

13 Deflections of beams

13.1 Introduction

In Chapter 7 we showed that the loading actions at any section of a simply-supported beam or cantilever can be resolved into a bending moment and a shearing force. Subsequently, in Chapters 9 and 10, we discussed ways of estimating the stresses due to these bending moments and shearing forces. There is, however, another aspect of the problem of bending which remains to be treated, namely, the calculation of the *stiffness* of a beam. In most practical cases, it is necessary that a beam should be not only strong enough for its purpose, but also that it should have the requisite stiffness, that is, it should not deflect from its original position by more than a certain amount. Again, there are certain types of beams, such as those carried by more than two supports and beams with their ends held in such a way that they must keep their original directions, for which we cannot calculate bending moments and shearing forces without studying the deformations of the axis of the beam; these problems are statically indeterminate, in fact.

In this chapter we consider methods of finding the deflected form of a beam under a given system of external loads and having known conditions of support.

13.2 Elastic bending of straight beams

It was shown in Section 9.2 that a straight beam of uniform cross-section, when subjected to end couples M applied about a principal axis, bends into a circular arc of radius R , given by

$$\frac{1}{R} = \frac{M}{EI} \quad (13.1)$$

where EI , which is the product of Young's modulus E and the second moment of area I about the relevant principal axis, is the flexural stiffness of the beam; equation (13.1) holds only for *elastic* bending.

Where a beam is subjected to shearing forces, as well as bending moments, the axis of the beam is no longer bent to a circular arc. To deal with this type of problem, we assume that equation (13.1) still defines the radius of curvature at any point of the beam where the bending moment is M . This implies that where the bending moment varies from one section of the beam to another, the radius of curvature also varies from section to section, in accordance with equation (13.1).

In the unstrained condition of the beam, Cz is the longitudinal centroidal axis, Figure 13.1, and Cx , Cy are the principal axes in the cross-section. The co-ordinate axes Cx , Cy are so arranged that the y -axis is vertically downwards. This is convenient as most practical loading conditions give rise to vertically downwards deflections. Suppose bending moments are applied about axes parallel to Cx , so that bending is restricted to the yz -plane, because Cx and Cy are principal axes.

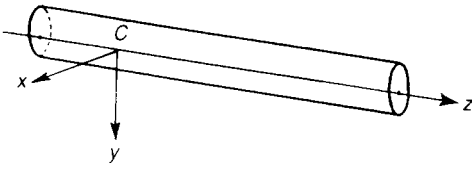


Figure 13.1 Longitudinal and principal centroidal axes for a straight beam.

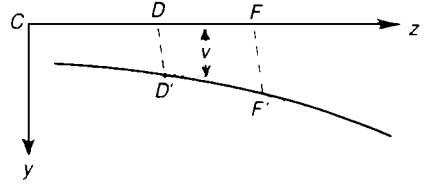


Figure 13.2 Displacements of the longitudinal axis of the beam.

Consider a short length of the unstrained beam, corresponding with DF on the axis Cz , Figure 13.2. In the strained condition D and F are displaced to D' and F' , respectively, which lies in the yz -plane. Any point such as D on the axis Cz is displaced by an amount v parallel to Cy ; it is also displaced a small, but negligible, amount parallel to Cz .

The radius of curvature R at any section of the beam is then given by

$$\frac{1}{R} = \frac{\frac{d^2v}{dz^2}}{\pm \left[1 + \left(\frac{dv}{dz} \right)^2 \right]^{3/2}} \tag{13.2}$$

We are concerned generally with only small deflections, in which v is small; this implies that (dv/dz) is small, and that $(dv/dz)^2$ is negligible compared with unity. Then, with sufficient accuracy, we may write

$$\frac{1}{R} = \pm \frac{d^2v}{dz^2} \tag{13.3}$$

The equations (13.1) and (13.3) give

$$\pm EI \frac{d^2v}{dz^2} = M \tag{13.4}$$

We must now consider whether the positive or negative sign is relevant in this equation; we have already adopted the convention in Section 7.4 that sagging bending moments are positive. When a length of the beam is subjected to sagging bending moments, as in Figure 13.3, the value of (dv/dz) along the length diminishes as z increases; hence a sagging moment implies that the curvature is negative. Then

$$EI \frac{d^2v}{dz^2} = - M \tag{13.5}$$

where M is the *sagging* bending moment.

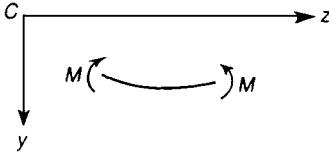


Figure 13.3 Curvature induced by sagging bending moment.

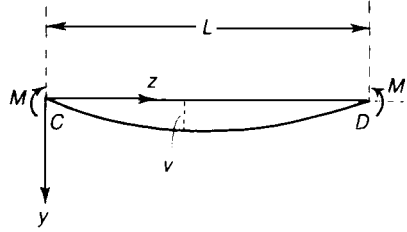


Figure 13.4 Deflected form of a beam in pure bending.

Where the beam is loaded on its axis of shear centres, so that no twisting occurs, M may be written in terms of shearing force F and intensity w of vertical loading at any section. From equation (7.9) we have

$$\frac{d^2M}{dz^2} = \frac{dF}{dz} = -w$$

On substituting for M from equation (13.5), we have

$$\frac{d^2}{dz^2} \left[-EI \frac{d^2v}{dz^2} \right] = \frac{dF}{dz} = -w \quad (13.6)$$

This relation is true if EI varies from one section of a beam to another. Where EI is constant along the length of a beam,

$$-EI \frac{d^4v}{dz^4} = \frac{dF}{dz} = -w \quad (13.7)$$

As an example of the use of equation (13.4), consider the case of a uniform beam carrying couples M at its ends, Figure 13.4. The bending moment at any section is M , so the beam is under a constant bending moment. Equation (13.5) gives

$$EI \frac{d^2v}{dz^2} = -M$$

On integrating once, we have

$$EI \frac{dv}{dz} = -Mz + A \quad (13.8)$$

where A is a constant. On integrating once more

$$EIv = -\frac{1}{2} Mz^2 + Az + B \quad (13.9)$$

where B is another constant. If we measure v relative to a line CD joining the ends of the beam, v is zero at each end. Then $v = 0$, for $z = 0$ and $z = L$.

On substituting these two conditions into equation (13.9), we have

$$B = 0 \quad \text{and} \quad A = \frac{1}{2}ML$$

The equation (13.9) may be written

$$EIv = \frac{1}{2}Mz(L - z) \quad (13.10)$$

At the mid-length, $z = \frac{1}{2}L$, and

$$v = \frac{ML^2}{8EI} \quad (13.11)$$

which is the greatest deflection. At the ends $z = 0$ and $z = L/2$,

$$\frac{dv}{dz} = \frac{ML}{2EI} \text{ at } C; \quad \frac{dv}{dz} = -\frac{ML}{2EI} \text{ at } D \quad (13.12)$$

It is important to appreciate that equation (13.3), expressing the radius of curvature R in terms of v , is only true if the displacement v is small.

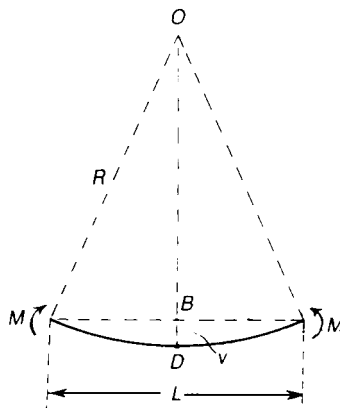


Figure 13.5 Distortion of a beam in pure bending.

We can study more accurately the pure bending of a beam by considering it to be deformed into the arc of a circle, Figure 13.5; as the bending moment M is constant at all sections of the beam, the radius of curvature R is the same for all sections. If L is the length between the ends, Figure 13.5, and D is the mid-point,

$$OB = \sqrt{R^2 - (L^2/4)}$$

Thus the central deflection v , is

$$v = BD = R - \sqrt{R^2 - (L^2/4)}$$

Then

$$v = R \left[1 - \sqrt{1 - \frac{L^2}{4R^2}} \right]$$

Suppose L/R is considerably less than unity; then

$$v = R \left[\frac{1}{2} \left(\frac{L^2}{4R^2} \right) + \frac{1}{8} \left(\frac{L^2}{4R^2} \right)^2 + \dots \right]$$

which can be written

$$v = \frac{L^2}{8R} \left[1 + \frac{L^2}{4R^2} + \dots \right]$$

But

$$\frac{1}{R} = \frac{M}{EI}$$

and so

$$v = \frac{ML^2}{8EI} \left[1 + \frac{M^2 L^2}{4(EI)^2} + \dots \right] \quad (13.13)$$

Clearly, if $(L^2/4R^2)$ is negligible compared with unity we have, approximately,

$$v = \frac{ML^2}{8EI}$$

which agrees with equation (13.11). The more accurate equation (13.13) shows that, when $(L^2/4R^2)$

is not negligible, the relationship between v and M is non-linear; for all practical purposes this refinement is unimportant, and we find simple linear relationships of the type of equation (13.11) are sufficiently accurate for engineering purposes.

13.3 Simply-supported beam carrying a uniformly distributed load

A beam of uniform flexural stiffness EI and span L is simply-supported at its ends, Figure 13.6; it carries a uniformly distributed lateral load of w per unit length, which induces bending in the yz plane only. Then the reactions at the ends are each equal to $\frac{1}{2}wL$; if z is measured from the end C , the bending moment at a distance z from C is

$$M = \frac{1}{2}wLz - \frac{1}{2}wz^2$$

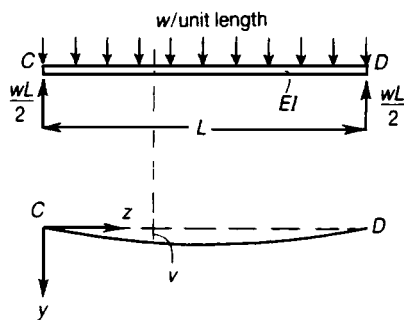


Figure 13.6 Simply-supported beam carrying a uniformly supported load.

Then from equation (13.5),

$$EI \frac{d^2v}{dz^2} = -M = -\frac{1}{2}wLz + \frac{1}{2}wz^2$$

On integrating,

$$EI \frac{dv}{dz} = -\frac{wLz^2}{4} + \frac{wz^3}{6} + A$$

and

$$EIv = -\frac{wLz^3}{12} + \frac{wz^4}{24} + Az + B \quad (13.14)$$

Suppose $v = 0$ at the ends $z = 0$ and $z = L$; then

$$B = 0, \quad \text{and} \quad A = wL^3/24$$

Then equation (13.14) becomes

$$EIv = \frac{wz}{24} [L^3 - 2Lz^2 + z^3] \quad (13.15)$$

The deflection at the mid-length, $z = \frac{1}{2}L$, is

$$v = \frac{5wL^4}{384EI} \quad (13.16)$$

13.4 Cantilever with a concentrated load

A uniform cantilever of flexural stiffness EI and length L carries a vertical concentrated load W at the free end, Figure 13.7. The bending moment a distance z from the built-in end is

$$M = -W(L - z)$$

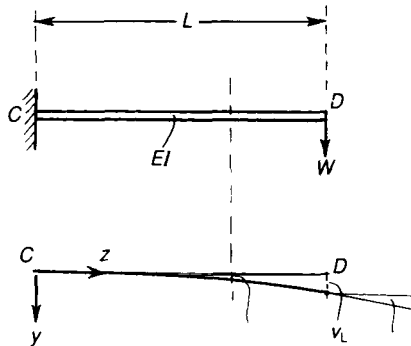


Figure 13.7 Cantilever carrying a vertical load at the remote end.

Hence equation (13.5) gives

$$EI \frac{d^2v}{dz^2} = W(L - z)$$

Then

$$EI \frac{dv}{dz} = W \left(Lz - \frac{1}{2}z^2 \right) + A \quad (13.17)$$

and

$$EIv = W \left(\frac{1}{2}Lz^2 - \frac{1}{6}z^3 \right) + Az + B$$

At the end $z = 0$, there is zero slope in the deflected form, so that $dv/dz = 0$; then equation (13.17) gives $A = 0$. Furthermore, at $z = 0$ there is also no deflection, so that $B = 0$. Then

$$EIv = \frac{Wz^2}{6} (3L - z)$$

At the free end, $z = L$,

$$v_L = \frac{WL^3}{3EI} \quad (13.18)$$

The slope of the beam at the free end is

$$\theta_L = \left(\frac{dv}{dz} \right)_{z=L} = \frac{WL^2}{2EI} \quad (13.19)$$

When the cantilever is loaded at some point between the ends, at a distance a , say, from the built-in support, Figure 13.8, the beam between G and D carries no bending moments and therefore remains straight. The deflection at G can be deduced from equation (13.18); for $z = a$,

$$v_a = \frac{Wa^3}{3EI} \quad (13.20)$$

and the slope at $z = a$ is

$$\theta_a = \frac{Wa^2}{2EI} \quad (13.21)$$

Then the deflection at the free end D of the cantilever is

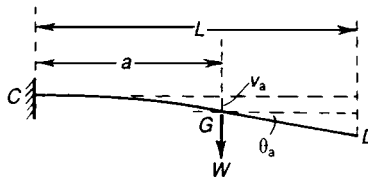


Figure 13.8 Cantilever with a load applied between the ends.

$$\begin{aligned}
 v_L &= \frac{Wa^3}{3EI} + (L - a) \frac{Wa^2}{2EI} \\
 &= \frac{Wa^2}{6EI} (3L - a)
 \end{aligned}
 \tag{13.22}$$

13.5 Cantilever with a uniformly distributed load

A uniform cantilever, Figure 13.9, carries a uniformly distributed load of w per unit length over the whole of its length. The bending moment at a distance z from C is

$$M = -\frac{1}{2}w(L - z)^2$$

Then, from equation (13.5),

$$EI \frac{d^2v}{dz^2} = \frac{1}{2}w(L - z)^2 = \frac{1}{2}w(L^2 - 2Lz + z^2)$$

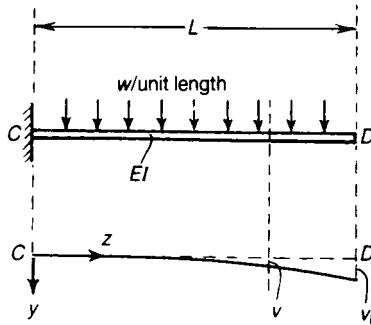


Figure 13.9 Cantilever carrying a uniformly distributed load.

Thus

$$EI \frac{dv}{dz} = \frac{1}{2}w \left(L^2z - Lz^2 + \frac{1}{3}z^3 \right) + A$$

and

$$EIv = \frac{1}{2}w \left(\frac{1}{2}L^2z^2 - \frac{1}{3}Lz^3 + \frac{1}{12}z^4 \right) + Az + B$$

At the built end, $z = 0$, and we have

$$\frac{dv}{dz} = 0 \quad \text{and} \quad v = 0$$

Thus $A = B = 0$. Then

$$EIv = \frac{1}{24}w(6L^2z^2 - 4Lz^3 + z^4)$$

At the free end, D , the vertical deflection is

$$v_L = \frac{wL^4}{8EI} \quad (13.23)$$

13.6 Propped cantilever with distributed load

The uniform cantilever of Figure 13.10(i) carries a uniformly distributed load w and is supported on a rigid knife edge at the end D . Suppose P is the force on the support at D . Then we regard Figure 13.10(i) as the superposition of the effects of P and w acting separately.

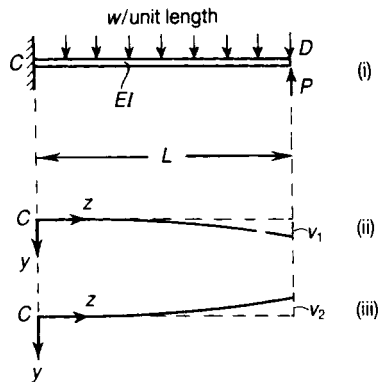


Figure 13.10 (i) Uniformly loaded cantilever propped at one end. (ii) Deflections due to w alone. (iii) Deflections due to P alone.

If w acts alone, the deflection at D is given by equation (13.23), and has the value

$$v_1 = \frac{wL^4}{8EI}$$

If the reaction P acted alone, there would be an upward deflection

$$v_2 = \frac{PL^3}{3EI}$$

at D . If the support maintains zero deflection at D ,

$$v_1 - v_2 = 0$$

This gives

$$\frac{PL^3}{3EI} = \frac{wL^4}{8EI}$$

or

$$P = \frac{3wL}{8} \quad (13.24)$$

Problem 13.1 A steel rod 5 cm diameter protrudes 2 m horizontally from a wall. (i) Calculate the deflection due to a load of 1 kN hung on the end of the rod. The weight of the rod may be neglected. (ii) If a vertical steel wire 3 m long, 0.25 cm diameter, supports the end of the cantilever, being taut but unstressed before the load is applied, calculate the end deflection on application of the load. Take $E = 200 \text{ GN/m}^2$. (RNEC)

Solution

(i) The second moment of area of the cross-section is

$$I_x = \frac{\pi}{64} (0.050)^4 = 0.307 \times 10^{-6} \text{ m}^4$$

The deflection at the end is then

$$v = \frac{PL^3}{3EI} = \frac{(1000)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})} = 0.0434 \text{ m}$$

(ii) Let T = tension in the wire; the area of cross-section of the wire is $4.90 \times 10^{-6} \text{ m}^2$. The elongation of the wire is then

$$e = \frac{Tl}{EA} = \frac{T(3)}{(200 \times 10^9)(4.90 \times 10^{-6})}$$

The load on the end of the cantilever is then $(1000 - T)$, and this produces a deflection of

$$v = \frac{(1000 - T)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})}$$

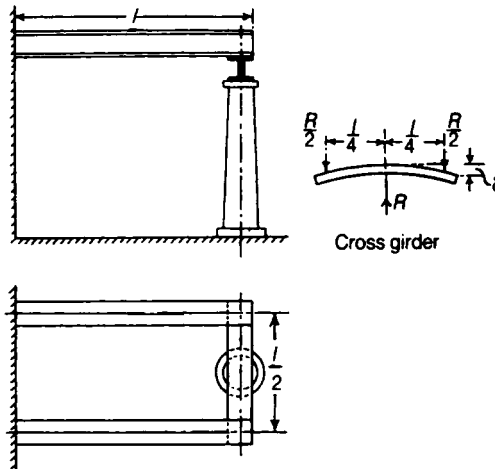
If this equals the stretching of the wire, then

$$\frac{(1000 - T)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})} = \frac{T(3)}{(200 \times 10^9)(4.90 \times 10^{-6})}$$

This gives $T = 934 \text{ N}$, and the deflection of the cantilever becomes

$$v = \frac{(66)(2)^3}{3(200 \times 10^9)(0.307 \times 10^{-6})} = 0.00276 \text{ m}$$

Problem 13.2 A platform carrying a uniformly distributed load rests on two cantilevers projecting a distance l m from a wall. The distance between the two cantilevers is $\frac{1}{2}l$. In what ratio might the load on the platform be increased if the ends were supported by a cross girder of the same section as the cantilevers, resting on a rigid column in the centre, as shown? It may be assumed that when there is no load on the platform the cantilevers just touch the cross girder without pressure. (Cambridge)



Solution

Let w_1 = the safe load per unit length on each cantilever when unsupported.

Then the maximum bending moment = $\frac{1}{2}w_1 l^2$.

Let w_2 = the safe load when supported,

δ = the deflection of the end of each cantilever,

$\frac{1}{2}R$ = the pressure between each cantilever and the cross girder.

Then the pressure is

$$\frac{R}{2} = \frac{3}{8} w_2 l - \frac{3EI\delta}{l^3}$$

We see from the figure above that

$$\delta = \frac{(R/2)(l/4)^3}{3EI} = \frac{Rl^3}{384EI}$$

l having the same value for the cantilevers and cross girder. Substituting this value of δ

$$\frac{R}{2} = \frac{3w_2l}{8} - \frac{R}{128}$$

or

$$R = \frac{48}{65}w_2l$$

The upward pressure on the end of each cantilever is $\frac{1}{2}R = 24w_2l/65$, giving a bending moment at the wall equal to $24w_2l^2/65$. The bending moment of opposite sign due to the distributed load is $\frac{1}{2}w_2l^2$. Hence it is clear that the maximum bending moment due to both acting together must occur at the wall and is equal to $(\frac{1}{2} - 24/65)w_2l^2 = (17/130)w_2l^2$. If this is to be equal to $\frac{1}{2}w_1l^2$, we must have $w_2 = (65/17)w_1$; in other words, the load on the platform can be increased in the ratio 65/17, or nearly 4/1. The bending moment at the centre of the cross girder is $6w_2l^2/65$, which is less than that at the wall.

13.7 Simply-supported beam carrying a concentrated lateral load

Consider a beam of uniform flexural stiffness EI and length L , which is simply-supported at its ends C and G , Figure 13.11. The beam carries a concentrated lateral load W at a distance a from C . Then the reactions at C and G are

$$V_c = \frac{W}{L}(L - a) \quad V_G = \frac{Wa}{L}$$

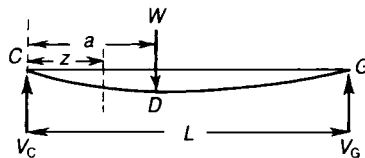


Figure 13.11 Deflections of a simply-supported beam carrying a concentrated lateral load.

Now consider a section of the beam a distance z from C ; if $z < a$, the bending moment at the section is

$$M = V_c z$$

and if $z > a$,

$$M = V_c z - W(z - a)$$

Then

$$EI \frac{d^2v}{dz^2} = -V_c z \quad \text{for } z < a$$

and

$$EI \frac{d^2v}{dz^2} = -V_c z + W(z - a) \quad \text{for } z > a$$

On integrating these equations, we have

$$EI \frac{dv}{dz} = -\frac{1}{2} V_c z^2 + A \quad \text{for } z < a \quad (13.25)$$

$$EI \frac{dv}{dz} = -\frac{1}{2} V_c z^2 + W \left(\frac{1}{2} z^2 - az \right) + A' \quad \text{for } z > a \quad (13.26)$$

and

$$EIv = -\frac{1}{6} V_c z^3 + Az + B \quad \text{for } z < a \quad (13.27)$$

$$EIv = -\frac{1}{6} V_c z^3 + W \left(\frac{1}{6} z^3 - \frac{1}{2} az^2 \right) + A'z + B' \quad \text{for } z > a \quad (13.28)$$

In these equations A , B , A' and B' are arbitrary constants. Now for $z = a$ the values of v given by equations (13.27) and (13.28) are equal, and the slopes given by equations (13.25) and (13.26) are also equal, as there is continuity of the deflected form of the beam through the point D . Then

$$-\frac{1}{6} V_c a^3 + Aa + B = -\frac{1}{6} V_c a^3 + W \left(\frac{1}{6} a^3 - \frac{1}{2} a^3 \right) + A'a + B'$$

and

$$-\frac{1}{2}V_c a^2 + A = -\frac{1}{2}V_c a^2 + W\left(\frac{1}{2}a^2 - a^2\right) + A'$$

These two equations give

$$A' = A + \frac{1}{2}Wa^2 \tag{13.29}$$

$$B' = B - \frac{1}{6}Wa^3$$

At the extreme ends of the beam $v = 0$, so that when $z = 0$ equation (13.27) gives $B = 0$, and when $z = L$, equation (13.28) gives

$$-\frac{1}{6}V_c L^3 + W\left(\frac{1}{6}L^3 - \frac{1}{2}aL^2\right) + A'L + B' = 0$$

We have finally,

$$A = \frac{1}{6}V_c L^2 - \frac{W}{6L}(L - a)^3$$

$$B = 0$$

$$A' = \frac{1}{6}V_c L^2 - \frac{W}{6L}(L - a)^3 + \frac{1}{2}Wa^2 \tag{13.30}$$

$$B' = -\frac{1}{6}Wa^3$$

But $V_c = W(L - a)/L$, so that equations (13.30) become

$$A = \frac{Wa}{6L}(L - a)(2L - a)$$

$$B = 0$$

$$A' = \frac{Wa}{6L}(2L^2 + a^2) \tag{13.31}$$

$$B' = -\frac{1}{6}Wa^3$$

Then equations (13.27) and (13.28) may be written

$$EIv = -\frac{W}{6L}(L-a)z^3 + \frac{Wa}{6L}(2L^2 - 3aL + a^2)z \quad \text{for } z < a \quad (13.32)$$

$$EIv = -\frac{W}{6L}(L-a)z^3 + \frac{W}{6}(z^3 - 3az^2) + \frac{Wa}{6L}(2L^2 + a^2)z - \frac{Wa^3}{6} \quad \text{for } z > a \quad (13.33)$$

The second relation, for $z > a$, may be written

$$EIv = -\frac{W}{6L}(L-a)z^3 + \frac{Wa}{6L}(2L^2 - 3aL + a^2)z + \frac{W}{6}(z-a)^3 \quad (13.34)$$

Then equations (13.32) and (13.33) differ only by the last term of equation (13.34); if the last term of equation (13.34) is discarded when $z < a$, then equation (13.34) may be used to define the deflected form in all parts of the beam.

On putting $z = a$, the deflection at the loaded point D is

$$v_D = \frac{Wa^2(L-a)^2}{3EI} \quad (13.35)$$

When W is at the centre of the beam, $a = \frac{1}{2}L$, and

$$v_D = \frac{WL^3}{48EI} \quad (13.36)$$

This is the maximum deflection of the beam only when $a = \frac{1}{2}L$.

13.8 Macaulay's method

The observation that equations (13.32) and (13.33) differ only by the last term of equation (13.34) leads to Macaulay's method, which ignores terms which are negative within the Macaulay brackets. That is, if the term $[z - a]$ in equation (13.34) is negative, it is ignored, so that equation (13.34) can be used for the whole beam. The method will be demonstrated by applying it to a few examples.

Consider the beam shown in Figure 13.12, which is simply-supported at its ends and loaded with a concentrated load W .

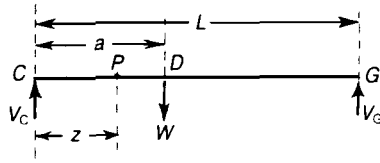


Figure 13.12 Form of step-function used in deflection analysis of a beam.

By taking moments, it can be seen that

$$V_c = W(L - a)/L \tag{13.37}$$

and the bending moment when $z < a$ is

$$M = V_c z \tag{13.38}$$

Then bending moment when $z > a$ is

$$M = V_c z - W(z - a) \tag{13.39}$$

Now

$$EI \frac{d^2v}{dz^2} = -M$$

hence, the Macaulay method allows us to express this relationship as follows

$$\begin{array}{c}
 \text{----- } z < = a \text{ -----} \quad \text{----- } a < z < L \text{ -----} \\
 EI \frac{d^2v}{dz^2} = -V_c z \qquad \qquad \qquad +W [z - a]
 \end{array} \tag{13.40}$$

On integrating equation (13.40), we get

$$EI \frac{dv}{dz} = -\frac{V_c z^2}{2} + A \qquad \qquad \qquad +\frac{W}{2} [z - a]^2 \tag{13.41}$$

$$\text{and} \qquad EIv = \frac{-V_c z^3}{6} + Az + B \qquad \qquad \qquad +\frac{W}{6} [z - a]^3 \tag{13.42}$$

The term on the right of equations (13.40) and (13.41) must be integrated by the manner shown, so that the arbitrary constants A and B apply when $z < a$ and also when $z > a$. The square brackets [] are called Macaulay brackets and *do not apply* when the term inside them is *negative*.

The two boundary conditions are:

$$\text{at } z = 0, \quad v = 0 \quad \text{and} \quad \text{at } z = L, \quad v = 0$$

Applying the first boundary condition to equation (13.42), we get

$$B = 0$$

Applying the second boundary condition to equation (13.42), we get

$$0 = -V_c L^3/6 + AL + W(L - a)^3/6$$

$$\text{or } AL = W(L - a)L^3/(6L) - W(L - a)^3/6$$

$$\text{or } A = W(L - a)L/6 - W(L - a)^3/(6L)$$

$$= \frac{W(L - a)}{6} \{L - (L - a)^2/L\}$$

$$\begin{aligned} \therefore EIv &= -W(L - a)z^3/(6L) \\ &+ W(L - a) \{L - (L - a)^2/L\}x/6 \\ &+ W[z - a]^3/6 \end{aligned}$$

On putting $z = a$, we get the deflection at D , namely v_D

$$\begin{aligned} \text{i.e. } v_D &= \frac{W(L - a)}{6EI} \{-a^3/L + (L - (L - a)^2/L)a + 0\} \\ &= \frac{W(L - a)}{6EI} \{-a^3/L + (L - (L^2 - 2aL + a^2)/L)a\} \\ &= \frac{W(L - a)}{6EI} (-a^3/L + La - La + 2a^2 - a^3/L) \\ &= \frac{W(L - a)}{6EI} \cdot (2a^2 - 2a^3/L) \end{aligned}$$

$$\text{or } v_D = \frac{W(L - a)^2 a^2}{3EIL}$$

If W is placed centrally, so that $a = L/2$,

$$v_D = \frac{W(L - L/2)^2 (L/2)^2}{3EI}$$

or
$$v_D = \frac{WL^3}{48EI} \quad (13.43)$$

13.9 Simply-supported beam with distributed load over a portion of the span

Suppose that the load is w per unit length over the portion DG , Figure 13.13; the reactions at the ends of the beam are

$$V_c = \frac{w}{2L} (L - a)^2$$

$$V_G = \frac{w}{2L} (L^2 - a^2)$$

The bending moment at a distance z from C is

$$M = V_c z - \frac{w}{2} [z - a]^2,$$

where the square brackets are Macaulay brackets, which only apply when the term inside them is positive.

i.e.
$$M = \frac{w}{2L} (L - a)^2 z - \frac{w}{2} [z - a]^2$$

----- $z < a$ ----- ----- $a < z < L$ -----

Hence
$$EI \frac{d^2v}{dz^2} = \frac{-w}{2L} (L - a)^2 z + \frac{w}{2} [z - a]^2 \quad (13.44)$$

so that
$$EI \frac{dv}{dz} = \frac{w}{4L} (L - a)^2 z^2 + A + \frac{w}{6} [z - a]^3 \quad (13.45)$$

and
$$EIv = \frac{-w}{12L} (L - a)^2 z^3 + Az + B + \frac{w}{24} [z - a]^4 \quad (13.46)$$

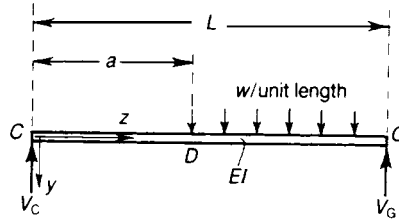


Figure 13.13 Load extending to one support.

The boundary conditions are that when

$$z = 0, \quad v = 0 \quad \text{and when} \quad z = L, \quad v = 0$$

Applying the first boundary condition to equation (13.46), we get

$$B = 0$$

Applying the second boundary condition to equation (13.46), we get

$$0 = -\frac{w}{12}(L-a)^2 L^2 + AL + \frac{w}{24}(L-a)^4$$

$$\begin{aligned} \therefore A &= \frac{w}{12}(L-a)^2 L - \frac{w}{24L}(L-a)^4 \\ &= \frac{w}{24L}(L-a)^2 \{2L^2 - (L-a)^2\} \\ &= \frac{w}{24L}(L-a)^2 \{2L^2 - L^2 - a^2 + 2aL\} \end{aligned}$$

$$\text{or} \quad A = \frac{w}{24L}(L-a)^2 (L^2 + 2La - a^2)$$

The equation for the deflection curve is then:

$$\begin{aligned} EIv &= \frac{-w}{2L}(L-a)^2 z^3 + \frac{w}{24L}(L-a)^2 (L^2 + 2La - a^2)z \\ &\quad + \frac{w}{24}[z-a]^4 \end{aligned} \quad (13.47)$$

where the square brackets in equation (13.47) are Macaulay brackets.

When the load does not extend to either support, Figure 13.14(i), the result of equation (13.47) may be used by superposing an upwards distributed load of w per unit length over the length GH

on a downwards distributed load of w per unit length over DH , Figure 13.14(ii). Due to the downwards distributed load alone

$$EIv = \frac{-w}{2L}(L-a)^2 z^3 + \frac{w}{24L}(L-a)^2(L^2 + 2La - a^2)z + \frac{w}{24}[z-a]^4 \quad (13.48)$$

where the square brackets in equation (13.48) are Macaulay brackets.

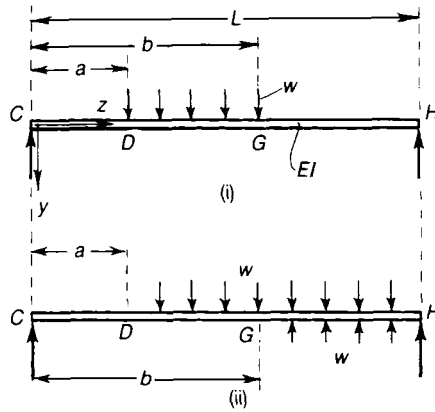


Figure 13.14 Load not extending to either support.

Due to the upwards distributed load

$$EIv = \frac{w}{2L}(L-b)^2 z^3 - \frac{w}{24L}(L-b)^2(L^2 + 2Lb - b^2)z - \frac{w}{24}[z-b]^4 \quad (13.49)$$

where the square brackets in equation (13.49) are Macaulay brackets.

On superposing the two deflected forms, the resultant deflection is given by

$$EIv = -\frac{wz^3}{2L}(b-a)(2L-a-b) + \frac{w}{24L} \left\{ (L-a)^2(L^2 + 2La - a^2) - (L-b)^2(L^2 + 2Lb - b^2) \right\} + \frac{w}{24}[z-a]^4 - \frac{w}{24}[z-b]^4 \quad (13.50)$$

where the square brackets of equation (13.50) are Macaulay brackets and must be ignored if the term inside them becomes negative.

13.10 Simply-supported beam with a couple applied at an intermediate point

The simply-supported beam of Figure 13.15 carries a couple M_a applied to the beam at a point a distance a from C . The vertical reactions at each end are (M_a/L) . The bending moment a distance z from C is

$$M = \frac{M_a z}{L} + M_a [z - a]^0 \tag{13.51}$$

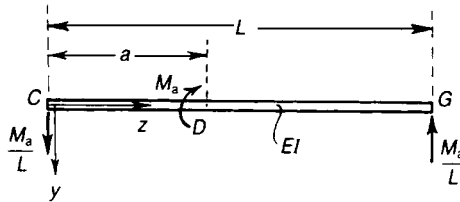


Figure 13.15 Beam with a couple applied at a point in the span.

The term on the right of equation (13.51) is so written, so that equation (13.51) applied over the whole length of the beam.

Hence,

----- $z < a$ ----- $a < z < L$ -----

$$EI \frac{d^2 v}{dz^2} = \frac{M_a z}{L} - M_a [z - a]^0$$

$$\therefore EI \frac{dv}{dz} = \frac{M_a z^2}{2L} + A - M_a [z - a] \tag{13.52}$$

and
$$EI v = \frac{M_a z^3}{6L} + Az + B - \frac{M_a}{2} [z - a]^2 \tag{13.53}$$

The boundary conditions are that

$$v = 0 \text{ at } z = 0 \text{ and at } z = L$$

From the first boundary condition, we get

$$B = 0$$

From the second boundary condition, we get

$$\begin{aligned}
 0 &= \frac{M_a L^2}{6} + AL \frac{-M_a}{2} (L - a)^2 \\
 \therefore A &= \frac{-M_a L}{6} + \frac{M_a}{2L} (L - a)^2 \\
 &= \frac{M_a}{6L} (-L^2 + 3L^2 + 3a^2 - 6aL) \\
 &= \frac{M_a}{6L} (2L^2 - 6La + 3a^2) \\
 \therefore EIv &= \frac{M_a z^3}{6L} + \frac{M_a z}{6L} (2L^2 - 6La + 3a^2) + \frac{-M_a}{2} [z - a]^2 \quad (13.54)
 \end{aligned}$$

where the square brackets in equation (13.54) are Macaulay brackets.

The deflection at D , when $z = a$, is

$$v_D = \frac{M_a a}{3EI} (L - a) (L - 2a) \quad (13.55)$$

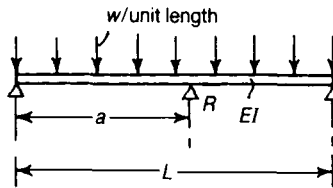
Problem 13.3 A steel beam rests on two supports 6 m apart, and carries a uniformly distributed load of 10 kN per metre run. The second moment of area of the cross-section is $1 \times 10^{-3} \text{ m}^4$ and $E = 200 \text{ GN/m}^2$. Estimate the maximum deflection.

Solution

The greatest deflection occurs at mid-length and has the value given by equation (13.16):

$$v = \frac{5wL^4}{384EI} = \frac{5(100 \times 10^3)(6)^4}{384(200 \times 10^9)(1 \times 10^{-3})} = 0.00844 \text{ m}$$

Problem 13.4 A uniform, simply-supported beam of span L carries a uniformly distributed lateral load of w per unit length. It is propped on a knife-edge support at a distance a from one end. Estimate the vertical force on the prop.



Solution

If the beam is unpropped, then, from equation (13.15), the downwards vertical deflection at the position of the prop is

$$(v)_{z=a} = \frac{wa}{24EI} (L^3 - 2La^2 + a^3)$$

If R is the reaction on the prop, then under the action of R alone the upwards vertical deflection at the prop is, from equation (13.35),

$$(v)_{z=a} = \frac{Ra^2(L-a)^2}{3EIL}$$

If there is no resultant deflection at the prop, we have

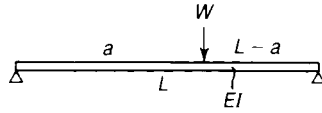
$$\frac{Ra^2(L-a)^2}{3EIL} = \frac{wa}{24EI} (L^3 - 2La^2 - a^3)$$

Thus, the reaction on the prop is

$$R = \frac{wL}{8} \left[\frac{1 - 2\left(\frac{a}{L}\right)^2 + \left(\frac{a}{L}\right)^3}{\frac{a}{L} \left(1 - \frac{a}{L}\right)^2} \right]$$

The propping force is least when the prop is at mid-span; in this case, $a/L = 0.5$ and $R = 5wL/8$.

Problem 13.5 A simply-supported, uniform beam, of span L and flexural stiffness EI , carries a vertical lateral load W at a distance a from one end. Calculate the greatest lateral deflection in the beam.

Solution

From section 13.7, the lateral deflection at any point is given by

$$EIv = -\frac{W}{6L}(L-a)z^3 + \frac{Wa}{6L}(2L^2 - 3aL + a^2)z \quad \text{for } z > a$$

$$EIv = -\frac{W}{6L}(L-a)z^3 + \frac{Wz^2}{6}(z - 3a) + \frac{Wa}{6L}(2L^2 + a^2)z - \frac{Wa^3}{6} \quad \text{for } z > a$$

Let us suppose first that $a > \frac{1}{2}L$, when we would expect the greatest deflection to occur in the range $z < a$; over this range

$$EI \frac{dv}{dz} = -\frac{W}{2L}(L-a)z^2 + \frac{Wa}{6L}(2L^2 - 3aL + a^2)$$

This is zero when

$$-\frac{W}{2L}(L-a)z^2 + \frac{Wa}{6L}(2L^2 - 3aL + a^2) = 0$$

i.e. when

$$(L-a)z^2 = \frac{1}{3}a(2L^2 - 3aL + a^2)$$

or when

$$z = \sqrt{\frac{a}{3}(2L - a)}$$

If this gives a root in the range $z < a$, then

$$\sqrt{\frac{a}{3}(2L - a)} < a$$

and $2L - a < 3a$, or $a > \frac{1}{2}L$. This is compatible with our earlier suppositions. Then, with $a > \frac{1}{2}L$, the greatest deflection occurs at the point

$z = \left[\frac{a}{3} (2L - a) \right]^{\frac{1}{2}}$ and has the value

$$v_{\max} = \frac{Wa}{9LEI} (2L - a) (L - a) \sqrt{\frac{a}{3} (2L - a)}$$

If $a < \frac{1}{2}L$, the greatest deflection occurs in the range $z > a$; in this case we replace a by $(L - a)$, whence the greatest deflection occurs at the point

$z = \sqrt{\frac{1}{3}(L^2 - a^2)}$, and has the value

$$v_{\max} = \frac{Wa}{9LEI} (L^2 - a^2) \sqrt{\frac{a}{3} (2L - a)}$$

13.11 Beam with end couples and distributed load

Suppose the ends of the beam CD , Figure 13.16, rest on knife-edges, and carry couples M_C and M_D . If, in addition, the beam carries a uniformly distributed lateral load w per unit length, the bending moment a distance z from C is

$$M = \frac{M_C}{L} (L - z) + M_D \frac{z}{L} + \frac{1}{2} wz (L - z)$$

The equation of the deflection curve is then given by

$$EI \frac{d^2v}{dz^2} = -\frac{M_C}{L} (L - z) - M_D \frac{z}{L} - \frac{1}{2} wz (L - z)$$

Then

$$EI \frac{dv}{dz} = -\frac{M_C}{L} \left(Lz - \frac{1}{2}z^2 \right) - \frac{M_D}{L} \left(\frac{z}{2} \right) - \frac{1}{2}w \left(\frac{Lz^2}{2} - \frac{z^3}{3} \right) + A$$

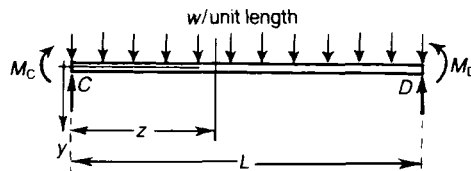


Figure 13.16 Simply-supported beam carrying a uniformly supported load.

and

$$EIv = \frac{M_C}{L} \left(\frac{Lz^2}{2} - \frac{z^3}{6} \right) - \frac{M_D}{L} \left(\frac{z^3}{6} \right) - \frac{1}{2}w \left(\frac{Lz^2}{6} - \frac{z^4}{12} \right) + Az + B \quad (13.56)$$

If the ends of the beam remain at the same level, $v = 0$ for $z = 0$ and $z = L$. Then $B = 0$ and

$$AL = \frac{1}{3}M_C L^2 + \frac{1}{6}M_D L^2 + \frac{1}{24}wL^4$$

Then

$$EIv = -\frac{M_C}{L} \left(\frac{Lz^2}{2} - \frac{z^3}{6} \right) - \frac{M_D}{L} \left(\frac{z^3}{6} \right) - \frac{1}{2}w \left(\frac{Lz^3}{6} - \frac{z^4}{12} \right) + z \left(\frac{M_C L}{3} + \frac{M_D L}{6} + \frac{wL^3}{24} \right)$$



The slopes at the ends are

$$\left(\frac{dv}{dz} \right)_{z=0} = \frac{L}{24EI} (8M_C + 4M_D + wL^2)$$

$$\left(\frac{dv}{dz} \right)_{z=L} = -\frac{L}{24EI} (4M_C + 8M_D + wL^2)$$

Suppose that the end D of the beam now sinks an amount δ downwards relative to C . Then at $v = L$ we have $v = \delta$, instead of $v = 0$. In equation (13.56), A is then given by

$$AL = EI\delta + \frac{1}{3}M_C L^2 + \frac{1}{6}M_D L^2 + \frac{1}{24}wL^4$$

For the slopes at the ends we have

$$\left(\frac{dv}{dz} \right)_{z=0} = \frac{L}{24EI} (8M_C + 4M_D + wL^2) + \frac{\delta}{L} \quad (13.57)$$

$$\left(\frac{dv}{dz} \right)_{z=L} = -\frac{L}{24EI} (4M_C + 8M_D + wL^2) + \frac{\delta}{L}$$

13.12 Beams with non-uniformly distributed load

When a beam carries a load which is not uniformly distributed the methods of the previous articles can still be employed if M and $\int M dz$ are both integrable functions of z , for we have in all cases

$$-EI \frac{d^2v}{dz^2} = M$$

which can be written in the form

$$\frac{d}{dz} \left(\frac{dv}{dz} \right) = -\frac{M}{EI}$$

If I is uniform along the beam the first integral of this is

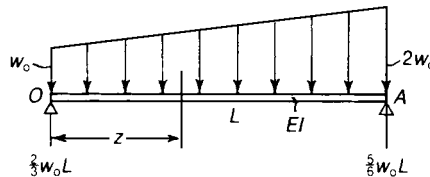
$$\frac{dv}{dz} = A - \frac{1}{EI} \int M dz \quad (13.58)$$

where A is a constant. The second integral is

$$v = Az + B - \frac{1}{EI} \iint M dz dz \quad (13.59)$$

If M and $\int M dz$ are integrable function of z the process of finding v can be continued analytically, the constants A and B being found from the terminal conditions. Failing this the integrations must be performed graphically or numerically. This is most readily done by plotting the bending-moment curve, and from that deducing a curve of areas representing $\int M dz$. From this curve a third is deduced representing $\iint M dz dz$.

Problem 13.6 A uniform, simply-supported beam carries a distributed lateral load varying in intensity from w_0 at one end to $2w_0$ at the other. Calculate the greatest lateral deflection in the beam.



Solution

The vertical reactions at O and A are $(2/3) w_0 L$ and $(5/6) w_0 L$. The bending moment at any section a distance z from O is then

$$M = \frac{2}{3}w_0Lz - \frac{1}{2}w_0z^2 - \frac{w_0z^3}{6L}$$

Then

$$EI \frac{d^2v}{dz^2} = - \left[\frac{2}{3}w_0Lz - \frac{1}{2}w_0z^2 - \frac{w_0z^3}{6L} \right]$$

On integrating once,

$$EI \frac{dv}{dz} = - \left[\frac{w_0Lz^2}{3} - \frac{w_0z^3}{6} - \frac{w_0z^4}{24L} + C_1 \right]$$

where C_1 is a constant. On integrating further,

$$EIv = - \left[\frac{w_0Lz^3}{9} - \frac{w_0z^4}{24} - \frac{w_0z^5}{120L} + C_1z + C_2 \right]$$

where C_2 is a further constant. If $v = 0$ at $z = L$, we have

$$C_1 = -\frac{11}{180} w_0L^3 \quad \text{and} \quad C_2 = 0$$

Then

$$EIv = \frac{11}{180} w_0L^3z - \frac{w_0Lz^3}{9} - \frac{w_0z^4}{24} + \frac{w_0z^5}{120L}$$

The greatest deflection occurs at $dv/dz = 0$, i.e. when

$$\frac{11}{180} w_0L^3 - \frac{w_0Lz^2}{3} + \frac{w_0z^3}{6} + \frac{w_0z^4}{24L} = 0$$

or when

$$15 \left(\frac{z}{L} \right)^4 + 60 \left(\frac{z}{L} \right)^3 - 120 \left(\frac{z}{L} \right)^2 + 22 = 0$$

The relevant root of this equation is $z/L = 0.506$ which gives the point of maximum deflection near to the mid-length. The maximum deflection is

$$v_{\max} \doteq \frac{7.03}{360} \frac{w_0 L^4}{EI} = 0.0195 \frac{w_0 L^4}{EI}$$

This is negligibly different from the deflection at mid-span, which is

$$(v)_z = L/2 = \frac{5w_0 L^4}{256EI}$$

13.13 Cantilever with irregular loading

In Figure 13.17(i) a cantilever is free at D and built-in to a rigid wall at C . The bending moment curve is DM of Figure 13.17(ii); the bending moments are assumed to be hogging, and are therefore negative. The curve CH represents $\int_0^z M dz$, and its ordinates are drawn downwards because M is negative. The curve CG is then constructed from CH by finding

$$\int_0^z \int_0^z M dz dz$$

In equation (13.51), the constants A and B are both zero as $v = 0$ and $dv/dz = 0$ at $z = 0$. Then CD is the base line for both curves.

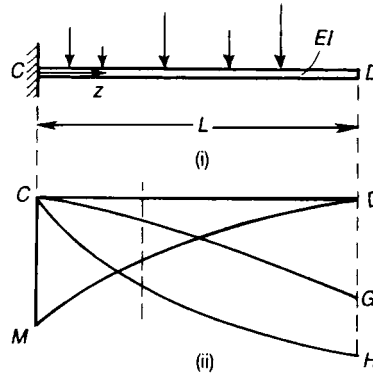


Figure 13.17 Cantilever carrying any system of lateral loads.

13.14 Beams of varying section

When the second moment of area of a beam varies from one section to another, equations (13.58) and (13.59) take the forms

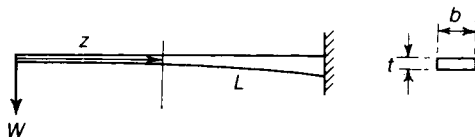
$$\frac{dv}{dz} = A - \frac{1}{E} \int \frac{M dz}{I}$$

and

$$v = Az + B - \frac{1}{E} \iint \frac{M}{I} dz dz$$

The general method of procedure follows the same lines as before. If (M/I) and $\int(M/I)dz$ are integrable functions of z , then (dv/dz) and v may be evaluated analytically; otherwise graphical or numerical methods must be employed, when a curve of (M/I) must be taken as the starting point instead of a curve of M .

Problem 13.7 A cantilever strip has a length L , a constant breadth b and thickness t varying in such a way that when the cantilever carries a lateral end load W , the centre line of the strip is bent into a circular arc. Find the form of variation of the thickness t .



Solution

The second moment of area, I , at any section is

$$I = \frac{1}{12} bt^3$$

The bending moment at any section is $(-Wz)$, so that

$$EI \frac{d^2v}{dz^2} = Wz$$

Then

$$\frac{d^2v}{dz^2} = \frac{Wz}{EI}$$

If the cantilever is bent into a circular arc, then d^2v/dz^2 is constant, and we must have

$$\frac{Wz}{EI} = \text{constant}$$

This requires that

$$\frac{z}{I} = \text{constant}$$

or $I \propto z$

Thus,

$$\frac{1}{12} bt^3 \propto z$$

or $t \propto z^{\frac{1}{3}}$

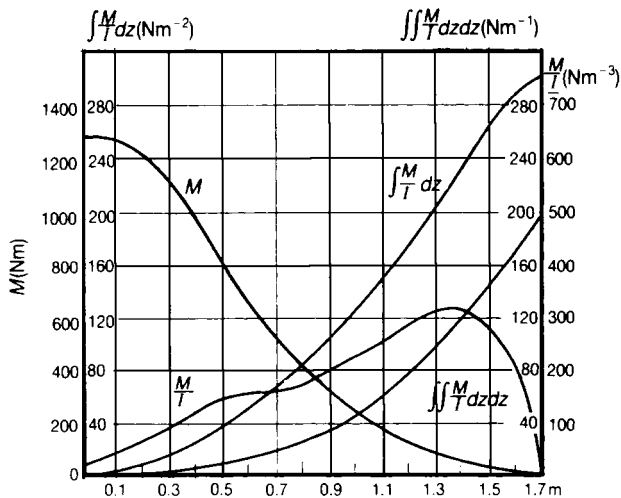
Any variation of the form

$$t = t_0 \left(\frac{z}{L} \right)^{\frac{1}{3}}$$

where t_0 is the thickness at the built-in end will lead to bending in the form of a circular arc.

Problem 13.8 The curve M , below, represents the bending moment at any section of a timber cantilever of variable bending stiffness. The second moments of area are given in the table below. Taking $E = 11 \text{ GN/m}^2$, deduce the deflection curve.

z (from supported end)(m)	0	0.1	0.2	0.3	0.5	0.7	0.9	1.1	1.3	1.5	1.6	1.7
I (m^4)	50.8	27.4	17.4	12.25	5.65	3.23	1.69	0.783	0.278	0.074	0.0298	0×10^{-4}



Solution

The first step is to calculate M/I at each section and to plot the M/I curve. We next plot the area under this curve at any section to give the curve

$$\int_0^z \frac{M}{I} dz$$

From this, the curve

$$\int_0^z \int \frac{M}{I} dz dz$$

is plotted to give the deflected form

$$v = \frac{1}{E} \int_0^z \int \frac{M}{I} dz dz$$

The maximum deflection at the free end of the cantilever is

$$v = \frac{1}{E} (300 \times 10^6) = \frac{300 \times 10^6}{11 \times 10^9} = 0.0272 \text{ m}$$

13.15 Non-uniformly distributed load and terminal couples; the method of moment-areas

Consider a simply-supported beam carrying end moments M_C and M_D , as in Figure 13.16, and a distributed load of varying intensity w . Suppose M_0 is the bending moment at any section due to the load w acting alone on the beam. Then

$$M = M_0 + \frac{M_C}{L} (L - z) + \frac{M_D}{L} z$$

The differential equation for the deflection curve is

$$EI \frac{d^2v}{dz^2} = -M_0 - \frac{M_C}{L} (L - z) - \frac{M_D}{L} z \quad (13.60)$$

The integral between the limits $z = 0$ and $z = L$ is

$$EI \left[\left(\frac{dv}{dz} \right)_{z=L} - \left(\frac{dv}{dz} \right)_{z=0} \right] = -\frac{1}{2} M_C L - \frac{1}{2} M_D L - \int_0^L M_0 dz \quad (13.61)$$

Again, on multiplying equation (13.60) by z , we have

$$EI \left(z \frac{d^2 v}{dz^2} \right) = -M_0 z - \frac{M_C z}{L} (L - z) - \frac{M_D}{L} z^2 \quad (13.62)$$

But

$$EI \left(z \frac{d^2 v}{dz^2} \right) = EI \frac{d}{dz} \left(z \frac{dv}{dz} - v \right)$$

Thus, on integrating equation (13.62),

$$EI \left[z \frac{dv}{dz} - v \right]_0^L = -\frac{M_D}{L} \left[\frac{z^3}{3} \right]_0^L - \frac{M_C}{L} \left[\frac{Lz^2}{2} - \frac{z^3}{3} \right]_0^L - \int_0^L M_0 z dz \quad (13.63)$$

But if $v = 0$ when $z = 0$ and $z = L$, then equation (13.63) becomes

$$EI \left[L \left(\frac{dv}{dz} \right)_{z=L} \right] = -\frac{1}{3} M_D L^2 - \frac{1}{6} M_C L^2 - \int_0^L M_0 z dz$$

Then

$$\left(\frac{dv}{dz} \right)_{z=L} = -\frac{M_D L}{3EI} - \frac{M_C L}{6EI} - \frac{1}{EIL} \int_0^L M_0 z dz \quad (13.64)$$

On substituting this value of $(dv/dz)_{z=L}$ into equation (13.61),

$$\left(\frac{dv}{dz} \right)_{z=0} = \frac{M_C L}{3EI} - \frac{M_D L}{6EI} - \frac{1}{EI} \int_0^L M_0 z dz - \frac{1}{EIL} \int_0^L M_0 z dz \quad (13.65)$$

the integral $\int_0^L M_0 dz$ is the area of the bending moment curve due to the load w alone; $\int_0^L M_0 z dz$ is the moment of this area about the end $z = 0$ of the beam. If A is the area of the bending moment diagram due to the lateral loads only, and \bar{z} is the distance of its centroid from $z = 0$, then

$$A = \int_0^L M_0 dz, \quad \bar{z} = \frac{1}{A} \int_0^L M_0 z dz$$

and equations (13.64) and (13.65) may be written

$$\left(\frac{dv}{dz}\right)_{z=0} = \frac{M_C L}{3EI} + \frac{M_D L}{6EI} + \frac{A(L - \bar{z})}{EIL} \tag{13.66}$$

$$\left(\frac{dv}{dz}\right)_{z=L} = -\frac{M_C L}{6EI} - \frac{M_D L}{3EI} - \frac{A\bar{z}}{EIL} \tag{13.67}$$

The method of analysis, making use of A and \bar{z} , is known as the *method of moment-areas*; it can be extended to deal with most problems of beam deflections.

When the section of the beam is not constant, equation (13.60) becomes

$$E \frac{d^2v}{dz^2} = -\frac{M_0}{I} - \frac{M_C}{I} + \frac{M_C - M_D}{L} \left(\frac{z}{L}\right)$$

The slopes at the ends of the beam are then given by

$$E \left[\left(\frac{dv}{dz}\right)_{z=L} - \left(\frac{dv}{dz}\right)_{z=0} \right] = -\int_0^L M_0 \frac{dz}{I} - M_C \int_0^L \frac{dz}{I} + \frac{1}{L} (M_C - M_D) \int_0^L \frac{zdz}{I}$$

and

$$E \left[\left(\frac{dv}{dz}\right)_{z=L} \right] = \frac{1}{L} \int_0^L M_0 \frac{zdz}{I} - \frac{M_C}{L} \int_0^L \frac{zdz}{I} + \frac{1}{L^2} (M_C - M_D) \int_0^L \frac{z^2 dz}{I}$$

It is necessary to plot five curves of (M_0/I) , $(1/I)$, (z/I) , (z^2/I) , $(M_0 z/I)$ and to find their areas.

As an example of the use of equations (13.66) and (13.67), consider the beam of Figure 13.18(i), which carries end couples, M_C and M_D , and a concentrated load W at a distance a from C .

The bending moment diagram for W acting alone is the triangle CBD , Figure 13.18(ii). The area of this triangle is

$$A = \frac{1}{2} L \left(\frac{Wa}{L} \right) (L - a) = \frac{Wa}{2} (L - a)$$

To evaluate its first moment about C , divide the triangle into two right-angled triangles, having centroids at G_1 and G_2 , respectively. Then

$$\begin{aligned} \bar{Az} &= \frac{1}{2}a \left[\frac{Wa}{L} (L - a) \right] \frac{2a}{3} + \frac{1}{2} [L - a] \left[\frac{Wa}{L} (L - a) \right] \left[\frac{1}{3} (L + 2a) \right] \\ &= \frac{1}{6} Wa (L^2 - a^2). \end{aligned}$$

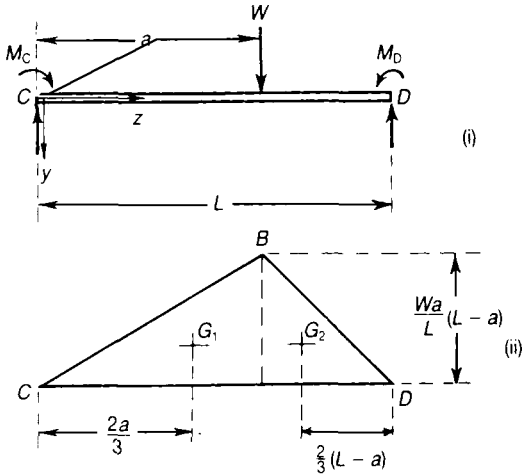


Figure 13.18 Moment-area solution of a beam carrying end couples and a concentrated load.

Then equations (13.66) and (13.67) give

$$\left(\frac{dv}{dz} \right)_{z=0} = \frac{M_C L}{3EI} + \frac{M_D L}{6EI} + \frac{Wa}{6EIL} (a^2 - 3aL + 2L^2)$$

$$\left(\frac{dv}{dz} \right)_{z=L} = -\frac{M_C L}{6EI} - \frac{M_D L}{3EI} - \frac{Wa}{6EIL} (L^2 - a^2)$$

Problem 13.9 Determine the deflection of the free end of the stepped cantilever shown in Figure 13.19(a).

Solution

The bending moment diagram is shown in Figure 13.19(b) and the M/I diagram is shown in Figure 13.19(c).

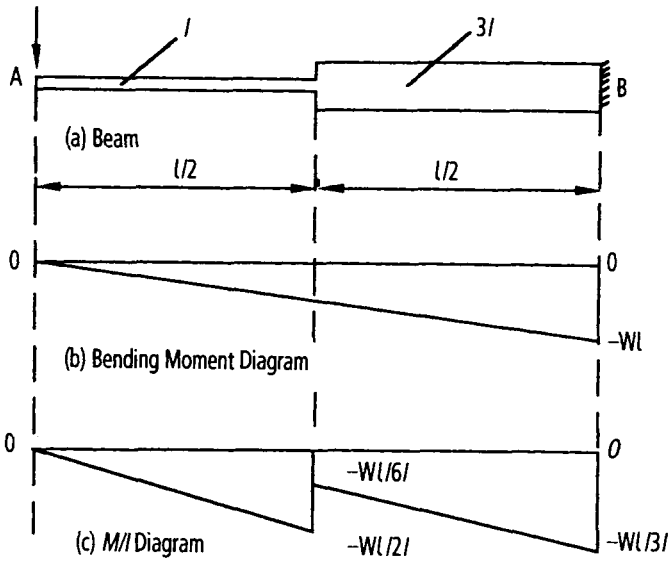


Figure 13.19 Stepped cantilever.

From equation (13.61)

$$EI \left(z \frac{dv}{dz} - v \right)_0^L = - \text{moment of area of the bending moment diagram}$$

$$\text{or } \left(z \frac{dv}{dz} - v \right)_0^L = -\frac{1}{E} \times \text{moment of area of the } M/I \text{ diagram}$$

Consider the moment of area of M/I about the point A , because we know that

$$\frac{dv}{dz} \text{ and } v = 0 \text{ at the point } B$$

$$\therefore \left[z \frac{dv}{dz} - v \right]_{z=L} - \left[0 \times \frac{dv}{dz} - v_A \right]_{z=0}$$

$$= \frac{1}{E} \left[\frac{WL}{2I} \times \frac{L}{4} \times \frac{2}{3} \times \frac{L}{2} + \frac{WL}{6I} \times \frac{L}{2} \times \frac{3L}{4} + \frac{WL}{6I} \times \frac{L}{4} \times \left(\frac{L}{2} + \frac{2}{3} \times \frac{L}{2} \right) \right]$$

or

$$0 + v_A = \frac{WL^3}{EI} \left[\frac{1}{24} + \frac{1}{16} + \frac{1}{24} \times \left(\frac{1}{2} + \frac{1}{3} \right) \right]$$

$$= \frac{WL^3}{EI} \left(\frac{1}{24} + \frac{1}{16} + \frac{5}{144} \right)$$

$$v_A = \frac{5WL^3}{36EI}$$

Problem 13.10 Determine the deflection of the free end of the varying depth cantilever shown in Figure 13.20(a)

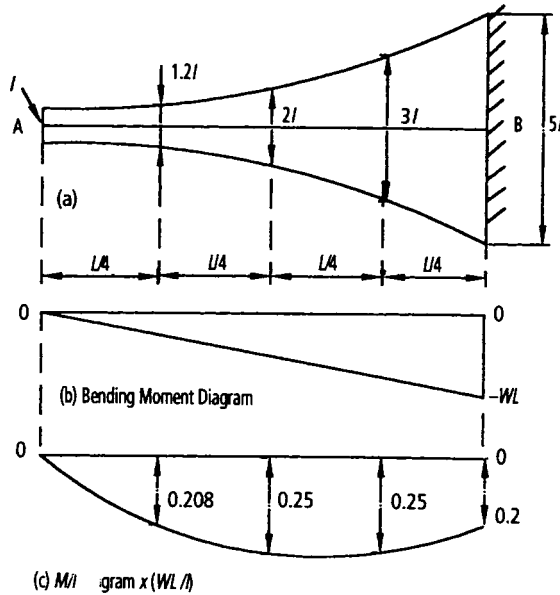


Figure 13.20 Varying depth cantilever.

Solution

Taking the moment of area of the M/I diagram about A , we eliminate v_B and dv/dz at B , because they are both zero. Additionally, as the M/I diagram is numerical, we can use numerical integration, namely Simpsons rule, as shown in Table 13.1.

Table 13.1 Numerical integration of the moment of M/I about A

Ordinate	M/I	z	zM/I	SM	$f(zM/I)$
1	0	0	0	1	0
2	0.208 WL/I	$L/4$	0.052 WL^2/I	4	0.208 WL^2/I
3	0.25 WL/I	$L/2$	0.125 WL^2/I	2	0.25 WL^2/I
4	0.25 WL/I	$3L/4$	0.188 WL^2/I	4	0.752 WL^2/I
5	0.2 WL/I	L	0.2 WL^2/I	1	0.2 WL^2/I
Σ					1.41 WL^2/I

From Table 13.1,

$$v_A = \frac{1}{3} \times \frac{L}{4} \times 1.41 WL^2/I$$

$$v_A = 0.1175 WL^2/I$$

13.16 Deflections of beams due to shear

In our simple theory of bending of beams, we assumed that plane sections remain plane during bending. The effect of shearing forces in a beam is to distort plane cross-sections into curved planes. In the cantilever of Figure 13.21, the cross-section DH warps as the force F is applied, due to the shearing strains in the fibres of the beam. We assume that the shearing stresses set up by F are distributed in the manner already discussed in Chapter 10. This is not true strictly, because shearing distortions no longer allow sections to remain plane; however, we assume these shearing effects are secondary, and we are justified therefore in estimating them on our original theory.

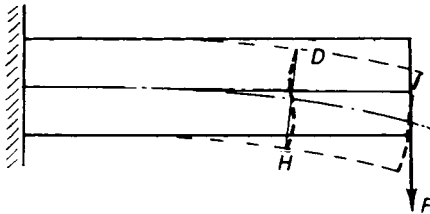


Figure 13.21 Shearing distortions in a cantilever.

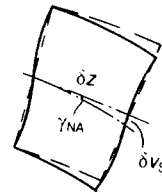


Figure 13.22 Shearing deflection at the neutral axis of a beam.

Suppose the shearing stress at the neutral axis of the beam is τ_{NA} , then the shearing strain at the neutral axis is

$$\gamma_{NA} = \frac{\tau_{NA}}{G} \quad (13.68)$$

where G is the shearing modulus. The additional deflection arising from shearing of the cross-section is then

$$\delta v_s = \gamma_{NA} \delta z = \frac{\tau_{NA}}{G} \delta z$$

Then

$$\frac{dv_s}{dz} = \frac{\tau_{NA}}{G} \quad (13.69)$$

For a cantilever of thin rectangular cross-section, Section 10.2,

$$\tau_{NA} = \frac{3F}{2ht} \quad (13.70)$$

where h is the depth of the cross-section, and t is the thickness. Then

$$\frac{dv_s}{dz} = \frac{3F}{2Ght}$$

Then

$$v_s = \frac{3Fz}{2Ght} + A \quad (13.71)$$

At $z = 0$, there is no shearing deflection, so $A = 0$. At the end $z = L$,

$$(v_s)_L = \frac{3FL}{2Ght} \quad (13.72)$$

The bending deflection at the free end, $z = L$, is

$$(v)_L = \frac{FL^3}{3EI} = \frac{4FL^3}{Eh^3t} \quad (13.73)$$

Then the total end deflection is

$$\begin{aligned} v_L &= \frac{4FL^3}{Eh^3t} + \frac{3FL}{2Ght} \\ &= \frac{4FL^3}{Eh^3t} \left[1 + \frac{3E}{8G} \left(\frac{h}{L} \right)^2 \right] \end{aligned} \quad (13.74)$$

For most materials $(3E/8G)$ is of order unity, so the contribution of the shear to the total deflection is equal approximately to $(h/L)^2$. Clearly, the shearing deflection is important only for deep beams.

Table 13.2 provides a summary of the maximum bending moments and lateral deflections for some statically determinate beams.

Problem 13.11 A 1.5 m length of the beam of Problem 11.2 is simply-supported at each end, and carries concentrated lateral load of 10 kN at the mid-span. Compare the central deflections due to bending and shearing.

Solution

From Problem 11.2, the second moment of area of the equivalent steel I-beam is $12.1 \times 10^{-6} \text{ m}^4$. The central deflection due to bending is, therefore,

$$v_B = \frac{WI^3}{48E_s I_s} = \frac{(10 \times 10^3) (1.5)^3}{48 (200 \times 10^9) (12.1 \times 10^{-6})} = 0.290 \times 10^{-3} \text{ m}$$

The average shearing stress in the timber is

$$\frac{5 \times 10^3}{(0.15) (0.075)} = 0.445 \text{ MN/m}^2$$

If the shearing modulus for timber is

$$4 \times 10^9 \text{ N/m}^2$$

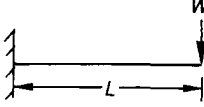
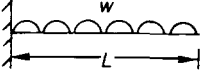
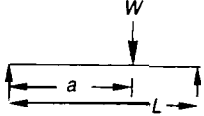
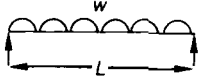
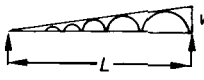
the shearing strain in the timber is

$$\gamma = \frac{0.445 \times 10^6}{4 \times 10^9} = 0.111 \times 10^{-3}$$

The resulting central deflection due to shearing is

$$v_s = \gamma \times 0.75 = (0.111 \times 10^{-3}) (0.75) = 0.0833 \times 10^{-3} \text{ m}$$

Table 13.2 Bending moment and deflections for some simple beams

Beam type and loading – length = L	M_{\max}	Maximum deflection
	$-WL$	$WL^3/3EI$ @ $z = L$
	$-wL^2/2$	$wL^4/8EI$ @ $z = L$
	$W(L-a) a/L$ @ $z = a$	$Wa^2(L-a)^2/(3EI)$ @ $z = L - \left[\frac{L^2 - a^2}{3} \right]^{1/2}$ when $a < L/2$
	$wL^2/8$ @ $z = L/2$	$5wL^4/384EI$ @ $z = L/2$
	$0.0641 wL^2$ @ $z = 0.5773 L$	$0.00653 wL^4/EI$ @ $z = 0.5195 L$

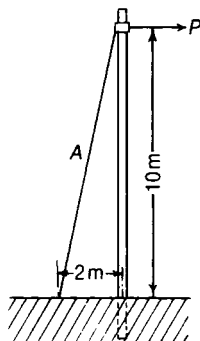
Thus, the shearing deflection is nearly 30% of the bending deflection. The estimated total central deflection is

$$v = v_B + v_s = 0.373 \times 10^{-3} \text{ m}$$

Further problems (answers on page 693)

- 13.12** A straight girder of uniform section and length L rests on supports at the ends, and is propped up by a third support in the middle. The weight of the girder and its load is w per unit length. If the central support does not yield, prove that it takes a load equal to $(5/8)wL$.
- 13.13** A horizontal steel girder of uniform section, 15 m long, is supported at its extremities and carries loads of 120 kN and 80 kN concentrated at points 3 m and 5 m from the two ends, respectively. I for the section of the girder is $1.67 \times 10^{-3} \text{ m}^4$ and $E = 200 \text{ GN/m}^2$. Calculate the deflections of the girder at points under the two loads. (Cambridge)
- 13.14** A wooden mast, with a uniform diameter of 30 cm, is built into a concrete block, and is subjected to a horizontal pull at point 10 m from the ground. The wire guy A is to be adjusted so that it becomes taut and begins to take part of the load when the mast is loaded to a maximum stress of 7 MN/m^2 .

Estimate the slack in the guy when the mast is unloaded. Take E for timber = 10 GN/m^2 . (Cambridge)



- 13.15** A bridge across a river has a span $2l$, and is constructed with beams resting on the banks and supported at the middle on a pontoon. When the bridge is unloaded the three supports are all at the same level, and the pontoon is such that the vertical displacement is equal to the load on it multiplied by a constant λ . Show that the load on the pontoon, due to a concentrated load W , placed one-quarter of the way along the bridge, is given by

$$\frac{11W}{16 \left(1 + \frac{6EI\lambda}{l^3} \right)}$$

where I is the second moment of area of the section of the beams. (Cambridge)

- 13.16** Two equal steel beams are built-in at one end and connected by a steel rod as shown. Show that the pull in the tie rod is

$$P = \frac{5Wl^3}{32 \left(\frac{6aI}{\pi d^2} + l^3 \right)}$$

where d is the diameter of the rod, and I is the second moment of area of the section of each beam about its neutral axis. (*Cambridge*)

