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Bridge Engineering Handbook.
Ed. Wai-Fah Chen and Lian Duan
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52

Effective Length of Compression Members

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52.1 Introduction *

The concept of *effective length factor* or *K factor* plays an important role in compression member design. Although great efforts have been made in the past years to eliminate the K factor in column design, K factors are still popularly used in practice for routine design [1].

Mathematically, the effective length factor or the *elastic K* factor is defined as

$$K = \sqrt{\frac{P_e}{P_{cr}}} = \sqrt{\frac{\pi^2 EI}{L^2 P_{cr}}} \tag{52.1}$$

where P_e is Euler load, elastic buckling load of a pin-ended column, P_{cr} is elastic buckling load of an end-restrained framed column, E is modulus of elasticity, I is moment of inertia in the flexural buckling plane, and L is unsupported length of column.

* Much of the material of this chapter was taken from Duan, L. and Chen, W. F., Chapter 17: Effective length factors of compression members, in *Handbook of Structural Engineering*, Chen, W. F., Ed., CRC Press, Boca Raton, FL, 1997.

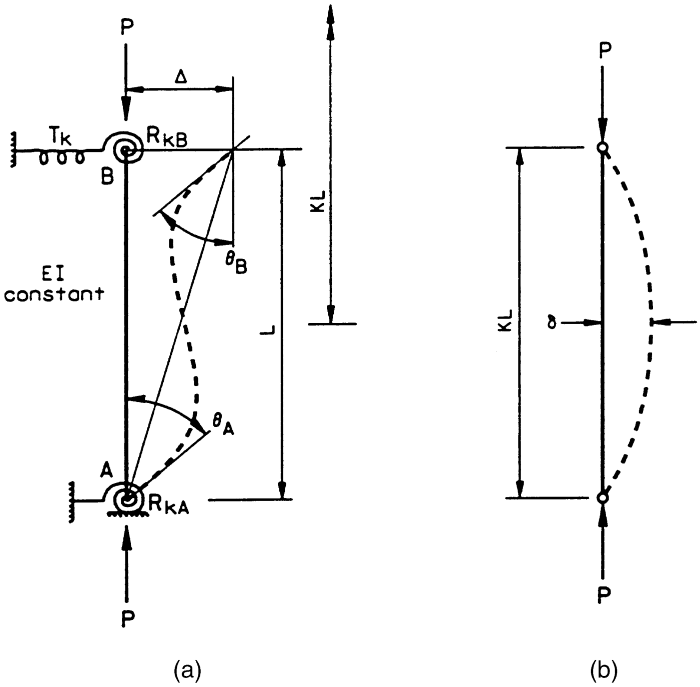

FIGURE 52.1 Isolated columns. (a) End-restrained columns; (b) pin-ended columns.

Physically, the K factor is a factor that, when multiplied by actual length of the end-restrained column (Figure 52.1a), gives the length of an equivalent pin-ended column (Figure 52.1b) whose buckling load is the same as that of the end-restrained column. It follows that the *effective length* KL of an end-restrained column is the length between adjacent inflection points of its pure flexural buckling shape.

Practically, design specifications provide the resistance equations for pin-ended columns, while the resistance of framed columns can be estimated through the K factor to the pin-ended column strength equations. Theoretical K factor is determined from an elastic eigenvalue analysis of the entire structural system, while practical methods for the K factor are based on an elastic eigenvalue analysis of selected subassemblages. This chapter presents the state-of-the-art engineering practice of the effective length factor for the design of columns in bridge structures.

52.2 Isolated Columns

From an eigenvalue analysis, the general K factor equation of an end-restrained column as shown in Figure 52.1 is obtained as

$$\det \begin{vmatrix} C + \dfrac{R_{kA}L}{EI} & S & -(C+S) \\ S & C + \dfrac{R_{kB}L}{EI} & -(C+S) \\ -(C+S) & -(C+S) & 2(C+S) - \left(\dfrac{\pi}{K}\right)^2 + \dfrac{T_k L^3}{EI} \end{vmatrix} = 0 \qquad (52.2)$$

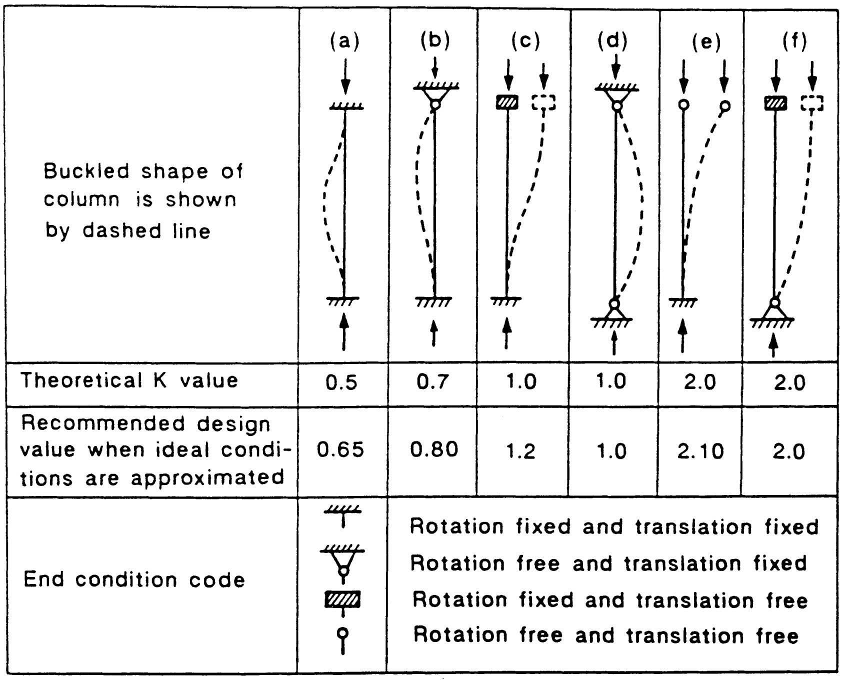

FIGURE 52.2 Theoretical and recommended K factors for isolated columns with idealized end conditions. (*Source:* American Institute of Steel Construction. *Load and Resistance Factor Design Specification for Structural Steel Buildings,* 2nd ed., Chicago, IL, 1993. With permission. Also from Johnston, B. G., Ed., Structural Stability Research Council, *Guide to Stability Design Criteria for Metal Structures,* 3rd ed., John Wiley & Sons, New York, 1976. With permission.)

where the stability function C and S are defined as

$$C = \frac{(\pi/K)\sin(\pi/K) - (\pi/K)^2\cos(\pi/K)}{2 - 2\cos(\pi/K) - (\pi/K)\sin(\pi/K)} \tag{52.3}$$

$$S = \frac{(\pi/K)^2 - (\pi/K)\sin(\pi/K)}{2 - 2\cos(\pi/K) - (\pi/K)\sin(\pi/K)} \tag{52.4}$$

The largest value of K satisfying Eq. (52.2) gives the elastic buckling load of an end-retrained column.

Figure 52.2 summarizes the theoretical K factors for columns with some idealized end conditions [2,3]. The recommended K factors are also shown in Figure 52.2 for practical design applications. Since actual column conditions seldom comply fully with idealized conditions used in buckling analysis, the recommended K factors are always equal or greater than their theoretical counterparts.

52.3 Framed Columns — Alignment Chart Method

In theory, the effective length factor K for any columns in a framed structure can be determined from a stability analysis of the entire structural analysis — eigenvalue analysis. Methods available for stability analysis include slope–deflection method [4], three-moment equation method [5], and energy methods [6]. In practice, however, such analysis is not practical, and simple models are often used to determine the effective length factors for farmed columns [7~10]. One such practical procedure that provides an approximate value of the elastic K factor is the alignment chart method [11]. This procedure has been adopted by the AASHTO [2] and AISC [3]. Specifications and the

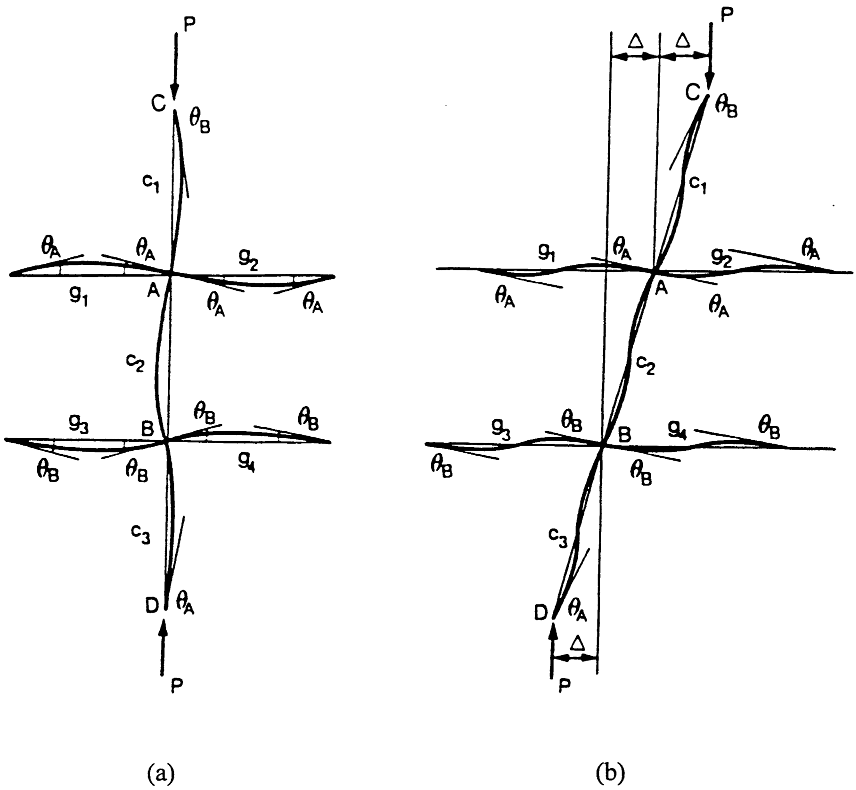

FIGURE 52.3 Subassemblage models for *K* factors of framed columns. (a) Braced frames; (b) unbraced frames.

ACI-318-95 Code [12], among others. At present, most engineers use the alignment chart method in lieu of an actual stability analysis.

52.3.1 Alignment Chart Method

The structural models employed for determination of *K* factors for framed columns in the alignment chart method are shown in Figure 52.3 The assumptions [2,4] used in these models are

1. All members have constant cross section and behave elastically.
2. Axial forces in the girders are negligible.
3. All joints are rigid.
4. For braced frames, the rotations at near and far ends of the girders are equal in magnitude and opposite in direction (i.e., girders are bent in single curvature).
5. For unbraced frames, the rotations at near and far ends of the girders are equal in magnitude and direction (i.e., girders are bent in double curvature).
6. The stiffness parameters $L\sqrt{P/EI}$, of all columns are equal.
7. All columns buckle simultaneously.

By using the slope–deflection equation method and stability functions, the effective length factor equations of framed columns are obtained as follows:

For columns in braced frames:

$$\frac{G_A G_B}{4}(\pi/K)^2+\left(\frac{G_A+G_B}{2}\right)\left(1-\frac{\pi/K}{\tan(\pi/K)}\right)+\frac{2\tan(\pi/2K)}{\pi/K}-1=0 \qquad (52.5)$$

For columns in unbraced frames:

$$\frac{G_A G_B (\pi / K)^2 - 36^2}{6(G_A + G_B)} - \frac{\pi / K}{\tan(\pi / K)} = 0 \tag{52.6}$$

where G is stiffness ratios of columns and girders, subscripts A and B refer to joints at the two ends of the column section being considered, and G is defined as

$$G = \frac{\sum (E_c I_c / L_c)}{\sum (E_g I_g / L_g)} \tag{52.7}$$

where Σ indicates a summation of all members rigidly connected to the joint and lying in the plane in which buckling of the column is being considered; subscripts c and g represent columns and girders, respectively.

Eqs. (52.5) and (52.6) can be expressed in form of alignment charts as shown in Figure 52.4. It is noted that for columns in braced frames, the range of K is $0.5 \leq K \leq 1.0$; for columns in unbraced frames, the range is $1.0 \leq K \leq \infty$. For column ends supported by but not rigidly connected to a footing or foundations, G is theoretically infinity, but, unless actually designed as a true friction-free pin, may be taken as 10 for practical design. If the column end is rigidly attached to a properly designed footing, G may be taken as 1.0.

Example 52.1

Given

A four-span reinforced concrete bridge is shown in Figure 52.5. Using the alignment chart, determine the K factor for Column DC. E = 25,000 MPa.

Section Properties are

Superstructure:	$I = 3.14\ (10^{12})\ \text{mm}^4$	$A = 5.86\ (10^6)\ \text{mm}^2$
Columns:	$I = 3.22\ (10^{11})\ \text{mm}^4$	$A = 2.01\ (10^6)\ \text{mm}^2$

Solution

1. **Calculate G factor for Column DC.**

$$G_D = \frac{\sum_D (E_c I_c / L_c)}{\sum_D (E_g I_g / L_g)} = \frac{3.22(10^{12}) / 12{,}000}{2(3.14)(10^{12}) / 55{,}000} = 0.235$$

$$G_B = 1.0 \quad \text{(Ref. [3])}$$

2. **From the alignment chart in Figure 52.4b, $K = 1.21$ is obtained**.

52.3.2 Requirements for Braced Frames

In stability design, one of the major decisions engineers have to make is the determination of whether a frame is braced or unbraced. The AISC-LRFD [3] states that a frame is braced when "lateral stability is provided by diagonal bracing, shear walls or equivalent means." However, there is no specific provision for the "amount of stiffness required to prevent sidesway buckling" in the AISC, AASHTO, and other specifications. In actual structures, a completely braced frame seldom exists.

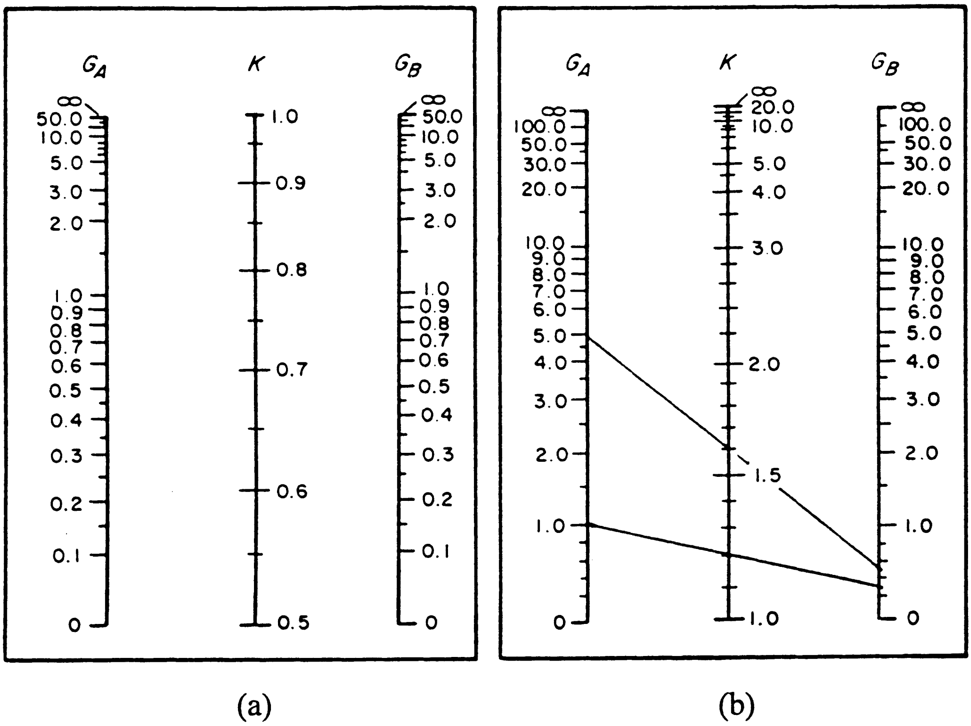

FIGURE 52.4 Alignment charts for effective length factors of framed columns. (a) Braced frames; (b) unbraced frames. (*Source:* American Institute of Steel Construction, *Load and Resistance Factor Design Specifications for Structural Steel Buildings,* 2nd ed., Chicago, IL, 1993. With permission. Also from Johnston, B. G., Ed., Structural Stability Research Council, *Guide to Stability Design Criteria for Metal Structures,* 3rd ed., John Wiley & Sons, New York, 1976. With permission.)

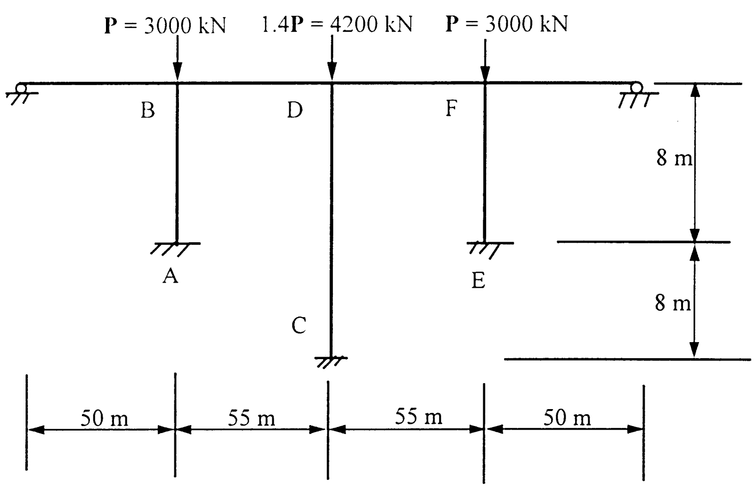

FIGURE 52.5 A four-span reinforced concrete bridge.

But in practice, some structures can be analyzed as braced frames as long as the lateral stiffness provided by bracing system is large enough. The following brief discussion may provide engineers with the tools to make engineering decisions regarding the basic requirements for a braced frame.

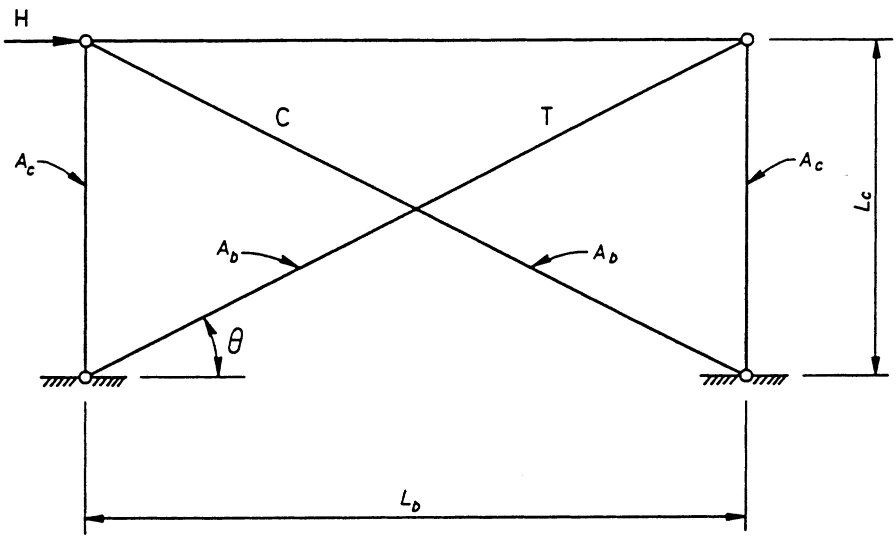

FIGURE 52.6 Diagonal cross-bracing system.

52.3.2.1 Lateral Stiffness Requirement

Galambos [13] presented a simple conservative procedure to estimate the minimum lateral stiffness provided by a bracing system so that the frame is considered braced.

$$\text{Required Lateral Stiffness} \quad T_k = \frac{\sum P_n}{L_c} \tag{52.8}$$

where $\sum$ represents summation of all columns in one story, P_n is nominal axial compression strength of column using the effective length factor $K = 1$, and L_c is unsupported length of the column.

52.3.2.2 Bracing Size Requirement

Galambos [13] employed Eq. (52.8) to a diagonal bracing (Figure 52.6) and obtained minimum requirements of diagonal bracing for a braced frame as

$$A_b = \frac{\left[1 + (L_b / L_c)^2\right]^{3/2} \sum P_n}{(L_b / L_c)^2 E} \tag{52.9}$$

where A_b is cross-sectional area of diagonal bracing and L_b is span length of beam.

A recent study by Aristizabal-Ochoa [14] indicates that the size of diagonal bracing required for a totally braced frame is about 4.9 and 5.1% of the column cross section for "rigid frame" and "simple farming," respectively, and increases with the moment inertia of the column, the beam span and with beam to column span ratio L_b/L_c.

52.3.3 Simplified Equations to Alignment Charts

52.3.3.1. Duan–King–Chen Equations

A graphical alignment chart determination of the K factor is easy to perform, while solving the chart Eqs. (52.5) and (52.6) always involves iteration. To achieve both accuracy and simplicity for design purpose, the following alternative K factor equations were proposed by Duan, King, and Chen [15].

For braced frames:

$$K = 1 - \frac{1}{5+9G_A} - \frac{1}{5+9G_B} - \frac{1}{10+G_A G_B} \tag{52.10}$$

For unbraced frames:

$$\text{For } K < 2 \qquad K = 4 - \frac{1}{1+0.2G_A} - \frac{1}{1+0.2G_B} - \frac{1}{1+0.01G_A G_B} \tag{52.11}$$

$$\text{For } K \geq 2 \qquad K = \frac{2\pi a}{0.9+\sqrt{0.81+4ab}} \tag{52.12}$$

where

$$a = \frac{G_A G_B}{G_A + G_B} + 3 \tag{52.13}$$

$$b = \frac{36}{G_A + G_B} + 6 \tag{52.14}$$

52.3.3.2 French Equations

For braced frames:

$$K = \frac{3G_A G_B + 1.4(G_A + G_B) + 0.64}{3G_A G_B + 2.0(G_A + G_B) + 1.28} \tag{52.15}$$

For unbraced frames:

$$K = \sqrt{\frac{1.6G_A G_B + 4.0(G_A + G_B) + 7.5}{G_A + G_B + 7.5}} \tag{52.16}$$

Eqs. (52.15) and (52.16) first appeared in the French Design Rules for Steel Structure [16] in 1966, and were later incorporated into the *European Recommendations for Steel Construction*[17]. They provide a good approximation to the alignment charts [18].

52.4 Modifications to Alignment Charts

In using the alignment charts in Figure 52.4 and Eqs. (52.5) and (52.6), engineers must always be aware of the assumptions used in the development of these charts. When actual structural conditions differ from these assumptions, unrealistic design may result [3,19,20]. SSRC Guide [19] provides methods enabling engineers to make simple modifications of the charts for some special conditions, such as, for example, unsymmetrical frames, column base conditions, girder far-end conditions, and flexible conditions. A procedure that can be used to account for far ends of restraining columns being hinged or fixed was proposed by Duan and Chen [21~23], and Essa [24]. Consideration of effects of material inelasticity on the K factor for steel members was developed originally by Yura

[25] and expanded by Disque [26]. LeMessurier [27] presented an overview of unbraced frames with or without leaning columns. An approximate procedure is also suggested by AISC-LRFD [3]. Several commonly used modifications for bridge columns are summarized in this section.

52.4.1 Different Restraining Girder End Conditions

When the end conditions of restraining girders are not rigidly jointed to columns, the girder stiffness (I_g/L_g) used in the calculation of G factor in Eq. (52.7) should be multiplied by a modification factor α_k given below:

For a braced frame:

$$\alpha_k = \begin{cases} 1.0 & \text{rigid far end} \\ 2.0 & \text{fixed far end} \\ 1.5 & \text{hinged far end} \end{cases} \tag{52.17}$$

For a unbraced frame:

$$\alpha_k = \begin{cases} 1.0 & \text{rigid far end} \\ 2/3 & \text{fixed far end} \\ 0.5 & \text{hinged far end} \end{cases} \tag{52.18}$$

52.4.2 Consideration of Partial Column Base Fixity

In computing the K factor for monolithic connections, it is important to evaluate properly the degree of fixity in foundation. The following two approaches can be used to account for foundation fixity.

52.4.2.1. Fictitious Restraining Beam Approach

Galambos [28] proposed that the effect of partial base fixity can be modeled as a fictitious beam. The approximate expression for the stiffness of the fictitious beam accounting for rotation of foundation in the soil has the form:

$$\frac{I_s}{L_B} = \frac{q\,BH^3}{72\,E_{\text{steel}}} \tag{52.19}$$

where q is modulus of subgrade reaction (varies from 50 to 400 lb/in.^{3}, 0.014 to 0.109 N/mm^3); B and H are width and length (in bending plane) of foundation, and E_{steel} is modulus of elasticity of steel.

Based on Salmon et al. [29] studies, the approximate expression for the stiffness of the fictitious beam accounting for the rotations between column ends and footing due to deformation of base plate, anchor bolts, and concrete can be written as

$$\frac{I_s}{L_B} = \frac{b\,d^2}{72\,E_{\text{steel}}/E_{\text{concrete}}} \tag{52.20}$$

where b and d are width and length of the base plate, subscripts concrete and steel represent concrete and steel, respectively. Galambos [28] suggested that the smaller of the stiffness calculated by Eqs. (52.25) and (52.26) be used in determining K factors.

52.4.2.2 AASHTO-LRFD Approach

The following values are suggested by AASHTO-LRFD [2]:

$G = 1.5$ footing anchored on rock
$G = 3.0$ footing not anchored on rock
$G = 5.0$ footing on soil
$G = 1.0$ footing on multiple rows of end bearing piles

Example 52.2

Given
Determine K factor for the Column AB as shown in Figure 52.5 by using the alignment chart with the necessary modifications. Section and material properties are given in Example 52.1 and spread footings are on soil.

Solution

1. **Calculate G factor with Modification for Column AB.**
   Since the far end of restraining girders are hinged, girder stiffness should be multiplied by 0.5. Using section properties in Example 52.1, we obtain:

$$G_B = \frac{\sum_B (E_c I_c / L_c)}{\sum_B \alpha_k (E_g I_g / L_g)}$$

$$= \frac{3.22(10^{12}) / 8,000}{(3.14)(10^{12}) / 55,000 + 0.5(3.14)(10^{12}) / 50,000} = 0.454$$

$$G \;\; = 5.0_A \quad (\text{Ref. [2]})$$

2. **From the alignment chart in Figure 52.4b, $K = 1.60$ is obtained.**

52.4.3 Column Restrained by Tapered Rectangular Girders

A modification factor α_T was developed by King et al. [30] for those framed columns restrained by tapered rectangular girders with different far-end conditions. The following modified G factor is introduced in connection with the use of alignment charts:

$$G = \frac{\sum (E_c I_c / L_c)}{\sum \alpha_T (E_g I_g / L_g)} \tag{52.21}$$

where I_g is moment of inertia of the girder at the near end. Both closed-form and approximate solutions for modification factor α_T were derived. It is found that the following two-parameter power-function can describe the closed-form solutions very well:

$$\alpha_T = \alpha_k (1 - r)^\beta \tag{52.22}$$

in which the parameter α_k is a constant (Eqs. 52.17 and 52.18) depending on the far-end conditions, and β is a function of far-end conditions and tapering factor a and r as defined in Figure 52.7.

1. For a linearly tapered rectangular girder (Figure 52.7a):

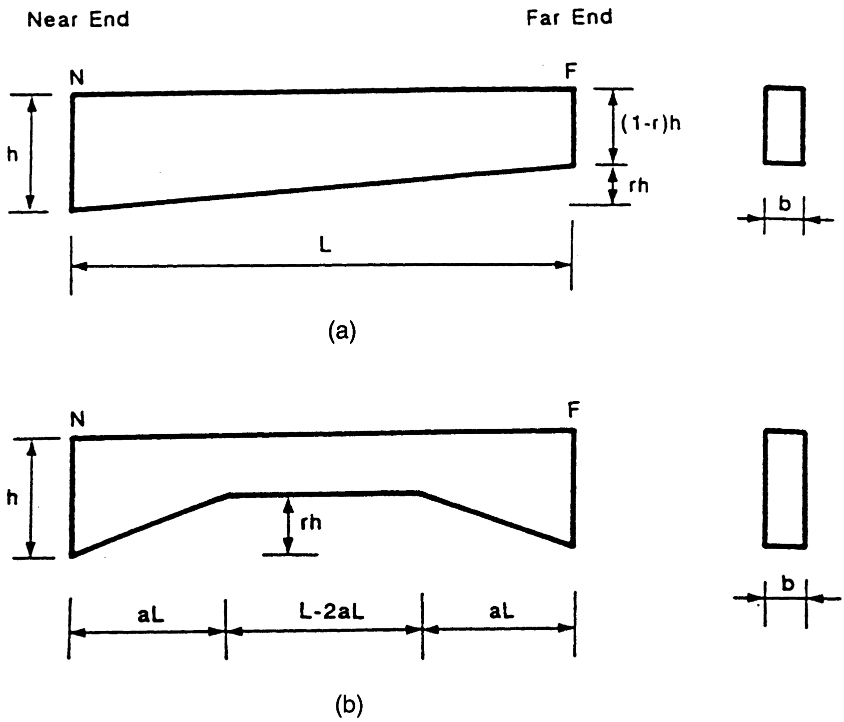

FIGURE 52.7 Tapered rectangular girders. (a) Linearly tapered girder. (b) symmetrically tapered girder.

For a braced frame:

$$\beta = \begin{cases} 0.02 + 0.4r & \text{rigid far end} \\ 0.75 - 0.1r & \text{fixed far end} \\ 0.75 - 0.1r & \text{hinged far end} \end{cases} \tag{52.23}$$

For an unbraced frame:

$$\beta = \begin{cases} 0.95 & \text{rigid far end} \\ 0.70 & \text{fixed far end} \\ 0.70 & \text{hinged far end} \end{cases} \tag{52.24}$$

2. For a symmetrically tapered rectangular girder (Figure 52.7b)
For a braced frame:

$$\beta = \begin{cases} 3 - 1.7a^2 - 2a & \text{rigid far end} \\ 3 + 2.5a^2 - 5.55a & \text{fixed far end} \\ 3 - a^2 - 2.7a & \text{hinged far end} \end{cases} \tag{52.25}$$

For an unbraced frame:

$$\beta = \begin{cases} 3 + 3.8a^2 - 6.5a & \text{rigid far end} \\ 3 + 2.3a^2 - 5.45a & \text{fixed far end} \\ 3 - 0.3a & \text{hinged far end} \end{cases} \tag{52.26}$$

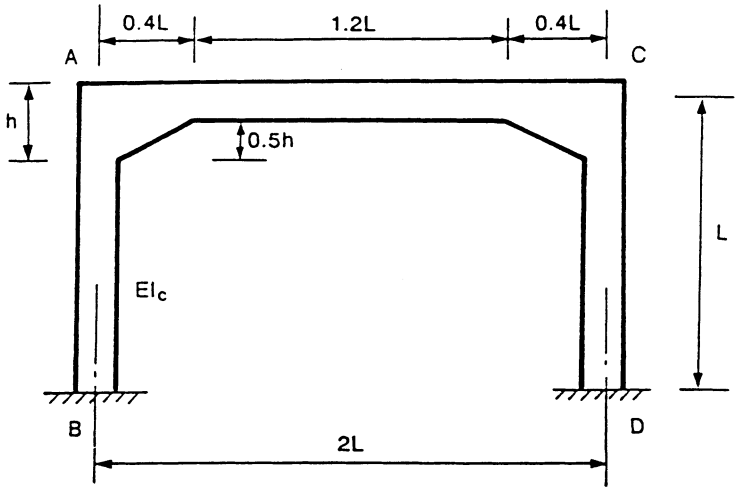

FIGURE 52.8 A simple frame with rectangular sections.

Example 52.3

Given

A one-story frame with a symmetrically tapered rectangular girder is shown in Figure 52.8. Assuming $r = 0.5$, $a = 0.2$, and $I_g = 2I_c = 2I$, determine K factor for Column AB.

Solution

1. **Use the Alignment Chart with Modification**

   For joint A, since the far end of girder is rigid, use Eqs. (52.26) and (52.22)

$$\beta = 3 + 3.8(0.2)^2 - 6.5(0.2) = 1.852$$

$$\alpha_T = (1 - 0.5)^{1.852} = 0.277$$

$$G_A = \frac{\sum E_c I_c / L_c}{\sum \alpha_T E_g I_g / L_g} = \frac{EI/L}{0.277\ E(2I)/2L} = 3.61$$

$$G_B = 1.0 \quad \text{(Ref. [3])}$$

   From the alignment chart in Figure 52.4b, $K = 1.59$ is obtained

2. **Use the Alignment Chart without Modification**

   A direct use of Eq. (52.7) with an average section ($0.75h$) results in

$$I_g = 0.75^3\,(2I) = 0.844I$$

$$G_A = \frac{EI/L}{0.844\,EI/2L} = 2.37$$

$$G_B = 1.0$$

   From the alignment chart in Figure 52.4b, $K = 1.50$, *or* $(1.50 - 1.59)/1.59 = -6\%$ in error on the less conservative side.

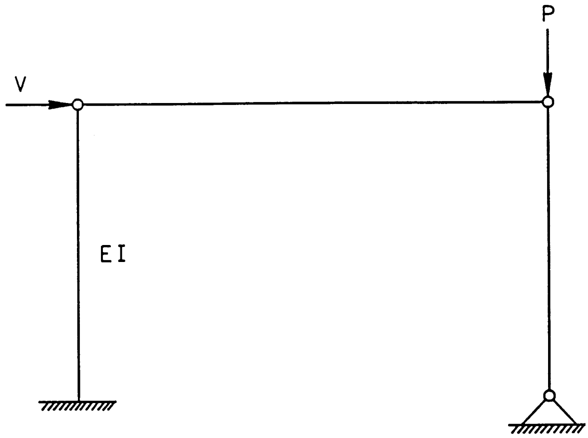

FIGURE 52.9 Subassemblage of LeMessurier method.

52.5 Framed Columns — Alternative Methods

52.5.1 LeMessurier Method

Considering that all columns in a story buckle simultaneously and strong columns will brace weak columns (Figure 52.9), a more accurate approach to calculate K factors for columns in a side-sway frame was developed by LeMessurier [27]. The K_i value for the ith column in a story can be obtained by the following expression:

$$K_i = \sqrt{\frac{\pi^2 EI_i}{L_i^2 P_i}\left(\frac{\sum P + \sum C_L P}{\sum P_L}\right)} \tag{52.27}$$

where P_i is axial compressive force for member i, and subscript i represents the ith column and ΣP is the sum of axial force of all columns in a story.

$$P_L = \frac{\beta EI}{L^2} \tag{52.28}$$

$$\beta = \frac{6(G_A + G_B) + 36}{2(G_A + G_B) + G_A G_B + 3} \tag{52.29}$$

$$C_L = \left(\beta \frac{K_o^2}{\pi^2} - 1\right) \tag{52.30}$$

in which K_o is the effective length factor obtained by the alignment chart for unbraced frames and P_L is only for those columns that provide side-sway stiffness.

Example 52.4

Given

Determine K factors for bridge columns shown in Figure 52.5 by using the LeMessurier method. Section and material properties are given in Example 52.1.

TABLE 52.1 Example 52.4 — Detailed Calculations by LeMessurier Method

Members	AB and EF	CD	Sum	Notes
I (mm^4 × 10^{11})	3.217	3.217	—	
L (mm)	8,000	12,000	—	
G_{top}	0.454	0.235	—	Eq. (52.7)
G_{bottom}	0.0	0.0	—	Eq. (52.7)
β	9.91	10.78	—	Eq. (52.29)
K_{io}	1.082	1.045	—	Alignment chart
C_L	0.176	0.193	—	Eq. (52.30)
P_L	50,813E	24,083E	123,709E	Eq. (52.28)
P	P	1.4P	3.4P	P = 3,000 kN
C_LP	0.176P	0.270P	0.622P	P = 3,000 kN

Solutions

The detailed calculations are listed in Table 52.1 By using Eq. (52.32), we obtain:

$$K_{AB} = \sqrt{\frac{\pi^2 EI_{AB}}{L_{AB}^2 P_{AB}}\left(\frac{\sum P + \sum C_L P}{\sum P_L}\right)}$$

$$= \sqrt{\frac{\pi^2 E(3.217)(10^{11})}{(8.000)^2 (P)}\left(\frac{3.4P + 0.622P}{123,709\,E}\right)} = 1.270$$

$$K_{CD} = \sqrt{\frac{\pi^2 EI_{CD}}{L_{CD}^2 P_{CD}}\left(\frac{\sum P + \sum C_L P}{\sum P_L}\right)}$$

$$= \sqrt{\frac{\pi^2 E(3.217)(10^{11})}{(12,000)^2 (1.4P)}\left(\frac{3.4P + 0.622P}{123,709\,E}\right)} = 0.715$$

52.5.2 Lui Method

A simple and straightforward approach for determining the effective length factors for framed columns without the use of alignment charts and other charts was proposed by Lui [31]. The formulas take into account both the member instability and frame instability effects explicitly. The K factor for the ith column in a story was obtained in a simple form:

$$K_i = \sqrt{\left(\frac{\pi^2 E\, I_i}{P_i\, L_i^2}\right)\left[\left(\sum \frac{P}{L}\right)\left(\frac{1}{5\sum \eta} + \frac{\Delta_1}{\sum H}\right)\right]} \tag{52.31}$$

where $\Sigma(P/L)$ represents the sum of axial-force-to-length ratio of all members in a story; ΣH is the story lateral load producing Δ_1, Δ_1 is the first-order interstory deflection; η is member stiffness index and can be calculated by

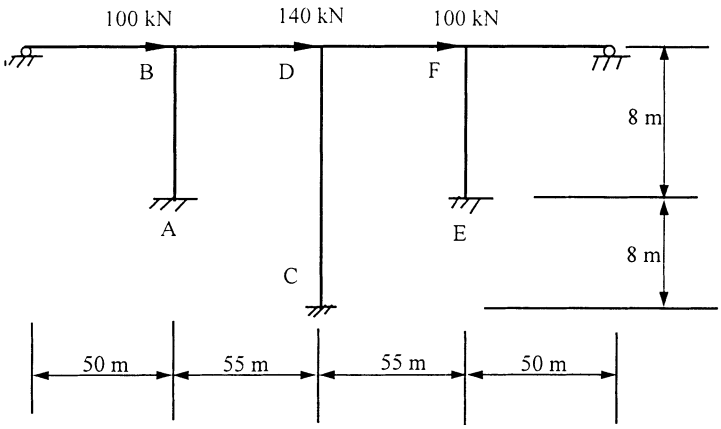

FIGURE 52.10 A bridge structure subjected to fictitious lateral loads.

TABLE 52.2 Example 52.5 — Detailed Calculations by Lui Method

Members	AB and EF	CD	Sum	Notes
I (mm^4 × 10^{11})	3.217	3.217	—	
L (mm)	8,000	12,000	—	
H (kN)	150	210	510	
Δ_I (mm)	0.00144	0.00146	—	
$\Delta_I/\Sigma H$ (mm/kN)	—	—	2.843 (10^{-6})	Average
M_{top} (kN-m)	–476.9	–785.5	—	
M_{bottom} (kN-m)	–483.3	–934.4	—	
m	0.986	0.841	—	
η (kN/mm)	185,606	46,577	417,789	Eq. (52.32)
P/L (kN/mm)	P/8,000	1.4 P/12,000	1.1P/3,000	P = 3,000 kN

$$\eta = \frac{(3 + 4.8m + 4.2m^2)EI}{L^3} \tag{52.32}$$

in which m is the ratio of the smaller to larger end moments of the member; it is taken as positive if the member bends in reverse curvature, and negative for single curvature.

It is important to note that the term ΣH used in Eq. (52.36) is not the actual applied lateral load. Rather, it is a small disturbing or fictitious force (taken as a fraction of the story gravity loads) to be applied to each story of the frame. This fictitious force is applied in a direction such that the deformed configuration of the frame will resemble its buckled shape.

Example 52.5

Given

Determine the K factors for bridge columns shown in Figure 52.5 by using the Lui method. Section and material properties are given in Example 52.1.

Solutions

Apply fictitious lateral forces at B, D, and F (Figure 52.10) and perform a first-order analysis. Detailed calculation is shown in Table 52.2.

By using Eq. (52.31), we obtain

$$K_{AB} = \sqrt{\left(\frac{\pi^2 E I_{AB}}{P_{AB} L_{AB}^2}\right)\left[\left(\sum \frac{P}{L}\right)\left(\frac{1}{5\sum \eta} + \frac{\Delta_1}{\sum H}\right)\right]}$$

$$= \sqrt{\left(\frac{\pi^2 (25,000)(3.217)(10^{11})}{P(8,000)^2}\right)\left[\left(\frac{1.1P}{3,000}\right)\left(\frac{1}{5(417,789)} + 2.843(10^{-6})\right)\right]}$$

$$= 1.229$$

$$K_{CD} = \sqrt{\left(\frac{\pi^2 E I_{CD}}{P_{CD} L_{CD}^2}\right)\left[\left(\sum \frac{P}{L}\right)\left(\frac{1}{5\sum \eta} + \frac{\Delta_1}{\sum H}\right)\right]}$$

$$= \sqrt{\left(\frac{\pi^2 (25,000)(3.217)(10^{11})}{1.4P(12,000)^2}\right)\left[\left(\frac{1.1P}{3,000}\right)\left(\frac{1}{5(417,789)} + 2.843(10^{-6})\right)\right]}$$

$$= 0.693$$

52.5.3 Remarks

For a comparison, Table 52.3 summarizes the K factors for the bridge columns shown in Figure 52.5 obtained from the alignment chart, LeMessurier and Lui methods, as well as an eigenvalue analysis. It is seen that errors of alignment chart results are rather significant in this case. Although the K factors predicted by Lui's formulas and LeMessurier's formulas are almost the same in most cases, the simplicity and independence of any chart in the case of Lui's formula make it more desirable for design office use [32].

TABLE 52.3 Comparison of K Factors for Frame in Figure 52.5

Columns	Theoretical	Alignment Chart	Lui Eq. (52.31)	LeMessurier Eq. (52.27)
AB	1.232	1.082	1.229	1.270
CD	0.694	1.045	0.693	0.715

52.6 Crossing Bracing Systems

Picard and Beaulieu [33,34] reported theoretical and experimental studies on double diagonal cross-bracings (Figure 52.6) and found that

1. A general effective length factor equation is given as

$$K = \sqrt{0.523 - \frac{0.428}{C/T}} \geq 0.50 \tag{52.33}$$

where C and T represent compression and tension forces obtained from an elastic analysis, respectively.

2. When the double diagonals are continuous and attached at an intersection point, the *effective length* of the compression diagonal is 0.5 times the diagonal length, i.e., $K = 0.5$, because the C/T ratio is usually smaller than 1.6.

El-Tayem and Goel [35] reported a theoretical and experimental study about the X-bracing system made from single equal-leg angles. They concluded that

1. Design of X-bracing system should be based on an exclusive consideration of one half diagonal only.
2. For X-bracing systems made from single equal-leg angles, an effective length of 0.85 times the half-diagonal length is reasonable, i.e., $K = 0.425$.

52.7 Latticed and Built-Up Members

It is a common practice that when a buckling model involves relative deformation produced by shear forces in the connectors, such as lacing bars and batten plates, between individual components, a modified effective length factor K_m or effective slenderness ratio $(KL/r)_m$ is used in determining the compressive strength. K_m is defined as

$$K_m = \alpha_v K \tag{52.34}$$

in which K is the usual effective length factor of a latticed member acting as a unit obtained from a structural analysis; and α_v is the shear factor to account for the effect of shear deformation on the buckling strength, Details of the development of the shear factor α_v can be found in textbooks by Bleich [5] and Timoshenko and Gere [36]. The following section briefly summarizes α_v formulas for various latticed members.

52.7.1 Latticed Members

By considering the effect of shear deformation in the latticed panel on buckling load, shear factor α_v of the following form has been introduced:

Laced Compression Members (Figures 52.11a and b)

$$\alpha_v = \sqrt{1 + \frac{\pi^2 EI}{(KL)^2} \frac{d^3}{A_d E_d a b^2}} \tag{52.35}$$

Compression Members with Battens (Figure 52.11c)

$$\alpha_v = \sqrt{1 + \frac{\pi^2 EI}{(KL)^2} \left(\frac{ab}{12 E_b I_b} + \frac{a^2}{24 EI_f} \right)} \tag{52.36}$$

Laced-Battened Compression Members (Figure 52.11d)

$$\alpha_v = \sqrt{1 + \frac{\pi^2 E I}{(KL)^2} \left(\frac{d^3}{A_d E_d a b^2} + \frac{b}{a A_b E_b} \right)} \tag{52.37}$$

Compression Members with Perforated Cover Plates (Figure 52.11e)

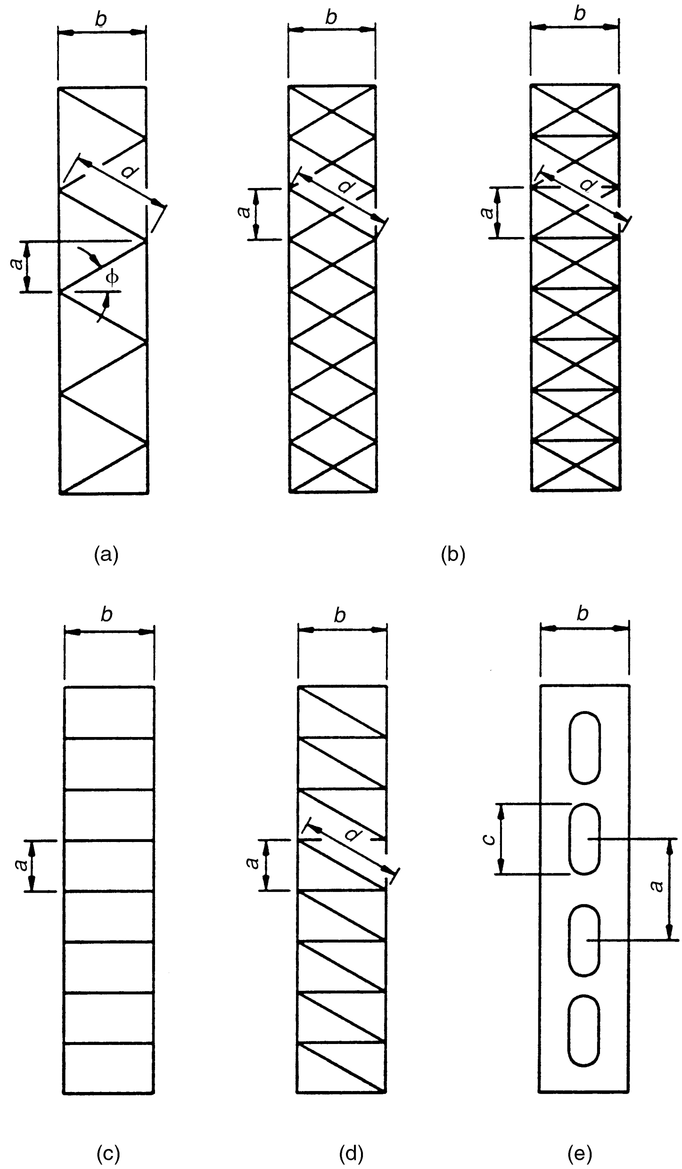

FIGURE 52.11 Typical configurations of latticed members. (a) Single lacing; (b) double lacing; (c) battens; (d) lacing-battens; (e) perforated cover plates.

$$\alpha_v = \sqrt{1 + \frac{\pi^2 EI}{(KL)^2}\left(\frac{9c^3}{64aEI_f}\right)} \tag{52.38}$$

where E_d is modulus of elasticity of materials for lacing bars; E_b is modulus of elasticity of materials for batten plates; A_d is cross-sectional area of all diagonals in one panel; I_b is moment inertia of all battens in one panel in the buckling plane, and I_f is moment inertia of one side of main components taken about the centroid axis of the flange in the buckling plane; a, b, d are height of panel, depth of member, and length of diagonal, respectively; and c is the length of a perforation.

The Structural Stability Research Council [37] suggested that a conservative estimating of the influence of 60° or 45° lacing, as generally specified in bridge design practice, can be made by modifying the overall effective length factor K by multiplying a factor α_v, originally developed by Bleich [5] as follows:

$$\text{For } \frac{KL}{r} > 40, \quad \alpha_v = \sqrt{1 + 300/(KL/r)^2} \tag{52.39}$$

$$\text{For } \frac{KL}{r} \leq 40, \quad \alpha_v = 1.1 \tag{52.40}$$

It should be pointed out that the usual K factor based on a solid member analysis is included in Eqs. (52.35) through (52.38). However, since the latticed members studied previously have pin-ended conditions, the K factor of the member in the frame was not included in the second terms of the square root of the above equations in their original derivations [5,36].

52.7.5 Built-Up Members

AISC-LRFD [3] specifies that if the buckling of a built-up member produces shear forces in the connectors between individual component members, the usual slenderness ratio KL/r for compression members must be replaced by the modified slenderness ratio $(KL/r)_m$ in determining the compressive strength.

1. *For snug-tight bolted connectors*:

$$\left(\frac{KL}{r}\right)_m = \sqrt{\left(\frac{KL}{r}\right)_o^2 + \left(\frac{a}{r_i}\right)^2} \tag{52.41}$$

2. *For welded connectors and for fully tightened bolted connectors*:

$$\left(\frac{KL}{r}\right)_m = \sqrt{\left(\frac{KL}{r}\right)_o^2 + 0.82\frac{\alpha^2}{(1+\alpha^2)}\left(\frac{a}{r_{ib}}\right)^2} \tag{52.42}$$

where $(KL/r)_o$ is the slenderness ratio of built-up member acting as a unit, $(KL/r)_m$ is modified slenderness ratio of built-up member, a/r_i is the largest slenderness ratio of the individual components, a/r_{ib} is the slenderness ratio of the individual components relative to its centroidal axis parallel to axis of buckling, a is the distance between connectors, r_i is the minimum radius of gyration of individual components, r_{ib} is the radius of gyration of individual components relative to its centroidal axis parallel to member axis of buckling, α is the separation ratio = $h/2r_{ib}$, and h is the distance between centroids of individual components perpendicular to the member axis of buckling.

Eq. (52.41) is the same as that used in the current Italian code, as well as in other European specifications, based on test results [38]. In this equation, the bending effect is considered in the first term in square root, and shear force effect is taken into account in the second term. Eq. (52.42) was derived from elastic stability theory and verified by test data [39]. In both cases, the end connectors must be welded or slip-critical-bolted.

52.8 Tapered Columns

The state-of-the-art design for tapered structural members was provided in the SSRC guide [37]. The charts as shown in Figure 52.12 can be used to evaluate the effective length factors for tapered

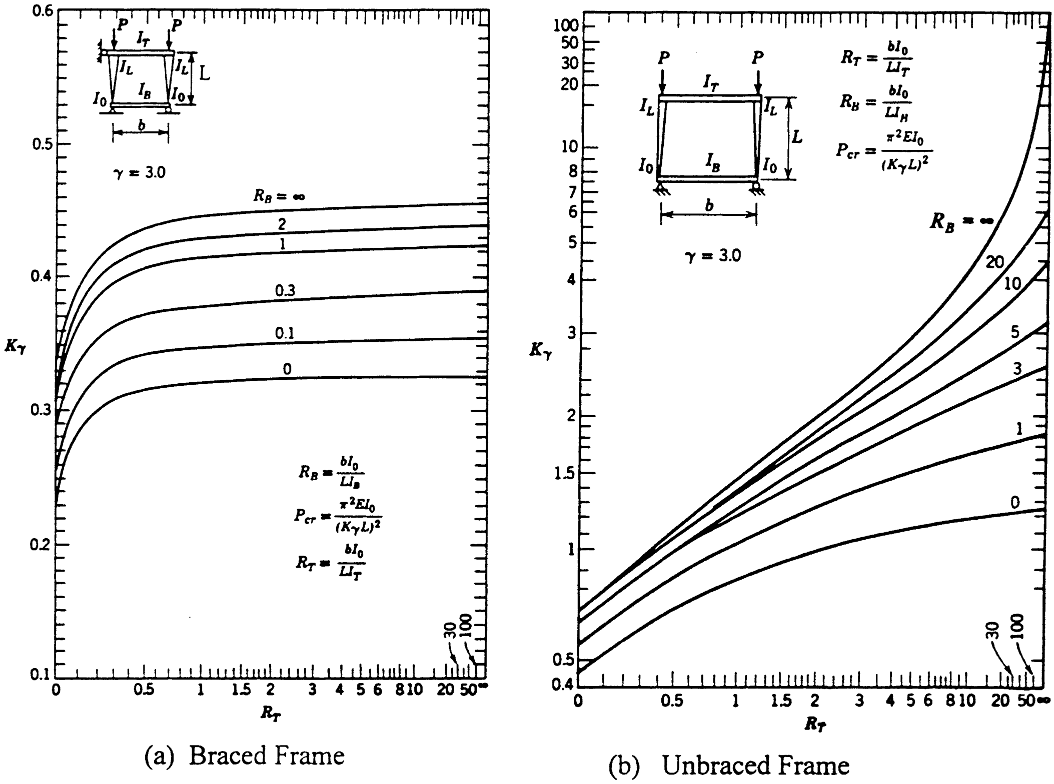

FIGURE 52.12 Effective length factor for tapered columns. (a) Braced frame; (b) unbraced frame. (*Source:* Galambos, T. V., Ed., Structural Stability Research Council Guide to Stability Design Criteria for Metal Structures, 4th ed., John Wiley & Sons, New York, 1988. With permission.)

column restrained by prismatic beams [37]. In these figures, I_T and I_B are the moment of inertia of top and bottom beam, respectively; b and L are length of beam and column, respectively; and γ is tapering factor as defined by

$$\gamma = \frac{d_1 - d_o}{d_o} \tag{52.43}$$

where d_o and d_1 are the section depth of column at the smaller and larger end, respectively.

52.9 Summary

This chapter summarizes the state-of-the-art practice of the effective length factors for isolated columns, framed columns, diagonal bracing systems, latticed and built-up members, and tapered columns. Design implementation with formulas, charts, tables, and various modification factors adopted in current codes and specifications, as well as those used in bridge structures, are described. Several examples are given to illustrate the steps of practical applications of these methods.

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