whole span causing an elastic bending stress of 150 MN/m^2.

Solution

The maximum bending moments occur at the built-in ends, and have value

$$M_{max} = \frac{wL^2}{12}$$

If the bending stress is 150 MN/m^2,

$$M_{max} = \frac{\sigma I}{y} = \sigma Z_e = \left(150 \times 10^6\right)\left(0.933 \times 10^{-3}\right) = 140 \text{kNm}$$

Then

$$w = \frac{12}{L^2}\left(M_{max}\right) = 46.7 \text{ kN/m}$$

14.6 Continuous beam

When the same beam runs across three or more supports it is spoken of as a *continuous beam.* Suppose we have three spans, as in Figure 14.9, each bridged by a separate beam; the beams will bend independently in the manner shown. In order to make the axes of the three beams form a single continuous curve across the supports B and C, we shall have to apply to each beam couples acting as shown by the arrows. When the beam is one continuous girder these couples, on any bay such as BC, are supplied by the action of the adjacent bays. Thus AB and CD, bending downwards under their own loads, try to bend BC upwards, as shown by the broken curve, thus applying the couples M_B and M_C to the bay BC. This upward bending is of course opposed by the down load on BC, and the general result is that the beam takes up a sinuous form, being, in general, concave upwards over the middle portion of each bay and convex upwards over the supports.

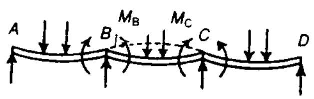


Figure 14.9 Bending moments at the supports of a continuous beam.

In order to draw the bending moment diagram for a continuous beam we must first find the couples such as M_B and M_C. In some cases there may also be external couples applied to the beam, at the supports, by the action of other members of the structure.

When the bending moments at the supports have been found, the bending moment and shearing force diagrams can be drawn for each bay according to the methods discussed in Chapter 7.

14.7 Slope-deflection equations for a single beam

In dealing with continuous beams we can make frequent use of the end slope and deflection properties of a single beam under any conditions of lateral loading. The uniform beam of Figure 14.10(i) carries any system of lateral loads; the ends are supported in an arbitrary fashion, the displacements and moments being as shown in the figure. In addition there are lateral forces at the supports. The rotations at the supports are θ_A and θ_B, respectively, reckoned positive if clockwise; M_A and M_B are also taken positive clockwise for our present purposes. The displacements δ_A and δ_B are taken positive downwards.

The loaded beam of Figure 14.10(i) may be regarded as the superposition of the loading conditions of Figures 14.10(ii) and (iii). In Figure 14.10(ii) the beam is built-in at each end; the moments at each end are easily calculable from the methods discussed in Sections 14.2 and 14.3. The fixed-end moments for this condition will be denoted by M_{FA} and M_{FB}. In Figure 14.10(iii) the beam carries no external loads between its ends, but end displacements and rotations are the same as those in Figure 14.10(i); the end couples for this condition are M_A' and M_B'. The superposition of Figures 14.10(ii) and (iii) gives the external loading and end conditions of Figure 14.10(i). We must find then the end couples in Figure 14.10(iii); from equations (13.49), putting $w = 0$, we have

$$\theta_A = \frac{M_A'L}{3EI} - \frac{M_B'L}{6EI} + \frac{1}{L}(\delta_B - \delta_A)$$

$$\theta_B = -\frac{M_A'L}{6EI} + \frac{M_B'L}{3EI} + \frac{1}{L}(\delta_B - \delta_A)$$

Then

$$\theta_A + \frac{1}{L}(\delta_A - \delta_B) = \frac{1}{6EI}(2M_A' - M_B')$$

$$\theta_B + \frac{1}{L}(\delta_A - \delta_B) = \frac{L}{6EI}(2M_B' - M_A')$$

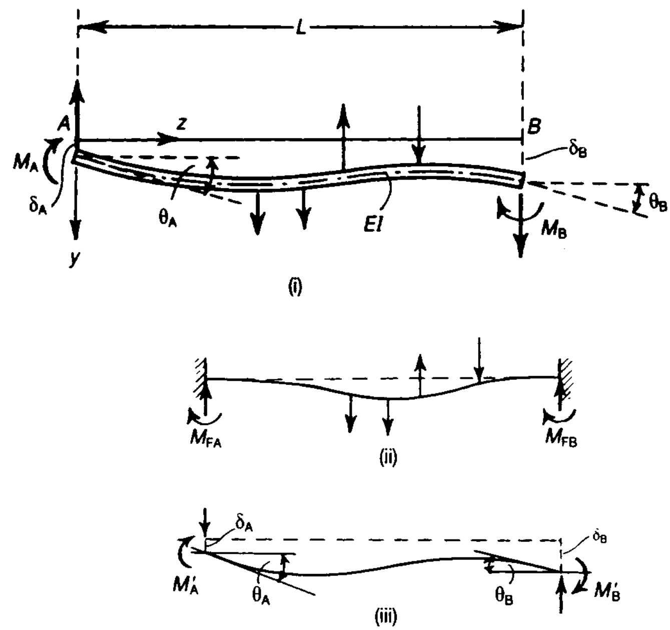


Figure 14.10 The single beam under any conditions of lateral load and end support shown in (i) can be regarded as the superposition of the built-in end beam of (ii) and the beam with end couples and end deformations of (iii).

But for the superposition we have

$$M_A' = M_A - M_{FA} \qquad M_B' = M_B - M_{FB}$$

Thus

$$\theta_A + \frac{1}{L}(\delta_A - \delta_B) = \frac{L}{6EI}\left[2(M_A - M_{FA}) - (M_B - M_{FB})\right] \tag{14.13}$$

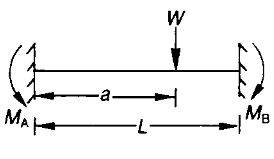
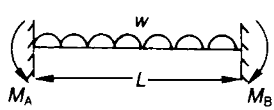
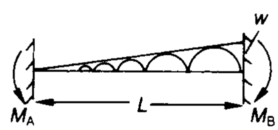
$$\theta_B + \frac{1}{L}(\delta_A - \delta_B) = \frac{L}{6EI}\left[2(M_B - M_{FB}) - (M_A - M_{FA})\right] \tag{14.14}$$

These are known as the *slope-deflection equations*; they give the values of the unknown moments,

M_A and M_B. These equations will be used in the matrix displacement method of Chapter 23.

Table 14.1 provides a summary of the end fixing moments and maximum deflections for some encastré beams.

Table 14.1 End fixing moments and maximum deflections for some encastré beams

Beam type and loading – length = L	M_A	M_B	δ Maximum deflection
	$-Wa(L-a)^2/L^2$	$-Wa^2(L-a)/L^2$	$\dfrac{-2W(L-a)^2a^3}{3EI(L+2a)^2}$ @ $z = 2aL/(L+2a)$ when $a > L/2$
	$-wL^2/12$	$-wL^2/12$	$\dfrac{wL^4}{384EI}$ @ $z = L/2$
	$-wL^2/30$	$-wL^2/20$	$\dfrac{0{\cdot}001309wL^4}{EI}$ @ $z = 0{\cdot}525L$

Further problems *(answers on page 693)*

14.2 A beam 8 m span is built-in at the ends, and carries a load of 60 kN at the centre, and loads of 30 kN, 2 m from each end. Calculate the maximum bending moment and the positions of the points of inflexion.

14.3 A girder of span 7 m is built-in at each end and carries two loads of 80 kN and 120 kN respectively placed at 2 m and 4 m from the left end. Find the bending moments at the ends and centre, and the points of contraflexure. *(Birmingham)*