11 Beams of two materials

11.1 Introduction

Some beams used in engineering structures are composed of two materials. A timber joist, for example, may be reinforced by bolting steel plates to the flanges. Plain concrete has little or no tensile strength, and beams of this material are reinforced therefore with steel rods or wires in the tension fibres. In beams of these types there is a composite action between the two materials.

11.2 Transformed sections

The composite beam shown in Figure 11.1 consists of a rectangular timber joist of breadth b and depth h, reinforced with two steel plates of depth h and thickness t.

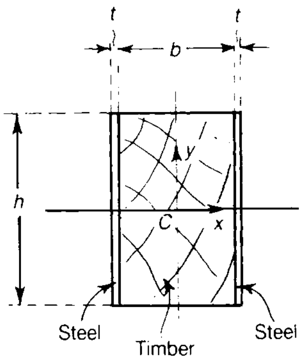


Figure 11.1 Timber beam reinforced with steel side plates.

Consider the behaviour of the composite beam under the action of a bending moment M applied about Cx; if the timber beam is bent into a curve of radius R, then, from equation (9.5), the bending moment carried by the timber beam is

$$M_t = \frac{(EI)_t}{R} \tag{11.1}$$

where $(EI)_t$ is the bending stiffness of the timber beam. If the steel plates are attached to the timber beam by bolting, or glueing, or some other means, the steel plates are bent to the same radius of curvature R as the timber beam. The bending moment carried by the two steel plates is then

$$M_s = \frac{(EI)_s}{R}$$

where $(EI)_s$ is the bending stiffness of the two steel plates. The total bending moment is then

$$M = M_t + M_s = \frac{1}{R}\left[(EI)_t + (EI)_s\right]$$

This gives

$$\frac{1}{R} = \frac{M}{(EI)_t + (EI)_s} \tag{11.2}$$

Clearly, the beam behaves as though the total bending stiffness EI were

$$EI = (EI)_t + (EI)_s \tag{11.3}$$

If E_t and E_s are the values of Young's modulus for timber and steel, respectively, and if I_t and I_s are the second moments of area about Cx of the timber and steel beams, respectively, we have

$$EI = (EI)_t + (EI)_s = E_t I_t + E_s I_s \tag{11.4}$$

Then

$$EI = E_t \left[I_t + \left(\frac{E_s}{E_t} \right) I_s \right] \tag{11.5}$$

If I_s is multiplied by (E_s/E_t), which is the ratio of Young's moduli for the two materials, then from equation (11.5) we see that the composite beam may be treated as wholly timber, having an equivalent second moment of area

$$I_t + \left(\frac{E_s}{E_t} \right) I_s \tag{11.6}$$

This is equivalent to treating the beam of Figure 11.2(i) with reinforcing plates made of timber, but having thicknesses

$$\left(\frac{E_s}{E_t} \right) \times t$$

as shown in Figure 11.2(ii); the equivalent timber beam of Figure 11.2(ii) is the *transformed section* of the beam. In this case the beam has been transformed wholly to timber. Equally the beam may be transformed wholly to steel, as shown in Figure 11.2(iii). For bending about Cx the *breadths* of the component beams are factored to find the transformed section; the depth h of the beam is unaffected.

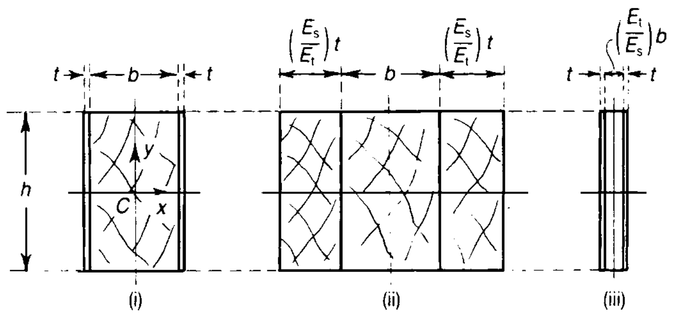


Figure 11.2 (i) Composite beam of timber and steel bent about Cx. (ii) Equivalent timber beam. (iii) Equivalent steel beam.

The bending stress σ_t in the fibre of the timber core of the beam a distance y from the neutral axis is

$$\sigma_t = M_t \frac{y}{I_t}$$

Now, from equations (11.1) and (11.2)

$$M_t = \frac{(EI)_t}{R}, \quad M = \frac{1}{R}\left[(EI)_t + (EI)_s\right]$$

and on eliminating R,

$$M_t = \frac{M}{1 + \frac{E_s I_s}{E_t I_t}} \tag{11.7}$$

Then

$$\sigma_t = \frac{My}{I_t\left(1 + \frac{E_s I_s}{E_t I_t}\right)} = \frac{My}{I_t + \left(\frac{E_s}{E_t}\right) I_s} \tag{11.8}$$

the bending stresses in the timber core are found therefore by considering the *total* bending moment M to be carried by the transformed timber beam of Figure 11.2(ii). The longitudinal strain at the distance y from the neutral axis Cx is

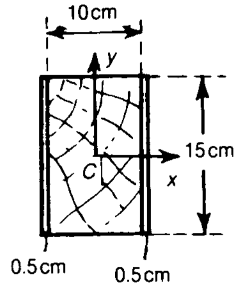
$$\varepsilon = \frac{\sigma_t}{E_t} = \frac{My}{E_t I_t + E_s I_s}$$

Then at the distance y from the neutral axis the stress in the steel reinforcing plates is

$$\sigma_s = E_s \varepsilon = \frac{My}{I_s + \left(\frac{E_t}{E_s}\right) I_t} \qquad (11.9)$$

because the strains in the steel and timber are the same at the same distance y from the neutral axis. This condition of equal strain is implied in the assumption made earlier that the steel and the timber components of the beam are bent to the same radius of curvature R.

Problem 11.1 A composite beam consists of a timber joist, 15 cm by 10 cm, to which reinforcing steel plates, ½ cm thick, are attached. Estimate the maximum bending moment which may be applied about Cx, if the bending stress in the timber is not to exceed 5 MN/m^2, and that in the steel 120 MN/m^2. Take $E_s/E_t = 20$.



Solution

The maximum bending stresses occur in the extreme fibres. If the stress in the timber is 5 MN/m^2, the stress in the steel at the same distance from Cx is

$$5 \times 10^6 \times \frac{E_s}{E_t} = 100 \times 10^6 \text{ N/m}^2 = 100 \text{ MN/m}^2$$

Thus when the maximum timber stress is attained, the maximum steel stress is only 100 MN/m^2. If the maximum permissible stress of 120 MN/m^2 were attained in the steel, the stress in the timber would exceed 5 MN/m^2, which is not permissible. The maximum bending moment gives therefore a stress in the timber of 5 MN/m^2. The second moment of area about Cx of the equivalent timber beam is

$$I_x = \frac{1}{12}(0.10)(0.15)^3 + \frac{1}{12}(0.010)(0.15)^3 \times 20$$

$$= 0.0842 \times 10^{-3}\ \text{m}^4$$

For a maximum stress in the timber of 5 MN/m^2, the moment is

$$M = \frac{(5 \times 10^6)(0.0842 \times 10^{-3})}{0.075} = 5610\ \text{Nm}$$

11.3 Timber beam with reinforcing steel flange plates

In Section 11.2 we discussed the composite bending action of a timber be.m reinforced with steel plates over the depth of the beam. A similar bending problem arises when the timber joist is reinforced on its upper and lower faces with steel plates, as shown if Figure 11.3(i); the timber web of the composite beams may be transformed into steel to give the equivalent steel section of Figure 11.3(ii); alternatively, the steel flanges may be replaced by equivalent timber flanges to give the equivalent timber beam of Figure 11.3(iii). The problem is then treated in the same way as the beam in Section 11.2; the stresses in the timber and steel are calculated from the second moment of area of the transformed timber and steel sections.

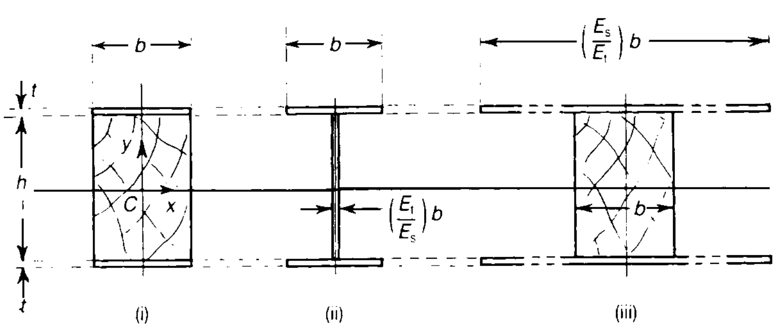


Figure 11.3 (i) Timber beam with reinforced steel flange plates. (ii) Equivalent steel I-beam. (iii) Equivalent timber I-beam.

An important difference, however, between the composite actions of the beams of Figures 11.2 and 11.3 lies in their behaviour under shearing forces. The two beams, used as cantilevers carrying end loads F, are shown in Figure 11.4; for the timber joist reinforced over the depth, Figure 11.4(i), there are no shearing actions between the timber and the steel plates, except near the loaded ends of the cantilever.

However, for the joist of Figure 11.4(ii), a shearing force is transmitted between the timber and the steel flanges at all sections of the beam. In the particular case of thin reinforcing flanges, it is sufficiently accurate to assume that the shearing actions in the cantilever of Figure 11.4(ii) are resisted largely by the timber joist; on considering the equilibrium of a unit length of the composite beam, equilibrium is ensured if a shearing force (F/h) per unit length of beam is transmitted between the timber joist and the reinforcing flanges, Figure 11.5. This shearing force must be carried by bolts, glue or some other suitable means. The end deflections of the cantilevers shown in Figure 11.4 may be difficult to estimate; this is due to the fact that account may have to be taken of the shearing distortions of the timber beams.

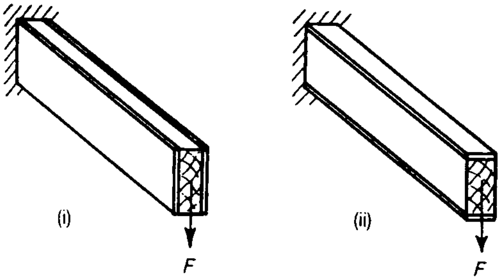


Figure 11.4 composite beams under shearing action, showing (i) steel and timber both resisting shear and (ii) timber alone resisting shear.

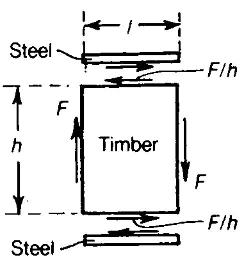
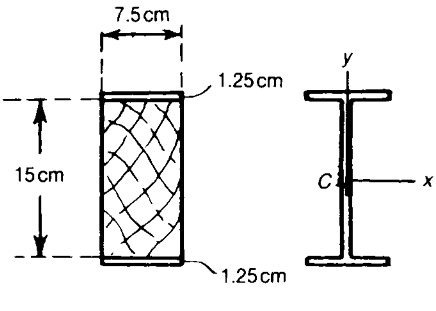


Figure 11.5 Shearing actions in a timber joist with reinforcing steel flanges.

Problem 11.2 A timber joist 15 cm by 7.5 cm has reinforcing steel flange plates 1.25 cm thick. The composite beam is 3 m long, simply-supported at each end, and carries a uniformly distributed lateral load of 10 kN. Estimate the maximum bending stresses in the steel and timber, and the intensity of shearing force transmitted between the steel plates and the timber. Take $E_s/E_t = 20$.



Solution

The second moment of area of the equivalent steel section is

$$I_x = \frac{1}{20}\left[\frac{1}{12}(0.075)(0.15)^3\right] + 2\left[(0.0125)(0.075)^3\right] = 11.6 \times 10^{-6} \text{ m}^4$$

The maximum bending moment is

$$\frac{(10 \times 10^3)(3)}{8} = 3750 \text{ Nm}$$

The maximum bending stress in the steel is then

$$\sigma_s = \frac{(3750)(0.0875)}{(11.6 \times 10^{-6})} = 28.3 \text{ MN/m}^2$$

The bending stress in the steel at the junction of web and flange is

$$\sigma_s = \frac{(3750)(0.0750)}{(11.6 \times 10^{-6})} = 24.2 \text{ MN/m}^2$$

The stress in the timber at this junction is then

$$\sigma_t = \frac{E_t}{E_s} \times \sigma_s = \frac{1}{20}(24.2) = 1.2 \text{ MN/m}^2$$

On the assumption that the shearing forces at any section of the beam area taken largely by the timber, the shearing force between the timber and steel plates is

$$(5 \times 10^3) / (0.15) = 33.3 \text{ kN/m}$$

because the maximum shearing force in the beam is 5 kN.

11.4 Ordinary reinforced concrete

It was noted in Chapter 1 that concrete is a brittle material which is weak in tension. Consequently a beam composed only of concrete has little or no bending strength since cracking occurs in the extreme tension fibres in the early stages of loading. To overcome this weakness steel rods are embedded in the tension fibres of a concrete beam; if concrete is cast around a steel rod, on setting the concrete shrinks and grips the steel rod. It happens that the coefficients of linear expansion of concrete and steel are very nearly equal; consequently, negligible stresses are set up by temperature changes.

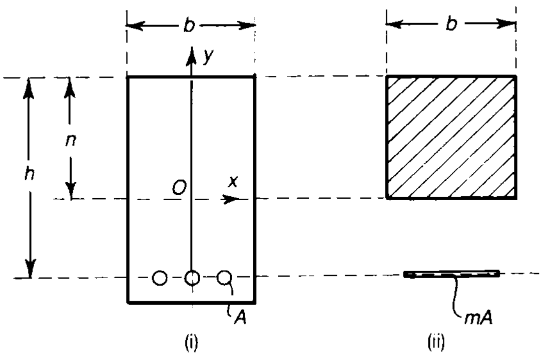


Figure 11.6 Simple rectangular concrete beam with reinforcing steel in the tension flange.

The bending of an ordinary reinforced concrete beam may be treated on the basis of transformed sections. Consider the beam of rectangular cross-section shown in Figure 11.6. The breadth of the concrete is b, and h is the depth of the steel reinforcement below the upper extreme fibres. The beam is bent so that tensile stresses occur in the lower fibres. The total area of cross-section of the steel reinforcing rods is A; the rods are placed longitudinally in the beam. The beam is now bent so that Ox becomes a neutral axis, compressive stresses being induced in the concrete above Ox. We assume that concrete below the neutral axis cracks in tension, and is therefore ineffectual; we neglect the contribution of the concrete below Ox to the bending strength of the beam. Suppose m is the ratio of Young's modulus of steel, E_s, to Young's modulus of concrete, E_c; then

$$m = \frac{E_s}{E_c} \tag{11.10}$$

If the area A of steel is transformed to concrete, its equivalent area is mA; the equivalent concrete beam then has the form shown in Figure 11.6(ii). The depth of the neutral axis Ox below the extreme upper fibres is n. The equivalent concrete area mA on the tension side of the beam is concentrated approximately at a depth h.

We have that the neutral axis of the beam occurs at the centroid of the equivalent concrete beam; then

$$bn \times \frac{1}{2}n = mA\,(h - n)$$

Thus n is the root of the quadratic equation

$$\frac{1}{2}bn^2 + mAn - mAh = 0 \tag{11.11}$$

The relevant root is

$$n = \frac{mA}{b}\left(\sqrt{1 + \frac{2bh}{mA}} - 1\right) \tag{11.12}$$

The second moment of area of the equivalent concrete beam about its centroidal axis is

$$I_c = \frac{1}{3}bn^3 + mA\left(h - n\right)^2 \tag{11.13}$$

The maximum compressive stress induced in the upper extreme fibres of the concrete is

$$\sigma_c = \frac{Mn}{I_c} \tag{11.14}$$

$$\sigma_s = \frac{M(h - n)}{I_c} \times \frac{E_s}{E_c} = \frac{mM(h - n)}{I_c} \tag{11.15}$$

Problem 11.3 A rectangular concrete beam is 30 cm wide and 45 cm deep to the steel reinforcement. The direct stresses are limited to 115 MN/m^2 in the steel and 6.5 MN/m^2 in the concrete, and the modular ratio is 15. What is the area of steel reinforcement if both steel and concrete are fully stressed? Estimate the permissible bending moment for this condition.

Solution

From equations (11.14) and (11.15)

$$\sigma_s = \frac{M(h - n)}{\dfrac{bn^3}{3m} + A\left(h - n\right)^2} = 115 \text{ MN/m}^2$$

and

$$\sigma_c = \frac{Mn}{\dfrac{1}{3}bn^3 + mA\left(h - n\right)^2} = 6.5 \text{ MN/m}^2$$

Then

$$\frac{M\,(h\ -\ n)}{115} = \frac{Mn}{6.5}\ \text{m}$$

Hence

$$h - n = 1.18n \quad \text{and} \quad \frac{n}{h} = \frac{1}{2.18} = 0.458$$

Then

$$n = 0.458 \times 0.45 = 0.206\ \text{m}$$

From equation (11.11)

$$\frac{2mA}{bh} = \frac{(n/h)^2}{1 - (n/h)} = \frac{(0.458)^2}{0.542} = 0.387$$

Then

$$A = 0.387\,\frac{bh}{2m} = \frac{0.387 \times 0.30 \times 0.45}{30} = 1.75 \times 10^{-3}\ \text{m}^2$$

As the maximum allowable stresses of both the steel and concrete are attained, the allowable bending moment may be elevated on the basis of either the steel or the concrete stress. The second moment of area of the equivalent concrete beam is

$$I_c = \frac{1}{3}\,bn^3 + mA\,(h - n)^2$$

$$= \frac{1}{3}\,(0.30)\,(0.206)^3 + 15(0.00174)\,(0.244)^2 = 2.42 \times 10^{-3}\ \text{m}^2$$

The permissible bending moment is

$$M = \frac{\sigma_c\, I_c}{y_c} = \frac{(6.5 \times 10^6)\,(2.42 \times 10^{-3})}{(0.206)} = 76.4\ \text{kNm}$$

Problem 11.4 A rectangular concrete beam has a breadth of 30 cm and is 45 cm deep to the steel reinforcement, which consists of two 2.5 cm diameter bars. Estimate the permissible bending moment if the stresses are limited to 115 MN/m^2 and 6.5 MN/m^2 in the steel and concrete, respectively, and if the modular ratio is 15.

Solution

The area of steel reinforcement is $A = 2(\pi/4)(0.025)^2 = 0.982 \times 10^{-3}\ \text{m}^2$. From equation (11.12)

$$\frac{n}{h} = \frac{mA}{bh}\left[\sqrt{1 + \frac{2bh}{mA}} - 1\right]$$

Now

$$\frac{mA}{bh} = \frac{(15)\,(0.982)\times 10^{-3}}{(30)\,(45)\times 10^{-4}} = 0.1091$$

Then

$$\frac{n}{h} = 0.1091\left[\left(1 + \frac{2}{0.1091}\right)^{\frac{1}{2}} - 1\right] = 0.370$$

Thus

$$n = 0.370h = 0.167\ \text{m}$$

The second moment of area of the equivalent concrete beam is

$$I_c = \frac{1}{3}\,bn^3 + mA\,(h - n)^2$$

$$= \frac{1}{3}\,(0.30)\,(0.167)^3 + 15\,\left(0.982 \times 10^{-3}\right)(0.283)^2$$

$$= (0.466 + 1.180)\,10^{-3}\ \text{m}^4$$

$$= 1.646 \times 10^{-3}\ \text{m}^4$$

If the maximum allowable concrete stress is attained, the permissible moment is

$$M = \frac{\sigma_c\, I_c}{n} = \frac{\left(6.5 \times 10^6\right)\left(1.646 \times 10^{-3}\right)}{0.167} = 64\ \text{kNm}$$

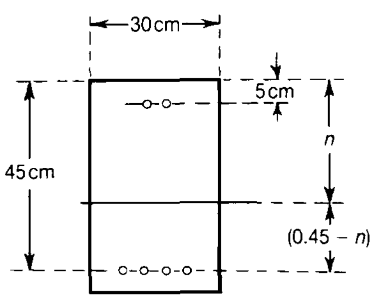
If the maximum allowable steel stress is attained, the permissible moment is

$$M = \frac{\sigma_s\, I_c}{m(h - n)} = \frac{\left(115 \times 10^6\right)\left(1.646 \times 10^{-3}\right)}{15(0.283)} = 44.6\ \text{kNm}$$

Steel is therefore the limiting material, and the permissible bending moment is

$$M = 44.6 \text{ kNm}$$

Problem 11.5 A rectangular concrete beam, 30 cm wide, is reinforced on the tension side with four 2.5 cm diameter steel rods at a depth of 45 cm, and on the compression side with two 2.5 diameter rods at a depth of 5 cm. Estimate the permissible bending moment if the stresses in the concrete are not to exceed 6.5 MN/m^2 and in the steel 115 MN/m^2. The modular ratio is 15.



Solution

The area of steel reinforcement is 1.964×10^{-3} m^2 on the tension side, and 0.982×10^{-3} m^2 on the compression side. The cross-sectional area of the equivalent concrete beam is

$$(0.30)n + (m - 1)(0.000982) + m(0.001964) = (0.30n + 0.0433)\text{m}^2$$

The position of the neutral axis is obtained by taking moments, as follows:

$$(0.30)n\left(\frac{1}{2}n\right) + (m - 1)(0.000982)(0.05) + m(0.001964)(0.45)$$

$$= (0.30n + 0.0433)n$$

This reduces to

$$n^2 - 0.288n - 0.093 = 0$$

giving

$$n = -0.144 \pm 0.337$$

The relevant root is $n = 0.193$ m

The second moment of area of the equivalent concrete beam is

$$I_c = \frac{1}{3}(0.30)n^3 + (m-1)(0.000982)(n-0.05)^2 + m(0.001964)(0.45-n)^2$$

$$= (0.720 + 0.281 + 1.950)10^{-3}$$

$$= 2.95 \times 10^{-3} \text{ m}^4$$

If the maximum allowable concrete stress is attained, the permissible moment is

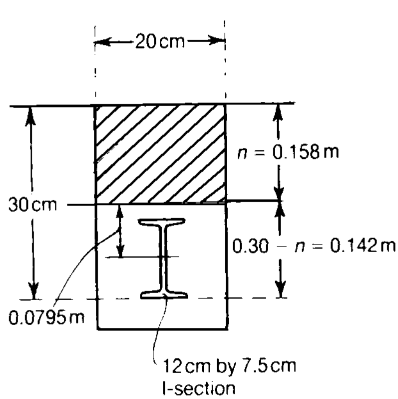
$$M = \frac{\sigma_c I_c}{n} = \frac{(6.5 \times 10^6)(2.95 \times 10^{-3})}{0.193} = 99.3 \text{ kNm}$$

If the maximum allowable steel stress is attained, the permissible moment is

$$M = \frac{\sigma_s I_c}{m(0.45 - n)} = \frac{(115 \times 10^6)(2.95 \times 10^{-3})}{15(0.257)} = 88.0 \text{ kNm}$$

Thus, steel is the limiting material, and the allowable moment is 88.0 kNm.

Problem 11.6 A steel I-section, 12.5 cm by 7.5 cm, is encased in a rectangular concrete beam of breadth 20 cm and depth 30 cm to the lower flange of the I-section. Estimate the position of the neutral axis of the composite beam, and find the permissible bending moment if the steel stress is not to exceed 115 MN/m^2 and the concrete stress 6.5 MN/m^2. The modular ratio is 15. The area of the steel beam is 0.00211 m^2 and its second moment of area about its minor axis is 5.70×10^{-6} m^4.



Solution

The area of the equivalent steel beam is

$$\frac{(0.20)n}{15} + 0.00211 \text{ m}^2$$

The position of the neutral axis is obtained by taking moments, as follows:

$$\left(\frac{0.20n}{15} + 0.00211\right)n = \left(\frac{0.20n}{15}\right)\left(\frac{1}{2}n\right) + (0.00211)(0.2375)$$

This reduces to

$$n^2 + 0.316n - 0.075 = 0$$

The relevant root of which is

$$n = 0.158 \text{ m}$$

The second moment of area of the equivalent steel beam is

$$I_s = \frac{1}{3}\left(\frac{0.20}{15}\right)(0.158)^3 + (0.00211)(0.0795)^2 = 0.0366 \times 10^{-3} \text{ m}^4$$

The allowable bending moment on the basis of the steel stress is

$$M = \frac{\sigma_s I_s}{(0.30 - n)} = \frac{(115 \times 10^6)(0.0366 \times 10^{-3})}{0.142} = 29.7 \text{ kNm}$$

If the maximum allowable concrete stress is 6.5 MN/m^2, the maximum allowable compressive stress in the equivalent steel beam is

$$m(6.5 \times 10^6) = 97.5 \text{ MN/m}^2$$

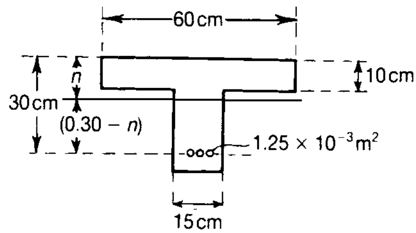
On this basis, the maximum allowable moment is

$$M = \frac{(97.5 \times 10^6)(0.0366 \times 10^{-3})}{0.158} = 22.6 \text{ kNm}$$

Concrete is therefore the limiting material, and the maximum allowable moment is

$$M = 22.6 \text{ kNm}$$

Problem 11.7 A reinforced concrete T-beam contains 1.25×10^{-3} m^2 of steel reinforcement on the tension side. If the steel stress is limited to 115 MN/m^2 and the concrete stress to 6.5 MN/m^2, estimate the permissible bending moment. The modular ratio is 15.



<u>*Solution*</u>

Suppose the neutral axis falls below the underside of the flange. The area of the equivalent concrete beam is

$$(0.60)n - 0.45(n - 0.10) + (0.00125)15 = 0.15n + 0.0638 \text{ m}^2$$

The position of the neutral axis is obtained by taking moments, as follows:

$$(0.60n)\left(\frac{1}{2}n\right) + (0.00125)(15)(0.30) - 0.45(n - 0.10)\left(\frac{1}{2}\right)(n + 0.10)$$

$$= (0.15n + 0.0638)n$$

This reduces to

$$n^2 + 0.850n - 0.1044 = 0$$

the relevant root of which is $n = 0.109$ m which agrees with our assumption earlier that the neutral axis lies below the flange.

The second moment of area of the equivalent concrete beam is

$$I_c = \frac{1}{3}(0.60)\left(n^3\right) - \frac{1}{3}(0.45)(n - 0.10)^3 + 0.00125\,(15)\,(0.30 - n)^2$$

$$= (0.259 + 0.000 + 0.685)10^{-3} \text{ m}^4$$

$$= 0.944 \times 10^{-3} \text{ m}^4$$

If the maximum allowable concrete stress is attained, the permissible moment is

$$M = \frac{\sigma_c I_c}{n} = \frac{\left(6.5 \times 10^6\right)\left(0.944 \times 10^{-3}\right)}{0.109} = 56.3 \text{ kNm}$$

If the maximum allowable steel stress is attained, the permissible moment is

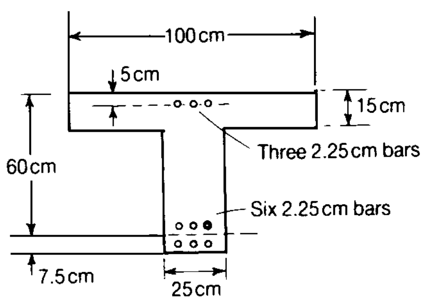
$$M = \frac{\sigma_s I_c}{m(0.30 - n)} = \frac{(115 \times 10^6)(0.944 \times 10^{-3})}{15(0.191)} = 37.9 \text{ kNm}$$

Steel is therefore the limiting material, and the permissible bending moment is 37.9 kNm.

Further Problems *(answers on page 693)*

11.8 A concrete beam of rectangular section is 10 cm wide and is reinforced with steel bars whose axes are 30 cm below the top of the beam. Estimate the required total area of the cross-section of the steel if the maximum compressive stress in the concrete is to be 7.5 MN/m^2 and the tensile stress in the steel is 135 MN/m^2 beam is subjected to pure bending. What bending moment would the beam withstand when in this condition? Assume that Young's modulus for steel is 15 times that for concrete and that concrete can sustain no tensile stresses. *(Cambridge)*

11.9 A reinforced concrete T-beam carries a uniformly distributed super-load on a simply-supported span of 8 m. The stresses in the steel and concrete are not to exceed 125 MN/m^2 and 7 MN/m^2, respectively. The modular ratio is 15, and the density of concrete is 2400 kg/m^3. Determine the permissible super-load. *(Nottingham)*



11.10 A wooden joist 15 cm deep by 7.5 cm wide is reinforced by glueing to its lower face a steel strip 7.5 cm wide by 0.3 cm thick. The joist is simply-supported over a span of 3 m, and carries a uniformly distributed load of 5000 N. Find the maximum direct stresses in the wood and steel and the maximum shearing stress in the glue. Take $E_s/E_t = 20$. *(Cambridge)*

11.11 A timber beam is 15 cm deep by 10 cm wide, and carries a central load of 30 kN at the centre of a 3 m span; the beam is simply-supported at each end. The timber is reinforced with flat steel plates 10 cm wide by 1.25 cm thick bolted to the upper and lower surfaces of the beam. Taking E for steel as 200 GN/m^2 and E for timber as 1 GN/m^2, estimate

(i) the maximum direct stress in the steel strips;
(ii) the average shearing stress in the timber;
(iii) the shearing load transmitted by the bolts;
(iv) the bending and shearing deflections at the centre of the beam.