

# 8 Geometrical properties of cross-sections

## 8.1 Introduction

The strength of a component of a structure is dependent on the geometrical properties of its cross-section in addition to its material and other properties. For example, a beam with a large cross-section will, in general, be able to resist a bending moment more readily than a beam with a smaller cross-section. Typical cross-sections of structural members are shown in Figure 8.1.

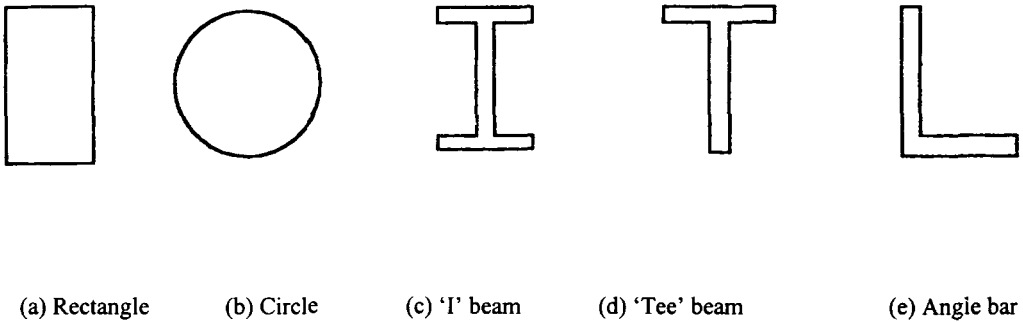


Figure 8.1 Some typical cross-sections of structural components.

The cross-section of Figure 8.1(c) is also called a *rolled steel joist (RSJ)*; it is used extensively in structural engineering. It is quite common to make cross-sections of metal structural members in the form of the cross-sections of Figure 8.1(c) to (e), as such cross-sections are structurally more efficient in bending than cross-sections such as Figures 8.1(a) and (b). Wooden beams are usually of rectangular cross-section and not of the forms shown in Figures 8.1(c) to (e). This is because wooden beams have grain and will have lines of weakness along their grain if constructed as in Figures 8.1(c) to (e).

## 8.2 Centroid

The position of the centroid of a cross-section is the centre of the moment of area of the cross-section. If the cross-section is constructed from a homogeneous material, its centroid will lie at the same position as its centre of gravity.

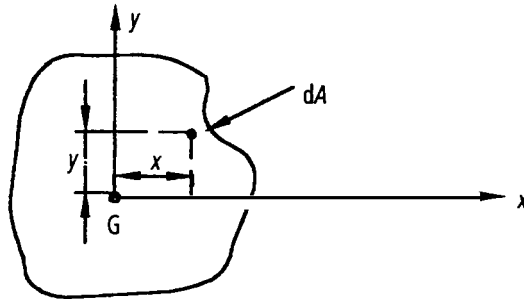


Figure 8.2 Cross-section.

Let  $G$  denote the position of the centroid of the plane lamina of Figure 8.2. At the centroid the moment of area is zero, so that the following equations apply

$$\Sigma x \, dA = \Sigma y \, dA = 0 \quad (8.1)$$

where  $dA$  = elemental area of the lamina

$x$  = horizontal distance of  $dA$  from  $G$

$y$  = vertical distance of  $dA$  from  $G$

### 8.3 Centroidal axes

These are the axes that pass through the centroid.

### 8.4 Second moment of area (I)

The second moments of area of the lamina about the  $x - x$  and  $y - y$  axes, respectively, are given by

$$I_{xx} = \Sigma y^2 \, dA = \text{second moment of area about } x - x \quad (8.2)$$

$$I_{yy} = \Sigma x^2 \, dA = \text{second moment of area about } y - y \quad (8.3)$$

Now from Pythagoras' theorem

$$x^2 + y^2 = r^2$$

$$\therefore \Sigma x^2 \, dA + \Sigma y^2 \, dA = \Sigma r^2 \, dA$$

$$\text{or } I_{yy} + I_{xx} = J \quad (8.4)$$

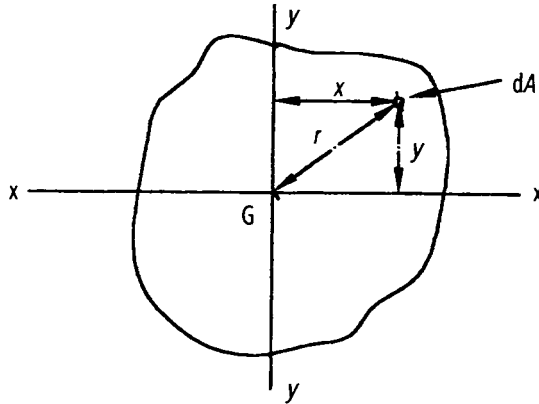


Figure 8.3 Cross-section.

where

$$\begin{aligned}
 J &= \text{polar second moment of area} \\
 &= \sum r^2 dA
 \end{aligned}
 \tag{8.5}$$

Equation (8.4) is known as the *perpendicular axes theorem* which states that the sum of the second moments of area of two mutually perpendicular axes of a lamina is equal to the polar second moment of area about a point where these two axes cross.

### 8.5 Parallel axes theorem

Consider the lamina of Figure 8.4, where the  $x-x$  axis passes through its centroid. Suppose that  $I_{xx}$  is known and that  $I_{XX}$  is required, where the  $X-X$  axis lies parallel to the  $x-x$  axis and at a perpendicular distance  $h$  from it.

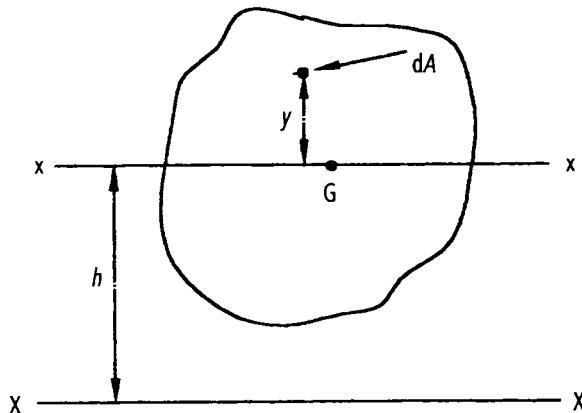


Figure 8.4 Parallel axes.

Now from equation (8.2)

$$I_{xx} = \Sigma y^2 dA$$

and

$$I_{XX} = \Sigma (y + h)^2 dA \quad (8.6)$$

$$= \Sigma (y^2 + h^2 + 2hy) dA, \quad (8.7)$$

but  $\Sigma 2hy dA = 0$ , as 'y' is measured from the centroid.

$$\therefore I_{XX} = \Sigma (y^2 + h^2) dA \quad (8.8)$$

but

$$I_{xx} = \Sigma y^2 dA$$

$$\therefore I_{XX} = I_{xx} + h^2 \Sigma dA$$

$$= I_{xx} + h^2 A \quad (8.9)$$

where

$$A = \text{area of lamina} = \Sigma dA$$

Equation (8.9) is known as the *parallel axes theorem*, which states that the second moment of area about the  $X-X$  axis is equal to the second moment of area about the  $x-x$  axis +  $h^2 \times A$ , where  $x-x$  and  $X-X$  are parallel.

$h$  = the perpendicular distance between the  $x-x$  and  $X-X$  axes.

$I_{xx}$  = the second moment of area about  $x-x$

$I_{XX}$  = the second moment of area about  $X-X$

The importance of the parallel axes theorem is that it is useful for calculating second moments of area of sections of RSJs, tees, angle bars etc. The geometrical properties of several cross-sections will now be determined.

**Problem 8.1** Determine the second moment of area of the rectangular section about its centroid ( $x-x$ ) axis and its base ( $X-X$ ) axis; see Figure 8.5. Hence or otherwise, verify the parallel axes theorem.

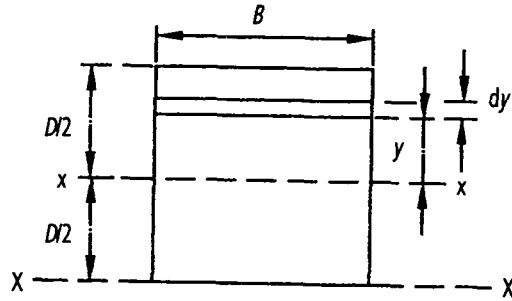


Figure 8.5 Rectangular section.

Solution

From equation (8.2)

$$\begin{aligned}
 I_{xx} &= \int y^2 dA = \int_{-D/2}^{D/2} y^2 (B dy) \\
 &= B \left[ \frac{y^3}{3} \right]_{-D/2}^{D/2} = \frac{2B}{3} [y^3]_0^{D/2} \quad (8.10)
 \end{aligned}$$

$$I_{xx} = BD^3/12 \text{ (about centroid)}$$

$$\begin{aligned}
 I_{XX} &= \int_{-D/2}^{D/2} (y + D/2)^2 B dy \\
 &= B \int_{-D/2}^{D/2} (y^2 + D^2/4 + Dy) dy \\
 &= B \left[ \frac{y^3}{3} + \frac{D^2 y}{4} + \frac{Dy^2}{2} \right]_{-D/2}^{D/2} \quad (8.11)
 \end{aligned}$$

$$I_{XX} = BD^3/3 \text{ (about base)}$$

To verify the parallel axes theorem,

from equation (8.9)

$$\begin{aligned}
 I_{xx} &= I_{xx} + h^2 \times A \\
 &= \frac{BD^3}{12} + \left(\frac{D}{2}\right)^2 \times BD \\
 &= BD^3 \left(\frac{1}{12} + \frac{1}{4}\right)
 \end{aligned}$$

$$I_{xx} = BD^3/3 \quad \text{QED}$$

**Problem 8.2** Determine the second moment of area about  $x-x$ , of the circular cross-section of Figure 8.6. Using the perpendicular axes theorem, determine the polar second moment of area, namely ' $J$ '.

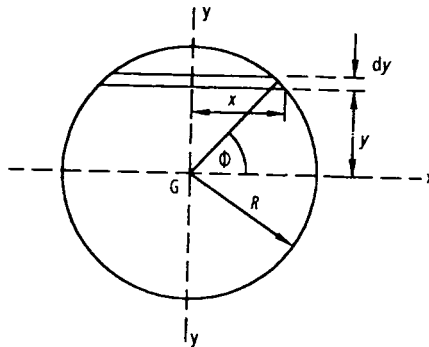


Figure 8.6 Circular section.

**Solution**

From the theory of a circle,

$$x^2 + y^2 = R^2$$

$$\text{or } y^2 = R^2 - x^2 \quad (8.12)$$

Let  $x = R \cos \phi$  (see Figure 8.6)

$$\therefore y^2 = R^2 - R^2 \cos^2 \phi \quad (8.13)$$

$$= R^2 \sin^2 \phi \quad (8.14)$$

or  $y = R \sin \phi$

and  $\frac{dy}{d\phi} = R \cos \phi$  (8.15)

or  $dy = R \cos \phi d\phi$  (8.16)

Now  $A =$  area of circle

$$\begin{aligned} &= 4 \int_0^R x dy \\ &= 4 \int_0^{\pi/2} R \cos \phi R \cos \phi d\phi \\ &= 4R^2 \int_0^{\pi/2} \cos^2 \phi d\phi \end{aligned}$$

but  $\cos^2 \phi = \frac{1 + \cos 2\phi}{2}$

$$\begin{aligned} \therefore A &= 2R^2 \int_0^{\pi/2} (1 + \cos 2\phi) d\phi \\ &= 2R^2 \left[ \phi + \frac{\sin 2\phi}{2} \right]_0^{\pi/2} \\ &= 2R^2 \left[ \left( \frac{\pi}{2} + 0 \right) - (0 + 0) \right] \end{aligned}$$

or  $A = \pi R^2$  QED (8.17)

Now  $I_{xx} = 4 \int_0^R y^2 x dy$  (8.18)

Substituting equations (8.14), (8.13) and (8.16) into equation (8.18), we get

$$\begin{aligned} I_{xx} &= 4 \int_0^{\pi/2} R^2 \sin^2 \phi R \cos \phi R \cos \phi d\phi \\ &= 4R^4 \int_0^{\pi/2} \sin^2 \phi \cos^2 \phi d\phi \end{aligned}$$

but  $\sin^2 \phi = (1 - \cos 2\phi)/2$

$$\text{and } \cos^2 \phi = (1 + \cos 2\phi)/2$$

$$\begin{aligned} \therefore I_{xx} &= R^4 \int_0^{\pi/2} (1 - \cos 2\phi) (1 + \cos 2\phi) d\phi \\ &= R^4 \int_0^{\pi/2} (1 - \cos^2 2\phi) d\phi \end{aligned}$$

$$\text{but } \cos^2 2\phi = \frac{1 + \cos 4\phi}{2}$$

$$\begin{aligned} I_{xx} &= R^4 \int_0^{\pi/2} \left[ 1 - \frac{1 + \cos 4\phi}{2} \right] d\phi \\ &= R^4 \left[ \phi - \phi/2 - \frac{\sin 4\phi}{8} \right]_0^{\pi/2} \\ &= R^4 [(\pi/2 - \pi/4 - 0) - (0 - 0 - 0)] \end{aligned}$$

$$\text{or } I_{xx} = \pi R^4/4 = \pi D^4/64 \quad (8.19)$$

where

$$D = \text{diameter} = 2R$$

As the circle is symmetrical about  $x-x$  and  $y-y$

$$I_{yy} = I_{xx} = \pi D^4/64$$

From the perpendicular axes theorem of equation (8.4),

$$\begin{aligned} J &= \text{polar second moment of area} \\ &= I_{xx} + I_{yy} = \pi D^4/64 + \pi D^4/64 \end{aligned} \quad (8.20)$$

$$\text{or } J = \pi D^4/32 = \pi R^4/2$$



**Problem 8.3** Determine the second moment of area about its centroid of the RSJ of Figure 8.7.

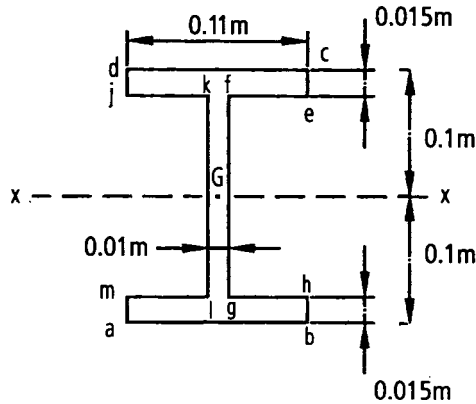


Figure 8.7 RSJ.

**Solution**

$I_{xx} =$  'I' of outer rectangle (abcd) about x-x minus the sum of the I's of the two inner rectangles (efgh and jklm) about x-x.

$$= \frac{0.11 \times 0.2^3}{12} - \frac{2 \times 0.05 \times 0.17^3}{12}$$

$$= 7.333 \times 10^{-5} - 4.094 \times 10^{-5}$$

or  $I_{xx} = 3.739 \times 10^{-5} \text{ m}^4$

**Problem 8.4** Determine  $I_{xx}$  for the cross-section of the RSJ as shown in Figure 8.8.

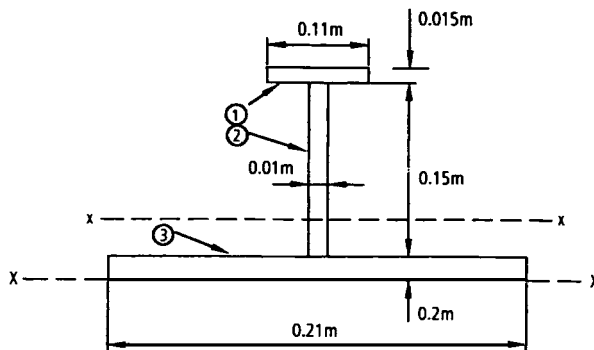


Figure 8.8 RSJ (dimensions in metres).

Solution

The calculation will be carried out with the aid of Table 8.1. It should be emphasised that this method is suitable for almost any *computer spreadsheet*. To aid this calculation, the RSJ will be subdivided into three rectangular elements, as shown in Figure 8.8.

Table 8.1

Col. 1	Col. 2	Col. 3	Col. 4	Col. 5	Col. 6
Element	$a = bd$	$y$	$ay$	$ay^2$	$i = bd^3/12$
1	$0.11 \times 0.015 = 0.00165$	0.1775	$2.929 \times 10^{-4}$	$5.199 \times 10^{-5}$	$0.11 \times 0.015^3/12 = 3 \times 10^{-8}$
2	$0.01 \times 0.15 = 0.0015$	0.095	$1.425 \times 10^{-4}$	$1.354 \times 10^{-5}$	$0.01 \times 0.15^3/12 = 2.812 \times 10^{-6}$
3	$0.02 \times 0.21 = 0.0042$	0.01	$4.2 \times 10^{-5}$	$4.2 \times 10^{-7}$	$0.21 \times 0.02^3/12 = 1.4 \times 10^{-7}$
$\Sigma$	$\Sigma a = 0.00735$	—	$\Sigma ay = 4.77 \times 10^{-4}$	$\Sigma ay^2 = 6.595 \times 10^{-5}$	$\Sigma i = 2.982 \times 10^{-6}$

$a$  = area of an element (column 2)

$y$  = vertical distance of the local centroid of an element from XX (column 3)

$ay$  = the product  $a \times y$  (column 4 = column 2  $\times$  column 3)

$ay^2$  = the product  $a \times y \times y$  (column 5 = column 3  $\times$  column 4)

$i$  = the second moment of area of an element about its own local centroid =  $bd^3/12$

$b$  = ‘width’ of element (horizontal dimension)

$d$  = ‘depth’ of element (vertical dimension)

$\Sigma$  = summation of the column

$\bar{y}$  = distance of centroid of the cross-section about XX

$$= \Sigma ay / \Sigma a \tag{8.21}$$

$$= 4.774 \times 10^{-4} / 0.00735 = 0.065 \text{ m} \tag{8.22}$$

Now from equation (8.9)

$$\begin{aligned} I_{xx} &= \sum ay^2 + \sum i \\ &= 6.595 \times 10^{-5} + 2.982 \times 10^{-6} \end{aligned} \quad (8.23)$$

$$I_{xx} = 6.893 \times 10^{-5} \text{ m}^4$$

From the parallel axes theorem (8.9),

$$\begin{aligned} I_{xx} &= I_{xx} - \bar{y}^2 \sum a \\ &= 6.893 \times 10^{-5} - 0.065^2 \times 0.00735 \end{aligned} \quad (8.24)$$

$$\text{or } I_{xx} = 3.788 \times 10^{-5} \text{ m}^4$$

### Further problems (for answers, see page 692)

**8.5** Determine  $I_{xx}$  for the thin-walled sections shown in Figures 8.9(a) to 8.9(c), where the wall thicknesses are 0.01 m.

**NB** Dimensions are in metres.  $I_{xx}$  = second moment of area about a horizontal axis passing through the centroid.

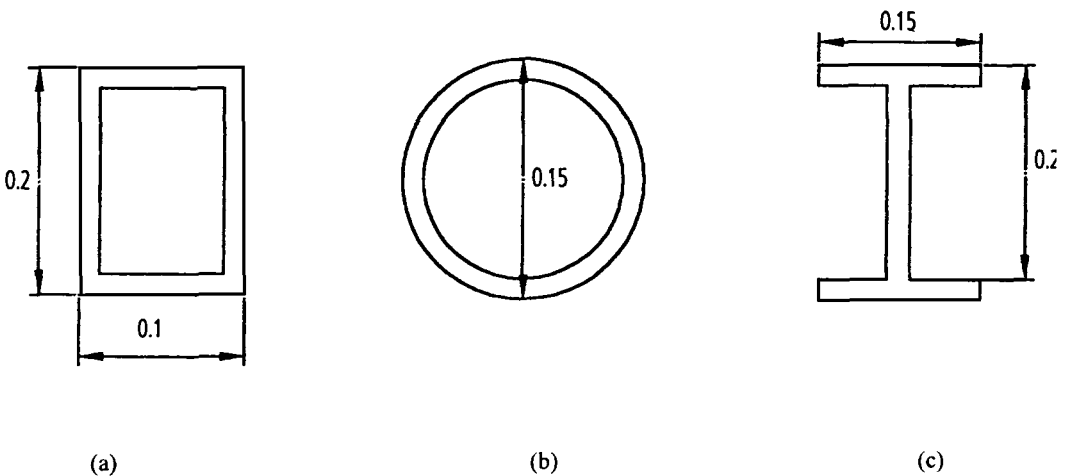


Figure 8.9 Thin-walled sections.

- 8.6 Determine  $I_{xx}$  for the thin-walled sections shown in Figure 8.10, which have wall thicknesses of 0.01 m.

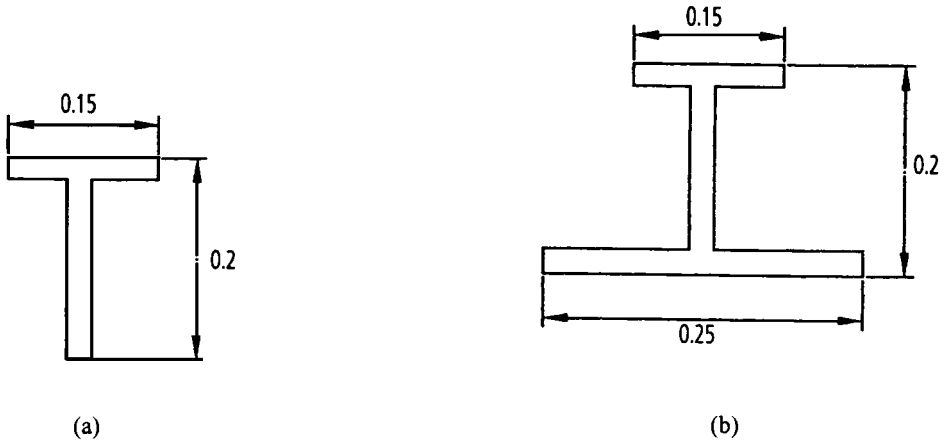


Figure 8.10

- 8.7 Determine the position of the centroid of the section shown in Figure 8.11, namely  $\bar{y}$ . Determine also  $I_{xx}$  for this section.

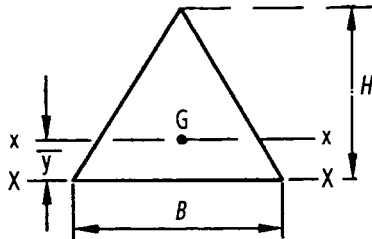


Figure 8.11 Isosceles triangular section.