7 Bending moments and shearing forces

7.1 Introduction

In Chapter 1 we discussed the stresses set up in a bar due to axial forces of tension and compression. When a bar carries lateral forces, two important types of loading action are set up at any section: these are a bending moment and a shearing force.

Consider first the simple case of a beam which is fixed rigidly at one end B and is quite free at its remote end D, Figure 7.1; such a beam is called a *cantilever*, a familiar example of which is a fishing rod held at one end. Imagine that the cantilever is horizontal, with one end B embedded in a wall, and that a lateral force W is applied at the remote end D. Suppose the cantilever is divided into two lengths by an imaginary section C; the lengths BC and CD must individually be in a state of statical equilibrium. If we neglect the mass of the cantilever itself, the loading actions over the section C of CD balance the actions of the force W at C. The length CD of the cantilever is in equilibrium if we apply an upwards vertical force F and an anti-clockwise couple M at C; F is equal in magnitude to W, and M is equal to $W(L - z)$, where z is measured from B. The force F at C is called a *shearing force*, and the couple M is a *bending moment.*

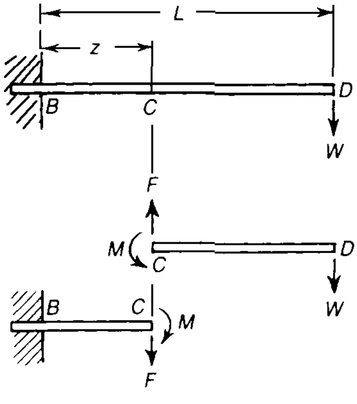


Figure 7.1 Bending moment and shearing force in a simple cantilever beam.

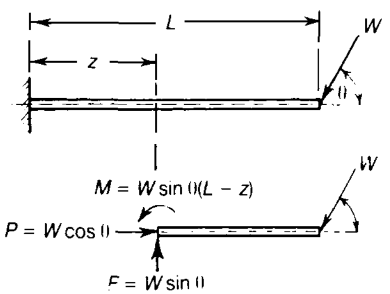


Figure 7.2 Cantilever with and inclined end load.

But at the imaginary section C of the cantilever, the actions F and M on CD are provided by the length BC of the cantilever. In fact, equal and opposite actions F and M are applied by CD to BC. For the length BC, the actions at C are a downwards shearing force F, and a clockwise couple M.

When the cantilever carries external loads which are not applied normally to the axis of the beam, Figure 7.2, axial forces are set up in the beam. If W is inclined at an angle θ to the axis of the beam, Figure 7.2, the axial thrust in the beam at any section is

$$P = W \cos \theta \tag{7.1}$$

The bending moment and shearing force at a section a distance z from the built-in end are

$$M = W(L-z) \sin \theta \qquad F = W \sin \theta \tag{7.2}$$

7.2 Concentrated and distributed loads

A concentrated load on a beam is one which can be regarded as acting wholly at one point of the beam. For the purposes of calculation such a load is localised at a point of the beam; in reality this would imply an infinitely large bearing pressure on the beam at the point of application of a concentrated load. All loads must be distributed in practice over perhaps only a small length of beam, thereby giving a finite bearing pressure. Concentrated loads arise frequently on a beam where the beam is connected to other transverse beams.

In practice there are many examples of distributed loads: they arise when a wall is built on a girder; they occur also in many problems of fluid pressure, such as wind pressure on a tall building, and aerodynamic forces on an aircraft wing.

7.3 Relation between the intensity of loading, the shearing force, and bending moment in a straight beam

Consider a straight beam under any system of lateral loads and external couples, Figure 7.3; an element length δz of the beam at a distance z from one end is acted upon by an external lateral load, and internal bending moments and shearing forces. Suppose external lateral loads are distributed so that the intensity of loading on the elemental length δz is w.

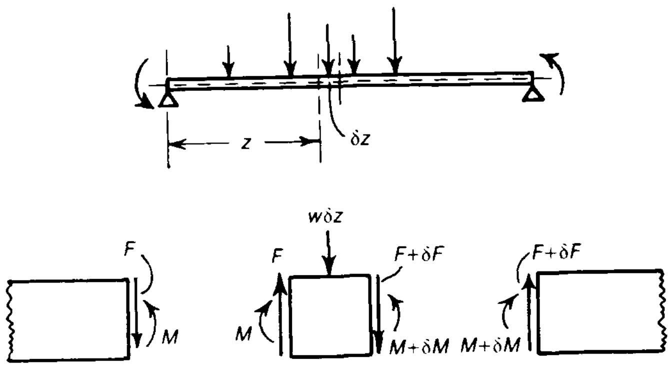


Figure 7.3 Shearing and bending actions on an elemental length of a straight beam.

Then the external vertical force on the element is $w\delta z$, Figure 7.3; this is reacted by an internal bending moment M and shearing force F on one face of the element, and $M + \delta M$ and $F + \delta F$ on the other face of the element. For vertical equilibrium of the element we have

$$(F + \delta F) - F + w\delta z = 0$$

If δz is infinitesimally small,

$$\frac{dF}{dz} = -w \tag{7.3}$$

Suppose this relation is integrated between the limits z_1 and z_2, then

$$\int_{z=z_1}^{z=z_2} dF = -\int_{z_1}^{z_2} w dz$$

If F_1 and F_2 are the shearing forces at $z = z_1$ and $z = z_2$ respectively, then

$$(F_2 - F_1) = -\int_{z_1}^{z_2} w dz$$

or

$$F_1 - F_2 = \int_{z_1}^{z_2} w dz \tag{7.4}$$

Then, the decrease of shearing force from z_1 to z_2 is equal to the area below the load distribution curve over this length of the beam, or the difference between F_1 and F_2 is the net lateral load over this length of the beam.

Furthermore, for rotational equilibrium of the elemental length δz,

$$(F + \delta F)\,\delta z - (M + \delta M) + M + wdz\left(\frac{1}{2}\,\delta z\right) = 0$$

Then, to the first order of small quantities,

$$F\delta z - \delta M = 0$$

Then, in the limit as δz approaches zero,

$$\frac{dM}{dz} = F \tag{7.5}$$

On integrating between the limits $z = z_1$ and $z = z_2$, we have

$$\int_{z=z_1}^{z=z_2} dM = \int_{z_1}^{z_2} F dz$$

Thus

$$M_2 - M_1 = \int_{z_1}^{z_2} F dz \tag{7.6}$$

where M_1 and M_2 are the values M at $z = z_1$ and $z = z_2$, respectively. Then the increase of bending moment from z_1 to z_2 is the area below the shearing force curve for that length of the beam.

Equations (7.4) and (7.6) are extremely useful for finding the bending moments and shearing forces in beams with irregularly distributed loads. From equation (7.4) the shearing force F at a section distance z from one end of the beam is

$$F = F_1 - \int_{z_1}^{z} w dz \tag{7.7}$$

On substituting this value of F into equation (7.6),

$$M_2 - M_1 = \int_{z_1}^{z_2} \left\{ F_1 - \int_{z_1}^{z} w dz \right\} dz$$

Thus

$$M_2 = M_1 + F_1 (z_2 - z_1) - \int_{z_1}^{z_2} \left\{ \int_{z_1}^{z} w dz \right\} dz \tag{7.8}$$

From equation (7.5) we have that the bending moment M has a stationary value when the shearing force F is zero. Equations (7.3) and (7.5) give

$$\frac{d^2 M}{dz^2} = \frac{dF}{dz} = -w \tag{7.9}$$

For the directions of M, F and w considered in Figure 7.3, M is *mathematically* a maximum, since d^2M/dz^2 is negative; the significance of the word mathematically will be made clearer in Section 7.8.

All the relations developed in this section are merely statements of statical equilibrium, and are therefore true independently of the state of the material of the beam.

7.4 Sign conventions for bending moments and shearing forces

The bending moments on the elemental length δz of Figure 7.3 tend to make the beam concave on its upper surface and convex on its lower surface; such bending moments are sometimes called *sagging* bending moments. The shearing forces on the elemental length tend to rotate the element in a *clockwise* sense. In deriving the equations in this section it is assumed implicitly, therefore, that

(i) *downwards* vertical loads are positive;

(ii) *sagging* bending moments are positive; and

(iii) *clockwise* shearing forces are positive.

These sign conventions are shown in Figure 7.4. Any other system of sign conventions can be used, provided the signs of the loads, bending moments and shearing forces are considered when equations (7.3) and (7.5) are applied to any particular problem.

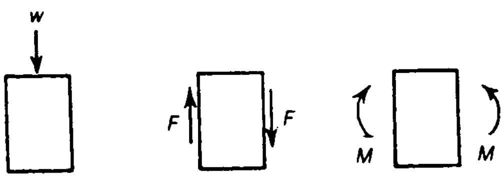


Figure 7.4 Positive values of w, F and M, (i) downward vertical loading, (ii) clockwise shearing forces, (iii) sagging bending-moment.

Figures that show graphically the variations of bending moment and shearing force along the length of a beam are called *bending moment diagrams* and *shearing force diagrams*. Sagging bending moments are considered positive, and clockwise shearing forces taken as positive. The two quantities are plotted above the centre line of the beam when positive, and below when negative. Before we can calculate the stresses and deformations of beams, we must be able to find the bending moment and shearing force at any section.

7.5 Cantilevers

A cantilever is a beam supported at one end only; for example, the beam already discussed in Section 7.1, and shown in Figure 7.1, is held rigidly at B. Consider first the cantilever shown in Figure 7.5(a), which carries a concentrated lateral load W at the free end. The bending moment at a section a distance z from B is

$$M = -W(L-z)$$

the negative sign occurring since the moment is hogging, as shown in Figure 7.5(b). The variation of bending moment is linear, as shown in Figure 7.5(c). The shearing force at any section is

$$F = +W$$

the shearing force being positive as it is clockwise, as shown in Figure 7.5(d). The shearing force is constant throughout the length of the cantilever. We note that

$$\frac{dM}{dz} = W = F$$

Further $dF/dz = 0$, as there are no lateral loads between B and D.

The bending moment diagram is shown in Figure 7.5(c) and the shearing force diagram is shown in Figure 7.5(e)

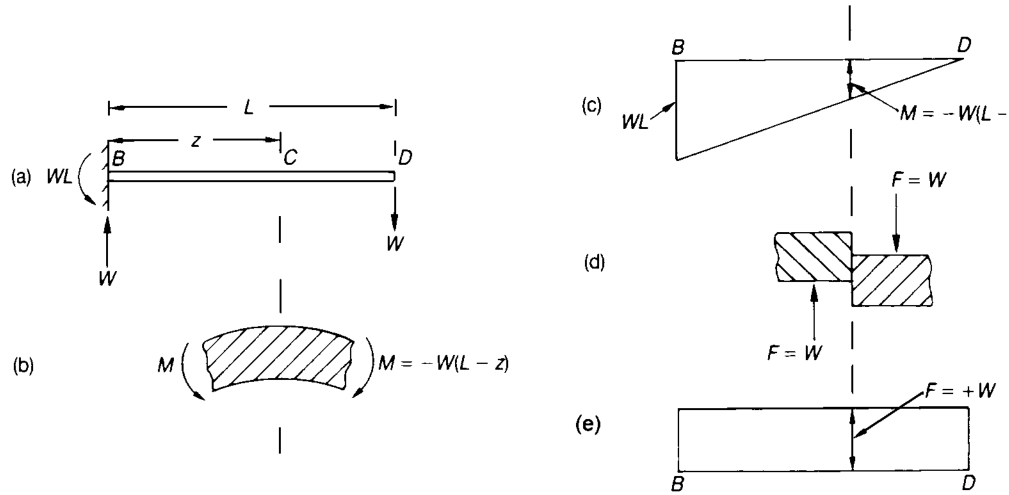


Figure 7.5 Bending-moment and shearing-force diagrams for a cantilever with a concentrated load at the free end.

Now consider a cantilever carrying a uniformly distributed downwards vertical load of intensity w, Figure 7.6(a). The shearing force at a distance z from B is

$$F = +w(L - z)$$

as shown in Figure 7.6 (d). The bending moment at a distance z from B is

$$M = -\frac{1}{2} w(L - z)^2$$

as shown in Figure 7.6(b). The shearing force varies linearly and the bending moment parabolically along the length of the beam, as shown in Figure 7.6(e) and 7.6(c), respectively. We see that

$$\frac{dM}{dz} = w(L - z) = +F$$

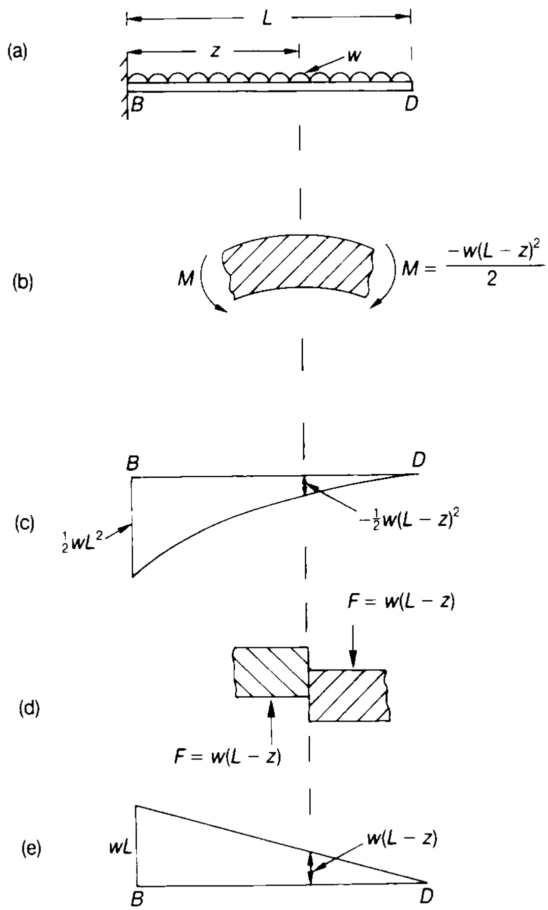
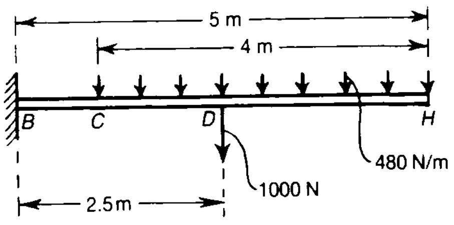


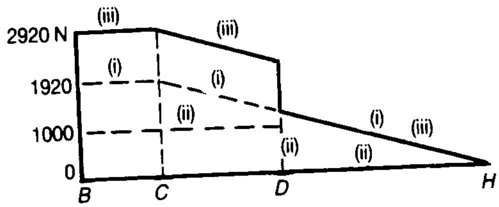
Figure 7.6 Bending-moment and shearing-force diagrams for a cantilever under uniformly distributed load.

Problem 7.1 A cantilever 5 m long carries a uniformly distributed vertical load 480 N per metre from *C* from *H*, and a concentrated vertical load of 1000 N at its mid-length, *D*. Construct the shearing force and bending moment diagrams.



<u>Solution</u>

The shearing force due to the distributed load increases uniformly from zero at H to +1920 N at C, and remains constant at +1920 N from C to B; this is shown by the lines (i). Due to the concentrated load at D, the shearing force is zero from H to D, and equal to +1000 N from D to B, as shown by lines (ii). Adding the two together we get the total shearing force shown by lines (iii).



The bending moment due to the distributed load increases parabolically from zero at H to

$$-\frac{1}{2}(480)(4)^2 = -3840 \text{ Nm}$$

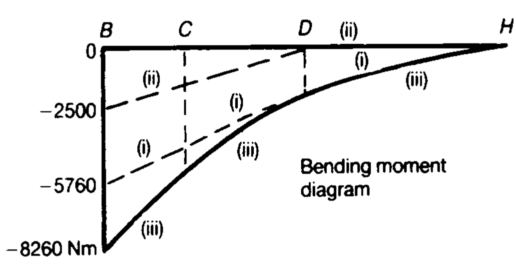
at C. The total load on CH is 1920 N with its centre of gravity 3 m from B; thus the bending moment at B due to this load is

$$-(1920)(3) = -5760 \text{ Nm}$$

From C to B the bending moment increases uniformly, giving lines (i). The bending moment due to the concentrated load increases uniformly from zero at D to

$$-(1000)(2.5) = -2500 \text{ Nm}$$

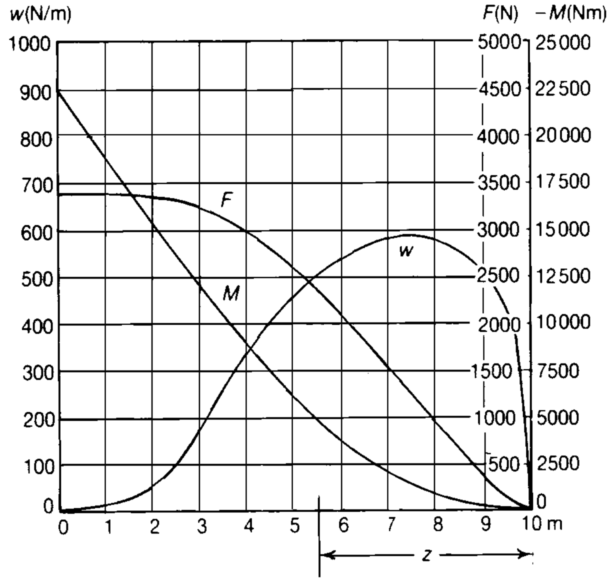
at B, as shown by lines (ii). Combining (i) and (ii), the total bending moment is given by (iii).



The method used here for determining shearing-force and bending-moment diagrams is known as the *principle of superposition*.

7.6 Cantilever with non-uniformly distributed load

Where a cantilever carries a distributed lateral load of variable intensity, we can find the bending moments and shearing forces from equations (7.4) and (7.6). When the loading intensity w cannot be expressed as a simple analytic function of z, equations (7.4) and (7.6) can be integrated numerically.



Problem 7.2 A cantilever of length 10 m, built in at its left end, carries a distributed lateral load of varying intensity w N per metre length. Construct curves of shearing force and bending moment in the cantilever.

Solution

If z is the distance from the free end of cantilever, the shearing force at a distance z from the free end is

$$F = \int_0^z w dz$$

We find first the shearing force F by numerical integration of the w-curve. The greatest force occurs at the built-in end, and has the value

$$F_{max} \doteq 3400 \text{ N}$$

The bending moment at a section a distance z from the free end is

$$M = - \int_0^z F dz$$

and is found therefore by numerical integration of the F-curve. The greatest bending moment occurs at the built-in end, and has the value

$$M_{max} = 22500 \text{ Nm}$$

NB It should be noted that by inspection the bending moment and the shearing force at the free end of the cantilever are zero; these are boundary conditions.

7.7 Simply-supported beams

By *simply-supported* we mean that the supports are of such a nature that they do not apply any resistance to bending of a beam; for instance, knife-edges or frictionless pins perpendicular to the plane of bending cannot transmit couples to a beam. The remarks concerning bending moments and shearing forces, which were made in Section 7.5 in relation to cantilevers, apply equally to beams simply-supported at each end, or with any conditions of end support.

As an example, consider the beam shown in Figure 7.7(a), which is simply-supported at B and C, and carries a vertical load W a distance a from B. If the ends are simply-supported no bending moments are applied to the beam at B and C. By taking moments about B and C we find that the reactions at these supports are

$$\frac{W}{L}(L - a) \text{ and } \frac{Wa}{L}$$

respectively. Now consider a section of the beam a distance z from B; if $z < a$, the bending moment and shearing force are

$$M = +\frac{Wz}{L}(L - a), \quad F = +\frac{W}{L}(L - a), \text{ as shown by Figures 7.7(b) and 7.7(d)}$$

If $z > a$,

$$M = +\frac{Wz}{L}(L - a) - W(z - a) = +\frac{Wa}{L}(L - z)$$

$$F = -\frac{Wa}{L}$$

The bending moment and shearing force diagrams show discontinuities at $z = a$; the maximum bending moment occurs under the load W, and has the value

$$M_{max} = \frac{Wa}{L}(L - a) \qquad (7.10)$$

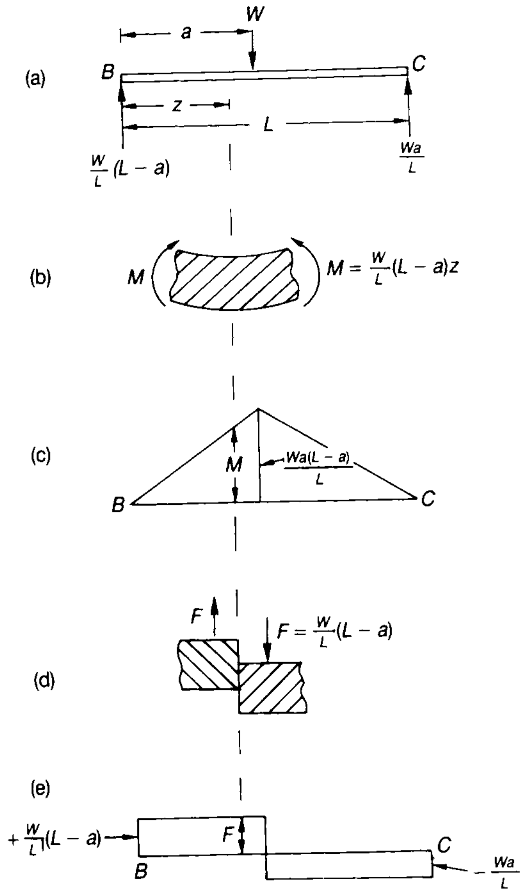


Figure 7.7 Bending-moment and shearing-force diagrams for a simply-supported beam with a single concentrated lateral load.

The simply-supported beam of Figure 7.8(a) carries a uniformly-distributed load of intensity w. The vertical reactions at B and C are $\frac{1}{2}wL$. Consider a section at a distance z from B. The bending moment at this section is

$$M = \frac{1}{2}wLz - \frac{1}{2}wz^2$$

$$= \frac{1}{2}wz\,(L - z)$$

as shown in Figure 7.8(b) and the shearing force is

$$F = +\frac{1}{2}wL - wz$$

$$= w\left(\frac{1}{2}L - z\right)$$

as shown in Figure 7.8(d).

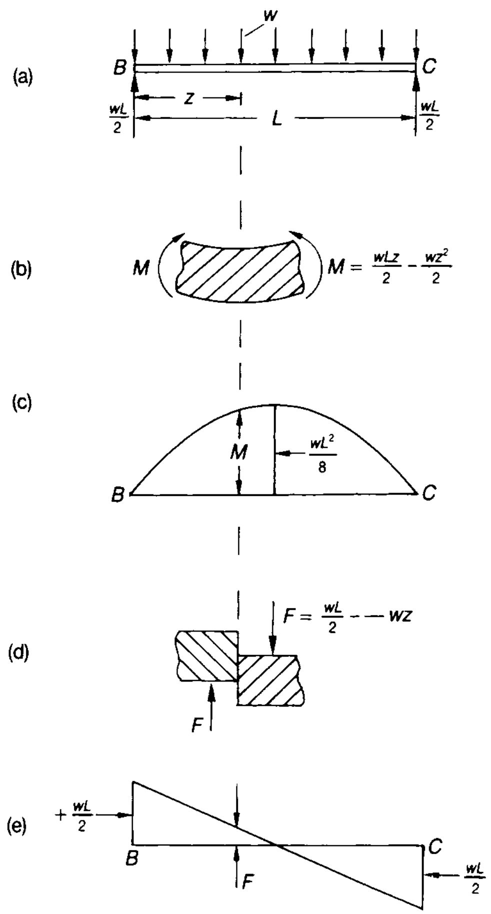


Figure 7.8 Bending-moment and shearing-force diagrams for a simply-supported beam with a uniformly distributed lateral load.

The bending moment is a maximum at $z = ½L$, where

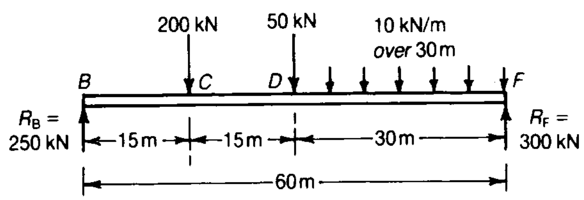
$$M_{max} = \frac{wL^2}{8} \tag{7.11}$$

At $z = \frac{1}{2}L$, we note that

$$\frac{dM}{dz} = +F = 0$$

The bending moment diagram is shown in Figure 7.8(c) and the shearing force diagram is shown in Figure 7.8(e).

Problem 7.3 A simply-supported beam carries concentrated lateral loads at C and D, and a uniformly distributed lateral load over the length DF. Construct the bending moment and shearing force diagrams.



Solution

First we calculate the vertical reactions at B and F. On taking moments about F,

$$60\,R_B = (200 \times 10^3)(45) + (50 \times 10^3)(30) + (300 \times 10^3)(15) = 15\,000 \times 10^3$$

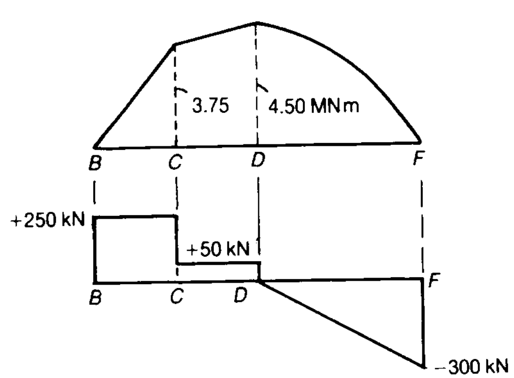
Then

$$R_B = 250 \text{ kN}$$

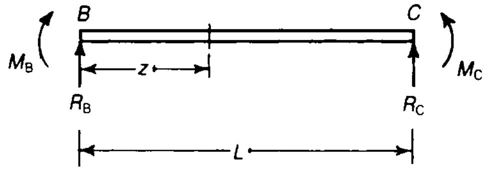
and

$$R_F = (200 \times 10^3) + (50 \times 10^3) + (300 \times 10^3) - R_B = 300 \text{ kN}$$

The bending moment varies linearly between B and C, and between C and D, and parabolically from D to F. The maximum bending moment is 4.5 MNm, and occurs at D. The maximum shearing force is 300 kN, and occurs at F.



Problem 7.4 A beam rests on knife-edges at each end, and carries a clockwise moment M_B at B, and an anticlockwise moment M_C at C. Construct bending moment and shearing force diagrams for the beam.



Solution

Suppose R_B and R_C are vertical reactions at B and C; then for statical equilibrium of the beam

$$R_B = -R_C = \frac{1}{L}\left(M_C - M_B\right)$$

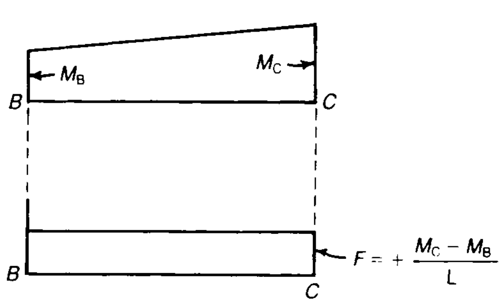
The shearing force at all sections is then

$$F = R_B = \frac{1}{L}\left(M_C - M_B\right)$$

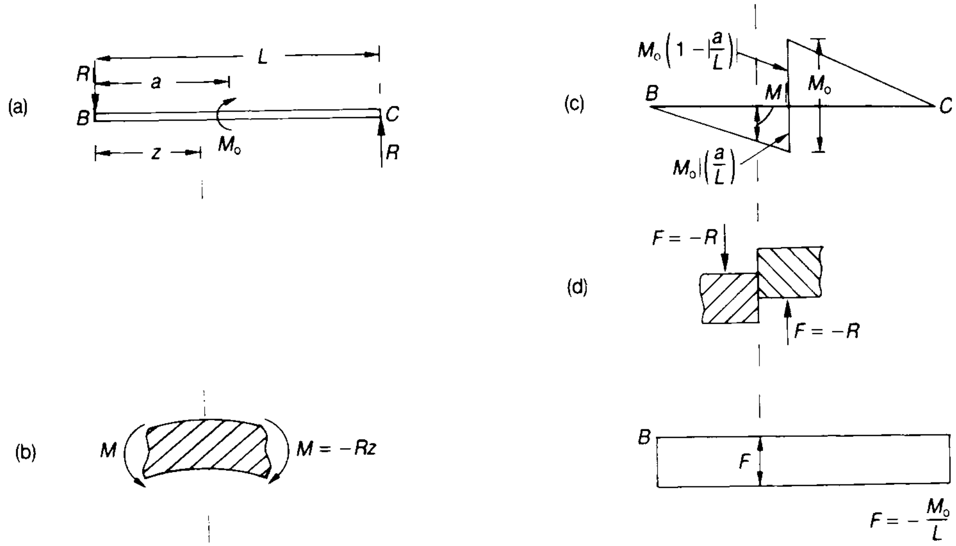
The bending moment a distance z from B is

$$M = M_B + R_B z = \frac{M_B}{L}\left(L - z\right) + \frac{M_C z}{L}$$

so M varies linearly between B and C.



Problem 7.5 A simply-supported beam carries a couple M_0 applied at a point distant a from B. Construct bending moment and shearing force diagrams for the beam.



Solution

The vertical reactions R at B and C are equal and opposite. For statical equilibrium of BC,

$$M_0 = RL, \text{ or } R = \frac{M_0}{L}$$

The shearing force at all sections is

$$F = -R = -\frac{M_0}{L}$$

as shown in Figure (d), above. The bending moment at $z < a$ is

$$M = -Rz = -\frac{M_0 z}{L}$$

as shown in Figure (c), above, and for $z > a$

$$M = -Rz + M_0 = M_0\left(1 - \frac{z}{L}\right)$$

as shown in Figure (c), above.

7.8 Simply-supported beam carrying a uniformly distributed load and end couples

Consider a simply-supported beam BC, carrying a uniformly distributed load w per unit length, and couples M_B and M_C applied to ends, Figure 7.9(i). The reactions R_B and R_C can be found directly by taking moments about B and C in turn; we have

$$R_B = \frac{wL}{2} - \frac{1}{L}(M_B - M_C) \tag{7.12}$$

$$R_C = \frac{wL}{2} + \frac{1}{L}(M_B - M_C)$$

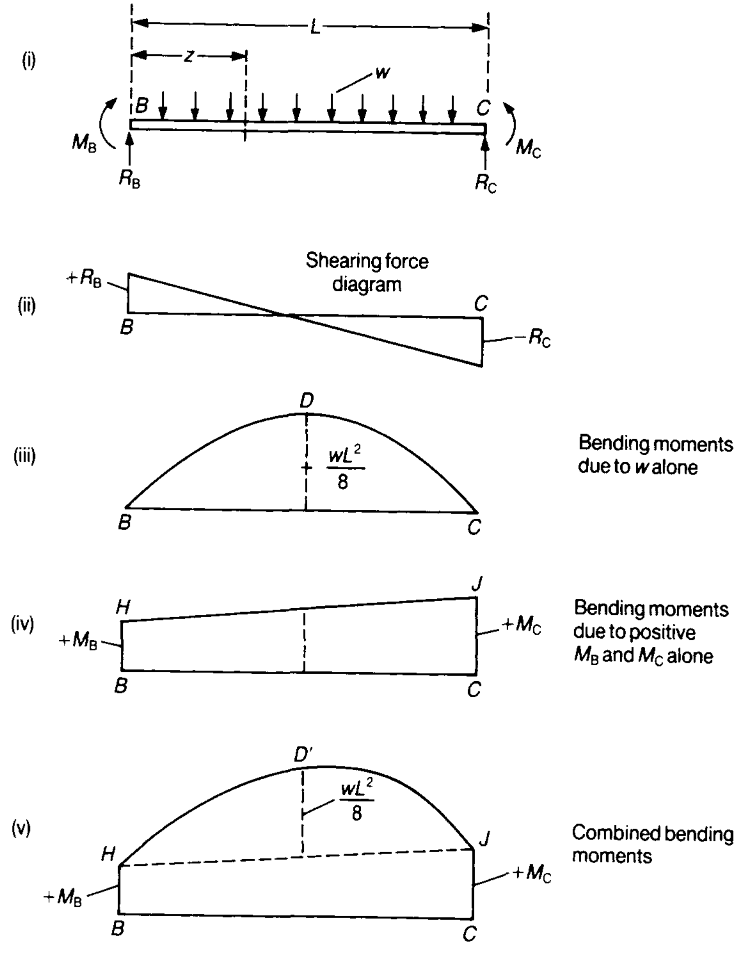


Figure 7.9 Simply-supported beam with uniformly distributed lateral load and end couples.

These give the shearing forces at the end of the beam, and the shearing force at any point of the beam can be deduced, Figure 7.9(ii). In discussing bending moments we consider the total loading actions on the beam as the superposition of a uniformly distributed load and end couples; the distributed load gives rise to a parabolic bending moment curve, *BDC* in Figure 7.9(iii), whereas the end couples M_B and M_C give the straight line *HJ*, Figure 7.9(iv). The combined effects of the lateral load and the end couples give the curve *BHD'JC*, Figure 7.9(v). The bending moment at a distance z from B is

$$M = \frac{1}{2}wz\,(L - z) + \frac{M_B}{L}(L - z) + \frac{M_C z}{L} \tag{7.13}$$

The 'maximum' bending moment occurs when

$$\frac{dM}{dz} = \frac{1}{2}w\,(L - 2z) - \frac{M_B}{L} + \frac{M_C}{L} = 0$$

that is, when

$$z = \frac{1}{2}L - \frac{1}{wL}(M_B - M_C)$$

The value of M for this value of z is

$$M_{\max} = \frac{1}{8}wL^2 + \frac{1}{2}(M_B + M_C) + \frac{1}{2wL^2}(M_B - M_C)^2 \tag{7.14}$$

This, however, is only a mathematical 'maximum'; if M_B or M_C is negative, the numerically greatest bending moment may occur at B or C. Care should therefore be taken to find the truly greatest bending moment in the beam.

7.9 Points of inflection

When either, or both, of the end couples in Figure 7.9 is reversed in direction, there is at least one section of the beam where the bending moment is zero.

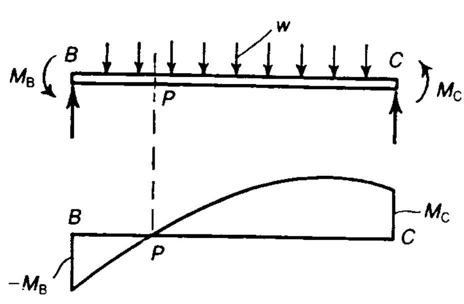


Figure 7.10 Single point of inflection in a beam.

In Figure 7.10 the end couple M_B is applied in an anticlockwise direction; the bending moment at a distance z from B is

$$M = \frac{1}{2}wz(L - z) - \frac{M_B}{L}(L - z) + \frac{M_C z}{L} \tag{7.15}$$

and this is zero when

$$z^2 - zL\left(1 + \frac{2}{wL^2}\left[M_B + M_C\right]\right) + \frac{2M_B}{w} = 0 \tag{7.16}$$

The distance PB is the relevant root of this quadratic equation.

When the end couple M_C is also reversed in direction, Figure 7.11, there are two points, P and Q, in the beam at which the bending moment is zero. The distances P and Q from B are given by the roots of the equation

$$z^2 - zL\left[1 + \frac{2}{wL^2}\left(M_B - M_C\right)\right] + \frac{2M_B}{w} = 0 \tag{7.17}$$

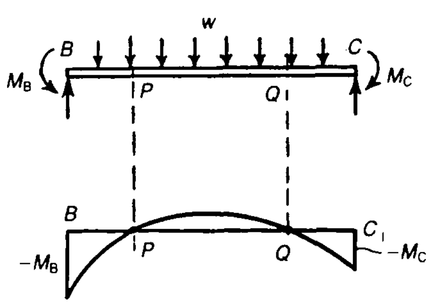


Figure 7.11 Beam with two points of inflection.

The distance PQ is

$$2\sqrt{\frac{L^2}{4} - \left(\frac{M_B - M_C}{w}\right) + \left(\frac{M_B - M_C}{wL}\right)^2} \tag{7.18}$$

The points P and Q are called *points of inflection*, or *points of contraflexure*; as we shall see later, the curvature of the deformed beam changes sign at these points.

7.10 Simply-supported beam with a uniformly distributed load over part of a span

The beam *BCDF*, shown in Figure 7.12, carries a uniformly distributed vertical load *w* per unit length over the portion *CD*. On taking moments about *B* and *F*,

$$V_B = \frac{bw}{2L}(b + 2c), \quad V_F = \frac{bw}{2L}(b + 2a) \tag{7.19}$$

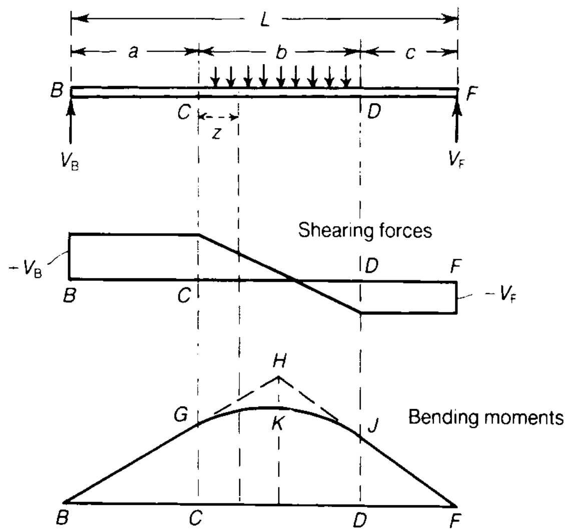


Figure 7.12 Shearing-force and bending-moment diagrams for simply-supported beam with distributed load over part of the span.

The bending moments at *C* and *D* are

$$M_C = aV_B = \frac{baw}{2L}(b + 2c)$$

$$M_D = cV_F = \frac{bcw}{2L}(b + 2a) \tag{7.20}$$

The bending moments in *BC* and *FD* vary linearly. The bending moment in *CD*, at a distance *z* from *C*, is

$$M = \left(1 - \frac{z}{b}\right) M_C + \frac{z}{b} M_D + \frac{1}{2} wz (b - z) \tag{7.21}$$

Then

$$\frac{dM}{dz} = \frac{1}{b}\left(M_D - M_C\right) + \frac{1}{2}w\left(b - 2z\right)$$

On substituting for M_C and M_D from equations (7.20)

$$\frac{dM}{dz} = \frac{bw}{2L}\left(c - a\right) + \frac{1}{2}w\left(b - 2z\right)$$

At C, $z = 0$, and

$$\frac{dM}{dz} = \frac{bw}{2L}\left(b + 2c\right) = V_B$$

But V_B is the slope of the line BG in the bending moment diagram, so the curve of equation (7.21) is tangential to BG at G. Similarly, the curve of equation (7.21) is tangential to FJ at J. Between C and D the bending moment varies parabolically; the simplest method of constructing the bending moment diagram for CD is to produce BG and FJ to meet at H, and then to draw a parabola between G and J, having tangents BG and FJ.

7.11 Simply-supported beam with non-uniformly distributed load

Suppose a simply-supported beam of span L, Figure 7.13, carries a lateral distributed load of variable intensity w. Then, from equation (7.4), if F is the shearing force a distance z from B,

$$F_0 - F = \int_0^z w\,dz$$

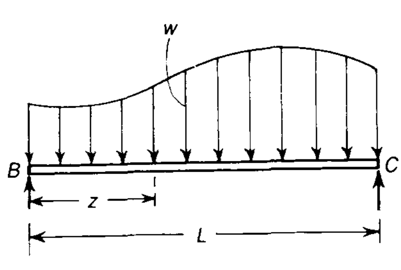


Figure 7.13 Simply-supported beam with lateral load of varying intensity.

where F_0 is the shearing force at $z = 0$. Then

$$F = F_0 - \int_0^z w dz \tag{7.22}$$

Furthermore, from equation (7.6), the bending moment a distance z from B is

$$M = M_0 + F_0 z - \int_0^z \int_0^z w dz\, dz \tag{7.23}$$

where M_0 is the bending moment at $z = 0$. However, as the beam is simply-supported at $z = 0$, we have $M_0 = 0$, and so

$$M = F_0 z - \int_0^z \int_0^z w dz\, dz$$

The end $z = L$ is also simply-supported, so for this end $M = 0$; then

$$F_0 L - \int_0^L \int_0^z w dz\, dz = 0$$

This gives

$$F_0 = \frac{1}{L} \int_0^L \int_0^z w dz\, dz \tag{7.24}$$

Equations (7.22), (7.23) and (7.24) may be used in the graphical solution of problems in which w is not an analytic function of z. The value of F_0 is found firstly from equation (7.24); numerical integrations then give the values of F and M, from equations (7.22) and (7.23), respectively.

7.12 Plane curved beams

Consider a beam BCD, Figure 7.14, which is curved in the plane of the figure. The beam is loaded so that no twisting occurs, and bending is confined to the plane of Figure 7.14. Suppose an imaginary cross-section of the beam is taken at C; statical equilibrium of the length CD of the beam is ensured if, in general, a force and a couple act at C; it is convenient to consider the resultant force at C as consisting of two components—an axial force P, acting along the centre line of the beam, and a lateral force F, acting along the normal to the centre line of the beam. The couple M at C acts about an axis perpendicular to the plane of bending and passing through the centre line of the beam. The actions at C on the length BC of the beam, are equal and opposite to those at C on the length CD.

As before the couple M is the *bending moment* in the beam at C, and the lateral force F is the *shearing force*.

As an example, consider the beam of Figure 7.15, which has a centre line of constant radius R. The beam carries a radial load W at its free end. Consider a section of the beam at some angular position θ: for statical equilibrium of the length of the bar shown in Figure 7.15(ii),

$$M = WR\sin\theta$$
$$F = W\cos\theta$$
$$P = W\sin\theta \qquad (7.25)$$

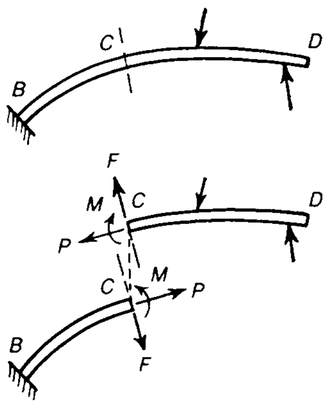


Figure 7.14 Bending and shearing actions in a plane curved beam.

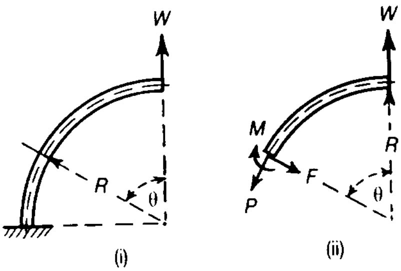


Figure 7.15 Plane curved beam of circular form carrying an end load.

Consider again, the beam shown in Figure 7.16, consisting of two straight limbs, BC and CD, connected at C. In CD the bending moment varies linearly, from zero at D to 70 000 Nm at C. In BC the bending moment is constant and equal to 70 000 Nm. In Figure 7.17 the bending moments are plotted on the concave sides of the bent limbs; this is equivalent to following the sign convention of Section 7.4, that sagging bending moments are positive.

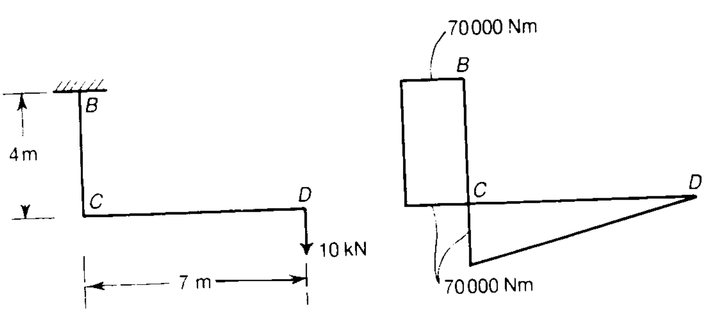
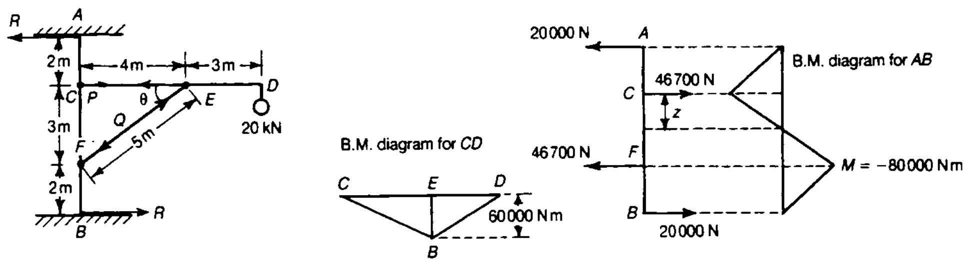


Figure 7.16 Bending moments in a bracket.

Problem 7.6 *AB* is a vertical post of a crane; the sockets at *A* and *B* offer no constraint against flexure. The horizontal arm *CD* is hinged to *AB* at *C* and supported by the strut *FE* which is freely hinged at its two extremities to *AB* and *CD*. Construct the bending moment diagrams for *AB* and *CD*. *(Cambridge)*



<u>*Solution*</u>

It is clear from considering the equilibrium of the whole crane that the horizontal reactions at *A* and *B* must be equal and opposite, and that the couple due to them must equal the moment of the 20 kN force. Let *R* be the magnitude of the horizontal reactions at *A* and *B*, then

$$7R = 7(20000)$$

and therefore

$$R = 20000\ N$$

Let P = the pull in *CE*, and Q = the thrust in *FE*. Then taking moments about *C* for the rod *CD* we have

$$4Q\ \sin\theta = 7(20000)$$

and therefore

$$Q = 58300\ N$$

Resolving horizontally for *AB* we have

$$P = Q\cos\theta = \frac{1}{2}(70000)\cot\theta = 46700\ N$$

The vertical reaction at $E = Q\sin\theta = 35\,000$ N.

We can now draw the bending moment diagrams for *AB* and *CD*, considering only the forces at right-angles to each beam; let us take *CD* first. *CD* is a beam freely supported at *C* and *E* and loaded at *D*. The bending moment at $E = 3 \times 20\,000 = 60\,000$ Nm, to which value it rises uniformly from zero at *D*; from *E* to *C* the bending moment decreases uniformly to zero.

AB is supported at A and B and loaded with equal and opposite loads at C and F.

The bending moment at C is

$$(2)(20\,000) = 40\,000 \text{ Nm.}$$

The bending moment at F is

$$(2)(-20\,000) = -40\,000 \text{ Nm.}$$

At any point z between C and F, the bending moment is

$$M = 20\,000\,(z+2) - 46\,700z = 40\,000 - 26\,700z$$

In the bending moment diagram positive bending moments are those which make the beam concave to the left, and are plotted to the left in the figure.

7.13 More general case of bending of a curved bar

In Figure 7.17, OBC represents the centre line of a beam of any shape; the line OBC is curved in space in general. Suppose the beam carries any system of external loads; consider the actions over a section of the beam at B. For statical equilibrium of BC we require at B a force and a couple.

The force is resolved into two components—an axial force P along the centre line of the beam, and a shearing force F normal to the centre line; the couple is resolved into two components—a torque T about the centre line of the beam, and a bending moment M about an axis perpendicular to the centre line. The axis of M is not necessarily coincident with the axis of F.

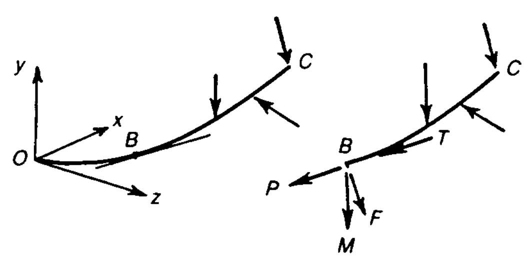
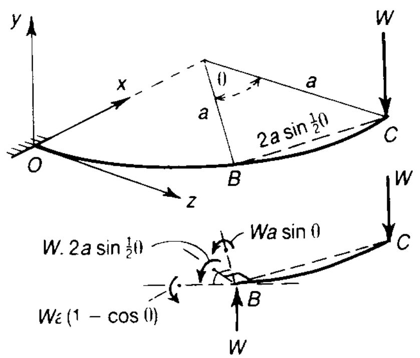


Fig. 7.17 Lateral loading of a curved beam.

Problem 7.7 The centre line of a beam is curved in the plane xz with a radius a. Find the loading actions at any section of the beam when a concentrated load W is applied at C in a direction parallel to yO.



Solution

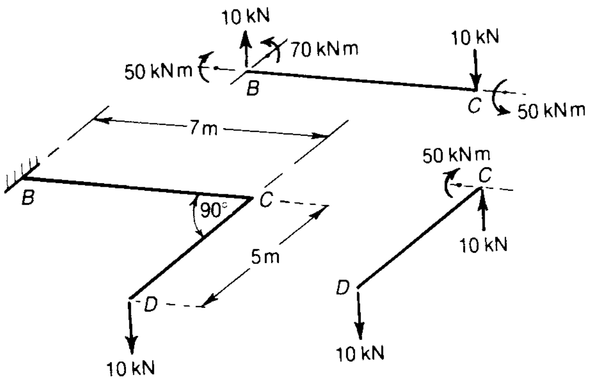
Consider any section at an angular position θ in the *xz*-plane; there is no axial force on the centre line, and the shearing force at any section is *W*. The torque about he centre line is

$$W(a - a\cos\theta) = Wa(1 - \cos\theta)$$

The bending moment acts about the radius, and has the value

$$Wa \sin\theta$$

Problem 7.8 The axis of a beam consists of two lines *BC* and *CD* in a horizontal plane and at right angles to each other. Estimate the greatest bending moment and torque when the beam carries a vertical load of 10 kN at *D*.



Solution

Consider the statical equilibrium of *DC* alone; there is no torque in *DC*, and the only internal actions at *C* in *DC* are a shearing force of 10 kN and a bending moment of 50 kNm. Now reverse the actions at C on DC and consider these reversed actions at C on BC. Equilibrium of BC is ensured if there is a shearing force of 10 kN at B, a bending moment of 70 kNm, and a torque of 50 kNm.

7.14 Rolling loads and influence lines

In the design of bridge girders it is frequently necessary to know the maximum bending moment and shearing force which each section will have to bear when a travelling load, such as a train, passes from one end of the bridge to the other. The diagrams which we have considered so far show the simultaneous values of the bending moment, or shearing force, for all sections of the beam with the loads in one fixed position; we shall now see how to construct a diagram which shows the greatest value of these quantities for all positions of the loads. These diagrams are called *maximum bending moment* or *maximum shearing force*, diagrams.

We assume that the loads on a beam are moving slowly; then there are negligible inertia effects from the mass of the beam and any moving masses.

7.15 A single concentrated load traversing a beam

Suppose a single concentrated vertical load W travels slowly along a beam BC, which is simply-supported at each end, Figure 7.18(i). If a is the distance of the load from B, the reactions at B and C are

$$R_B = \frac{W}{L}(L - a) \qquad R_C = \frac{Wa}{L}$$

The bending moment at a distance z from B, is

$$M = \frac{Wz}{L}(L - a) \text{ for } z < a \tag{7.26}$$

$$M = \frac{Wa}{L}(L - z) \text{ for } z > a \tag{7.27}$$

Consider the load rolling slowly from C to B: initially $z < a$, and the bending moment, given by equation 7.26, increases as a decreases; when $a = z$,

$$M = \frac{Wz}{L}(L - z) \tag{7.28}$$

As W proceeds further, we have $z > a$, and the bending moment, given by equation (7.27), decreases as a decreases further.

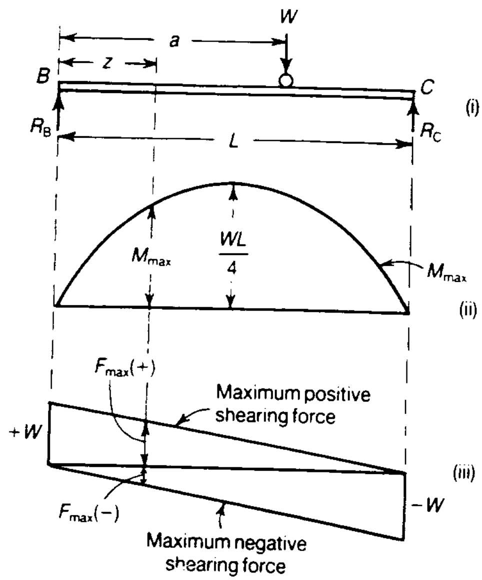


Figure 7.18 Bending moments and shearing forces due to a rolling load traversing a simply-supported beam.

Clearly, equation 7.28 is the greatest bending moment which can occur at the section; thus, for any section a distance z from B, the maximum bending moment that can be induced is

$$M_{max} = \frac{Wz}{L}(L - z) \tag{7.29}$$

and this occurs when the load W is at that section of the beam. The variation of M_{max} for different values of z is shown in Figure 7.18(ii); the curve of M_{max} is a parabola, attaining a peak value when $z = \frac{1}{2}L$, for which

$$M_{max} = \frac{WL}{4}$$

The shearing force a distance z from B is

$$F = R_B = \frac{W}{L}(L - a) \quad \text{for} \quad z < a \tag{7.30}$$

$$F = -R_C = -\frac{Wa}{L} \quad \text{for} \quad z > a \tag{7.31}$$

Consider again a load rolling slowly from C to B; initially $z < a$, and the shearing force, given by equation (7.30), is positive and increases as a diminishes. The greatest positive shearing force occurs just before the load W passes the section under consideration; it has the value

$$F_{max}(+) = \frac{W}{L}(L - z) \tag{7.32}$$

After the load has passed the section being considered, that is, when $z > a$, the shearing force, given by equation (7.31) is negative and decreases as a diminishes further. The greatest negative shearing force occurs when the load W has just passed the section at a distance z; it has the value

$$F_{max}(-) = -\frac{Wz}{L} \tag{7.33}$$

The variations of maximum positive and negative shearing forces are shown in Figure 7.18(iii).

7.16 Influence lines of bending moment and shearing force

A curve that shows the value of the bending moment at a given section of a beam, for all positions of a travelling load, is called the bending-moment *influence line* for that section; similarly, a curve that shows the shearing force at the section for all positions of the load is called the shearing force *influence line* for the section. The distinction between influence lines and maximum bending-moment (or shearing force) diagrams must be carefully noted: for a given load there will be only one maximum bending-moment diagram for the beam, but an infinite number of bending-moment influence lines, one for each section of the beam.

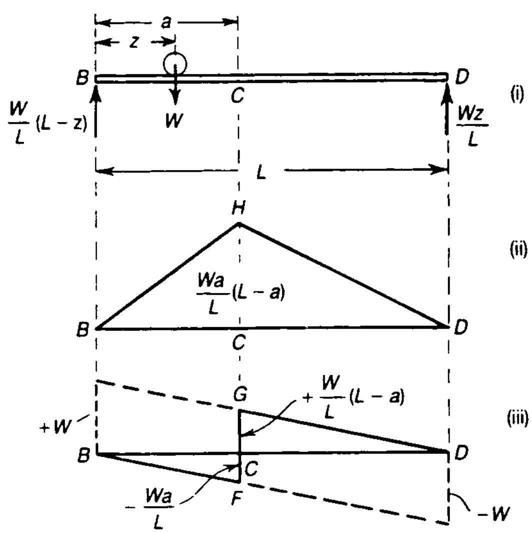


Figure 7.19 (i) Single rolling load on a simply-supported beam. (ii) Bending-moment influence line for section C. (iii) Shearing force influence line for Section C.

Consider a simply-supported beam, Figure 7.19, carrying a single concentrated load, W. As the load rolls across the beam, the bending moments at a section C of the beam vary with the position of the load. Suppose W is a distance z from B; then the bending moment at a section C is given by

$$M = \frac{Wz}{L}\left(L - a\right) \quad \text{for} \quad z < a$$

and

$$M = \frac{Wa}{L}\left(L - z\right) \quad \text{for} \quad z > a$$

The first of these equations gives the straight line BH in Figure 7.19(ii), and the second the line HD. The influence line for bending moments at C is then BHD; the bending moment is greatest when the load acts at the section.

Again, the shearing force at C is

$$F = -\frac{Wz}{L} \quad \text{for} \quad z < a$$

and

$$F = +\frac{W}{L}\left(L - z\right) \quad \text{for} \quad z > a$$

These relationships give the lines $BFCGD$ for the shearing force influence line for C. There is an abrupt change of shearing force as the load W crosses the section C.

Further problems *(answers on page 692)*

7.9 Draw the shearing-force and bending-moment diagrams for the following beams:

(i) A cantilever of length 20 m carrying a load of 10 kN at a distance of 15 m from the supported end.

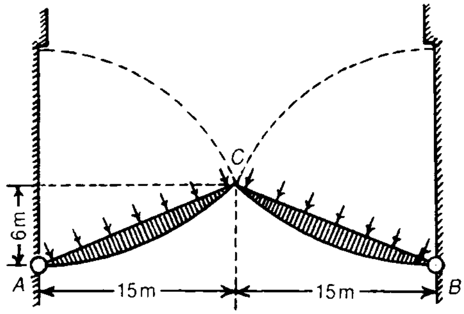
(ii) A cantilever of length 20 m carrying a load of 10 kN uniformly distributed over the inner 15 m of its length.

(iii) A cantilever of length 12 m carrying a load of 8 kN, applied 5 m from the supported end, and a load of 2kN/m over its whole length.

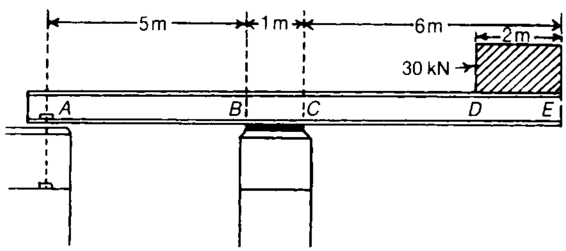
(iv) A beam, 20 m span, simply-supported at each end and carrying a vertical load of 20 kN at a distance 5 m from one support.

(v) A beam, 16 m span, simply-supported at each end and carrying a vertical load of 2.5 kN at a distance of 4 m from one support and the beam itself weighing 500 N per metre.

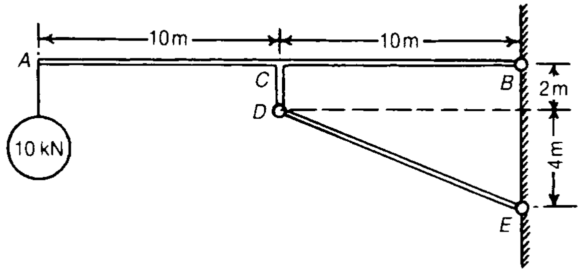
7.10 A pair of lock gates are strengthened by two girders AC and BC. If the load on each girder amounts to 15 kN per metre run, find the bending moment at the middle of each girder. *(Cambridge)*



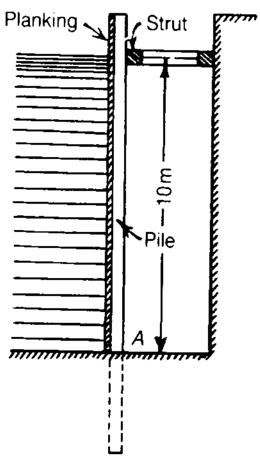
7.11 A girder $ABCDE$ bears on a wall for a length BC and is prevented from overturning by a holding-down bolt at A. The packing under BC is so arranged that the pressure over the bearing is uniformly distributed and the 30 kN load may also be taken as a uniformly distributed load. Neglecting the mass of the beam, draw its bending moment and shearing force diagrams. *(Cambridge)*



7.12 Draw the bending moment and shearing force diagrams for the beam shown. The beam is supported horizontally by the strut DE, hinged at one end to a wall, and at the other end to the projection CD which is firmly fixed at right angles to AB. The beam is freely hinged to the wall at B. The masses of the beam and strut can be neglected. *(Cambridge)*



7.13 A timber dam is made of planking backed by vertical piles. The piles are built-in at the section A where they enter the ground and they are supported by horizontal struts whose centre lines are 10 m above A. The piles are spaced 1 m apart between centres and the depth of water against the dam is 10 m.



Assuming that the thrust in the strut is two-sevenths the total water pressure resisted by each pile, sketch the form of the bending moment and shearing force diagrams for a pile. Determine the magnitude of the bending moment at A and the position of the section which is free from bending moment. *(Cambridge)*

7.14 The load distribution (full lines) and upward water thrust (dotted lines) for a ship are given, the numbers indicating kN per metre run. Draw the bending moment diagram for the ship. *(Cambridge)*